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# Direct-sum decompositions of modules with semilocal endomorphism rings

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**Abstract** According to the classical Krull–Schmidt Theorem, any module of finite composition length decomposes as a direct sum of indecomposable modules in an essentially unique way, that is, unique up to isomorphism of the indecomposable summands and a permutation of the summands. Modules that do not have finite composition length can have completely different behaviors. In this survey, we consider in particular the case of the modules $M_R$ whose endomorphism ring $E := \mathrm{End}(M_R)$ is a semilocal ring, that is, $E/J(E)$ is a semisimple artinian ring. For instance, modules of finite composition length have a semilocal endomorphism ring, but several other classes of modules also have a semilocal endomorphism ring, for example artinian modules, finite direct sums of uniserial modules, finitely generated modules over commutative semilocal rings, and finitely presented modules over arbitrary semilocal rings. Several interesting phenomena appear in these cases. For instance, modules with a semilocal endomorphism ring have very regular direct-sum decompositions into indecomposables, their direct summands can be described via lattices, and direct-sum decompositions into indecomposables (=uniserial submodules) of finite direct sums of

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uniserial modules are described via their monogeny classes and their epigeny classes up to two permutations of the factors.

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## 1 Introduction, historical background, motivations

In this survey, rings will be associative rings $R$ with an identity, and modules will be unital right $R$-modules, unless otherwise stated.

We want to study direct-sum decompositions of a module $M_R$ into finitely many direct summands $M_1, M_2, \dots, M_t$:

$$M_R = M_1 \oplus M_2 \oplus \cdots \oplus M_t.$$

According to the *"Classical Krull–Schmidt Theorem",* if the module $M_R$ is of finite composition length (that is, satisfies both the ascending chain condition and the descending chain condition on submodules), then $M_R$ is a direct sum of indecomposable modules in an essentially unique way. Here "essentially unique" means that if

$$M_R = M_1 \oplus M_2 \oplus \cdots \oplus M_t = N_1 \oplus N_2 \oplus \cdots \oplus N_s$$

are two direct-sum decompositions of $M_R$ as direct sums of (necessarily finitely many) indecomposable direct summands $M_1, M_2, \dots, M_t, N_1, N_2, \dots, N_s$, then $t = s$ and there exists a permutation $\sigma$ of the indices $1, 2, \dots, t$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \dots, t$. Arbitrary modules do not have this property in general. There are modules that are not direct sums of indecomposable modules, or modules that are direct sums of indecomposable modules in essentially different ways (see Bergman and Dicks' Theorem 2.1). The easiest example of "non-uniqueness" is probably the following. Let $R$ be a commutative integral domain with at least two distinct maximal ideals $M$ and $N$ that are not principal ideals. Then the morphism $M \oplus N \to R, (x, y) \mapsto x + y$, is an $R$-module epimorphism, which necessarily splits because $R$ is a projective $R$-module. The kernel of this morphism is isomorphic to $M \cap N$, so that we have a splitting short exact sequence $0 \to M \cap N \to M \oplus N \to R \to 0$. Thus $M \oplus N \cong R \oplus (M \cap N)$. Since $M$ and $N$ are not principal ideals, they are not isomorphic to $R$. Thus the two direct-sum decompositions $M \oplus N \cong R \oplus (M \cap N)$ are essentially different. In this example, $R$ is a commutative integral domain, hence $R$ is an $R$-module of Goldie dimension 1. Thus $R$ and its submodules $M$, $N$ and $M \cap N$ are modules of Goldie dimension 1. This proves that the two essentially different direct-sum decompositions $M \oplus N \cong R \oplus (M \cap N)$ are direct-sum decompositions into indecomposable modules.

The origins of the Krull–Schmidt Theorem can be dated back to 1879, when Frobenius and Stickelberger [47] proved that any finite abelian group is a direct sum of cyclic groups whose orders are powers of prime, and this powers of primes are uniquely determined by the group. Their result was then generalized in different directions to various algebraic structures, as follows.

## 1.1 Groups

A first generalization to finite non-commutative groups is due to the Scottish mathematician Maclagan-Wedderburn [66] (he mentions some credit is due to G.A. Miller.). Wedderburn's result can be stated by saying that if a finite group $G$ has two direct-product decompositions $G = G_1 \times G_2 \times \cdots \times G_t = H_1 \times H_2 \times \cdots \times H_s$, then $t = s$ and there is an automorphism $\varphi$ of $G$ such that $\varphi(G_i) = H_{\sigma(i)}$ for all $i$'s. Remak (1911) proved that $\varphi$ can be chosen a central automorphism. The Soviet mathematician Schmidt [79] then simplified and improved Remark's results.

The present form of the Krull–Schmidt Theorem in Group Theory is the following.

**Theorem 1.1** *If a group $G$ satisfies both the ascending chain condition and the descending chain condition on normal subgroups, then $G$ is a direct product $G_1 \times G_2 \times \cdots \times G_t$ of finitely many indecomposable groups in an essentially unique way in the following sense. If*

$$G = G_1 \times G_2 \times \cdots \times G_t = H_1 \times H_2 \times \cdots \times H_s$$

*with $G_1, G_2, \ldots, G_t, H_1, H_2, \ldots, H_s$ indecomposable groups, then $t = s$ and there is a permutation $\sigma$ of the indices $1, 2, \ldots, t$ such that $G_i \cong H_{\sigma(i)}$ for every $i = 1, 2, \ldots, t$.*

## 1.2 Modules

Krull [61] extended the results to the case of "abelian operator groups" (i.e., modules) with the ascending and descending chain conditions, and the theory was subsequently further deepened by Schmidt [80]. Recall that a ring $R$ is *local* if it has a unique maximal right ideal, equivalently a unique maximal left ideal, that is, if $R/J(R)$ is a division ring. (Here, and in the rest of this survey, if $R$ is any ring, $J(R)$ will denote the Jacobson radical of $R$.) Any module with a local endomorphism ring is necessarily indecomposable [4, p. 144] (We refer the reader to the book [4] by Anderson and Fuller for the standard terminology of Module Theory we make use of in this survey.)

Azumaya [8] extended the classical Krull–Schmidt Theorem to the case of possibly infinite direct sums of modules with local endomorphism rings. Notice that any indecomposable module of finite composition length has a local endomorphism ring (Fitting's Lemma).

**Theorem 1.2** (Krull–Schmidt–Remak–Azumaya Theorem) *Let $M$ be a module that is a direct sum of modules with local endomorphism rings. Then $M$ is a direct sum of indecomposable modules in an essentially unique way in the following sense. If*

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j,$$

*where all the $M_i$'s ($i \in I$) and all the $N_j$'s ($j \in J$) are indecomposable modules, then there exists a bijection $\varphi \colon I \to J$ such that $M_i \cong N_{\varphi(i)}$ for every $i \in I$.*

Going back to Krull, he knew that the ascending chain condition was not sufficient for the "Classical Krull–Schmidt Theorem" to hold. More precisely, recall that a module is *noetherian* if it satisfies the ascending chain condition (on submodules) and *artinian* if it satisfies the descending chain condition, so that, as we have already said, a module is of finite composition length if and only if it is both noetherian and artinian. Krull knew that noetherian modules decompose as a direct-sum of indecomposable modules, but that their direct-sum decompositions are not essentially unique. That is, he had found noetherian modules with non-isomorphic direct-sum decompositions into indecomposables. Therefore, in 1932 [62, pp. 37–38], Krull asked whether the "Classical Krull–Schmidt Theorem" holds for artinian modules. That is, any artinian module is a direct sum of indecomposable modules, but is the direct-sum decomposition of an artinian module into indecomposables essentially unique? The negative answer to this question was given only in 1995 in [39]. In that paper, examples of the failure of the Krull–Schmidt Theorem for artinian modules were constructed from examples of the failure of the Krull–Schmidt Theorem for suitable noetherian modules, making use of a technique due to Camps and Menal [16]. We will come back to this aspect of the construction of artinian modules for which the Krull–Schmidt Theorem fails at the end of Section 9.

As far as a Krull–Schmidt Theorem for noetherian modules is concerned, see Corollary 9.4 and Brookfield [13, Theorem 2.8].

In this survey, we will present the behavior of uniserial modules relatively to direct-sum decompositions (Sects. 5.1, 15, 16). Other classes of modules have the same behavior (Sects. 5.4–5.6). More generally, we want to show what happens for modules whose endomorphism ring has at most two maximal right ideals (Section 14) and modules with a semilocal endomorphism ring (Sections 6–13). Sections 2–4 contain preliminary notions and results.

### 1.3 Modular lattices

Ore [68] extended Wedderburn's result to modular lattices with the ascending and the descending chain conditions.

### 1.4 Additive categories

We will denote as $\mathrm{Ob}(\mathcal{C})$ the class of all objects of a category $\mathcal{C}$. A category is *preadditive* if each of its Hom sets is endowed with an abelian group structure in such a way that composition is $\mathbb{Z}$-bilinear. An *additive category* is a preadditive category with a zero object in which any pair of objects has a coproduct (equivalenty, a product — in a preadditive category, product, coproduct and biproduct of any two objects coincide, and we will use the term *direct sum* as well). If $\mathcal{C}$ is a category, *idempotents split* in $\mathcal{C}$ if for every object $A$ of $\mathcal{C}$ and every endomorphism $f$ of $A$ with $f^2 = f$, there exist an object $B$ and morphisms $g\colon A \to B$ and $h\colon B \to A$ such that $hg = f$ and $gh = 1_B$, where $1_B$ denotes the identity morphism of $B$. It is possible to prove [28, Lemma 2.1] that, if $\mathcal{C}$ is an additive category, idempotents split in $\mathcal{C}$ if and only if every idempotent has a kernel in $\mathcal{C}$, that is, for every object $A$, any morphism $f\colon A \to A$ with $f^2 = f$ has a kernel. The link between splitting idempotents and direct-sum decompositions in an additive category is clear: if an object $A$ decomposes as a direct sum $B \oplus C$, that is, as a biproduct, then the composite morphism of the projection of $A$ onto $B$ and the inclusion of $B$ into $A$ is a splitting idempotent. Conversely, if $g\colon A \to B$ and $h\colon B \to A$ are morphisms such that $gh = 1_B$, then $hg$ is an idempotent endomorphism of $A$ and $A \cong B \oplus \ker g$ [28, Lemma 2.1].

Any full subcategory $\mathcal{C}$ of the category Mod-$R$ of all right modules over a ring $R$ is preadditive. A full subcategory $\mathcal{C}$ of Mod-$R$ is an additive category if and only if it contains a trivial (=null=zero) module and, for any pair of modules $A, B \in \mathrm{Ob}(\mathcal{C})$, there is a module in $\mathrm{Ob}(\mathcal{C})$ isomorphic to the direct sum $A \oplus B$. Idempotents split in $\mathcal{C}$ if and only if, for every $A \in \mathrm{Ob}(\mathcal{C})$ and every direct summand $B$ of $A$, there exists in $\mathrm{Ob}(\mathcal{C})$ a module isomorphic to $B$.

**Theorem 1.3** ([9, p. 20]) *Let $\mathcal{C}$ be an additive category in which idempotent split. Let $A_1, \dots, A_n$ be nonzero objects of $\mathcal{C}$ and assume that the endomorphism rings of the $A_i$'s are all local rings. Then:*

(1) *Every direct summand of $A_1 \oplus \cdots \oplus A_n$ is a direct sum of finitely many indecomposable objects.*
(2) *If $A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots \oplus B_m$ and the $B_j$'s are indecomposable objects, then $n = m$ and there exists a permutation $\sigma$ of $\{1, 2, \dots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for all $i = 1, 2, \dots, n$.*

This theorem was generalized to the infinite case by Walker and Warfield [82] (cf. [7], where the case in which the endomorphism rings of the $A_i$ are principal ideal domains is also considered).

Nowadays the name "Krull–Schmidt" is given to any theorem concerning uniqueness of direct direct-sum decompositions into indecomposables. We say that *the Krull–Schmidt property holds* in an additive category $\mathcal{C}$ if every object of $\mathcal{C}$ is a direct sum (=coproduct) of a finite number of indecomposable objects and, if $A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots \oplus B_m$, where the $A_i$'s and the $B_j$'s are indecomposable objects of $\mathcal{C}$, then $n = m$ and there is a permutation $\sigma$ of $\{1, 2, \dots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for every $i = 1, 2, \dots, n$ [26].

## 2 The commutative monoids $V(\mathcal{C})$ and $V(R)$

In order to avoid set-theoretical problems, in this survey we will mostly consider skeletally small categories, that is, categories $\mathcal{C}$ in which the class $\mathrm{Ob}(\mathcal{C})$ contains a *set* of representatives of the objects up to isomorphism. Equivalently, a category $\mathcal{C}$ is skeletally small if it has a skeleton whose class of objects is a set.

## 2.1 Construction of $V(\mathcal{C})$ and $V(R)$

A commutative monoid is a set endowed with a binary operation that is associative, commutative and has an identity element. *All the monoids in this survey will be commutative and additive,* that is, their operation will be denoted as an addition $+$, and their identity element will be denoted as 0. For a commutative monoid $M$, let $U(M)$ denote the group of all $a \in M$ with an opposite $-a$ in $M$. A commutative additive monoid $M$ is *reduced* if $U(M) = \{0\}$, that is, if $x + y = 0$ implies $x = y = 0$ for every $x, y \in M$. For every monoid $M$, the monoid $M_{\text{red}} := M/U(M)$ is reduced.

Given any skeletally small additive category $\mathcal{C}$, let $V(\mathcal{C})$ be the set of objects of a skeleton of $\mathcal{C}$. For any $A \in \text{Ob}(\mathcal{C})$, the unique element of $V(\mathcal{C})$ isomorphic to $A$ will be denoted by $\langle A \rangle$. Every pair of objects $A, B$ of the skeleton of $\mathcal{C}$ has a unique direct sum $\langle A \oplus B \rangle$ in the skeleton. Define an addition in $V(\mathcal{C})$ setting $A + B := \langle A \oplus B \rangle$ for every $A$ and $B$ in $V(\mathcal{C})$. Then $V(\mathcal{C})$ with this addition turns out to be a reduced commutative monoid. It is easy to prove that the Krull–Schmidt property holds in the additive category $\mathcal{C}$ if and only if the monoid $V(\mathcal{C})$ is a *free* monoid, that is, isomorphic to the direct sum $\mathbb{N}_0^{(I)}$ for some set $I$.

A *translation-invariant pre-order* on a monoid $M$ is a reflexive and transitive relation $\leq$ on the set $M$ such that $x, y, z \in M$ and $x \leq y$ imply $x + z \leq y + z$. There is a natural translation-invariant pre-order $\leq$ on any commutative additive monoid $M$, defined by $x \leq y$ if there exists $z \in M$ such that $x + z = y$. It is called the *algebraic pre-order* on $M$. An element $u$ of a commutative monoid $M$ is called an *order-unit* in $M$ if, for every $x \in M$, there exists an integer $n \geq 0$ such that $x \leq nu$; that is, if for every $x \in M$ there exist $y \in M$ and a non-negative integer $n$ such that $x + y = nu$.

For any ring $R$, let proj-$R$ denote the full subcategory of Mod-$R$ whose objects are all finitely generated projective right $R$-modules. The category proj-$R$ is always skeletally small. The monoid $V(\text{proj-}R)$ is usually denoted as $V(R)$. (The reason of the importance of these monoids $V(R)$ will be explained in Sect. 2.2). Thus, for any ring $R$, $V(R)$ consists of a set of representatives of the finitely generated projective right $R$-modules up to isomorphism, and $\langle R_R \rangle$ is an order-unit in the commutative reduced monoid $V(R)$. More generally, an element $P_R$ of $V(R)$ is an order-unit if and only if $P_R$ is a progenerator [4, § 22].

It is possible to define the category of commutative monoids with order-unit. Its objects are the pairs $(M, u)$, where $M$ is any commutative monoid and $u \in M$ is an order-unit. The morphisms $f : (M, u) \to (M', u')$ are the monoid homomorphisms $f : M \to M'$ such that $f(u) = u'$. Then $V$ turns out to be a functor of the category of all rings with identity into the category of commutative monoids with order-unit. It associates to any ring $R$ the monoid with order-unit $(V(R), \langle R_R \rangle)$ and to any ring morphism $f : R \to S$ the morphism $V(f) : (V(R), \langle R_R \rangle) \to (V(S), \langle S_S \rangle)$ defined by $V(f)(P_R) = \langle P \otimes_R S \rangle$ for every $P_R \in V(R)$.

The construction of the monoid $V(R)$ can be generalized from the module $R_R$ to any other module $A_R$. If $R$ is an arbitrary ring and $A_R$ is a right $R$-module, let $\text{add}(A_R)$ denote the full subcategory of Mod-$R$ whose objects are all right $R$-modules that are isomorphic to a direct summand of $A_R^n$ for some integer $n \geq 0$. For example, proj-$R = \text{add}(R_R)$. The category $\text{add}(A_R)$ is an additive category in which idempotents split, and we can construct from the category $\mathrm{add}(A_R)$ the commutative monoid $V(\mathrm{add}(A_R))$ and the monoid with order-unit $(V(\mathrm{add}(A_R)), \langle A_R\rangle)$.

Notice that the commutative monoid $V(\mathcal{C})$ is the algebraic object that naturally describes the direct-sum decompositions of the objects in an additive category $\mathcal{C}$, and the commutative monoid with order-unit $(V(\mathrm{add}(A_R)), \langle A_R\rangle)$ is the algebraic object that naturally describes the direct-sum decompositions of a module $A_R$.

If, in the construction of the monoid $V(R)$, we consider finitely generated projective left $R$-modules instead of finitely generated projective right $R$-modules, we get a monoid isomorphic to $V(R)$, because if $R$-proj denotes the full subcategory of $R$-Mod whose objects are all finitely generated projective left $R$-modules, then the contravariant functors

$$\mathrm{Hom}_R(-, {}_RR_R)\colon \text{Mod-}R \to R\text{-Mod} \quad \text{and} \quad \mathrm{Hom}_R(-, {}_RR_R)\colon R\text{-Mod} \to \text{Mod-}R$$

define a duality between the categories proj-$R$ and $R$-proj. This duality induces an isomorphism of monoids $V(\text{proj-}R) \cong V(R\text{-proj})$ and an isomorphism of monoids with order-unit $(V(\text{proj-}R), \langle R_R\rangle) \cong (V(R\text{-proj}), \langle {}_RR\rangle)$.

The monoids that can be realized as $V(R)$ for a ring $R$ with identity can be extremely arbitrary, as the following wonderful theorem, due to Bergman and Dicks, shows. Recall that a ring $R$ is *hereditary* if all its right ideals and all its left ideals are projective modules.

**Theorem 2.1** ([10] and [11]) *Let $k$ be a field and let $(M, u)$ be a commutative reduced monoid with order-unit. Then there exists a hereditary $k$-algebra $R$ such that $(M, u)$ and $(V(R), \langle R_R\rangle)$ are isomorphic monoids with order-unit.*

Theorem 2.1 was first proved by Bergman for finitely generated monoids with order-unit [10, Theorems 6.2 and 6.4]. Then it was extended by Bergman and Dicks to arbitrary monoids with order-unit [11, p. 315].

**Corollary 2.2** *Let $k$ be a field and let $M$ be a commutative reduced monoid. Then there exist a hereditary $k$-algebra $R$ and an additive full subcategory $\mathcal{C}$ of* proj-$R$ *such that $M \cong V(\mathcal{C})$.*

To prove the Corollary, it suffices to notice that if $M$ is a commutative reduced monoid, it is possible to add a further element $\infty$ to $M$, getting a commutative monoid $M \cup \{\infty\}$ with $x + \infty = \infty$ for every $x \in M \cup \{\infty\}$. The new element $\infty$ is an order-unit in the monoid $M \cup \{\infty\}$, and it is therefore possible to apply Theorem 2.1 to the monoid with order-unit $(M \cup \{\infty\}, \infty)$, obtaining a hereditary algebra $R$ with $(M \cup \{\infty\}, \infty) \cong (V(R), \langle R\rangle)$. Now it suffices to take as $\mathcal{C}$ the full subcategory of proj-$R$ whose objects are all finitely generated projective right $R$-modules not isomorphic to $R_R$.

### 2.2 Importance of $V(R)$ and projective modules

The particular importance of the monoid $V(R)$ is given by the following theorem, due to Dress [21]. Fix a right module $A_S$ over a ring $S$. Let $R := \mathrm{End}(A_S)$ be the endomorphism ring of $A_S$, so that ${}_RA_S$ is a bimodule.

**Theorem 2.3** *The functors*

$$\mathrm{Hom}_S(A,-)\colon \text{Mod-}S \to \text{Mod-}R \quad \textit{and} \quad -\otimes_R A\colon \text{Mod-}R \to \text{Mod-}S$$

*induce a categorical equivalence between the full subcategory* $\mathrm{add}(A_S)$ *of* Mod-$S$ *and the full subcategory* proj-$R$ *of* Mod-$R$. *In particular, the monoids* $V(\mathrm{add}(A_S))$ *and* $V(R)$ *are isomorphic, and the monoids with order-unit* $(V(\mathrm{add}(A_R)), \langle A_R\rangle)$ *and* $(V(R), \langle R_R\rangle)$ *are isomorphic.*

In the categorical equivalence of Theorem 2.3, the $S$-module $A_S$ corresponds to the projective $R$-module $R_R$. More generally, if $e \in R$ is idempotent, so the we have a direct sum decomposition $A_S = eA_S \oplus (1-e)A_S$, then the direct summand $eA_S$ of $A_S$, which is an object of $\mathrm{add}(A_S)$, corresponds to the object $eR$ of proj-$R$.

The idea of Theorem 2.3, that the direct-sum decompositions of any module $A_S$ can be described via the finitely generated projective modules over its endomorphism ring, can be further extended to study the direct-sum decompositions of any object $A$ of a preadditive category $\mathcal{P}$. Recall that a ring $R$ (possibly without 1) is said to be a ring with *enough idempotents* if there is a set $\{e_i \mid i \in I\}$ of pair-wise orthogonal idempotents of $R$ with $R = \oplus_{i\in I} e_i R = \oplus_{i\in I} Re_i$ [49]. Let $\mathcal{P}$ be a skeletally small preadditive category and let $V(\mathcal{P})$ be the set of objects of a skeleton of $\mathcal{P}$. We can associate to $\mathcal{P}$ the *(Gabriel) functor ring* $F(\mathcal{P}) := \oplus_{A\in V(\mathcal{P})} \oplus_{B\in V(\mathcal{P})} \mathrm{Hom}_{\mathcal{P}}(A,B)$ [50]. In this ring, the product $ff'$ of $f \in \mathrm{Hom}_{\mathcal{P}}(A,B)$ and $f' \in \mathrm{Hom}_{\mathcal{P}}(A',B')$ is 0 if $A \neq B'$ and is the composite morphism $f \circ f'$ if $A = B'$. If we choose different skeletons of $\mathcal{P}$, the functor rings we obtain are isomorphic. The ring $F(\mathcal{P})$ is a ring with enough idempotents (a suitable set of idempotents is given by the identities $1_A$, where $A$ ranges in $V(\mathcal{P})$).

Now we are dealing with rings (possibly) without identity, so that some care is needed as far as modules are concerned. Recall that a right module $M_R$ over a ring $R$ (possibly without identity) is said to be a *unitary* module if $M = MR$. This definition coincides with the definition we are used to in the case where $R$ has an identity (a module $M_R$ over a ring $R$ with identity $1_R$ is *unitary* if $x \cdot 1_R = x$ for every $x \in M_R$), because if $R$ is a ring with identity $1_R$ and $M_R$ is a module unitary in the sense that $M = MR$, then, for every $x \in M_R$, there exist an integer $n \geq 1$ and elements $x_1, \dots, x_n \in M_R$ and $r_1, \dots, r_n \in R$ such that $x = \sum_{i=1}^n x_i r_i$, so that $x \cdot 1_R = \left(\sum_{i=1}^n x_i r_i\right) \cdot 1_R = \sum_{i=1}^n x_i \cdot (r_i \cdot 1_R) = \sum_{i=1}^n x_i r_i = x$.

Let Mod-$R$ denote the category whose objects are all unitary right modules over a ring $R$ possibly without identity. Two rings $R$ and $S$ with enough idempotents are said to be *Morita equivalent* if their categories Mod-$R$ and Mod-$S$ of unitary right modules are equivalent. Similarly to the case of rings with identity, it can be proved that two rings $R$ and $S$, possibly without identity, are Morita equivalent if and only if there exists a surjective Morita context $(R, S, M, N, \varphi, \psi)$, that is, if there exist unital bimodules ${}_RM_S$ and ${}_SN_R$ and surjective $S$-bimodule and $R$-bimodule morphisms $\varphi\colon N \otimes_R M \to S$ and $\psi\colon M \otimes_S N \to R$, satisfying the compatibility conditions $\varphi(n \otimes m)n' = n\psi(m \otimes n')$ and $m'\varphi(n \otimes m) = \psi(m' \otimes n)m$ for every $m, m' \in M$ and every $n, n' \in N$ [5]. If $R$ is a ring with enough idempotents, then $R_R$ is a projective right $R$-module, that is, a projective object in the category Mod-$R$ of unitary right

$R$-modules, but $R_R$ is not a finitely generated object in general. It is possible to prove that the finitely generated projective objects in Mod-$R$ are the unitary right $R$-modules isomorphic to the direct summands of the modules $(\oplus_{i\in F} e_i R)^n$, where $F$ is a finite subset of $I$ and $n \geq 0$ is an integer.

Notice that non-equivalent skeletally small preadditive categories can have the same functor ring. For instance, let $\mathcal{P}$ be any skeletally small preadditive category with at least two non-isomorphic objects, let $R$ be its functor ring and let $R^*$ be the category with only one object with endomorphism ring $R$. Then $\mathcal{P}$ and $R^*$ are two non-equivalent categories with the same functor ring $R$. If, instead of arbitrary skeletally small preadditive categories, we consider skeletally small additive categories in which idempotents split, then the correspondence {skeletally small additive categories in which idempotents split} $\to$ {rings with enough idempotents} that associates to each category its functor ring, is a one-to-one correspondence between the equivalence classes of skeletally small additive categories in which idempotents split, that is, the skeletally small additive categories in which idempotents split up to category equivalence, and the Morita equivalence classes of rings with enough idempotents. The inverse correspondence is the correspondence that associates to each ring $R$ with enough idempotents the category proj-$R$ of all finitely generated projective unitary right $R$-modules. Every skeletally small additive category $\mathcal{A}$ in which idempotents split is equivalent to the category proj-$R$ of all finitely generated projective unitary right modules over the functor ring $R = F(\mathcal{A})$.

From now on, all the rings $R$ we will consider in the rest of the survey will be rings with identity.

## 3 Some notions of category theory

In this Section, we will present some notions concerning (preadditive) categories needed in the sequel.

### 3.1 Closure under direct sums and idempotent completion

If $\mathcal{P}$ is any preadditive category, we can "close" $\mathcal{P}$ under finite direct sums, getting an additive category Mat($\mathcal{P}$) generated by $\mathcal{P}$. That is, if $\mathcal{P}$ is a preadditive category, we can construct the "free additive category" Mat($\mathcal{P}$). The objects of Mat($\mathcal{P}$) are the $n$-tuples $(P_1, \dots, P_n)$ of objects of $\mathcal{P}$ for $n \geq 0$, and the morphisms in Mat($\mathcal{P}$) from an $n$-tuple $(P_1, \dots, P_n)$ to an $m$-tuple $(Q_1, \dots, Q_m)$ of objects of $\mathcal{P}$ is an $m \times n$ matrix $(f_{ij})_{ij}$ of morphisms in $\mathcal{P}$ with $f_{ij} \in \mathrm{Hom}_{\mathcal{P}}(P_j, Q_i)$ ([67, Exercise 6(a), p. 198], [65, p. 11]). The category Mat($\mathcal{P}$) is an additive category. There is an obvious canonical embedding functor $E\colon \mathcal{P} \to \mathrm{Mat}(\mathcal{P})$ that sends an object $P$ of $\mathcal{P}$ into the 1-tuple $(P)$.

**Proposition 3.1** ([65, p. 11], [3, Lemma A.1]) *Let $\mathcal{P}$ be a preadditive category and $E\colon \mathcal{P} \to \mathrm{Mat}(\mathcal{P})$ be the canonical embedding functor. For any additive category $\mathcal{A}$ and additive functor $F\colon \mathcal{P} \to \mathcal{A}$, there is an additive functor $F'\colon \mathrm{Mat}(\mathcal{P}) \to \mathcal{A}$ with $F = F'E$. The additive functor $F'$ is unique up to natural isomorphism, in the following sense. An additive functor $F''\colon \mathrm{Mat}(\mathcal{P}) \to \mathcal{A}$ has the property that $F = F''E$ if* *and only if there is a natural isomorphism* $\eta\colon F' \to F''$ *such that* $\eta_{E(U)} = 1_{F(U)}$ *for every* $U \in \mathrm{Ob}(\mathcal{P})$.

Now that we have the additive category $\mathrm{Mat}(\mathcal{P})$ "freely generated" by the preadditive category $\mathcal{P}$, we can construct the idempotent completion $\widehat{\mathrm{Mat}(\mathcal{P})}$ of $\mathrm{Mat}(\mathcal{P})$. If $\mathcal{A}$ is any additive category, the *idempotent completion* $\widehat{\mathcal{A}}$ of $\mathcal{A}$ is the additive category in which idempotents split defined as follows. The objects of $\widehat{\mathcal{A}}$ are the pairs $(B, e)$, where $B$ is an object of $\mathcal{A}$ and $e$ is an idempotent element of $\mathrm{End}_{\mathcal{A}}(B)$. The morphisms $(B, e) \to (B', e')$ in $\widehat{\mathcal{A}}$ are the morphisms $\varphi\colon B \to B'$ in $\mathcal{A}$ such that $e'\varphi e = \varphi$. Thus $\mathrm{Hom}_{\widehat{\mathcal{A}}}((B, e), (B', e'))$ is a subgroup of $\mathrm{Hom}_{\mathcal{A}}(B, B')$. The idempotent completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ has the following property. For every functor $G\colon \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into an additive category $\mathcal{B}$ in which idempotents split, there exists an additive functor $H\colon \widehat{\mathcal{A}} \to \mathcal{B}$ such that $G = HF$. Here $F\colon \mathcal{A} \to \widehat{\mathcal{A}}$ is the embedding functor defined by $F(A) := (A, 1_A)$ for every object $A$ of $\mathcal{A}$ [28, Section 7]. The additive functor $H$ is unique up to natural isomorphism, in the following sense. An additive functor $H'\colon \widehat{\mathcal{A}} \to \mathcal{B}$ has the property that $G = H'F$ if and only if there is a natural isomorphism $\eta\colon H \to H'$ such that $\eta_{(A,1_A)} = 1_{G(A)}$ for every $A \in \mathrm{Ob}(\mathcal{A})$.

Thus, if $\mathcal{P}$ is any preadditive category and we construct the additive category $\mathrm{Mat}(\mathcal{P})$ first and then its idempotent completion $\widehat{\mathrm{Mat}(\mathcal{P})}$, we embed $\mathcal{P}$ into an additive category $\widehat{\mathrm{Mat}(\mathcal{P})}$ in which idempotents split, and this is in turn equivalent to the category proj-$R$ of finitely generated projective modules over the functor ring $R$. This shows that any skeletally small preadditive category is equivalent to a full subcategory of the category proj-$R$ for a suitable ring $R$ with enough idempotents. Notice that if $R$ does not have an identity, then the commutative monoid $V(\text{proj-}R)$ does not have an order-unit necessarily.

## 3.2 Products and coproducts

The notion of product of categories is well known. Given categories $\mathcal{C}_i$, where $i$ ranges in an index set $I$, the *product category* $\prod_{i\in I} \mathcal{C}_i$ has as objects all $I$-tuples $(A_i)_{i\in I}$, where $A_i$ is an object of $\mathcal{C}_i$ for every $i \in I$, and as Hom sets the sets $\mathrm{Hom}_{\prod_{i\in I}\mathcal{C}_i}((A_i)_{i\in I}, (B_i)_{i\in I}) := \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}_i}(A_i, B_i)$. It is interesting to notice that in suitable set theories (for instance in the theory called $\mathsf{NFU}_{\mathsf{P}}$; cf. [46]) it is possible to the define the product $\prod_{i\in I} \mathcal{C}_i$ also when $I$ is a class, that is, it is possible to construct the product of a class of categories.

For every index $j \in I$, there is a canonical projection functor

$$P_j\colon \prod_{i\in I} \mathcal{C}_i \to \mathcal{C}_j.$$

Clearly, the product category with the family of functors $P_i$ has the following universal property.

**Proposition 3.2** *Let* $\{\mathcal{C}_i \mid i \in I\}$ *be a family of categories indexed in a set* $I$. *If* $\mathcal{B}$ *is an arbitrary category and* $\{F_i : \mathcal{B} \to \mathcal{C}_i \mid i \in I\}$ *is a family of functors, then there exists a unique functor* $F : \mathcal{B} \to \prod_{i\in I} \mathcal{C}_i$ *with* $P_i F = F_i$ *for every* $i \in I$.

In other words, the product category is the product in the category **Cat** of all categories (which exists in $\mathsf{NFU}_\mathsf{P}$). Moreover, if all the categories $\mathcal{C}_i$'s are preadditive (or additive), then the product category $\prod_{i\in I}\mathcal{C}_i$ is also a preadditive (or additive, respectively) category, and the functors $P_i$ turn out to be additive functors. That is, we can consider the subcategory **Preadd** of **Cat**, whose objects are all preadditive categories and whose morphisms are all additive functors. In the category **Preadd**, we have the full subcategory **Add**, whose objects are all additive categories. Then the product category is also the product in the categories **Preadd** and **Add**.

If we dualize the construction of the product of categories, the situation becomes more intricate. In the category **Cat** of all categories, the coproduct of a family of categories $\mathcal{C}_i$, $i\in I$, is the disjoint union $\dot{\bigcup}_{i\in I}\mathcal{C}_i$ of the categories $\mathcal{C}_i$'s, $i\in I$. This is the category where the class of objects is the disjoint union $\dot{\bigcup}_{i\in I}\operatorname{Ob}(\mathcal{C}_i)$ of the classes $\operatorname{Ob}(\mathcal{C}_i)$, and the set $\operatorname{Hom}_{\dot{\bigcup}_{i\in I}\mathcal{C}_i}(C_i, C'_j)$ of all morphisms of an object $C_i\in\operatorname{Ob}(\mathcal{C}_i)$ into an object $C'_j\in\operatorname{Ob}(\mathcal{C}_j)$ $(i,j\in I)$ is $\operatorname{Hom}_{\mathcal{C}_i}(C_i, C'_j)$ if $i=j$ and is the empty set $\emptyset$ if $i\neq j$.

In the category **Preadd** of all preadditive categories, the coproduct of a family of preadditive categories $\mathcal{C}_i$ $(i\in I)$ is the category $\dot{\coprod}_{i\in I}\mathcal{C}_i$ where the class of objects is the disjoint union $\dot{\bigcup}_{i\in I}\operatorname{Ob}(\mathcal{C}_i)$ of the classes $\operatorname{Ob}(\mathcal{C}_i)$, and the set $\operatorname{Hom}_{\dot{\coprod}_{i\in I}\mathcal{C}_i}(C_i, C'_j)$ of all morphisms of an object $C_i\in\operatorname{Ob}(\mathcal{C}_i)$ into an object $C'_j\in\operatorname{Ob}(\mathcal{C}_j)$, $i,j\in I$, is $\operatorname{Hom}_{\mathcal{C}_i}(C_i, C'_j)$ if $i=j$ and is the trivial group with one element if $i\neq j$. Both in **Cat** and in **Preadd**, we have functors $E_i$ of $\mathcal{C}_i$ into the corresponding coproduct category satisfying the dual of Proposition 3.2.

In the category **Add** of all additive categories, a coproduct category satisfying the dual of Proposition 3.2 does not exist in general for a family of additive categories $\mathcal{C}_i$, $i\in I$, but it exists if instead of requiring uniqueness in the dual of Proposition 3.2, we require uniqueness only up to natural isomorphism like in Proposition 3.1 and for idempotent completion. If $\mathcal{C}_i$ is an additive category for every $i\in I$, it suffices to take its coproduct in **Preadd** first, getting the preadditive category $\dot{\coprod}_{i\in I}\mathcal{C}_i$, and then form the free additive category $\operatorname{Mat}(\dot{\coprod}_{i\in I}\mathcal{C}_i)$. We will denote this additive category $\operatorname{Mat}(\dot{\coprod}_{i\in I}\mathcal{C}_i)$ as $\coprod^w_{i\in I}\mathcal{C}_\lambda$, and call it the *weak coproduct* of the family of additive categories $\mathcal{C}_i$, $i\in I$. Notice that this construction is possible also when the categories $\mathcal{C}_i$ are preadditive only, it is not necessary that they are additive categories. In this case, the weak coproduct $\coprod^w_{\lambda\in\Lambda}\mathcal{C}_\lambda$ of preadditive categories $\mathcal{C}_\lambda$ turns out to be always an additive category. For every index $j\in I$, there is a canonical embedding functor $F_j\colon\mathcal{C}_j\to\coprod^w_{i\in I}\mathcal{C}_i$. It is the composition of the canonical functor $E_j\colon\mathcal{C}_j\to\dot{\coprod}_{i\in I}\mathcal{C}_i$ and the canonical functor $\dot{\coprod}_{i\in I}\mathcal{C}_i\to\operatorname{Mat}(\dot{\coprod}_{i\in I}\mathcal{C}_i)$.

Since the uniqueness in Proposition 3.1 is only up to natural isomorphism, the weak coproduct enjoys the following weak form of universal property.

**Proposition 3.3** *Let $\mathcal{C}_i$ be an additive category for every $i$ in a class $I$. Let $\coprod^w_{i\in I}\mathcal{C}_i$ be the weak coproduct and $F_j\colon\mathcal{C}_j\to\coprod^w_{i\in I}\mathcal{C}_i$, $j\in I$, be the embedding functors. If $G_j\colon\mathcal{C}_j\to\mathcal{D}$ is an additive functor into a fixed additive category $\mathcal{D}$ for all $j\in I$, then there exists an additive functor $G\colon\coprod^w_{i\in I}\mathcal{C}_i\to\mathcal{D}$ such that $GF_j=G_j$ for all $j\in I$. Such functor $G$ is unique up to natural isomorphisms that are the identity on all the objects $F_i(U)$, where $i\in I$ and $U\in\operatorname{Ob}(\mathcal{C}_i)$.*

Mathematical objects satisfying universal properties are uniquely determined up to isomorphism. In this case, where we have only this weak form of universal property, the categories satisfying Proposition 3.3 are determined only up to category equivalence, as stated in the next Proposition.

**Proposition 3.4** *Let $I$ be a class and $\mathcal{C}_i$ be an additive category for every $i \in I$. The following conditions are equivalent for an additive category $\mathcal{C}$ and additive functors $F'_i : \mathcal{C}_i \to \mathcal{C}$, where $i$ ranges in $I$.*

(a) *For every additive category $\mathcal{D}$ and additive functors $G_i : \mathcal{C}_i \to \mathcal{D}$, there exists an additive functor $G : \mathcal{C} \to \mathcal{D}$ with $GF'_i$ and $G_i$ naturally isomorphic, and, for every additive functor $G' : \mathcal{C} \to \mathcal{D}$ with $G'F'_i$ and $G_i$ naturally isomorphic for all $i \in I$, there exists a natural isomorphism $G \to G'$.*
(b) *There exists a category equivalence $E : \coprod^w_{i \in I} \mathcal{C}_i \to \mathcal{C}$ such that $F'_i = EF_i$ for every $i \in I$.*

For instance, let $\mathcal{C}_i$ be an additive category for every $i \in I$. Consider the full subcategory $\prod^f_{i \in I} \mathcal{C}_i$ of the category product $\prod_{i \in i} \mathcal{C}_i$ whose objects are the $I$-tuples $S = (A_i)_{i \in I}$ in which $A_i$ is a null object of $\mathcal{C}_i$ for all $i \in I$ except for finitely many indices $i \in I$, that is, such that there exists a finite subset $I_S$ of $I$ with $A_i$ a null object for all $i \in I \setminus I_S$. Then the category $\prod^f_{i \in I} \mathcal{C}_i$ is equivalent to the weak coproduct $\coprod^w_{i \in I} \mathcal{C}_i$, so that the category $\prod^f_{i \in I} \mathcal{C}_i$ also satisfies the weak form of the universal property of Proposition 3.3. One finds a similar behavior also for idempotent completion. Idempotent completion is determined only up to category equivalence by its weak form of universal property.

Sometimes, in the construction of coproduct categories, we need categories $\mathcal{C}_i$, $i \in I$, where $I$ is a proper class. Sets are not sufficients. For instance, there is a proper class of pair-wise non-isomorphic abelian groups with local endomorphism rings [42, Section 4]. Let $\mathcal{C}$ be the full subcategory of the category Ab of all abelian groups whose objects are the abelian groups that are direct sums of finitely many subgroups each of which has a local endomorphism ring. The Krull–Schmidt property holds in the additive category $\mathcal{C}$. Recall that the *Jacobson radical* $J(\mathcal{C})$ of a preadditive category $\mathcal{C}$ is the ideal of $\mathcal{C}$ defined, for every pair of objects $A, B$ of $\mathcal{C}$, by $J(\mathcal{C})(A, B) := \{f : A \to B \mid 1_A - gf$ has a right inverse for every morphism $g : B \to A$ in $\mathcal{C}\}$. In particular, $\mathcal{J}(A, A) = J(\mathrm{End}_R(A))$. If $J(\mathcal{C})$ denotes the Jacobson radical of a preadditive category $\mathcal{C}$, then: (1) the factor category $\mathcal{C}/J(\mathcal{C})$ turns out to be an abelian category in which idempotents split, (2) in $\mathcal{C}/J(\mathcal{C})$ the endomorphism rings of all objects are semisimple artinian rings, (3) $\mathcal{C}/J(\mathcal{C})$ is not skeletally small, and (4) there exist division rings $k_i$ indexed in a class $I$ with $\mathcal{C}/J(\mathcal{C})$ equivalent to the weak coproduct $\coprod^w_{i \in I}$vect-$k_i$ of the categories of all finite dimensional vector spaces over the $k_i$'s. Here we can take as class $I$ a class of representatives up to isomorphism of all abelian groups with a local endomorphism ring.

### 3.3 Spectral categories, amenable semisimple categories, and dual construction

The construction of the spectral category is due to Gabriel and Oberst [51]. Let $\mathcal{A}$ be a Grothendieck category. Write $A' \leq_e A$ for "$A'$ is an essential subobject of $A$". For any object $A$ of $\mathcal{A}$, the intersection of two essential subobjects of $A$ is an essential subobject of $A$, so that the set of all the essential subobjects of $A$ is downwards directed. If $B$ is another object of $\mathcal{A}$ and we apply the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, B)$, we get an upwards directed family of abelian groups $\mathrm{Hom}_{\mathcal{A}}(A', B)$, with $A' \leq_e A$. The *spectral category* $\mathrm{Spec}\,\mathcal{A}$ of $\mathcal{A}$ is defined as the category that has the same objects as $\mathcal{A}$ and, for objects $A$ and $B$ of $\mathcal{A}$, with

$$\mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(A, B) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(A', B).$$

Here, as we have already said, the direct limit is taken over all essential subobjects $A'$ of $A$. The composition in $\mathrm{Spec}\,\mathcal{A}$ is defined as follows. If $f \in \mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(A, B)$ and $g \in \mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(B, C)$ are morphisms in $\mathrm{Spec}\,\mathcal{A}$, then $f$ is represented by a morphism $f' \colon A' \to B$ for some essential subobject $A'$ of $A$ and $g$ is represented by a morphism $g' \colon B' \to C$ for some essential subobject $B'$ of $B$. Then $f'^{-1}(B')$ is an essential subobject of $A$. The composite morphism $gf$ in $\mathrm{Spec}\,\mathcal{A}$ is the image in $\mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(A, C) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(A', C)$ of the composite morphism of the restriction $f'^{-1}(B') \to B'$ of $f'$ to $f'^{-1}(B')$ and $g'$. There is a canonical functor $P \colon \mathcal{A} \to \mathrm{Spec}\,\mathcal{A}$ which is the identity on objects and sends a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ to its image in the direct limit $\mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(A, B)$. The spectral category $\mathrm{Spec}\,\mathcal{A}$ of any Grothendieck category $\mathcal{A}$ is a Grothendieck category in which every object is injective. More generally, a category is called a *spectral category* if it is a Grothendieck category in which every object is injective. The *discrete part* of a spectral category $\mathcal{A}$ is the full subcategory of $\mathcal{A}$ whose objects are all semisimple objects of $\mathcal{A}$. A spectral category is *discrete* if all its objects are semisimple. For any division ring $k$, the category Vect-$k$ of all right vector spaces over $k$ is a discrete spectral category. A category $\mathcal{A}$ is a discrete spectral category if and only if there exists an indexed set $\{k_i \mid i \in I\}$ of division rings $k_i$ with $\mathcal{A}$ equivalent to the product $\prod_{i \in I}$Vect-$k_i$ [81, Proposition V.6.7].

The construction of the spectral category can be dualized [38]. Let $\mathcal{A}$ be a Grothendieck category. Let $\mathcal{A}'$ be the category with the same objects as $\mathcal{A}$ and, for all objects $A, B \in \mathrm{Ob}(\mathcal{A})$ with

$$\mathrm{Hom}_{\mathcal{A}'}(A, B) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(A, B/B'),$$

where the direct limit is taken over the upwards directed family of all superfluous subobjects $B'$ of $B$. The category $\mathcal{A}'$ is an additive category in which every morphism has a cokernel, but not necessarily a kernel [29, Example 4.7]. There is a canonical functor $F \colon \mathcal{A} \to \mathcal{A}'$ that is the identity on objects and sends a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ to its image in the direct limit $\mathrm{Hom}_{\mathcal{A}'}(A, B) = \varinjlim \mathrm{Hom}_{\mathcal{A}}(A, B/B')$.

Additive categories in which idempotents split have been called *amenable* by Freyd [65, p. 12].

**Definition 3.5** For any division ring $k$, let vect-$k$ denote the category of all finitely dimensional right vector spaces over $k$. A category $\mathcal{A}$ is *amenable semisimple* if there exists a class $\{k_i \mid i \in I\}$ of division rings $k_i$ with $\mathcal{A}$ equivalent to the weak coproduct $\coprod^{w}_{i \in I}$vect-$k_i$ [65, p. 20]. Equivalently, a category is amenable semisimple if and only if it is an additive category in which idempotents split and the endomorphism rings $\mathrm{End}_{\mathcal{A}}(A)$ of all non-zero objects $A$ of $\mathcal{A}$ are semisimple artinian rings [28, Theorem 7.1].

An amenable semisimple category is necessarily abelian.

## 4 Krull–Schmidt property and IBN rings

The aim of this Section is to show that in the setting of categories in which the Krull–Schmidt property holds, the notion of IBN ring appears in a way more natural than that of local ring. Recall that a ring $R$ has the *invariant basis property* or *invariant basis number* (IBN) if $R_R^n \cong R_R^m$ implies $n = m$ for every integer $n, m \geq 0$. Notice that this is a two-sided condition, because $R_R^n \cong R_R^m$ if and only if ${}_RR^n \cong {}_RR^m$ (the contravariant functor $\mathrm{Hom}_R(-, {}_RR_R)$ gives a duality between free modules).

Let $\mathcal{C}$ be a (skeletally small) additive category in which the Krull–Schmidt property holds. Then the monoid $V(\mathcal{C})$ is a free monoid, that is, isomorphic to $\mathbb{N}_0^{(I)}$ for some set $I$. The set $I$ is necessarily in one-to-one correspondence with a set of representatives of the indecomposable objects of $\mathcal{C}$ up to isomorphism. In general, there is not a decomposition of the category $\mathcal{C}$ corresponding to the direct-sum decomposition $V(\mathcal{C}) \cong \mathbb{N}_0^{(I)}$. That is, $V(\mathcal{C}) \cong \mathbb{N}_0^{(I)}$ does not imply that there must exist a category equivalence of $\mathcal{C}$ into $\coprod_{i\in I}^{w} \mathcal{C}_i$ for suitable additive categories $\mathcal{C}_i$ with $V(\mathcal{C}_i) \cong \mathbb{N}_0$. Clearly, if there is an equivalence between $\mathcal{C}$ and $\coprod_{i\in I}^{w} \mathcal{C}_i$, where $V(\mathcal{C}_i) \cong \mathbb{N}_0$ for all $i$, then $V(\mathcal{C}) \cong \mathbb{N}_0^{(I)}$, but the converse is not true. For instance, if $R$ is a ring and $\mathcal{C}$ is the full subcategory of Mod-$R$ whose objects are all right $R$-modules of finite composition length, then the Krull–Schmidt property holds in $\mathcal{C}$ by the Classical Krull–Schmidt Theorem. Thus $V(\mathcal{C}) \cong \mathbb{N}_0^{(I)}$ where $I$ is in one-to-one correspondence with a set of representatives of the indecomposable modules in $\mathcal{C}$ up to isomorphism, but this decomposition of $V(\mathcal{C})$ does not correspond to a decomposition of the category $\mathcal{C}$, essentially because there can be non-zero morphisms between non-isomorphic modules of finite composition length.

**Proposition 4.1** [36] *The following conditions are equivalent for an additive category $\mathcal{C}$:*

(a) *Every object of $\mathcal{C}$ is a direct sum of finitely many indecomposable objects; the category $\mathcal{C}$ has exactly one indecomposable object up to isomorphism; and if $A$ is an indecomposable object of $\mathcal{C}$ and $A^n \cong A^m$, then $n = m$*
(b) *The monoid $V(\mathcal{C})$ is isomorphic to the additive monoid $\mathbb{N}_0$.*
(c) *There exists a ring $R$ with IBN such that $\mathcal{C}$ is equivalent to the full subcategory $\mathcal{F}_R$ of* Mod-$R$ *whose objects are all finitely generated free right $R$-modules.*

We call IBN category every (necessarily skeletally small) additive category $\mathcal{C}$ in which idempotents split satisfying the equivalent conditions of Proposition 4.1.

As we have remarked in the second paragraph of this Section, category equivalence is a concept that is too strong to characterize additive categories in which the Krull–Schmidt property holds. Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ is:

– *dense* if every object $B$ of $\mathcal{B}$ is isomorphic to $F(A)$ for some object $A$ of $\mathcal{A}$;
– a (*category*) *equivalence* if it is full, faithful and dense;
– *isomorphism reflecting* if, for every pair $A, A'$ of objects of $\mathcal{A}$, $F(A) \cong F(A')$ implies $A \cong A'$;
– *direct-summand reflecting* if, for every pair $A, A'$ of objects of $\mathcal{A}$ with $F(A)$ isomorphic to a direct summand of $F(B)$, $A$ is isomorphic to a direct summand of $B$.

We say that an additive functor $F: \mathcal{A} \to \mathcal{B}$ between preadditive categories $\mathcal{A}$ and $\mathcal{B}$ is a *weak equivalence* if it is isomorphism reflecting and dense.

**Theorem 4.2** [36] *Let $\mathcal{C}$ be a skeletally small additive category in which idempotents split. The following conditions are equivalent.*

(a) *The Krull–Schmidt property holds in $\mathcal{C}$.*
(b) *There exist a family $\{\mathcal{C}_i \mid i \in I\}$ of IBN categories $\mathcal{C}_i$ and a weak equivalence $F: \mathcal{C} \to \coprod^{w}_{i \in I} \mathcal{C}_i$.*

Hence the notion of IBN ring is related to the concept of category with the Krull–Schmidt property in a much more natural way than that of local ring.

## 5 Weak Krull–Schmidt Theorems

### 5.1 Uniserial modules

A module $U_R$ is *uniserial* if the lattice $\mathcal{L}(U_R)$ of its submodules is linearly ordered under inclusion. That is, if, for any submodules $V$ and $W$ of $U$, either $V \subseteq W$ or $W \subseteq V$. A module is *serial* if it is a direct sum of uniserial submodules. Thus a module is serial and has finite Goldie dimension if and only if it is a direct sum of a finite family of uniserial submodules. (For the definition of *Goldie dimension*, see Sect. 5.2.) A ring $R$ is *serial* if the two modules $R_R$ and ${}_RR$ are both serial modules. For instance, semisimple artinian rings, commutative valuation rings, and rings of $n \times n$ triangular matrices with entries in a field are serial rings.

The next result describes the endomorphism ring of a uniserial module, showing that it has at most two maximal right (left) ideals. Recall that a *completely prime ideal* of a ring $R$ is a proper ideal $P$ of $R$ such that, for every $x, y \in R$, $xy \in P$ implies that either $x \in P$ or $y \in P$.

**Theorem 5.1** [25, Theorem 9.1] *Let $U_R$ be a uniserial module over an arbitrary ring $R$, and let $E := \mathrm{End}(U_R)$ denote its endomorphism ring. Set $I := \{f \in E \mid f$ is not injective$\}$ and $K := \{f \in E \mid f$ is not surjective$\}$. Then $I$ and $K$ are two two-sided completely prime ideals of $E$, and every proper right ideal of $E$ and every proper left ideal of $E$ is contained either in $I$ or in $K$. Moreover, exactly one of the following two conditions holds:*

(a) *Either $I$ and $K$ are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case $E$ is a local ring with maximal ideal $I \cup K$, or*
(b) *$I$ and $K$ are not comparable, and in this case $E/I$ and $E/K$ are division rings, and $E/J(E) \cong E/I \times E/K$.*

Two modules $U$ and $V$ are said to have

(1) *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;
(2) *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

For instance, it is well known that two injective modules have the same monogeny class if and only if they are isomorphic [14].

**Theorem 5.2** (Weak Krull–Schmidt Theorem, [24, Theorem 1.9]) *Let $U_1, \dots, U_n$, $V_1, \dots, V_t$ be $n + t$ non-zero uniserial right modules over a ring $R$. Then the direct sums $U_1 \oplus \cdots \oplus U_n$ and $V_1 \oplus \cdots \oplus V_t$ are isomorphic $R$-modules if and only if $n = t$ and there exist two permutations $\sigma$ and $\tau$ of $\{1, 2, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.*

As we will see in Theorem 15.2, Příhoda proved in [70] that every direct summand of a direct sum of finitely many uniserial modules is a direct sum of finitely many uniserial modules. Thus Theorem 5.2 describes the behaviour of *all* direct-sum decompositions of a direct sum of finitely many uniserial modules.

Theorem 5.2 allowed us to solve in [24] a problem posed by Warfield in 1975 [84]. In that paper, Warfield described the structure of serial rings and proved that every finitely presented module over a serial ring is serial, that is, a direct sum of uniserial modules, necessarily finitely many. The problem he posed in that paper was the uniqueness question for decompositions of a finitely presented module into uniserial summands. In other words, Warfield asked whether the Krull–Schmidt Theorem holds for direct sums of (finitely many) uniserial modules.

Warfield's question was answered completely in the negative in [24] via a counterexample: Krull–Schmidt fails for serial modules. If we fix an integer $n \geq 2$, then there exist a serial ring $R$ and $n^2$ pairwise non-isomorphic finitely presented uniserial right $R$-modules $U_{i,j}$ $(i, j = 1, 2, \dots, n)$ such that:

(a) for every $i, j, k, \ell = 1, 2, \dots, n$, $[U_{i,j}]_m = [U_{k,\ell}]_m$ if and only if $i = k$;
(b) for every $i, j, k, \ell = 1, 2, \dots, n$, $[U_{i,j}]_e = [U_{k,\ell}]_e$ if and only if $j = \ell$.

Thus the $n^2$ pairwise non-isomorphic modules $U_{i,j}$ can be arranged in an $n \times n$ square matrix in such a way that the modules on the same row have the same monogeny class, and the modules on the same column have the same epigeny class. Thus

$$U_{1,1} \oplus U_{2,2} \oplus \cdots \oplus U_{n,n} \cong U_{\sigma(1),\tau(1)} \oplus U_{\sigma(2),\tau(2)} \oplus \cdots \oplus U_{\sigma(n),\tau(n)}$$

for every pair of permutations $\sigma, \tau$ of $\{1, 2, \dots, n\}$; that is, the module $U_{1,1} \oplus U_{2,2} \oplus \cdots \oplus U_{n,n}$ is a direct sum of $n$ uniserial modules with $n!$ essentially different direct-sum decompositions into non-zero uniserial submodules, which is the maximum allowed by Theorem 5.2. (By Theorem 15.2, this is equivalent to saying that the module has $n!$ essentially different direct-sum decompositions into *indecomposable* submodules, because every indecomposable direct summand of a finite direct sum of uniserial modules is a uniserial module.)

## 5.2 Goldie dimension and dual Goldie dimension: biuniform modules

Goldie dimension appears for the first time in the studies of Alfred Goldie about the existence of quotient rings of fractions of non-commutative rings [52], under the name of *dimension* of a ring. In the literature, it is also called *Goldie rank*, *uniform rank*, or simply *rank*, or *uniform dimension*. From rings, the notion of Goldie dimension immediately passed to modules, and then to modular lattices. In the case of a module $M_R$, the Goldie dimension $\dim(M_R)$ of $M_R$ is the supremum of all cardinals $\lambda$ such that $M_R$ contains a submodule that is the internal direct sum of a family $\mathcal{F}$ of non-zero submodules of $M_R$ such that $\mathcal{F}$ has cardinality $\lambda$. Let us see how this notion generalizes to modular lattices.

Let $(L, \vee, \wedge)$ be a *modular lattice* with 0. That is, $L$ is a lattice with a smallest element 0 and such that $a \wedge (b \vee c) = (a \wedge b) \vee C$ for every $a, b, c \in L$ with $c \leq a$ (*modular identity*). If $a, b \in L$ and $a \leq b$, the *interval* between $a$ and $b$ is the subset $[a, b] := \{x \in L \mid a \leq x \leq b\}$ of $L$.

**Lemma 5.3** [53] *Let* $(L, \vee, \wedge)$ *be a modular lattice with a smallest element* 0. *The following conditions are equivalent for a finite subset* $A = \{a_i \mid i \in I\}$ *of* $L$*:*

(a) *For every* $i \in I$, *one has* $a_i \wedge (\bigvee_{j \neq i} a_j) = 0$.
(b) *If* $a_1, \dots, a_n$ *are the distinct elements of* $A$, *then* $a_\ell \wedge (\bigvee_{j<\ell} a_j) = 0$ *for every* $\ell = 2, 3, \dots, n$.
(c) $(\bigvee_{b \in B} b) \wedge (\bigvee_{c \in C} c) = 0$ *for every pair* $B, C$ *of disjoint subsets of* $A$.

A subset of $L \setminus \{0\}$ is said to be *join-independent* if all its finite subsets satisfy the equivalent conditions of Lemma 5.3. The *Goldie dimension* $\dim L$ of the modular lattice $L$ is the supremum of all cardinals $\lambda$ such that $L$ contains an independent subset of cardinality $\lambda$. In this survey, we are mainly interested in the case of *finite* Goldie dimension. This will be considered in Theorem 5.4.

We need some further terminology about modular lattices. Let $L$ be a modular lattice with 0. An element $a \in L$ is *essential* in $L$ if $a \wedge x \neq 0$ for every non-zero element $x \in L$. If $a, b$ are elements of $L$, $a \leq b$ and $a$ is essential in the lattice $[0, b]$, then $a$ is said to be *essential* in $b$.

**Theorem 5.4** [54] *Let* $L$ *be a modular lattice with* 0. *The following conditions are equivalent:*

(a) *Every join-independent subset of* $L$ *is finite.*
(b) $L$ *contains a finite join-independent subset* $\{a_1, a_2, \dots, a_n\}$ *with all the* $a_i$*'s uniform elements of* $L$ *and* $a_1 \vee a_2 \vee \dots \vee a_n$ *essential in* $L$.
(c) *The Goldie dimension* $\dim L$ *of* $L$ *is finite.*
(d) *If* $a_0 \leq a_1 \leq a_2 \leq \cdots$ *is an ascending chain of elements of* $L$, *then there exists* $i \geq 0$ *such that* $a_i$ *is essential in* $a_j$ *for every* $j \geq i$.

*Moreover, if these equivalent conditions hold and* $\{a_1, a_2, \dots, a_n\}$ *is a finite join-independent subset of* $L$ *satisfying Condition* (b)*, that is, with* $a_i$ *uniform for every* $i = 1, 2, \dots, n$ *and* $a_1 \vee a_2 \vee \ldots \vee a_n$ *essential in* $L$, *then* $\dim L = n$.

Thus, if $L$ has finite Goldie dimension $n$, then $L$ contains a join-independent subset of cardinality $n$. This property does not necessarily hold for lattices $L$ of infinite Goldie dimension $\lambda$ [19].

Our main application of Goldie dimension of modular lattices will be to the lattice of all submodules of a module $M_R$. The Goldie dimension of the lattice $\mathcal{L}(M_R)$ of all submodules of a module $M_R$ is called the *Goldie dimension* $\dim M_R$ of $M_R$. Thus, as we have said at the beginning of this Sect. 5.2, the Goldie dimension of a module $M_R$ turns out to be the supremum of all cardinals $\lambda$ such that $M_R$ contains a direct sum of $\lambda$ non-zero submodules.

The *dual Goldie dimension* $\operatorname{codim}(M_R)$ of a module $M_R$ is the Goldie dimension of the dual (=opposite) lattice of the lattice $\mathcal{L}(M)$.

We say that a module is *uniform* if it has Goldie dimension 1, *couniform* if it has dual Goldie dimension 1, *biuniform* if it is both uniform and couniform. Clearly, every non-zero uniserial module is biuniform. Theorems 5.1 and 5.2 hold not only for non-zero uniserial modules, but more generally for arbitrary biuniform modules [25, Theorems 9.1 and 9.13].

## 5.3 Another uniqueness theorem

Let us see how the concept of having the same monogeny class or the same epigeny class can be extended to a much broader setting. Assume that we have an additive functor $F: \mathcal{C}' \to \mathcal{A}$ of an additive category $\mathcal{C}'$ into an amenable semisimple category $\mathcal{A}$. Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}'$ such that every object of $\mathcal{C}'$ is a direct sum of finitely many objects of $\mathcal{C}$, and the objects of $\mathcal{C}$ are indecomposable objects in $\mathcal{C}'$. Further assume that $F(C)$ is a simple object of $\mathcal{A}$ for every object $C$ of $\mathcal{C}$.

Under these hypotheses, for every $A, B \in \mathrm{Ob}(\mathcal{C})$ and every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, either $F(f) = 0$ or $F(f)$ is an isomorphism, because $F(A)$ and $F(B)$ are simple objects of $\mathcal{A}$ (Schur's Lemma). If $A, B \in \mathrm{Ob}(\mathcal{C}')$, we write $[A]_F = [B]_F$, and say that *$A$ and $B$ belong to the same $F$-class,* if there exist morphisms $f: A \to B$ and $g: B \to A$ in $\mathcal{C}'$ with $F(f)$ and $F(g)$ isomorphisms in $\mathcal{A}$.

For example, let $R$ be a ring and $F: \mathcal{C}' \to \mathcal{A}$ be the restriction of the canonical functor $P\colon$ Mod-$R \to$ Spec(Mod-$R$) to the subcategory $\mathcal{C}'$ of Mod-$R$ whose objects are all serial right $R$-modules of finite Goldie dimension and the subcategory $\mathcal{A}$ of Spec(Mod-$R$) whose objects are all semisimple objects of Spec(Mod-$R$) of finite composition length. Let $\mathcal{C}$ be the full subcategory of $\mathcal{C}'$ whose objects are all uniserial right $R$-modules. Then the hypotheses in the first paragraph of this Sect. 5.3 are satisfied. For instance, for every object $C$ of $\mathcal{C}$, $F(C)$ is a simple object of $\mathcal{A}$ whose endomorphism ring is the division ring $\mathrm{End}_{\mathcal{A}}(F(C)) = \mathrm{End}_{\mathrm{Spec(Mod}\text{-}R)}(P(C)) \cong \mathrm{End}_{\mathrm{Spec(Mod}\text{-}R)}(P(E(C))) \cong \mathrm{End}_R(E(C))/J(\mathrm{End}_R(E(C)))$. Here $E(C)$ is the injective envelope of the $R$-module $C$, and is an indecomposable injective $R$-module. If $A, B \in \mathrm{Ob}(\mathcal{C}')$, then $[A]_F = [B]_F$ if and only if there exist morphisms $f: A \to B$ and $g: B \to A$ in $\mathcal{C}'$ with $F(f)$ and $F(g)$ isomorphisms in $\mathcal{A}$, that is, if and only if there exist morphisms $f: A \to B$ and $g: B \to A$ in $\mathcal{C}'$ with $f$ and $g$ essential monomorphisms in Mod-$R$, i.e., if and only if $A$ and $B$ have the same monogeny class.

The following result has been proved in [3] and is a generalization of [20, Theorem 1.1].

**Theorem 5.5** *Let $A_1, \dots, A_n, B_1, \dots, B_t$ be objects of $\mathcal{C}$. Then $[A_1 \oplus \cdots \oplus A_n]_F = [B_1 \oplus \cdots \oplus B_t]_F$ if and only if $n = t$ and there is a permutation $\sigma$ of $\{1, 2, \dots, n\}$ such that $[A_i]_F = [B_{\sigma(i)}]_F$ for every $i = 1, 2, \dots, n$.*

We can now sketch a proof of the necessity in the statement of Theorem 5.2. Let $U_1, \dots, U_n, V_1, \dots, V_t$ be non-zero uniserial right modules (or, more generally, biuniform modules) over a ring $R$. Assume $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$. As all the modules $U_i$ are uniform, that is, of Goldie dimension 1, the Goldie dimension of $U_1 \oplus \cdots \oplus U_n$ is $n$. Similarly, the Goldie dimension of $V_1 \oplus \cdots \oplus V_t$ is $t$. Thus $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$ implies $n = t$. In order to prove the existence of the permutation $\sigma$, apply Theorem 5.5 to the functor $F$ restriction of the functor $P\colon$ Mod-$R \to$ Spec(Mod-$R$) described above. To prove the existence of the permutation $\tau$, apply Theorem 5.5 to the restriction of the dual functor $F\colon$ Mod-$R \to$ (Mod-$R$)$'$.

Notice that the description of the functor $P$ only uses that the modules are uniform, and that the proof of the necessity in Theorem 5.2 works also for biuniform modules.

## 5.4 Cyclically presented modules over local rings

The behavior of uniserial modules described in Sect. 5.1 is enjoyed by other classes of modules, which we will present in this and the following two subsections. Here is a first example, which was studied in [1]. A right module over a ring $R$ is *cyclically presented* if it is isomorphic to $R/aR$ for some element $a \in R$. For any ring $R$, we will denote with $U(R)$ the group of all invertible elements of $R$.

If $R/aR$ and $R/bR$ are cyclically presented modules over a local ring $R$, we say that $R/aR$ and $R/bR$ *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$. It is possible to prove, though we will not need it here, that two cyclically presented modules over a local ring have the same lower part if and only if their Auslander–Bridger transposes have the same epigeny class [1].

We have a description of the endomorphism ring of a cyclically presented module over a local ring similar to that of the endomorphism ring of a uniserial module. It is easy to check that the endomorphism ring $\mathrm{End}_R(R/aR)$ of a non-zero cyclically presented module $R/aR$ is isomorphic to $E/aR$, where $E := \{r \in R \mid ra \in aR\}$ is the *idealizer* of $aR$. The next theorem is the analog of Theorem 5.1 for cyclically presented modules over a local ring.

**Theorem 5.6** *Let $a$ be a non-zero non-invertible element of an arbitrary local ring $R$, let $E$ be the idealizer of $aR$, and let $E/aR$ be the endomorphism ring of the cyclically presented right $R$-module $R/aR$. Set $I := \{r \in R \mid ra \in aJ(R)\}$ and $K := J(R) \cap E$. Then $I$ and $K$ are two two-sided completely prime ideals of $E$ containing $aR$, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of $E/aR$, and every proper right ideal of $E/aR$ and every proper left ideal of $E/aR$ is contained either in $I/aR$ or in $K/aR$. Moreover, exactly one of the following two conditions holds:*

(a) *Either $I$ and $K$ are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case $E/aR$ is a local ring, or*

(b) *$I$ and $K$ are not comparable, and in this case $E/I$ and $E/K$ are division rings, $J(E/aR) = (I \cap K)/aR$, and $(E/aR)/J(E/aR)$ is canonically isomorphic to the direct product $E/I \times E/K$.*

Notice that the statement of Theorem 5.6 does not consider the two cases $a = 0$ and $a$ invertible, but this is not a problem because these two cases are trivial. (If $a = 0$, then $R/aR \cong R_R$ and its endomorphism ring is isomorphic to $R$, which is a local ring, hence $R_R$ has a good behavior as far as direct sums are concerned. Also, if $a$ is invertible, then $R/aR = 0$, and there is not much to say in this case.)

**Theorem 5.7** (Weak Krull–Schmidt Theorem) *Let $a_1, \dots, a_n, b_1, \dots, b_t$ be $n + t$ non-invertible elements of a local ring $R$. Then the direct sums $R/a_1R \oplus \cdots \oplus R/a_nR$ and $R/b_1R \oplus \cdots \oplus R/b_tR$ are isomorphic right $R$-modules if and only if $n = t$ and there exist two permutations $\sigma, \tau$ of $\{1, 2, \dots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \dots, n$.*

This theorem has an immediate consequence as far as equivalence of matrices is concerned. Recall that two $m \times n$ matrices $A$ and $B$ with entries in a ring $R$ are said to be *equivalent* matrices, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix $P$ and an $n \times n$ invertible matrix $Q$ with entries in $R$ (that is, matrices invertible in the rings $M_m(R)$ and $M_n(R)$, respectively) such that $B = PAQ$. We denote by $\mathrm{diag}(a_1, \dots, a_n)$ the $n \times n$ diagonal matrix whose $(i, i)$ entry is $a_i$ and whose other entries are zero.

It is easy to check that if $R$ is a *commutative* local ring and $a_1, \dots, a_n, b_1, \dots, b_n$ are elements of $R$, then $\mathrm{diag}(a_1, \dots, a_n) \sim \mathrm{diag}(b_1, \dots, b_n)$ if and only if there exists a permutation $\sigma$ of $\{1, 2, \dots, n\}$ with $a_i$ and $b_{\sigma(i)}$ associate elements of $R$ for every $i = 1, 2, \dots, n$. Here $a, b \in R$ are *associate elements* if they generate the same principal ideal of $R$. (For the proof, assume $\mathrm{diag}(a_1, \dots, a_n) \sim \mathrm{diag}(b_1, \dots, b_n)$. Then the modules $R/a_1R \oplus \cdots \oplus R/a_nR$ and $R/b_1R \oplus \cdots \oplus R/b_tR$ are isomorphic. But the endomorphism rings of the modules $R/a_iR$, $R/b_jR$ are the rings $R/a_iR$, $R/b_jR$, which are local rings, so that we can apply the Krull–Schmidt-Remak-Azumaya Theorem 1.2, and find that there exists a permutation $\sigma$ of $\{1, 2, \dots, n\}$ such that $R/a_iR \cong R/b_{\sigma(i)}R$ for every $i \in I$. Now two isomorphic modules have the same annihilator, so that $a_iR = b_{\sigma(i)}R$, that is, $a_i$ and $b_{\sigma(i)}$ are associate elements. The opposite implication is trivial)

If the ring $R$ is local, but non-necessarily commutative, we have the following result, which is a corollary of Theorem 5.7:

**Proposition 5.8** [1] *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be elements of a local ring $R$. Then* $\mathrm{diag}(a_1, \dots, a_n) \sim \mathrm{diag}(b_1, \dots, b_n)$ *if and only if there exist two permutations $\sigma, \tau$ of $\{1, 2, \dots, n\}$ with*

$$[R/a_iR]_l = [R/b_{\sigma(i)}R]_l \quad \text{and} \quad [R/a_iR]_e = [R/b_{\tau(i)}R]_e$$

*for every $i = 1, 2, \dots, n$.*

## 5.5 Kernels of morphisms between indecomposable injective modules

Here is another class of modules with the same behavior as uniserial modules and cyclically presented modules over local rings. This class of modules was studied in [30].

For a right module $A_R$ over a ring $R$, let $E(A_R)$ denote the injective envelope of $A_R$. We say that two modules $A_R$ and $B_R$ *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi\colon E(A_R) \to E(B_R)$ and a homomorphism $\psi\colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$.

There is a standard technique of homological algebra to extend a morphism between two modules to their injective resolutions. We need fix the notation. Assume that $E_0, E_1, E_0', E_1'$ are indecomposable injective right modules over a ring $R$, and that $\varphi\colon E_0 \to E_1, \varphi'\colon E_0' \to E_1'$ are two right $R$-module morphisms. A morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to a morphism $f_0\colon E_0 \to E_0'$. Now $f_0$ induces a morphism $\widetilde{f}_0\colon E_0/\ker\varphi \to E_0'/\ker\varphi'$, which extends to a morphism $f_1\colon E_1 \to E_1'$. Thus we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker\varphi & \longrightarrow & E_0 & \xrightarrow{\ \varphi\ } & E_1 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} \\
0 & \longrightarrow & \ker\varphi' & \longrightarrow & E_0' & \xrightarrow[\ \varphi'\ ]{} & E_1'.
\end{array} \tag{1}$$

The morphisms $f_0$ and $f_1$ are not uniquely determined by $f$.

Here is the description of the endomorphism rings of these modules that are kernels of morphisms $\varphi\colon E_0 \to E_1$ between indecomposable injective modules $E_0$ and $E_1$. Notice that if $\varphi$ is the zero morphism, then the kernel of $\varphi$ is $E_0$, whose endomorphism ring is a local ring, hence has a good behavior as far as direct sums are concerned. Also, if $\varphi$ is an injective morphism, that is, a monomorphism, then the kernel of $\varphi$ is 0, and there is not much to say in this case. Hence we can consider only morphisms $\varphi\colon E_0 \to E_1$ between injective indecomposable modules $E_0$ and $E_1$ with $\varphi$ non-zero and non-injective.

**Theorem 5.9** *Let $E_0$ and $E_1$ be indecomposable injective right modules over a ring $R$, and let $\varphi\colon E_0 \to E_1$ be a non-zero non-injective morphism. Let $S := \mathrm{End}_R(\ker\varphi)$ denote the endomorphism ring of $\ker\varphi$. Set $I := \{f \in S \mid$ the endomorphism $f$ of $\ker\varphi$ is not a monomorphism$\}$ and $K := \{f \in S \mid$ the endomorphism $f_1$ of $E_1$ is not a monomorphism$\} = \{f \in S \mid \ker\varphi \subset f_0^{-1}(\ker\varphi)\}$. Then $I$ and $K$ are two two-sided completely prime ideals of $S$, and every proper right ideal of $S$ and every proper left ideal of $S$ is contained either in $I$ or in $K$. Moreover, exactly one of the following two conditions holds:*

(a) *Either $I$ and $K$ are comparable (that is, $I \subseteq K$ or $K \subseteq I$), in which case $S$ is a local ring with maximal ideal $I \cup K$, or*
(b) *$I$ and $K$ are not comparable, and in this case $S/I$ and $S/K$ are division rings and $S/J(S) \cong S/I \times S/K$.*

**Theorem 5.10** (Weak Krull–Schmidt Theorem) *Let* $\varphi_i : E_{i,0} \to E_{i,1}$ $(i = 1, 2, \ldots, n)$ *and* $\varphi'_j : E'_{j,0} \to E'_{j,1}$ $(j = 1, 2, \ldots, t)$ *be* $n + t$ *non-injective morphisms between indecomposable injective right modules* $E_{i,0}$, $E_{i,1}$, $E'_{j,0}$, $E'_{j,1}$ *over an arbitrary ring* $R$. *Then the direct sums* $\oplus_{i=0}^{n} \ker \varphi_i$ *and* $\oplus_{j=0}^{t} \ker \varphi'_j$ *are isomorphic* $R$*-modules if and only if* $n = t$ *and there exist two permutations* $\sigma, \tau$ *of* $\{1, 2, \ldots, n\}$ *such that* $[\ker \varphi_i]_m = [\ker \varphi'_{\sigma(i)}]_m$ *and* $[\ker \varphi_i]_u = [\ker \varphi'_{\tau(i)}]_u$ *for every* $i = 1, 2, \ldots, n$.

Thus also in this case we find exactly the same behavior: at most two maximal ideals and the same weak form of the Krull–Schmidt Theorem.

## 5.6 Couniformly presented modules

We will now present a class of modules that has been introduced and studied in [31]. Again, we will find that it has the same behavior: two maximal ideals and a weak Krull–Schmidt Theorem. This further class of modules is over arbitrary rings and properly contains the class of cyclically presented modules over local rings we have seen in Sect. 5.4.

Let us first introduce the class of couniform projective right modules, that is, projective modules of dual Goldie dimension one (the projective modules where the sum of any two proper submodules is a proper submodule).

**Lemma 5.11** [1, Lemma 8.7] *Let* $R$ *be an arbitrary ring. The following conditions are equivalent for a projective right module* $P_R$*:*

(1) $P_R$ *is couniform.*
(2) *The module* $P_R$ *is the projective cover of a simple module.*
(3) $\mathrm{End}(P_R)$ *is a local ring.*
(4) *There exists an idempotent* $e \in R$ *with* $eRe$ *a local ring and* $P_R \cong eR$.
(5) $P_R$ *is a finitely generated module with a unique maximal submodule.*
(6) $P_R$ *is finitely generated, nonzero, and all its proper submodules are superfluous.*
(7) *The projective left* $R$*-module* $\mathrm{Hom}(P_R, R)$ *is couniform.*

We will say that a module $M_R$ is *couniformly presented* if it is non-zero and there exists an exact sequence

$$0 \longrightarrow C_R \xrightarrow{\iota} P_R \longrightarrow M_R \longrightarrow 0 \tag{2}$$

with both $C_R$ and $P_R$ couniform and $P_R$ projective. Under these hypotheses, we will say that (2) is a *couniform presentation* of the couniformly presented module $M_R$. The setting is dual to that of Sect. 5.5. The dual of the notion of kernel of a morphism between indecomposable injective (=uniform injective) right modules is that of cokernel of a morphism between couniform projective right modules, that is, couniformly presented modules.

We can now dualize the construction of Diagram (1), as follows. Let $M_R$ and $M'_R$ be couniformly presented modules and $0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0$, $0 \to C'_R \xrightarrow{\iota'} P'_R \to M'_R \to 0$ be two couniform presentations. Every morphism $f : M_R \to M'_R$

lifts to a morphism $f_0 \colon P_R \to P'_R$ of the projective covers. Let $f_1 \colon C_R \to C'_R$ be the restriction of $f_0$. We get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_R & \xrightarrow{\iota} & P_R & \longrightarrow & M_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C'_R & \xrightarrow{\iota'} & P'_R & \longrightarrow & M'_R & \longrightarrow & 0. \end{array}$$

In the case where $M'_R = M_R$, we obtain the following description of the endomorphism ring of a couniformly presented right module $M_R$.

**Theorem 5.12** *Let $M_R$ be a couniformly presented module over an arbitrary ring $R$ and let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of $M_R$. Set $K := \{f \in \mathrm{End}(M_R) \mid f$ is not surjective$\}$ and $I := \{f \in \mathrm{End}(M_R) \mid f_1 \colon C_R \to C_R$ is not surjective$\}$. Then $K$ and $I$ are two two-sided completely prime ideals of $\mathrm{End}(M_R)$, and every proper right ideal of $\mathrm{End}(M_R)$ and every proper left ideal of $\mathrm{End}(M_R)$ is contained either in $K$ or in $I$, so that $K \cup I$ is the set of all zero-divisors of $\mathrm{End}(M_R)$. Moreover, exactly one of the following two conditions holds:*

(a) *Either $K$ and $I$ are comparable, in which case $\mathrm{End}(M_R)$ is a local ring with maximal ideal $K \cup I$, or*
(b) *$I$ and $K$ are not comparable, and in this case $\mathrm{End}(M_R)/I$ and $\mathrm{End}(M_R)/K$ are division rings, $J(\mathrm{End}(M_R)) = K \cap I$, and $\mathrm{End}(M_R)/J(\mathrm{End}(M_R))$ is canonically isomorphic to the direct product $\mathrm{End}(M_R)/K \times \mathrm{End}(M_R)/I$.*

The definition of "having the same lower part" we gave in Sect. 5.4 for cyclically presented modules over a local ring can now be extended to arbitrary couniformly presented modules over an arbitrary ring $R$, as follows. Let $M_R$ and $M'_R$ be two couniformly presented modules and let $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$ be two couniform presentations of $M, M'_R$ respectively. We say that $M_R$ and $M'_R$ *have the same lower part,* and write $[M_R]_\ell = [M'_R]_\ell$, if there exist two homomorphisms $f_0 \colon P_R \to P'_R$ and $f'_0 \colon P'_R \to P_R$ with $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$. This notion is clearly dual to the notion of "having the same upper part" given at the beginning of Sect. 5.5.

**Theorem 5.13** (Weak Krull–Schmidt Theorem for couniformly presented modules) *Let $M_1, \dots, M_n, N_1, \dots, N_t$ be $n + t$ couniformly presented right $R$-modules. Then the direct sums $M_1 \oplus \cdots \oplus M_n$ and $N_1 \oplus \cdots \oplus N_t$ are isomorphic if and only if $n = t$ and there exist two permutations $\sigma, \tau$ of $\{1, 2, \dots, n\}$ such that $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ and $[M_i]_e = [N_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.*

## 5.7 Completely prime ideals

We conclude this Section describing one of the possible general patterns that allow to treat all the previous examples at the same time. Let $\mathcal{C}$ be a full subcategory of the category Mod-$R$ for some ring $R$ and assume that every object of $\mathcal{C}$ is an indecomposable right $R$-module. Define a *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ as an assignement of a subgroup $\mathcal{P}(A, B)$ of the additive abelian group $\mathrm{Hom}_R(A, B)$ to every pair $(A, B)$ of objects of $\mathcal{C}$ with the following two properties: (1) for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, every $f : A \to B$ and every $g : B \to C$, one has that $gf \in \mathcal{P}(A, C)$ if and only if either $f \in \mathcal{P}(A, B)$ or $g \in \mathcal{P}(B, C)$; (2) $\mathcal{P}(A, A)$ is a proper subgroup of $\mathrm{Hom}_R(A, A)$ for every object $A \in \mathrm{Ob}(\mathcal{C})$. Thus, if $\mathcal{P}$ is a completely prime ideal of $\mathcal{C}$, then, in the quotient category $\mathcal{C}/\mathcal{P}$, the endomorphism ring of every object is a (non-necessarily commutative) integral domain. We need a further definition. Let $\mathcal{P}$ be a completely prime ideal of $\mathcal{C}$. If $A, B$ are objects of $\mathcal{C}$, we say that $A$ and $B$ *have the same* $\mathcal{P}$ *class*, and write $[A]_\mathcal{P} = [B]_\mathcal{P}$, if there exist right morphisms $f : A \to B$ and $g : B \to A$ with $f \notin \mathcal{P}(A, B)$ and $g \notin \mathcal{P}(B, A)$.

**Theorem 5.14** (Weak Krull–Schmidt Theorem [43, Theorem 6.2]) *Let $\mathcal{C}$ be a full subcategory of* Mod-$R$ *and $\mathcal{P}, \mathcal{Q}$ be two completely prime ideals of $\mathcal{C}$. Assume that all objects of $\mathcal{C}$ are indecomposable right $R$-modules and that, for every $A \in$* $\mathrm{Ob}(\mathcal{C})$, *$f : A \to A$ is an automorphism of $A$ if and only if $f \notin \mathcal{P}(A, A) \cup \mathcal{Q}(A, A)$. Then, for every $A_1, \dots, A_n, B_1, \dots, B_t \in$* $\mathrm{Ob}(\mathcal{C})$, *the modules $A_1 \oplus \cdots \oplus A_n$ and $B_1 \oplus \cdots \oplus B_t$ are isomorphic if and only if $n = t$ and there exist two permutations $\sigma, \tau$ of* $\{1, 2, \dots, n\}$ *such that $[A_i]_\mathcal{P} = [B_{\sigma(i)}]_\mathcal{P}$ and $[A_i]_\mathcal{Q} = [B_{\tau(i)}]_\mathcal{Q}$ for all $i = 1, \dots, n$.*

For all the classes $\mathcal{C}$ of modules described in this Section, the fact that the weak form of the Krull–Schmidt Theorem holds can be described saying that the corresponding monoid $V(\mathcal{C})$ is a subdirect product of two free monoids.

Another class of modules that can be described via two invariants is that of Auslander–Bridger modules [32]. For Auslander–Bridger modules, the two invariants are epi-isomorphism and lower-isomorphism. All these modules (uniserial modules, couniformly presented modules, Auslander–Bridger modules, and modules that can be obtained from these closing under finite direct sums and direct summands) have a semilocal endomorphism ring. The rest of this survey will be devoted to the study of arbitrary modules with a semilocal endomorphism ring.

## 6 Semilocal rings

In commutative Algebra, a commutative ring is *semilocal* if it has only finitely many maximal ideals. In non-commutative Algebra, an arbitrary (associative) ring $R$ is said to be *semilocal* if $R/J(R)$ is semisimple artinian. Luckily, the two definitions agree for commutative rings, as is easy to prove. Every local ring is clearly semilocal. Every semisimple artinian ring is semilocal.

It is possible to prove that semilocal rings are exactly rings of finite dual Goldie dimension. More precisely, it is possible to prove that:

**Proposition 6.1** *The following conditions are equivalent for a ring $R$.*

(1) *The ring $R$ is semilocal.*
(2) *The right $R$-module $R_R$ has finite dual Goldie dimension.*
(3) *The left $R$-module ${}_RR$ has finite dual Goldie dimension.*

*Moreover, if these equivalent conditions hold, then*

$$\operatorname{codim}(R_R) = \operatorname{codim}({}_R R) = \dim(R/J(R)).$$

### 6.1 Modules with a semilocal endomorphism ring

We will be mainly interested in modules $M_R$ whose endomorphism ring $\operatorname{End}(M_R)$ is a semilocal ring. For instance, if $M_R$ is a module that is either uniserial, or more generally biuniform, or cyclically presented with $R$ local, or more generally couniformly presented over an arbitrary ring $R$, or the kernel of a morphism between two indecomposable injective modules, then either $M_R = 0$, or $M_R$ has a local endomorphism ring $\operatorname{End}(M_R)$, or $\operatorname{End}(M_R)$ has exactly two maximal ideals. Thus, except for the trivial case $M_R = 0$, we find that $\operatorname{End}(M_R)/J(\operatorname{End}(M_R))$ is either a division ring or the direct product of two division rings. In both cases, $\operatorname{End}(M_R)/J(\operatorname{End}(M_R))$ is semisimple artinian, so that all these modules $M_R$ have a semilocal endomorphism ring.

"Having a semilocal endomorphism ring $\operatorname{End}(M_R)$" is a finiteness condition on the module $M_R$. For instance, it is possible to prove that:

(1) Any module with a semilocal endomorphism ring is a direct sum of finitely many indecomposable modules.
(2) A module with a semilocal endomorphism ring is not a direct sum of infinitely many non-zero submodules.
(3) Every module with a semilocal endomorphism ring is directly finite. That is, if $M_R, N_R$ are modules and $\operatorname{End}(M_R)$ is semilocal, then $M_R \oplus N_R \cong M_R$ implies $N_R = 0$.
(4) More generally, modules with a semilocal endomorphism ring cancel from direct sums. That is, if $M_R, N_R, N'_R$ are modules and $\operatorname{End}(M_R)$ is semilocal, then $M_R \oplus N_R \cong M_R \oplus N'_R$ implies $N_R \cong N'_R$.
(5) *($n$-th root property)* If $n \geq 1$ is an integer, $M_R, N_R$ are modules and $\operatorname{End}(M_R)$ is semilocal, then $M_R^n \cong N_R^n$ implies $M_R \cong N_R$.
(6) The class of all modules with semilocal endomorphism rings is closed under direct summands. That is, if $M_R, N_R$ are modules, $N_R$ is isomorphic to a direct summand of $M_R$ and $\operatorname{End}(M_R)$ is semilocal, then $\operatorname{End}(N_R)$ is semilocal.
(7) The class of all modules with semilocal endomorphism rings is closed under finite direct sums. That is, if $M_R, N_R$ are modules and $\operatorname{End}(M_R)$, $\operatorname{End}(N_R)$ are both semilocal rings, then $\operatorname{End}(M_R \oplus N_R)$ is a semilocal ring.
(8) If $M_R$ is a module with $\operatorname{End}(M_R)$ semilocal, then $M_R$ has only finitely many direct summands up to isomorphism. In particular, if $M_R$ is a module with $\operatorname{End}(M_R)$ semilocal, then $M_R$ has only finitely many direct-sum decompositions up to isomorphism. That is, essentially finitely many direct-sum decompositions in the sense of Theorem 1.3.

Some of the proofs of (1)–(8) are immediate. For the others, see [25], in particular Section 4.2.

### 6.2 Examples

Answering a question posed by Menal in [64, Question 16], Camps and Dicks proved in [15] that:

**Corollary 6.2** *The endomorphism ring of an artinian module is a semilocal ring.*

As we have anticipated in Section 1 (Sect. 1.2), the Krull–Schmidt theorem does not holds for artinian modules, but we will come back to this in Corollary 9.4.

Apart from artinian modules, there are several other classes of modules whose endomorphism ring is semilocal. The main sources of examples are the papers [59] and [38]. The following modules have a semilocal endomorphism rings:

(1) Finitely generated modules over a commutative semilocal ring [85, Lemma 2.3].
(2) Finitely presented modules over a semilocal ring [38, Theorem 3.3]. (Notice that there exist finitely generated modules over non-commutative semilocal rings whose endomorphism rings are not semilocal [38, Example 3.5].)
(3) Modules of finite Goldie dimension and finite dual Goldie dimension [59]
(4) Every submodule of a quotient finite dimensional injective module [38, Corollary 5.8]. (A module $M$ is *quotient finite dimensional* if every homomorphic image of $M$ has finite Goldie dimension.)
(5) Every submodule of an injective serial right module of finite Goldie dimension [38, Corollary 5.10].
(6) Finite-rank torsion-free modules over any semilocal commutative noetherian domain of Krull dimension 1 [38, Corollary 5.9].
(7) Finite-rank torsion-free modules over any valuation domain [85, Theorem 5.4].
(8) Linearly compact modules [59].

## 7 Local morphisms

We will now describe one of the main techniques to prove that a module has a semilocal endomorphism ring. For instance, with this technique, it can be proved that most of the modules of the examples in Sect. 6.2 have semilocal endomorphism rings. We need the concept of local morphism.

Let $R$ and $S$ be local commutative rings with maximal ideals $\mathcal{M}$ and $\mathcal{N}$ respectively. In Algebraic Geometry and Commutative Algebra, a ring morphism $\varphi\colon R \to S$ is called a *local morphism* if $\varphi(\mathcal{M}) \subseteq \mathcal{N}$. We need this concept when $R$ and $S$ are arbitrary associative rings with identity (not necessarily commutative and not necessarily local). In this case, a ring morphism $\varphi\colon R \to S$ is said to be a *local* morphism if, for every $r \in R$, $\varphi(r)$ invertible in $S$ implies $r$ invertible in $R$. It is immediate to check that the two definitions coincide when $R$ and $S$ are local commutative rings. The notion of local morphism for non-commutative rings was introduced, in the case in which $S$ was a division ring, by Cohn [18].

The reason why local morphisms enter the picture is the following characterization of semilocal rings:

**Theorem 7.1** (Camps and Dicks [15]) *A ring $R$ is semilocal if and only if there exists a local morphism of $R$ into a semilocal ring, if and only if there exists a local morphism of $R$ into a semisimple artinian ring.*

More precisely, in view of Proposition 6.1:

**Proposition 7.2** *For any local morphism $R \to S$,* $\operatorname{codim}(R_R) \leq \operatorname{codim}(S_S)$.

Let us see how these results allow us to prove that the endomorphism of a module $M_R$ with suitable properties is semilocal.

(1) Let us begin with the case in which $M_R$ is an artinian module. Let $\operatorname{soc}(M_R)$ be the *socle* of $M_R$, that is, the sum of all the simple submodules of $M_R$. For any artinian module $M_R$, $\operatorname{soc}(M_R)$ is an essential semisimple submodule of $M_R$ of finite composition length. Every endomorphism of $M_R$ restricts to an endomorphism of the socle $\operatorname{soc}(M_R)$. Hence restriction gives a ring morphism $\Psi\colon \operatorname{End}(M_R) \to \operatorname{End}(\operatorname{soc}(M_R))$. Let us prove that the morphism $\Psi$ is local. If $f$ is an endomorphism of $M_R$ and its restriction to $\operatorname{soc}(M_R)$ is invertible in the ring $\operatorname{End}(\operatorname{soc}(M_R))$, then $f$ is injective because $\operatorname{soc}(M_R)$ is essential in $M_R$. But every injective endomorphism of an artinian module is an automorphism. Thus $f$ is an automorphism, that is, an invertible element of $\operatorname{End}(M_R)$. This proves that $\Psi$ is a local morphism. Now $\operatorname{soc}(M_R)$ is a semisimple module of finite composition length, so that its endomorphism ring $\operatorname{End}(\operatorname{soc}(M_R))$ is semisimple artinian. Theorem 7.1 now tells us that $\operatorname{End}(M_R)$ is a semilocal ring.

(2) Generalizing (1), let us prove that if $M_R$ is any module of finite Goldie dimension and all injective endomorphisms of $M_R$ are automorphisms, then the endomorphism ring $\operatorname{End}(M_R)$ of $M_R$ is semilocal. For this, consider the canonical functor $P\colon \text{Mod-}R \to \operatorname{Spec}(\text{Mod-}R)$ of Mod-$R$ into its spectral category (Sect. 3.3). The functor $P$ induces a ring morphism $P_{M_R}\colon \operatorname{End}(M_R) \to \operatorname{End}_{\operatorname{Spec}(\text{Mod-}R)}(P(M_R))$. Since $M_R$ is of finite Goldie dimension, $P(M_R) = P(E(M_R))$ is a semisimple object of finite composition length in the category Spec(Mod-$R$). Therefore its endomorphism ring $\operatorname{End}_{\operatorname{Spec}(\text{Mod-}R)}(P(M_R))$ is a semisimple artinian ring. The ring morphism $P_{M_R}$ is a local morphism, because if $f$ is an endomorphism of $M_R$ and $P(f)$ is an automorphism of $P(M_R) = P(E(M_R))$, then $f$ is an essential monomorphism. As injective endomorphisms of $M_R$ are automorphisms by hypothesis, we get that $f$ is invertible in $\operatorname{End}(M_R)$. Thus $\operatorname{End}(M_R)$ is a semilocal ring by Theorem 7.1.

(3) Let us see how one proves that every module of finite Goldie dimension $n$ and finite dual Goldie dimension $m$ has a semilocal endomorphism ring of dual Goldie dimension $\leq n+m$. In the previous paragraph, we have already considered the canonical functor $P\colon \text{Mod-}R \to \operatorname{Spec}(\text{Mod-}R)$. Now consider its dual $F\colon \text{Mod-}R \to (\text{Mod-}R)'$ also, where $(\text{Mod-}R)'$ is the category obtained from Mod-$R$ with the construction dual to the construction of the spectral category, that is, the category obtained inverting all superfluous epimorphisms of Mod-$R$ (Sect. 3.3). For any module $M_R$ with $\dim(M_R) = n$ and $\operatorname{codim}(M_R) = m$, the product functor $P \times F\colon \text{Mod-}R \to \operatorname{Spec}(\text{Mod-}R) \times (\text{Mod-}R)'$ induces a ring morphism

$$\operatorname{End}(M_R) \to \operatorname{End}_{\operatorname{Spec}(\text{Mod-}R)}(P(M_R)) \times \operatorname{End}_{(\text{Mod-}R)'}(F(M_R)),$$

which turns out to be a local morphism. The rings $\operatorname{End}_{\operatorname{Spec}(\text{Mod-}R)}(P(M_R))$ and $\operatorname{End}_{(\text{Mod-}R)'}(F(M_R))$ are semisimple artinian rings of Goldie dimension $n$ and $m$

respectively. In particular, they are semilocal rings of dual Goldie dimension $n$ and $m$ respectively. Thus $\operatorname{codim}(\operatorname{End}(M_R)) \leq n + m$ by Proposition 7.2.

(4) We will now sketch the proof that every finitely presented module over a semilocal ring has a semilocal endomorphism ring. Let $M_R$ be a finitely presented right module over a semilocal ring $R$. Let $\mathcal{C}$ be the full subcategory of Mod-$R$ whose objects are all right $R$-modules with a projective cover. Let $F\colon$ Mod-$R \to$ (Mod-$R$)$'$ be the canonical functor (Sect. 3.3). The kernel of the functor $F$ is the ideal $\mathcal{K}$ of Mod-$R$ of all morphisms with a superfluous image, so that $F$ induces a faithful functor $\overline{F}\colon$ Mod-$R/\mathcal{K} \to$ (Mod-$R$)$'$. For every object $M_R$ of $\mathcal{C}$, fix a projective cover $\pi_M\colon P(M) \to M_R$. Let $K\colon \mathcal{C} \to$ Mod-$R/\mathcal{K}$ be the functor that maps the object $M_R$ of $\mathcal{C}$ to the object $\ker \pi_M$ of Mod-$R/\mathcal{K}$. If $f\colon M_R \to N_R$ is a morphism in $\mathcal{C}$, $f$ lifts to a morphism $f_0\colon P(M) \to P(N)$, which restricts to a morphism $f_1\colon \ker \pi_M \to \ker \pi_N$ in Mod-$R$. Let $K$ map the morphism $f$ of $\mathcal{C}$ to the image of $f_1$ in Mod-$R/\mathcal{K}$. Then $K$ turns out to be a well defined functor.

Let $F_{(1)}\colon \mathcal{C} \to$ (Mod-$R$)$'$ be the composite functor of the functors $K\colon \mathcal{C} \to$ Mod-$R/\mathcal{K}$ and $\overline{F}\colon$ Mod-$R/\mathcal{K} \to$ (Mod-$R$)$'$. For every object $M_R$ of $\mathcal{C}$, the product functor $F \times F_{(1)}\colon \mathcal{C} \to$ (Mod-$R$)$' \times$ (Mod-$R$)$'$ induces a ring morphism $\operatorname{End}(M_R) \to \operatorname{End}_{(\text{Mod-}R)'}(F(M_R)) \times \operatorname{End}_{(\text{Mod-}R)'}(F_{(1)}(M_R))$, which turns out to be a local morphism. Thus, if $K_R$ is a superfluous submodule of a projective module $P_R$ and $K_R$, $P_R$ have finite dual Goldie dimension, then $\operatorname{End}(P_R/K_R)$ is a semilocal ring of dual Goldie dimension $\leq \operatorname{codim}(K_R) + \operatorname{codim}(P_R)$. Now every finitely presented module over a semilocal ring is isomorphic to a direct summand of such a quotient $P_R/K_R$. Since the class of modules with semilocal endomorphism rings is closed under direct summands, it is possible to conclude that the endomorphism ring of any finitely presented right module over a semilocal ring is a semilocal ring.

## 8 Krull monoids

### 8.1 Divisor homomorphisms and pullbacks

Let $M$, $M'$ be (commutative additive) monoids and $\leq$ be their algebraic pre-order (Section 2). A monoid homomorphism $f\colon M \to M'$ is a *divisor homomorphism* if, for every $x, y \in M$, $f(x) \leq f(y)$ implies $x \leq y$.

In order to give our main example of divisor homomorphism, recall that $V$ can be viewed as a functor of the category of rings with identity into the category of commutative monoids with order-unit (Sect. 2.1). (It can also be seen as a functor of the category of rings with identity into the category of commutative monoids). If $R$ is a ring and we apply the functor $V$ to the canonical projection $\pi\colon R \to R/J(R)$, we get a morphism of monoids with order-unit $V(\pi)\colon (V(R), \langle R\rangle) \to (V(R/J(R)), \langle R/J(R)\rangle)$.

**Proposition 8.1** ([36, Lemma 2.10 and Proposition 2.11], [4, Lemma 17.17]) *For any ring $R$, the monoid morphism $V(\pi)\colon (V(R), \langle R\rangle) \to (V(R/J(R)), \langle R/J(R)\rangle)$ is a divisor homomorphism and is an injective mapping.*

More generally, any additive functor $F\colon \mathcal{A} \to \mathcal{B}$ induces a monoid homomorphism $V(F)\colon V(\mathcal{A}) \to V(\mathcal{B})$ (notice that these monoids are large if the categories are not skeletally small). Clearly, the functor $F$ is direct-summand reflecting if and only if the monoid homomorphism $V(F)$ is a divisor homomorphism. Also, the functor $F$ is isomorphism reflecting if and only if $V(F)$ is an injective mapping, dense if and only if $V(F)$ is a surjective mapping, a weak equivalence if and only if $V(F)$ is a bijection.

Recall that for any commutative additive monoid $M$, the *enveloping group* (or *Grothendieck group*, or *group of differences*) of $M$ is "the abelian group $G(M)$ generated by $M$". More precisely, let $M$ be a monoid and consider the cartesian product $M \times M$, that is, the set of all pairs $(x, y)$ with $x, y \in M$. Define an equivalence relation $\equiv$ on $M \times M$ setting $(x, y) \equiv (x', y')$ if there exists $z \in M$ such that $x + y' + z = x' + y + z$. Let $x - y$ denote the equivalence class of $(x, y)$ modulo the equivalence relation $\equiv$. The enveloping group $G(M)$ of $M$ is the abelian group whose elements are all $x - y$, with $x, y \in M$, and in which the addition is defined by

$$(x - y) + (x' - y') = (x + x') - (y + y').$$

There is a canonical monoid morphism $d_M : M \to G(M)$, defined by $d_M(x) = x - 0$ for every $x \in M$. The kernel of $d_M$ is the smallest of the congruences $\sim$ on the monoid $M$ with $M/\!\sim$ cancellative. In particular, if $M$ is cancellative, the mapping $d_M$ is injective, that is, a monoid embedding. Every monoid morphism $f : M \to M'$ induces a group morphism $G(f): G(M) \to G(M')$, so that $G$ is a functor of the category CMon of commutative monoids into the category Ab of abelian groups. (In fact, $d$ is a natural transformation of the identity functor CMon $\to$ CMon into the composite functor CMon $\to$ Ab $\to$ CMon of the functor $G$: CMon $\to$ Ab and the forgetful functor Ab $\to$ CMon, but we don't want to insist on this.)

We need a further remark as far as pullbacks of monoids are concerned. It is the following. If

$$\begin{array}{ccc} & & M' \\ & & \big\downarrow{\scriptstyle g} \\ M & \xrightarrow[f]{} & M'' \end{array} \tag{3}$$

are morphisms of commutative monoids (or of commutative monoids with order-unit), the pullbacks of Diagram (3) in the category of sets and in the category of comutative monoids (or in the category of commutative monoids with order-unit) coincide. That is, they correspond to the subset (submonoid, submonoid with order-unit) of the product $M \times M'$ whose elements are all pairs $(x, x')$ with $x \in M, x' \in M'$ and $f(x) = g(x')$.

For any ring $R$, the enveloping group of the monoid $V(R)$ is denoted $K_0(R)$.

**Theorem 8.2** ([6]) *Let $R$ be a ring, $J(R)$ its Jacobson radical, and $\pi : R \to R/J(R)$ the canonical projection. Then the commutative diagram*

$$\begin{array}{ccc} V(R) & \xrightarrow{V(\pi)} & V(R/J(R)) \\ \Big\downarrow{\scriptstyle d_{V(R)}} & & \Big\downarrow{\scriptstyle d_{V(R/J(R))}} \\ K_0(R) & \xrightarrow[K_0(\pi)]{} & K_0(R/J(R)) \end{array}$$

*is a pullback of monoids.*

## 8.2 Krull monoids

Let $\mathbb{N}_0$ be the additive monoid of nonnegative integers. A *discrete valuation* of a commutative additive monoid $M$ is a non-zero monoid homomorphism $v\colon M \to \mathbb{N}_0$. In this case, $e(v) := \gcd(v(M))$ is called the *index* of the discrete valuation $v$. A commutative additive monoid $M$ is said to be a *Krull monoid* if there exists a divisor homomorphism of $M$ into a free commutative monoid. Equivalently, a monoid $M$ is a Krull monoid if and only if there exists a set $\{v_i \mid i \in I\}$ of discrete valuations $v_i$ of $M$ such that: (1) if $x, y \in M$ and $v_i(x) \leq v_i(y)$ for every $i \in I$, then $x \leq y$; (2) for every $x \in M$, the set $\{i \in I \mid v_i(x) \neq 0\}$ is finite.

Krull monoids were introduced by Chouinard in [17]. We will now show that reduced Krull monoids have a theory that is entirely similar to the theory of commutative Krull domains of commutative algebra. One can even prove that:

**Theorem 8.3** (Krause [60]) *A commutative integral domain $R$ is a Krull domain if and only if the monoid $R^* := R\setminus\{0\}$ is a Krull monoid.*

Let us touch on the similarity with Krull domains. Let $M$ be an additive, commutative, cancellative monoid $M$, let $G(M)$ be its enveloping group and assume that $G(M)$ is a torsion-free abelian group. These further hypotheses are not restrictive for us, because we are mainly interested in the monoids $V(\mathcal{C})$ (Section 2), which are always reduced monoid, and it is easily seen that:

**Lemma 8.4** *The following conditions are equivalent for a divisor homomorphism $f\colon M \to F$ of a commutative monoid $M$ into a free commutative monoid $F$.*

(a) *The monoid $M$ is reduced and cancellative.*
(b) *The monoid $M$ is reduced and directly finite.*
(c) *The monoid $M$ is reduced.*
(d) *The homomorphism $f$ is injective.*

Thus, whenever our monoid $V(\mathcal{C})$ is a Krull monoid, it is necessarily cancellative and its enveloping group is a free abelian group.

Slightly modifying the definition given above, we can call *discrete valuation* of an abelian group $G$ any surjective homomorphism $v\colon G \to \mathbb{Z}$. For any discrete valuation $v\colon G \to \mathbb{Z}$, one has that $G \cong \mathbb{Z} \oplus \ker v$. The *valuation submonoid* of $v$ consists of all $x \in G$ with $v(x) \geq 0$. Thus, the valuation submonoid of $v$ is isomorphic to $\mathbb{N}_0 \oplus \ker v$.

A *discrete valuation monoid* is a monoid $M$ with $M_{\mathrm{red}} \cong \mathbb{N}$. A cancellative monoid $M$ is a Krull monoid if and only if there exists a family $\{v_i \mid i \in I\}$ of discrete valuations $v_i\colon G(M) \to \mathbb{Z}$ such that: (1) $M = \{x \in G(M) \mid v_i(x) \geq 0$ for every $i \in I\}$;

(2) for every $x \in G(M)$ the set $\{i \in I \mid v_i(x) \neq 0\}$ is finite. For cancellative Krull monoids, like for comutative Krull domains, it is possible to define principal fractional ideals, divisorial fractional ideals, and the set $D(M)$ of all divisorial fractional ideals turns out to be a commutative monoid with respect to the operation $*$ defined, for every $I, J \in D(M)$, by $I * J :=$ "the intersection of all the principal fractional ideals containing $I + J$". The monoid $D(M)$ contains a subgroup $\mathrm{Prin}(M)$, consisting of all non-zero principal fractional ideals. It is possible to define the divisor class semigroup $\mathrm{Cl}(M) := D(M)/\mathrm{Prin}(M)$, and essential valuations. (A valuation $v\colon M \to \mathbb{N}_0$ is *essential* if, for all $x, y \in M$ with $v(x) \leq v(y)$, there exists $s \in M$ such that $x \leq y+s$ and $v(s) = 0$. Clearly, $v$ is essential if and only if $e(v)^{-1}v$ is essential, where $e(v)$ denotes the index of the valuation; see [33, Section 4].)

**Proposition 8.5** *The following conditions are equivalent for an additive, commutative, cancellative monoid $M$:*

(a) *$M$ is a Krull monoid.*
(b) *There exist a group $G$, a set $I$ and a subgroup $H$ of the free abelian group $\mathbb{Z}^{(I)}$ such that $M \cong G \oplus (H \cap \mathbb{N}_0^{(I)})$.*

In particular, any reduced Krull monoid is isomorphic to $H \cap \mathbb{N}_0^{(I)}$, where $I$ is a set, $\mathbb{N}_0^{(I)}$ is the positive cone of the free abelian group $\mathbb{Z}^{(I)}$ with the component-wise order, and $H$ is a subgroup of $\mathbb{Z}^{(I)}$. If the reduced Krull monoid is also finitely generated, e.g., for the monoids $V(R)$ that are Krull monoids, we can even suppose that the set $I$ is finite. Recall that, in the language of Minkowski's Geometry of Numbers, a subgroup $G$ of $\mathbb{Z}^{(I)}$ is represented by a lattice, that is, a structure with a very regular geometric pattern. Here the term "lattice" is used in a sense completely different from that used until now in this survey. If $V(R)$ is a Krull monoid, then $V(R) \cong \mathbb{N}_0^t \cap H$ is the intersection of the lattice $H \subseteq \mathbb{Z}^t$ with the positive cone $\mathbb{N}_0^t$. Thus the failure of the Krull–Schmidt theorem is minimal in this case, due only to the presence of the border of $\mathbb{N}_0^t \cap G$. In other words, when $V(R)$ is a Krull monoid, Krull–Schmidt uniqueness does not hold in general, but direct-sum decompositions have a very regular geometric pattern.

In Fig. 1, there are two examples of submonoids of $\mathbb{N}_0^2$ that are Krull monoids. Notice their regular geometric pattern. The monoid represented on the left is $H_1 \cap \mathbb{N}_0^2$, where $H_1$ is the subgroup of $\mathbb{Z}^2$ freely generated by $(1, 2)$ and $(3, 0)$. The monoid $H_1 \cap \mathbb{N}_0^2$ is generated by the set $\{(1, 2), (3, 0), (0, 6)\}$. This is the least set of generators of $H_1 \cap \mathbb{N}_0^2$, is the set of all atoms of the monoid, and is contained in any other set of generators of $H_1 \cap \mathbb{N}_0^2$.

The monoid on the right is $H_2 \cap \mathbb{N}_0^2$, where $H_2$ is the subgroup of $\mathbb{Z}^2$ freely generated by $(-1, 2)$ and $(4, 0)$. The monoid $H_2 \cap \mathbb{N}_0^2$ is generated by the set $\{(0, 8), (1, 6), (2, 4), (3, 2), (4, 0)\}$. Also in this case, this is the least set of generators of $H_2 \cap \mathbb{N}_0^2$, is the set of all atoms of the monoid, and is contained in any other set of generators of $H_2 \cap \mathbb{N}_0^2$.

If $R$ is a semilocal ring, every finitely generated projective module has a semilocal endomorphism ring, hence cancels from direct sums. It follows that $V(R)$ is a cancellative monoid, so that $d_{V(R)}\colon V(R) \to K_0(R)$ embeds the commutative monoid $V(R)$ into the abelian group $K_0(R)$. In this case, the ring $R/J(R)$

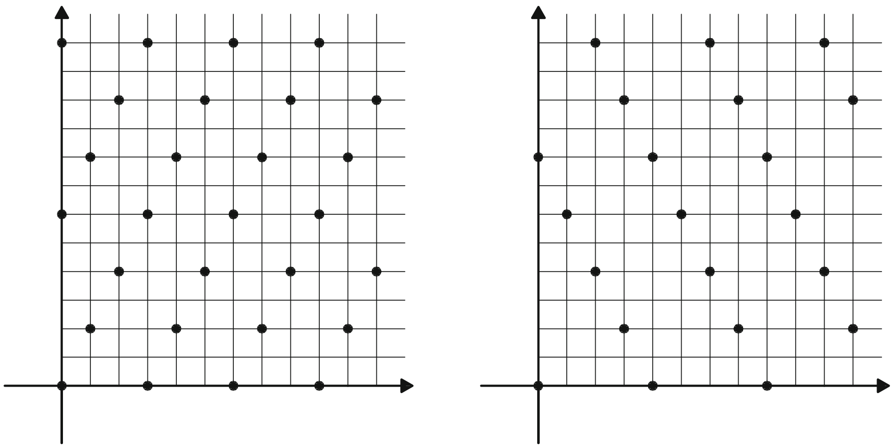

**Fig. 1** The monoids $H_1 \cap \mathbb{N}_0^2$ and $H_2 \cap \mathbb{N}_0^2$, two submonoids of $\mathbb{N}_0^2$ that are Krull monoids.

is semisimple artinian, hence $V(R/J(R))$ is a finitely generated free commutative monoid and $K_0(R/J(R))$ is a finitely generated free abelian group. Now $V(\pi)\colon (V(R), \langle R\rangle) \to (V(R/J(R)), \langle R/J(R)\rangle)$ is an injective divisor homomorphism (Proposition 8.1), so that $V(R)$ is a reduced Krull monoid. The group morphism $K_0(\pi)\colon K_0(R) \to K_0(R/J(R))$ is an injective mapping, so that $K_0(R)$ is a finitely generated free abelian group.

We anticipate a result whose proof will be sketched in Section 11. We call *semilocal* category any preadditive category with a non-zero object such that the endomorphism ring of every non-zero object is a semilocal ring. Artinian modules, all the examples in Section 5 (uniserial modules, cyclically presented modules over local rings, ...) and Sect. 6.2 (finitely generated modules over commutative semilocal rings, finitely presented modules over semilocal rings, ...) give examples of semilocal categories, and we can always close them under direct summands and finite direct sums getting semilocal additive categories in which idempotents split.

**Theorem 8.6** *If $\mathcal{C}$ a semilocal additive category in which idempotents split, the monoid $V(\mathcal{C})$ is a Krull monoid.*

Notice that here we allow $\mathcal{C}$ to be non-necessarily skeletally small, in which case the monoid $V(\mathcal{C})$ is a large monoid.

By Theorem 8.6, for every semilocal category $\mathcal{C}$, the monoid $V(\mathcal{C})$ can be embedded in a free monoid $\mathbb{N}_0^{(I)}$, so that every object of a semilocal category can be described up to isomorphism by finitely many non-zero positive integers. For instance, artinian modules can be described up to isomorphism by mappings with values in $\mathbb{N}_0$. This is an important non-trivial result.

We will come back again to these concepts in Section 11.

## 9 Some realization theorems

We have already seen in Corollary 2.2 that any commutative reduced monoid can be realized as $V(\mathcal{C})$ for some additive full subcategory $\mathcal{C}$ of proj-$R$, where $R$ is a suitable ring. Any commutative reduced monoid with order-unit $(M, u)$ can be realized as $(V(R), \langle R_R \rangle)$ for some ring $R$ (Theorem 2.1). The following results tell us what happens if we restrict our attention to semilocal rings. Notice that if $R$ is a semilocal ring, then the category proj-$R$ is an additive category in which idempotents split, so that the monoid $V(R)$ is a finitely generated commutative reduced Krull monoid (Theorem 8.6). More precisely, we saw at the end of Section 8 that, if $R$ is a semilocal ring, $V(\pi)\colon (V(R), \langle R \rangle) \to (V(R/J(R)), \langle R/J(R) \rangle)$ is an injective morphism of monoids with order-unit, is a divisor homomorphism of $V(R)$ into the finitely generated free commutative monoid $V(R/J(R))$, and $K_0(R)$ is a finitely generated free abelian group. The next result shows that these conditions characterize the monoids that can be realized as $V(R)$ for some $R$ semilocal.

**Theorem 9.1** ([35]) *Let k be a field, $(M, u)$ an additive monoid with order-unit, n a positive integer, v an order-unit in the free monoid $\mathbb{N}_0^n$ and $f\colon (M, u) \to (\mathbb{N}_0^n, v)$ a morphism of commutative monoids with order-unit that is a divisor homomorphism and an injective mapping. Then there exist a semilocal hereditary k-algebra R and isomorphisms of monoids with order-unit $(M, u) \to (V(R), \langle R \rangle)$ and $(\mathbb{N}_0^n, v) \to (V(R/J(R)), \langle R/J(R) \rangle)$, such that if $\pi\colon R \to R/J(R)$ denotes the canonical projection, then*

$$\begin{array}{ccc} (M, u) & \xrightarrow{\quad f \quad} & (\mathbb{N}_0^n, v) \\ \cong\downarrow & & \downarrow\cong \\ (V(R), \langle R \rangle) & \xrightarrow{V(\pi)} & (V(R/J(R)), \langle R/J(R) \rangle) \end{array}$$

*is a commutative diagram of monoids with order-unit.*

Theorem 9.1 has been extended in a number of directions. For instance, the next theorem tells us what happens passing from finitely generated reduced Krull monoids to arbitrary reduced Krull monoids. For any ring $R$, let $\mathcal{S}_R$ be the full subcategory of Mod-$R$ whose objects are all finitely generated projective right $R$-modules $P_R$ whose endomorphism ring $\mathrm{End}(P_R)$ is semilocal. Also for the submonoid $V(\mathcal{S}_R)$ of $V(R)$, the canonical projection $\pi\colon R \to R/J(R)$ induces a monoid morphism $V(\mathcal{S}_\pi)\colon V(\mathcal{S}_R) \to V(\mathcal{S}_{R/J(R)})$.

**Theorem 9.2** ([44]) *Let k be a field, M an additive monoid, I a set and $f\colon M \to \mathbf{N}^{(I)}$ a divisor homomorphism that is an injective mapping. Then there exist a k-algebra R and two monoid isomorphisms $M \to V(\mathcal{S}_R)$ and $\mathbb{N}_0^{(I)} \to V(\mathcal{S}_{R/J(R)})$ such that if $V(\mathcal{S}_\pi)\colon V(\mathcal{S}_R) \to V(\mathcal{S}_{R/J(R)})$ is the homomorphism induced by the canonical projection $\pi\colon R \to R/J(R)$, then the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{N}_0^{(I)} \\ \cong\downarrow & & \downarrow\cong \\ V(\mathcal{S}_R) & \xrightarrow{V(\mathcal{S}_\pi)} & V(\mathcal{S}_{R/J(R)}) \end{array}$$

*is a commutative diagram.*

Recall that a right $R$-module $M_R$ is *reflexive* if it is canonically isomorphic to its bidual $\mathrm{Hom}(\mathrm{Hom}(M_R, R_R), {}_RR)$, that is, if the canonical mapping $M_R \to \mathrm{Hom}(\mathrm{Hom}(M_R, R_R), {}_RR)$ is an isomorphism.

**Theorem 9.3** (Wiegand [88]) *Let $(M, u)$ be an additive monoid with order-unit, $n$ a positive integer, $v$ an order-unit in the free monoid $\mathbb{N}_0^n$ and $f: (M, u) \to (\mathbb{N}_0^n, v)$ a morphism of commutative monoids with order-unit that is a divisor homomorphism and an injective mapping. Then there exist a semilocal ring $R$ and isomorphisms of monoids with order-unit $(M, u) \to (V(R), \langle R\rangle)$ and $(\mathbb{N}_0^n, v) \to (V(R/J(R)), \langle R/J(R)\rangle)$, such that:*

(a) *$R$ is the endomorphism ring of a finitely generated reflexive module over a commutative noetherian local unique factorization domain of Krull dimension 2.*
(b) *If $\pi: R \to R/J(R)$ denotes the canonical projection, then*

$$\begin{array}{ccc} (M, u) & \xrightarrow{f} & (\mathbb{N}_0^n, v) \\ \cong\downarrow & & \downarrow\cong \\ (V(R), \langle R\rangle) & \xrightarrow{V(\pi)} & (V(R/J(R)), \langle R/J(R)\rangle) \end{array}$$

*is a commutative diagram of monoids with order-unit.*

The following nice corollary has been obtained by Roger Wiegand. It is the last result in [88].

**Corollary 9.4** *The following conditions on a commutative monoid $M$ are equivalent:*

(a) *$M$ is a finitely generated reduced Krull monoid.*
(b) *$M \cong V(\mathrm{add}(A_R))$ for some (cyclic) artinian module $A_R$ over a suitable ring $R$.*
(c) *$M \cong V(S)$ for some semilocal ring $S$.*
(d) *$M \cong V(\mathrm{add}(N_T))$ for some finitely generated module $N_T$ over a commutative noetherian semilocal ring $T$.*

Corollary 9.4 has as an immediate consequence another proof that the Krull–Schmidt theorem does not hold for artinian modules. To see it, fix any finitely generated reduced Krull monoid $M$ that is not free, for instance the submonoid of $\mathbb{N}_0^2$ generated by $(2, 0)$, $(1, 1)$ and $(0, 2)$. It is not free because $(2, 0) + (0, 2) = 2(1, 1)$. (These three elements are all the atoms of $M$.) By Corollary 9.4, there exists an artinian module $A_R$ over a suitable ring $R$ with $M \cong V(\mathrm{add}(A_R))$. This isomorphism of $M$ onto $V(\mathrm{add}(A_R))$ assigns to any element $(x, y) \in M$ an object $A_{(x,y)}$

of $\mathrm{add}(A_R)$ unique up to isomorphism. In particular, the modules $A_{(2,0)}$, $A_{(1,1)}$ and $A_{(0,2)}$ turns out to be indecomposable pair-wise non-isomorphic artinian $R$-modules with $A_{(2,0)} \oplus A_{(0,2)} \cong A_{(1,1)}^2$. This is a solution of the problem posed by Krull and mentioned at the end of Sect. 1.2. The first example of artinian modules with two non-isomorphic indecomposable decompositions was given in [39]. Other examples were given in [90] and [78]. In this last paper, Ringel constructs examples of artinian $R$-modules with two non-isomorphic indecomposable decompositions with $R$ a *local* ring.

As we have said in Sect. 1.2, the examples of the failure of the Krull–Schmidt Theorem for artinian modules constructed in [39] are obtained from examples of suitable noetherian modules for which the Krull–Schmidt Theorem fails, making use of a construction, due to Camps and Menal [16], which allows to transfer direct-sum decompositions. All the examples of the failure of the Krull–Schmidt Theorem for artinian modules constructed until now are built in this way: not only the first examples of artinian modules for which Krull–Schmidt fails in [39], but also the results of Wiegand and the examples of Yakovlev [90] and Ringel [78] are essentially constructed starting from other classes of modules for which Krull–Schmidt is known to fail. For instance, in the examples of Yakovlev and Ringel, the technique that allows to transfer the direct-sum decompositions is that developed later by Pimenov and Yakovlev in [69]. This partially explains why Corollary 9.4 holds: it is a corollary of Theorems 9.1 and 9.3 (or Theorem 8.6).

Roger Wiegand extensively extended the theory applying it to the study of modules over commutative rings. For instance, he considered the case of finitely generated torsion-free modules over one-dimensional local noetherian domains [87], finitely generated modules over one-dimensional noetherian Cohen-Macaulay local rings [34] and finitely generated modules that are free on the punctured spectrum [56]. For a nice survey on this direction of research, see [89] and the forthcoming book [63] about maximal Cohen-Macaulay modules over local rings.

## 10 Maximal ideals in preadditive categories

The content of this Section is taken from [40]. In this Section, when we say "maximal ideal" in a ring, we mean "maximal two-sided ideal", that is, "maximal in the set of all proper two-sided ideals".

Let $\mathcal{C}$ be a preadditive category. For any object $A$ of $\mathcal{C}$ and any two-sided ideal $I$ of $\mathrm{End}_{\mathcal{C}}(A)$, we will now define an ideal $\mathcal{A}_I$ of the category $\mathcal{C}$ called the *ideal of* $\mathcal{C}$ *associated to* $I$ [41,42]. It is defined as follows: a morphism $f\colon X \to Y$ in $\mathcal{C}$ is in $\mathcal{A}_I(X, Y)$ if and only if $\beta f \alpha \in I$ for every pair of morphisms $\alpha\colon A \to X$ and $\beta\colon Y \to A$ in $\mathcal{C}$. It is easily seen that the ideal $\mathcal{A}_I$ is the greatest of the ideals $\mathcal{I}'$ of $\mathcal{C}$ such that $\mathcal{I}'(A, A) \subseteq I$. Moreover, $\mathcal{A}_I(A, A) = I$. Notice that, dually, the ideal of $\mathcal{C}$ generated by $I$ is the smallest of the ideals $\mathcal{I}'$ of $\mathcal{C}$ such that $I \subseteq \mathcal{I}'(A, A)$.

The ideals of the category $\mathcal{C}$ associated to two distinct ideals of $\mathrm{End}_{\mathcal{C}}(A)$ are distinct. The next proposition describes when two maximal ideals of the endomorphism rings of two objects $A \neq A'$ of the preadditive category $\mathcal{C}$ have the same associated ideal. If $A$ is an object of $\mathcal{C}$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, Y)\,\mathrm{Hom}_{\mathcal{C}}(X, A)$ will denote the subgroup of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ generated by all composite morphisms $fg$ with $f \in \mathrm{Hom}_{\mathcal{C}}(A, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, A)$.

**Proposition 10.1** *Let $\mathcal{C}$ be a preadditive category, $A, A'$ two non-zero objects of $\mathcal{C}$, $M, M'$ maximal ideals of the rings $\mathrm{End}_{\mathcal{C}}(A)$, $\mathrm{End}_{\mathcal{C}}(A')$ respectively, and $\mathcal{A}_M, \mathcal{A}_{M'}$ the ideals of $\mathcal{C}$ associated to $M, M'$, respectively. The following conditions are equivalent:*

(1) $\mathcal{A}_M = \mathcal{A}_{M'}$.
(2) $\mathcal{A}_M(B, B) = \mathcal{A}_{M'}(B, B)$ *for every* $B \in \mathrm{Ob}(\mathcal{C})$.
(3) $M = \mathcal{A}_{M'}(A, A)$.
(3′) $M' = \mathcal{A}_M(A', A')$.
(4) $\mathcal{A}_M(C, C) = \mathcal{A}_{M'}(C, C) \neq \mathrm{End}_{\mathcal{C}}(C)$ *for some* $C \in \mathrm{Ob}(\mathcal{C})$.
(5) $\mathrm{Hom}_{\mathcal{C}}(A, A')M\,\mathrm{Hom}_{\mathcal{C}}(A', A) \subseteq M'$ *and* $\mathrm{Hom}_{\mathcal{C}}(A, A')\,\mathrm{Hom}_{\mathcal{C}}(A', A) \nsubseteq M'$.
(5′) $\mathrm{Hom}_{\mathcal{C}}(A', A)M'\,\mathrm{Hom}_{\mathcal{C}}(A, A') \subseteq M$ *and* $\mathrm{Hom}_{\mathcal{C}}(A', A)\,\mathrm{Hom}_{\mathcal{C}}(A, A') \nsubseteq M$.
(6) *There exist two morphisms* $\varphi\colon A \to A'$, $\psi\colon A' \to A$ *such that*

$$\psi\mathrm{End}_{\mathcal{C}}(A')\varphi \nsubseteq M, \quad \varphi\mathrm{End}_{\mathcal{C}}(A)\psi \nsubseteq M', \quad \psi M'\varphi \subseteq M \quad \text{and} \quad \varphi M\psi \subseteq M'.$$

An ideal $\mathcal{M}$ of a preadditive category $\mathcal{C}$ is a *maximal* ideal if it is properly contained in exactly one ideal of $\mathcal{C}$ (which must be necessarily the improper ideal of $\mathcal{C}$). Clearly, if all objects of $\mathcal{C}$ are zero objects, maximal ideals cannot exist in $\mathcal{C}$, but this is a trivial example. There exist non-trivial examples of additive small categories without maximal ideals [40, Example 4.1].

**Proposition 10.2** *Let $\mathcal{M}$ be a proper ideal of a preadditive category $\mathcal{C}$. The ideal $\mathcal{M}$ is a maximal ideal of $\mathcal{C}$ if and only if, for every object $A$ of $\mathcal{C}$ with $\mathcal{M}(A, A) \neq \mathrm{End}_{\mathcal{C}}(A)$, one has that:* (1) *$\mathcal{M}(A, A)$ is a maximal ideal of* $\mathrm{End}_{\mathcal{C}}(A)$, *and* (2) *$\mathcal{M}$ is the ideal of $\mathcal{C}$ associated to $\mathcal{M}(A, A)$.*

In some cases, maximal ideals of a preadditive category are easy to describe. For instance, for any ring $R$, the maximal ideals of the category proj-$R$ of all finitely generated projective $R$-modules are exactly the ideals of proj-$R$ associated to the maximal two-sided ideals of the ring $R$ [40, Proposition 2.5].

A preadditive category is *simple* if it has exactly two ideals, necessarily the trivial ones. Notice that a simple category must necessarily have a non-zero object. The next theorem describes simple categories.

**Theorem 10.3** *Let $\mathcal{C}$ be a preadditive category. The following conditions are equivalent:*

(a) *The category $\mathcal{C}$ is a simple category.*
(b) *$\mathcal{C}$ has a non-zero object, the endomorphism rings of all non-zero objects of $\mathcal{C}$ are simple rings and, for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$ with $A \neq 0$ and every $f\colon B \to C$, if $\beta f\alpha = 0$ for every $\alpha\colon A \to B$ and every $\beta\colon C \to A$, then $f = 0$.*
(c) *$\mathcal{C}$ has a non-zero object and is equivalent to a full subcategory of the category* proj-$R$ *of all finitely generated projective right $R$-modules for some simple ring $R$.*

Notice that, by Condition (3), every simple preadditive category is skeletally small. Moreover, by Condition (3) again, every full subcategory of a simple preadditive category containing a non-zero object is a simple category.

Simple additive categories with splitting idempotents have a particularly clear description, because:

**Proposition 10.4** *An additive category with splitting idempotents is a simple category if and only if it is equivalent to the category* proj-$R$ *for some simple ring* $R$.

Maximal ideals of a preadditive category $\mathcal{C}$ coincide with kernels of non-zero functors $F\colon \mathcal{C} \to$ proj-$R$, where $R$ ranges in the class of simple rings.

Semilocal categories have plenty of maximal ideals, as the next Proposition shows.

**Proposition 10.5** *Let* $\mathcal{C}$ *be a semilocal category. Then:*

(1) *Every ideal of* $\mathcal{C}$ *associated to a maximal ideal of the endomorphism ring of a non-zero object of* $\mathcal{C}$ *is a maximal ideal of* $\mathcal{C}$.
(2) *Every proper ideal of* $\mathcal{C}$ *is contained in a maximal ideal of* $\mathcal{C}$. *In particular, maximal ideals exist in any semilocal category* $\mathcal{C}$.

As we have done above, dim and codim will denote the Goldie dimension and the dual Goldie dimension. For the next corollary, recall that if $R$ is a simple artinian ring, then $R$ has a unique simple right module $S_R$ up to isomorphism and all finitely generated right $R$-modules $M_R$ are semisimple and isomorphic to $S_R^n$, where $n$ is the Goldie dimension $\dim(M_R)$ of $M_R$. We will denote by mod-$R$ the full subcategory of Mod-$R$ whose objects are all finitely generated right $R$-modules. Notice that if $R$ is a simple artinian ring and $S_R$ is its simple module, then mod-$R$ is equivalent to the category of all finitely generated right vector spaces over the division ring $\mathrm{End}(S_R)$ via the equivalence $\mathrm{Hom}(S_R, -)\colon$ mod-$R \to$ vect-$\mathrm{End}(S_R)$.

**Corollary 10.6** *If* $\mathcal{C}$ *is a semilocal category and* $\mathcal{M}$ *is a maximal ideal of* $\mathcal{C}$, *then there exist a simple artinian ring* $R$ *and a full and faithful functor* $F\colon \mathcal{C}/\mathcal{M} \to$ mod-$R$ *of the factor category* $\mathcal{C}/\mathcal{M}$ *into the category* mod-$R$. *Moreover, for every object* $B$ *of* $\mathcal{C}$, *the Goldie dimension of the semisimple right* $R$*-module* $F(B)$ *is* $\mathrm{codim}(\mathrm{End}_{\mathcal{C}}(B)/\mathcal{M}(B, B))$.

Let $\mathcal{C}$ be a semilocal category. In the class of all pairs $(A, M)$, where $A$ is any non-zero object of $\mathcal{C}$ and $M$ is any maximal ideal in the endomorphism ring $\mathrm{End}_{\mathcal{C}}(A)$, we can consider the equivalence relation $\sim$ defined by $(A, M) \sim (A', M')$ if $\mathcal{A}_M = \mathcal{A}_{M'}$. This equivalence relation on the class of pairs $(A, M)$ has been described in Proposition 10.1. Fix a class $\mathrm{Max}(\mathcal{C})$ of representatives modulo $\sim$. The class $\mathrm{Max}(\mathcal{C})$ will be called the *maximal spectrum* of $\mathcal{C}$.

For instance, it can be proved [40, Example 4.5] that if $\mathcal{C}$ is a preadditive category in which $\mathrm{End}_{\mathcal{C}}(A)$ is a local ring for every $A \in \mathrm{Ob}(\mathcal{C})$, so that, in particular, $\mathcal{C}$ has no zero object, then there is a bijection between $\mathrm{Max}(\mathcal{C})$ and $V(\mathcal{C})$ that associates to any pair $(A, M) \in \mathrm{Max}(\mathcal{C})$ the unique object $\langle A \rangle \in V(\mathcal{C})$ isomorphic to $A$.

Another example is the following. Let $R$ be a ring and consider the full subcategory $\mathcal{S}_R$ of Mod-$R$ whose objects are all finitely generated projective right $R$-modules $P_R$

whose endomorphism ring $\mathrm{End}(P_R)$ is semilocal. This category $\mathcal{S}_R$ has already been considered in Theorem 9.2. Let $\mathcal{H}$ be the full subcategory of Mod-$R$ whose objects are all simple homomorphic images of finitely generated projective modules with a semilocal endomorphism ring. It can be proved that there is a bijection $\mathrm{Max}(\mathcal{S}_R) \to V(\mathcal{H})$, where $V(\mathcal{H})$ is the class of objects of a skeleton of $\mathcal{H}$ [40, Theorem 4.7].

**Theorem 10.7** *If $\mathcal{C}$ is a semilocal category, then:*

(1) *The Jacobson radical of $\mathcal{C}$ is the intersection of all maximal ideals of $\mathcal{C}$.*
(2) *For each object $A$ in $\mathcal{C}$, $A$ is a non-zero object in $\mathcal{C}/\mathcal{M}$ for only finitely many maximal ideals $\mathcal{M}$ of $\mathcal{C}$.*
(3) *The collection of canonical functors $\mathcal{C} \to \mathcal{C}/\mathcal{M}$, $\mathcal{M} \in \mathrm{Max}(\mathcal{C})$, induces a functor $F\colon \mathcal{C} \to \coprod^{w}_{\mathcal{M}\in\mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M}$, and $F$ is isomorphism reflecting.*
(4) *Moreover, if $\mathcal{C}$ is also additive with splitting idempotents, the functor $F$ is direct-summand reflecting.*

The theorem does not hold without the hypothesis that the category is semilocal. For instance, let $\mathcal{C}$ be the category of all vector spaces of dimension $\leq \aleph_1$ over a field $k$. Then $\mathcal{C}$ has a unique maximal ideal $\mathcal{M}$ consisting of all morphisms of rank $\leq \aleph_0$, and all vector spaces of dimension $\leq \aleph_0$ turn out to be isomorphic modulo $\mathcal{M}$. Cf. [40].

## 11 The monoid $V(\mathcal{C})$ for a semilocal category $\mathcal{C}$

Now we will interpret the results of the previous Section in terms of commutative monoids. We need some further notions about Krull monoids. A divisor homomorphism $f\colon M \to F$ of $M$ into a free commutative monoid $F$ is called a *divisor theory* if, for every $u \in F$, there exist finitely many elements $x_1, \dots, x_m \in M$ such that $u = \min\{f(x_1), \dots, f(x_m)\}$, where the minimum is taken with respect to the algebraic pre-order on $F$ (notice that the partially ordered set $(F, \leq)$ is a lattice). Every Krull monoid $M$ has a divisor theory, and if $f\colon M \to F$ and $f'\colon M \to F'$ are two divisor theories, then there exists a unique isomorphism $\Phi\colon F \to F'$ such that $\Phi \circ f = f'$ [55, Theorems 20.4 and 23.4]. More precisely:

**Proposition 11.1** ([33, Satz 1, Satz 2 and Korollar]) *Let $M$ be a Krull monoid. Let $\varphi = (\varphi_i)_{i\in I}\colon M \to \mathbb{N}_0^{(I)}$ be a divisor homomorphism such that $\varphi_i \neq 0$ for all $i \in I$ and $e(\varphi_i)^{-1}\varphi_i \neq e(\varphi_j)^{-1}\varphi_j$ for $i \neq j$. Let $J$ be the set of all indices $j \in I$ for which the valuation $\varphi_j$ is essential. Then:*

(a) *The mapping $\varphi^* = (e(\varphi_j)^{-1}\varphi_j)_{j\in J}\colon M \to \mathbb{N}_0^{(J)}$ is a divisor theory.*
(b) *The divisor homomorphism $\varphi$ is a divisor theory if and only if $I = J$ and $e(\varphi_i) = 1$ for all $i \in I$.*
(c) *For every essential valuation $v\colon M \to \mathbb{N}_0$ of $M$, there exists an index $j \in J$ such that $e(v)^{-1}v = e(\varphi_j)^{-1}\varphi_j$.*

Probably, the easiest example of valuation of a Krull monoid that is not essential is the following [40, Example 5.1]. Let $M := \mathbb{N}_0^{(X)}$ be a free commutative monoid with a free set $X$ of generators such that $|X| \geq 2$. Let $v\colon M \to \mathbb{N}_0$ be the valuation defined, for every $(n_x)_{x\in X} \in \mathbb{N}_0^{(X)}$, by $v((n_x)_{x\in X}) = \sum_{x\in X} n_x$. Clearly, $v(s) = 0$ if and only if $s = 0$, and it is easy to construct elements $m_1$ and $m_2$ in $M$ that are not comparable with respect to the algebraic pre-order, but have the same valuation. Thus $v(m_1) = v(m_2)$, but there does not exist an $s \in M$ with $v(s) = 0$ and $m_1 \leq m_2 + s$.

Since the additive monoids $V(\mathcal{C})$ are large when the category $\mathcal{C}$ is not skeletally small, in order to describe additive semilocal categories that are not skeletally small, we need free monoids with a *class* of generators, that is, direct sums of *classes* of copies of $\mathbb{N}_0$. Let $M_i$ be a small commutative additive monoid for every index $i$ ranging in a class $I$. The *direct sum* of the monoids $M_i$ is the large monoid $\oplus_{i\in I} M_i$ whose elements are all mappings $m: I \to \bigcup_{i\in I} M_i$ of $I$ into the union $\bigcup_{i\in I} M_i$ of the sets $M_i$, such that $m(i) \in M_i$ for all $i \in I$ and $m(i) = 0_{M_i}$ for all $i \in I$ except for finitely many $i$'s. The class $\oplus_{i\in I} M_i$ has an obvious addition with respect to which it becomes a large commutative monoid. When all the monoids $M_i$ are equal to a unique monoid $M$, we will denote the direct sum $\oplus_{i\in I} M$ by $M^{(I)}$.

Now we can sketch a proof of Theorem 8.6. Let $\mathcal{C}$ be an additive semilocal category in which idempotents split. By Theorem 10.7(4), there is a canonical direct-summand reflecting functor $F: \mathcal{C} \to \coprod^w_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M}$. It induces a monoid homomorphism $V(F): V(\mathcal{C}) \to V(\coprod^w_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M})$, which is a divisor homomorphism because $F$ is direct-summand reflecting. By Corollary 10.6, $V(\coprod^w_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M}) \cong \oplus_{\mathcal{M}\in \mathrm{Max}(\mathcal{C})} V(\mathcal{C}/\mathcal{M}) \cong \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))}$. Thus $V(F)$ is a divisor homomorphism of $V(\mathcal{C})$ into a free commutative monoid, so that the monoid $V(\mathcal{C})$ turns out to be a Krull monoid. This proves Theorem 8.6.

We conclude this Section, showing that the argument of the last paragraph can be inverted, which will give us a further realization theorem. In order to avoid set theoretical problems, we will only consider the case of small monoids. Since the monoid we are interested in is the monoid $V(\mathcal{C})$, this means that we will only consider skeletally small additive categories $\mathcal{C}$. Let $I$ be a set and $\mathbb{N}_0^{(I)}$ be the free commutative monoid with free set of generators $I$. Let $\mathbb{Z}^{(I)}$ be the free abelian group with free set of generators $I$. As above, we denote the elements of $\mathbb{N}_0^{(I)}$ as functions $m: I \to \mathbb{N}_0$ with $s(i) = 0$ for almost all $i \in I$. More precisely, for any function $m: I \to \mathbb{N}_0$, the *support* of $m$ is the set $\mathrm{supp}(m) := \{i \in I \mid s(i) \neq 0\}$, and $\mathbb{N}_0^{(I)}$ is the set of all functions of $I$ into $\mathbb{N}_0$ of finite support. Like in Sect. 3.1, if $\mathcal{C}$ is any preadditive category, we will denote by $\mathrm{Mat}(\mathcal{P})$ and $\widehat{\mathrm{Mat}(\mathcal{P})}$, respectively, the free additive category generated by $\mathcal{C}$ and the idempotent completion of $\mathrm{Mat}(\mathcal{P})$.

**Theorem 11.2** ([40]) *Let $I$ be a set and $S$ be a subset of the monoid $\mathbb{N}_0^{(I)}$ such that $\bigcup_{s\in S} \mathrm{supp}(s) = I$. Let $\mathbb{N}_0 S$ be the submonoid of $\mathbb{N}_0^{(I)}$ generated by $S$ and $\mathbb{Z}S$ be the subgroup of $\mathbb{Z}^{(I)}$ generated by $S$. Then there exists a preadditive category $\mathcal{C}$ such that the full and faithful embeddings $\mathcal{C} \hookrightarrow \mathrm{Mat}(\mathcal{C}) \hookrightarrow \widehat{\mathrm{Mat}(\mathcal{C})}$ induce a commutative diagram of sets and mappings*

$$\begin{array}{ccccccc} S & \hookrightarrow & \mathbb{N}_0 S & \hookrightarrow & \mathbb{Z}S \cap \mathbb{N}_0^{(I)} & \hookrightarrow & \mathbb{N}_0^{(I)} \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ V(\mathcal{C}) & \hookrightarrow & V(\mathrm{Mat}(\mathcal{C})) & \hookrightarrow & V(\widehat{\mathrm{Mat}(\mathcal{C})}) & \hookrightarrow & \mathbb{N}_0^{(\mathrm{Max}(\mathcal{C}))}. \end{array}$$

*In this diagram, the vertical arrows represent bijections, and the squares in the middle and on the right are commutative squares of monoids and monoid homomorphisms.*

## 12 Rings and modules of type *n*

Most of the results in this Section are taken from [42]. We say that a ring $R$ *has type* $n$ if the factor ring $R/J(R)$ is a direct product of $n$ division rings. We will say that $R$ is a *ring of finite type* if it has type $n$ for some integer $n \geq 1$. Thus if a ring $R$ has finite type, then the type $n$ of $R$ necessarily coincides with the dual Goldie dimension $\operatorname{codim}(R_R)$ of $R_R$ (Proposition 6.1), but not conversely: the ring of $n \times n$ matrices with entries in a field is semilocal of finite dual Goldie dimension $n$, but it has not finite type for $n \geq 2$. Notice that a ring $R$ has type 1 if and only if it is a local ring.

**Proposition 12.1** *Let $R$ be a ring, $J(R)$ its Jacobson radical and $n$ a positive integer. The following conditions are equivalent:*

(1) *$R$ is a ring of type $n$.*
(2) *$n$ is the smallest of the positive integers $m$ for which there is a local morphism of $R$ into a direct product of $m$ division rings.*
(3) *$R$ has exactly $n$ distinct maximal right ideals, and they are all two-sided ideals in $R$.*
(4) *$R$ has exactly $n$ distinct maximal left ideals, and they are all two-sided ideals in $R$.*

For instance, if $k$ is a field, the ring of all $n \times n$ upper triangular matrices with entries in $k$ is a ring of type $n$. A commutative ring has finite type if and only if it is semilocal.

*Remark 12.2* If a ring $R$ has at most two maximal right ideals, then its maximal right ideals are two-sided. See [23, Proof of Lemma 2.3]. This is not true for $n \geq 3$, that is, for $n \geq 3$ there are rings with $n$ maximal right ideals not all two-sided. For instance, the ring of all $2 \times 2$ matrices with entries in the field with 2 elements has exactly three maximal ideals, and all of them are not two-sided.

We say that a right module $M_R$ over a ring $R$ *has type* $n$ if its endomorphism ring $\operatorname{End}(M_R)$ is a ring of type $n$. The zero module will be considered to be the unique module of type 0. Moreover, we will say that a module $M_R$ *has finite type* if it has type $n$ for some integer $n \geq 0$. In Section 5, we have already met with several classes of modules of type $\leq 2$. A module has type 1 if and only if its endomorphism ring is local. For other examples of modules of finite type, see [42, Section 5].

Now let $R$ be a ring and let $\mathcal{C}$ be a full subcategory of Mod-$R$. Let $M_R$ be a module of type $n$ that is an object of $\mathcal{C}$, let $P_1, \dots, P_n$ be the $n$ maximal right ideals of $\operatorname{End}(M_R)$ and $\mathcal{P}_i$ be the ideal of the category $\mathcal{C}$ associated to $P_i$ for every $i = 1, 2, \dots, n$. We will denote by $I(M_R)$ the set whose elements are the ideals $\mathcal{P}_1, \dots, \mathcal{P}_n$. The ideals of $\mathcal{C}$ associated to two distinct ideals $P_i$ of $\operatorname{End}(M_R)$ are two distinct ideals of the category $\mathcal{C}$, so that the set $I(M_R)$ has cardinality exactly $n$.

The class of $R$-modules of finite type is not closed under finite direct sums. The following proposition is more precise in this sense.

**Proposition 12.3** *Let $M$ and $N$ be two objects of an additive full subcategory $\mathcal{C}$ of* Mod-$R$. *Assume that $M, N$ are $R$-modules of type $m, n$, respectively.*

(a) *If the sets $I(M)$ and $I(N)$ are not disjoint, then $M \oplus N$ does not have finite type.*
(b) *If the sets $I(M)$ and $I(N)$ are disjoint, then $M \oplus N$ has type $m + n$ and the set $I(M \oplus N)$ is the disjoint union of the sets $I(M)$ and $I(N)$.*

In particular, if $M = N_1 \oplus \cdots \oplus N_t$, then $M$ has finite type if and only if $N_1, \ldots, N_t$ have finite type and $I(N_1), \ldots, I(N_t)$ are pair-wise disjoint. If this happens, the type of $M$ turns out to be the sum of the types of the $N_i$'s, and $I(M)$ turns out to be the disjoint union of the sets $I(N_i)$.

**Theorem 12.4** [42, Theorem 4.2] *Let $M$ and $N$ be two objects of a full subcategory $\mathcal{C}$ of* Mod-$R$. *Assume that $M, N$ are $R$-modules of finite type. Then:*

(1) *The module $M$ is isomorphic to a direct summand of $N$ if and only if $I(M)$ is a subset of $I(N)$.*
(2) *The modules $M$ and $N$ are isomorphic if and only if $I(M) = I(N)$.*

For any ring $R$, let FT-$R$ denote the full subcategory of Mod-$R$ whose objects are all right $R$-modules of finite type, and let SFT-$R$ denote the full subcategory of Mod-$R$ whose objects are all right $R$-modules that are direct sums of a finite number of right $R$-modules of finite type. For any full subcategory $\mathcal{C}$ of FT-$R$, set $I(\mathcal{C}) := \bigcup_{M_R \in \mathrm{Ob}(\mathcal{C})} I(M_R)$, so that $I(\mathcal{C})$ turns out to be a class of ideals of the category $\mathcal{C}$.

In the rest of this Section, if $\mathcal{P}$ is an ideal of a subcategory $\mathcal{C}$ of Mod-$R$, we will denote by $F : \mathcal{C} \to \mathcal{C}/\mathcal{P}$ the canonical projection functor.

**Lemma 12.5** *Let $\mathcal{C}$ be any full subcategory of* FT-$R$. *Let $M_R$ be an object of $\mathcal{C}$, $\mathcal{P}$ an element of $I(M_R)$ and $F : \mathcal{C} \to \mathcal{C}/\mathcal{P}$ the canonical projection functor. Then, for every object $N_R$ of $\mathcal{C}$, either $F(N_R) = 0$ or $F(N_R) \cong F(M_R)$. Moreover, $F(N_R) = 0$ if and only if $\mathcal{P}(N_R, N_R) = \mathrm{End}(N_R)$, and $F(N_R) \cong F(M_R)$ if and only if $\mathcal{P}(N_R, N_R)$ is a maximal ideal of* $\mathrm{End}(N_R)$.

If $\mathcal{C}$ is a full subcategory of FT-$R$, for every module $M_R$ of type $n$ there are exactly $n$ ideals $\mathcal{P}$ in the class $I(\mathcal{C})$ for which the object $M_R$ is non-zero in the factor category $\mathcal{C}/\mathcal{P}$.

Let $\mathcal{C}$ be a full subcategory of FT-$R$, and $\mathcal{S}(\mathcal{C})$ be the full subcategory of Mod-$R$ whose objects are all right $R$-modules that are isomorphic to direct summands of modules belonging to $\mathrm{Ob}(\mathcal{C})$. Let $\mathcal{F}$ be the class of all the canonical projection functors $F : \mathcal{S}(\mathcal{C}) \to \mathcal{S}(\mathcal{C})/\mathcal{P}$, where $\mathcal{P} \in I(M_R)$ for some $M_R \in \mathrm{Ob}(\mathcal{C})$.

**Theorem 12.6** *Let $\mathcal{C}$ be a full subcategory of* FT-$R$ *and $M, N$ be objects of $\mathcal{S}(\mathcal{C})$. Then $M \cong N$ if and only if $F(M) \cong F(N)$ for every $F \in \mathcal{F}$.*

**Proposition 12.7** *Let $M$ be a right $R$-module of finite type, let $P$ be a maximal ideal of* $\mathrm{End}(M_R)$, *let $\mathcal{P}$ be the ideal of* SFT-$R$ *associated to $P$ and let $\mathcal{K}$ be the ideal of $\mathcal{S}$(SFT-$R$) associated to $P$. Then the categories* SFT-$R/\mathcal{P}$, $\mathcal{S}$(SFT-$R$)$/\mathcal{K}$ *and* mod-$\mathrm{End}(M_R)/P$ *of all finite dimensional right vector spaces over the division ring* $\mathrm{End}(M_R)/P$ *are equivalent.*

For every $F \in \mathcal{F}$, let $\dim_F(M_R)$ be the dimension of the vector space over $\mathrm{End}(M_R)/P$ corresponding to $F(M_R)$. Theorem 12.6 implies that:

**Corollary 12.8** *If $M_R$ and $N_R$ are objects of $\mathcal{S}(\text{SFT-}R)$, then $M_R$ and $N_R$ are isomorphic $R$-modules if and only if $\dim_F(M_R) = \dim_F(N_R)$ for every $F \in \mathcal{F}$.*

Thus the monoid $V(\mathcal{S}(\text{SFT-}R))$ embeds as a submonoid in the free monoid with free class of generators $\mathcal{F}$. It is also possible to construct the weak coproduct categories

$$\coprod_{\mathcal{P}}^{w} \text{SFT-}R/\mathcal{P} \quad \text{and} \quad \coprod_{\mathcal{K}}^{w} \mathcal{S}(\text{SFT-}R)/\mathcal{K}.$$

Generalizing the definition of local morphism between rings, we say that a functor $F: \mathcal{A} \to \mathcal{B}$ is *local* if for every morphism $f: A \to A'$ in $\mathcal{A}$, $F(f)$ isomorphism in $\mathcal{B}$ implies $f$ isomorphism in $\mathcal{A}$.

**Proposition 12.9** *The canonical functors*

$$U: \text{SFT-}R \to \coprod_{\mathcal{P}}^{w} \text{SFT-}R/\mathcal{P} \quad \text{and} \quad V: \mathcal{S}(\text{SFT-}R) \to \coprod_{\mathcal{K}}^{w} \mathcal{S}(\text{SFT-}R)/\mathcal{K}$$

*are full, and their kernels are the Jacobson radicals of the categories* SFT-$R$ *and* $\mathcal{S}(\text{SFT-}R)$, *respectively. The functors $U$ and $V$ are local and isomorphism reflecting.*

It is possible to prove that the functor $V$ is direct-summand reflecting, but the functor $U$ is not necessarily direct-summand reflecting [42].

**Corollary 12.10** *Let $\mathcal{C}$ be a full subcategory of* FT-$R$ *and* $\mathrm{add}(\mathcal{C})$ *be the full additive subcategory of* Mod-$R$ *whose objects are all $R$-modules that are isomorphic to direct summands of direct sums of finitely many right $R$-modules belonging to* $\mathrm{Ob}(\mathcal{C})$. *Then the monoid $V(\mathrm{add}(\mathcal{C}))$ is a Krull monoid.*

## 13 Cyclically presented modules over rings of finite type

In this Section, we will present results that appear in [2]. We say that a right $R$-module is a DCP module if it is isomorphic to a direct summand of a cyclically presented right $R$-module. In the study of DCP right module over rings of finite type, it is convenient to use the functors $- \otimes_R S$ and $\mathrm{Tor}_1^R(-, S)$, where $S$ is a simple left $R$-module

**Theorem 13.1** *Let $A_R$ be a* DCP *right module over a ring $R$ of type $n$. Let $S_1, \dots, S_n$ be a set of representatives of the simple left $R$-modules up to isomorphism. Set $X := \{i \mid i = 1, \dots, n,\ A \otimes_R S_i \neq 0\}$ and $Y := \{i \mid i = 1, \dots, n,\ \mathrm{Tor}_1^R(A_R, S_i) \neq 0\}$. Then:*

(1) *The endomorphism ring $\mathrm{End}(A_R)$ of $A_R$ is a ring of type $\leq s + t$, where $s$ is the cardinality of the set $X$ and $t$ is the cardinality of the set $Y$.*
(2) *Every maximal right ideal of $\mathrm{End}(A_R)$ is either of the form $K_i := \{f \in \mathrm{End}(A_R) \mid f \otimes_R S_i = 0\}$ for some $i \in X$ or of the form $L_i := \{f \in \mathrm{End}(A_R) \mid \mathrm{Tor}_1^R(f, S_i) = 0\}$ for some $i \in Y$.*

We fix the notation for the rest of this Section. Let $R$ be a ring of type $n$ and $S_1, \ldots, S_n$ a set of representatives of its simple left modules up to isomorphism. Fix an index $i = 1, \ldots, n$. We say that two right $R$-modules $A_R$ and $B_R$ have *the same $i$-th $\otimes$ class* (notation: $[A_R]_{\otimes,i} = [B_R]_{\otimes,i}$) if there exist two morphisms $f\colon A_R \to B_R$ and $g\colon B_R \to A_R$ such that the morphisms $f \otimes_R S_i\colon A \otimes_R S_i \to B \otimes_R S_i$ and $g \otimes_R S_i\colon B \otimes_R S_i \to A \otimes_R S_i$ are two isomorphisms. Similarly, we say that two right $R$-modules $A_R, B_R$ have *the same $i$-th* $\mathrm{Tor}_1$ *class* ($[A_R]_{T,i} = [B_R]_{T,i}$) if there are two morphisms $f\colon A_R \to B_R$ and $g\colon B_R \to A_R$ such that the morphisms $\mathrm{Tor}_1^R(f, S_i)\colon \mathrm{Tor}_1^R(A_R, S_i) \to \mathrm{Tor}_1^R(B_R, S_i)$ and $\mathrm{Tor}_1^R(g, S_i)\colon \mathrm{Tor}_1^R(B_R, S_i) \to \mathrm{Tor}_1^R(A_R, S_i)$ are isomorphisms. Thus we have $2n$ invariants, the $n\otimes$ classes and the $n$Tor classes, which describe up to isomorphism DCP right modules over the ring $R$ of finite type $n$, as the following theorem shows.

**Theorem 13.2** *Let $A_1, \ldots, A_m, B_1, \ldots, B_t$ be fixed* DCP *right modules over a ring $R$ of type $n$. For every $i = 1, \ldots, n$, set*

$$\begin{aligned} X_i &:= \{j \mid j = 1, \ldots, m,\ [A_j]_{\otimes,i} \neq [0]_{\otimes,i}\},\\ Y_i &:= \{k \mid k = 1, \ldots, t,\ [B_k]_{\otimes,i} \neq [0]_{\otimes,i}\},\\ Z_i &:= \{j \mid j = 1, \ldots, m,\ [A_j]_{T,i} \neq [0]_{T,i}\},\\ W_i &:= \{k \mid k = 1, \ldots, t,\ [B_k]_{T,i} \neq [0]_{T,i}\}. \end{aligned}$$

*Then the modules $A_1 \oplus \cdots \oplus A_m$ and $B_1 \oplus \cdots \oplus B_t$ are isomorphic if and only if there are $2n$ bijections $\varphi_i\colon X_i \to Y_i$ and $\psi_i\colon Z_i \to W_i (i = 1, 2, \ldots, n)$ with $[A_j]_{\otimes,i} = [B_{\varphi_i(j)}]_{\otimes,i}$ for every $i = 1, \ldots, n$ and $j \in X_i$, and $[A_j]_{T,i} = [B_{\psi_i(j)}]_{T,i}$ for every $i = 1, \ldots, n$ and $j \in Z_i$.*

It is possible to dualize these results [2, Section 6]. Define a module $E_R$ to be an fdHI module (*heterogeneous injective module of finite Goldie dimension,* cf. [42]) if $E_R$ is a direct sum of finitely many pair-wise non-isomorphic indecomposable injective modules. The endomorphism ring $S := \mathrm{End}(E_R)$ of such a module is a semiperfect ring of finite type, so that we can apply the previous results to DCP$S$-modules. Now the contravariant functors

$$\begin{aligned} H &:= \mathrm{Hom}_R(-, {}_SE_R)\colon\ \text{Mod-}R \to S\text{-Mod}\\ H' &:= \mathrm{Hom}_S(-, {}_SE_R)\colon S\text{-Mod} \to \text{Mod-}R. \end{aligned}$$

define a duality between the full subcategory $\mathcal{K}$ of Mod-$R$ whose objects are all finite direct sums of kernels of morphisms between direct summands of $E_R$ and the full subcategory $\mathcal{C}$ of $S$-Mod whose objects are all finite direct sums of DCP left $S$-modules. For further details, see [2, Section 6].

## 14 The Krull–Schmidt Theorem in the case two

In this Section, we will present some results that appear in [43]. Let us begin with some preliminary results concerning the Weak Krull–Schmidt Theorem in a very general setting. Let $\mathcal{C}$ be a class of right modules over a ring $R$. For simplicity, we will also denote by $\mathcal{C}$ the full subcategory of Mod-$R$ whose class of objects is $\mathcal{C}$. Following [12, Definition 1.13], we say that $\mathcal{C}$ *satisfies condition* (*DSP*) if for any four right $R$-modules $A, B, C, D$ with $A \oplus B \cong C \oplus D$ and $A, B, C \in \mathcal{C}$, one has that $D \in \mathcal{C}$ also. Every class $\mathcal{C}$ of right $R$-modules has a (*DSP*)-*closure*, that is a smallest class $\mathcal{C}'$ of right $R$-modules containing $\mathcal{C}$, closed under isomorphism and satisfying condition (*DSP*). To see this, it suffices to define $\mathcal{C}'_0 := \mathcal{C}$ and $\mathcal{C}'_{n+1} := \mathcal{C}'_n \cup \{D \mid$ there exist $A, B, C \in \mathcal{C}'_n$ with $A \oplus B \cong C \oplus D\}$ for every integer $n \geq 0$. Then the union $\bigcup_{n\geq 0} \mathcal{C}'_n$ is the (*DSP*)-closure $\mathcal{C}'$ of $\mathcal{C}$.

Let $\mathcal{C}$ be a class of indecomposable right modules over a ring $R$. Following [43], we will say that *the Weak Krull–Schmidt Theorem holds for* $\mathcal{C}$ if there are two equivalences $\sim$ and $\equiv$ on the class $\mathcal{C}$ with the following property. If $U_1, \dots, U_n, V_1, \dots, V_m$ are modules in $\mathcal{C}$, then $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_m$ if and only if $m = n$ and there exist two permutations $\sigma, \tau$ of $\{1, \dots, n\}$ with $U_i \sim V_{\sigma(i)}$ and $U_i \equiv V_{\tau(i)}$ for every $i = 1, \dots, n$.

Let $\mathcal{C}$ be a class of indecomposable $R$-modules of type 2 and $V(\mathcal{C})$ be a class of representatives of the modules in $\mathcal{C}$ up to isomorphism. We associate a graph $G(\mathcal{C})$ to the class $\mathcal{C}$. The graph $G(\mathcal{C})$ is defined as follows. If $M_R$ is a module in $\mathcal{C}$ and $P$ is a two-sided ideal of End$(M_R)$, let $\mathcal{A}_P$ be the ideal of the full subcategory $\mathcal{C}$ of Mod-$R$ associated to $P$. The (large) graph $G(\mathcal{C}) = (V, E)$ associated to the class $\mathcal{C}$ has as its class $V$ of vertices the class of all the ideals $\mathcal{A}_P$ of the category $\mathcal{C}$, where $P$ is one of the two maximal right (two-sided) ideals of End$(M_R)$ and $M_R$ ranges in the class $\mathcal{C}$. The class of edges of $G(\mathcal{C})$ is $V(\mathcal{C})$. For every $M_R \in V(\mathcal{C})$, the endomorphism ring End$(M_R)$ has two maximal ideals $P$ and $Q$. Following the notation of Section 12, set $I(M_R) := \{\mathcal{A}_P, \mathcal{A}_Q\}$. The class of vertices is $I(\mathcal{C}) := \bigcup_{M_R \in \mathcal{C}} I(M_R)$. Since every module $M_R$ in $\mathcal{C}$ is indecomposable of type 2, its endomorphism ring End$(M_R)$ has two maximal ideals $P$ and $Q$, and the edge $M_R$ joins the vertices $\mathcal{A}_P$ and $\mathcal{A}_Q$. The graph $G(\mathcal{C})$ has no multiple edges and no loops by Theorem 12.4.

Recall that the *complete graph* $K_n$ is the graph with $n$ vertices and in which any two vertices are adjacent. A graph $G = (V, E)$ is *bipartite* if its vertex set $V$ is the disjoint union $V = X \dot{\cup} Y$ of two non-empty subsets $X$ and $Y$, in such a way that no two vertices of $X$ are adjacent and no two vertices of $Y$ are adjacent. A *complete bipartite graph* is a graph $G = (V, E)$ in which the vertex set $V$ is the disjoint union $V = X \dot{\cup} Y$ of two non-empty subsets $X$ and $Y$, and such that no two vertices in $X$ are adjacent, no two vertices in $Y$ are adjacent, but any vertex in $X$ is adjacent to any vertex in $Y$. If we deal with categories that are not skeletally small, we must allow "large graphs", that is, graphs $G = (V, E)$ in which $V$ and $E$ can be proper classes. For instance, any such graph is a bipartite complete graph if and only if it is isomorphic to the graph $K_{\alpha,\beta}$ for suitable classes $\alpha$ and $\beta$. In this graph $K_{\alpha,\beta}$, the class of vertices is the disjoint union $\alpha \dot{\cup} \beta$, the vertices in $\alpha$ are pair-wise non-adjacent, the vertices in $\beta$ are pair-wise non-adjacent, and each vertex in $\alpha$ is adjacent to every vertex in $\beta$.

**Theorem 14.1** *Let $R$ be a ring, $\mathcal{C}$ be a non-empty class of indecomposable modules of type* 2 *and $\mathcal{C}'$ its* (*DSP*)*-closure. Then exactly one of the following two conditions hold:*

(a) *The graph $G(\mathcal{C})$ is bipartite, the connected components of $G(\mathcal{C}')$ are complete bipartite graphs, the Weak Krull–Schmidt Theorem holds for both $\mathcal{C}$ and $\mathcal{C}'$, and $\mathcal{C}'$ consists only of indecomposable modules of type* 2.
(b) *The graph $G(\mathcal{C})$ is not bipartite and $\mathcal{C}'$ contains an indecomposable module that is not of type* 2.

**Theorem 14.2** *Let $R$ be a ring and $\mathcal{C}_2$ be the class of all right $R$-modules of type* 2. *The Weak Krull–Schmidt Theorem holds for $\mathcal{C}_2$ if and only if $G(\mathcal{C}_2)$ does not have subgraphs isomorphic to $K_4$, if and only if each connected component of $G(\mathcal{C}_2)$ is either isomorphic to $K_3$ or a complete bipartite graph.*

**Theorem 14.3** (Dichotomy) *Let $R$ be a ring. Exactly one of the following two conditions holds.*

(a) *There are two right $R$-modules $U_1, U_2$ of type* 2 *such that $U_1 \oplus U_2$ has three pair-wise non-isomorphic direct-sum decompositions.*
(b) *There exist two ideals $\mathcal{I}, \mathcal{K}$ of the full subcategory $\mathcal{S}$ of* Mod-$R$ *whose objects are all finite direct sums of right $R$-modules of type 2 such that the canonical functor $F: \mathcal{S} \to \mathcal{S}/\mathcal{I} \times \mathcal{S}/\mathcal{K}$ is isomorphism reflecting and both $\mathcal{S}/\mathcal{I}$ and $\mathcal{S}/\mathcal{K}$ are amenable semisimple categories.*

We say that a class $\mathcal{C}$ of indecomposable modules of type 2 *satisfies weak* (*DSP*) if for every $U, V, W \in \mathcal{C}$ such that the edges $I(U)$ and $I(U')$ are not incident in the graph $G(\mathcal{C})$ and for every module $X$, $U \oplus U' \cong W \oplus X$ implies $X \in \mathcal{C}$. The class of all indecomposable modules of type 2 satisfies weak (*DSP*) [43, Lemma 5.1].

**Lemma 14.4** *If a class $\mathcal{C}$ of indecomposable modules of type* 2 *satisfies weak* (*DSP*), *then every connected component of the graph $G(\mathcal{C})$ is either a complete graph or a complete bipartite graph.*

We are ready to present the Krull–Schmidt theorem for the modules of type 2. Let $R$ be a ring and $C_\lambda$, $\lambda \in \Lambda$, be the connected components of the graph $G(\mathcal{C}_2)$. Then $V(\mathcal{C}_2) = \oplus_{\lambda \in \Lambda} V(C_\lambda)$ [42, Theorem 4.10], that is, every element of $V(\mathcal{C}_2)$ is a sum of elements in the $V(C_\lambda)$'s in a unique way. Thus every module in $\mathcal{S}$ has a direct-sum decomposition, unique up to isomorphism, indexed in the connected components of $G(\mathcal{C}_2)$. Hence it is sufficient to describe what happens for direct sums of modules, all in the same connected component of $G(\mathcal{C}_2)$. Any connected component of $G(\mathcal{C}_2)$ is either a complete graph or a complete bipartite graph (Lemma 14.4). The next two propositions describe these two cases.

**Proposition 14.5** *Let $M_1, \dots, M_m, N_1, \dots, N_n$ be right $R$-modules of type* 2. *Assume that these $m+n$ modules are all in the same connected component of $G(\mathcal{C}_2)$ and that this connected component is a complete graph. Let $P_1, P_2$ be the two maximal ideals of* $\mathrm{End}(M_1), \dots, P_{2m-1}, P_{2m}$ *the two maximal ideals of* $\mathrm{End}(M_m)$, $Q_1, Q_2$ *the two maximal ideals of* $\mathrm{End}(N_1), \dots, Q_{2n-1}, Q_{2n}$ *the two maximal ideals of* $\mathrm{End}(N_n)$. *Then $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$ if and only if $m = n$ and there exists a permutation $\sigma$ of $\{1, 2, \dots, 2n\}$ such that $\mathcal{A}_{P_i} = \mathcal{A}_{Q_{\sigma(i)}}$ for all $i = 1, 2, \dots, 2n$.*

Thus, under the hypotheses of Proposition 14.5, the module $M_1 \oplus \cdots \oplus M_m$ has at most $\frac{(2m)!}{2^m \cdot m!}$ non-isomorphic direct-sum decompositions into direct sums of modules of type 2. The module $M_1 \oplus \cdots \oplus M_m$ has exactly $\frac{(2m)!}{2^m \cdot m!}$ non-isomorphic direct-sum decompositions into direct sums of modules of type 2 if the edges $\langle M_1 \rangle, \ldots, \langle M_m \rangle$ are pair-wise non-incident.

**Proposition 14.6** *Let $M_1, \ldots, M_m, N_1, \ldots, N_n$ be right $R$-modules of type 2. Assume that these $m+n$ modules are all in the same connected component $C = (V_C, E_C)$ of $G(\mathcal{C}_2)$ and that $C$ is a complete bipartite graph. Let $V_C = X_C \dot{\cup} Y_C$ be a corresponding bipartition of $C$, so that it is possible to label the maximal ideals $P_i, Q_i$ of $\mathrm{End}_R(M_i)$ and $P'_j, Q'_j$ of $\mathrm{End}_R(N_j)$ in such a way that their associated ideals $\mathcal{A}_{P_1}, \ldots, \mathcal{A}_{P_m}, \mathcal{A}_{P'_1}, \ldots, \mathcal{A}_{P'_n}$ are in $X$ and the associated ideals $\mathcal{A}_{Q_1}, \ldots, \mathcal{A}_{Q_m}, \mathcal{A}_{Q'_1}, \ldots, \mathcal{A}_{Q'_n}$ are in $Y$. Then $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$ if and only if $m = n$ and there exist two permutations $\sigma, \tau$ of $\{1, \ldots, n\}$ with $\mathcal{A}_{P_i} = \mathcal{A}_{P'_{\sigma(i)}}$ and $\mathcal{A}_{Q_i} = \mathcal{A}_{Q_{\tau(i)}}$ for every $i = 1, \ldots, n$.*

Thus, under the hypotheses of Proposition 14.6, the module

$$M_1 \oplus \cdots \oplus M_m$$

has at most $m!$ non-isomorphic direct-sum decompositions into direct sums of modules of type 2. The module $M_1 \oplus \cdots \oplus M_m$ has exactly $m!$ non-isomorphic direct-sum decompositions into direct sums of modules of type 2 if the edges $\langle M_1 \rangle, \ldots, \langle M_m \rangle$ are pair-wise non-incident. This is what happens for uniserial modules. Uniserial modules of type 2 are all on connected components that are all complete bipartite graphs.

We conclude the Section with a Proposition that describes when the graph $G(\mathcal{C})$ is bipartite.

**Proposition 14.7** ([3, Proposition 2.3], [43, Proposition 6.1]) *Let $R$ be a ring and $\mathcal{C}$ be a full subcategory of* Mod-$R$ *whose objects are indecomposable right $R$-modules. Let* $\mathrm{add}(\mathcal{C})$ *be the full subcategory of* Mod-$R$ *whose objects are the right $R$-modules that are isomorphic to direct summands of direct sums of finitely many modules in* $\mathrm{Ob}(\mathcal{C})$. *Then the following conditions are equivalent:*

(a) *The graph $G(\mathcal{C})$ is bipartite and the modules in $\mathcal{C}$ have all type $\leq 2$.*

(b) *There exist two additive functors $F_i \colon \mathrm{add}(\mathcal{C}) \to \mathcal{A}_i$, $i = 1, 2$, of the category $\mathrm{add}(\mathcal{C})$ into two amenable semisimple categories $\mathcal{A}_i$, such that $F_i(U)$ is a simple object of $\mathcal{A}_i$ for every $U \in \mathrm{Ob}(\mathcal{C})$ and both $i = 1$ and $i = 2$, and the product functor $F_1 \times F_2 \colon \mathrm{add}(\mathcal{C}) \to \mathcal{A}_1 \times \mathcal{A}_2$ is a local functor.*

(c) *There exist two additive functors $G_i \colon \mathcal{C} \to \mathcal{A}_i$, $i = 1, 2$, of the category $\mathcal{C}$ into two amenable semisimple categories $\mathcal{A}_i$, such that, for every $U \in \mathrm{Ob}(\mathcal{C})$:* (1) *$G_i(U)$ is a simple object of $\mathcal{A}_i$, and* (2) *for every $f \in \mathrm{End}_R(U)$, $f$ is an automorphism of $U$ if and only if $G_1(f)$ and $G_2(f)$ are automorphisms of $G_1(U)$ and $G_2(U)$, respectively.*

## 15 Direct summands of serial modules and other direct sums of modules of type 2

A natural question is whether every direct summand of a serial module is serial. There are three main results that answer this question. The first two are given by the following two theorems.

**Theorem 15.1** ([23, Theorem 2.7]) *Let $U$ be a uniserial right module over an arbitrary ring $R$ and let $n \geq 0$ be an integer. Then every direct summand of $U^n$ is isomorphic to $U^m$ for some $m \leq n$.*

The second result is the following splendid theorem due to Příhoda.

**Theorem 15.2** ([70]) *Let $R$ be any ring. Then the class of all serial right $R$-modules of finite Goldie dimension is closed under direct summands.*

The third result wonderful result is due to Puninski [75]. He proved that there exist direct summands of serial modules that are not serial.

Let us see how it is possible to extend to pairs of functors the idea of the modules whose endomorphism rings have two maximal ideals, the direct sums classified by two invariants, hence by two permutations, and the Weak Krull–Schmidt Theorems (Section 5). It can be shown that almost all the examples we have given in this survey can be modeled with the pattern that follows using a suitable pair of functors. See [3, Section 6].

We have already defined *local functors* as the additive functors $F: \mathcal{A} \to \mathcal{B}$ between preadditive categories $\mathcal{A}$ and $\mathcal{B}$ such that, for every morphism $f: A \to A'$ in $\mathcal{A}$, $F(f)$ isomorphism in $\mathcal{B}$ implies $f$ isomorphism in $\mathcal{A}$. We say that the functor $F$ is *weakly local* if, for every object $A \in \mathrm{Ob}(\mathcal{A})$ and every endomorphism $f$ of $A$, $F(f)$ automorphism in $\mathcal{B}$ implies $f$ automorphism in $\mathcal{A}$.

Let us fix the notation for the rest of this Section. The symbol $\mathcal{C}$ will denote a full subcategory of Mod-$R$, $V(\mathcal{C})$ the class of objects of a skeleton of $\mathcal{C}$ and $\mathrm{add}(\mathcal{C})$ will be the full additive subcategory of Mod-$R$ with splitting idempotents generated by $\mathrm{Ob}(\mathcal{C})$. That is, $\mathrm{add}(\mathcal{C})$ is the full subcategory of Mod-$R$ whose objects are all right $R$-modules that are isomorphic to direct summands of direct sums of finitely many modules in $\mathrm{Ob}(\mathcal{C})$. The category $\mathrm{add}(\mathcal{C})$ is clearly equivalent to the category $\widehat{\mathrm{Mat}(\mathcal{C})}$.

For each $i = 1, 2$, let $F_i: \mathrm{add}(\mathcal{C}) \to \mathcal{A}_i$ be an additive functor of $\mathrm{add}(\mathcal{C})$ into an amenable semisimple category $\mathcal{A}_i$. Assume that $F_i(U)$ is a simple object of $\mathcal{A}_i$ for every object $U$ of $\mathcal{C}$ and every $i = 1, 2$. Suppose that the product functor $F_1 \times F_2: \mathcal{C} \to \mathcal{A}_1 \times \mathcal{A}_2$ is a local functor and its extension $\mathrm{add}(\mathcal{C}) \to \mathcal{A}_1 \times \mathcal{A}_2$ is a weakly local functor. Finally, assume that if $A$ is an object of $\mathrm{add}(\mathcal{C})$ with $F_2(A) = 0$, then $A = 0$. It is easily seen that, from these hypotheses, it follows that the modules in $\mathcal{C}$ are indecomposable and of type $\leq 2$.

For every pair of objects $A, B$ of $\mathrm{add}(\mathcal{C})$, define $[A]_i = [B]_i$ if there are two morphisms $f: A \to B$ and $g: B \to A$ such that $F_i(f)$ and $F_i(g)$ are two isomorphisms. Notice that $[A]_i = [B]_i$ if and only if $A$ and $B$ belong to the same $F_i$-class in the terminology we introduced in Sect. 5.3. Trivially, $[A]_i = [B]_i$ implies $F_i(A) \cong F_i(B)$.

It is possible to associate to the pair of functors $(F_1, F_2)$ a bipartite graph $(V, E)$, called the *associated bipartite graph* and denoted $B(F_1, F_2)$, which is defined as follows. Set $X_i := \{[U]_i \mid U \in \mathrm{Ob}(\mathcal{C})\}$. The class of vertices $V$ of $B(F_1, F_2)$ is the disjoint union $X_1 \dot\cup X_2$. The class $E$ of edges of $B(F_1, F_2)$ is $E := \{\langle U \rangle \mid U \in V(\mathcal{C})\}$. Thus the edge $\langle U \rangle$ connects the vertex $[U]_1$ in $X_1$ to the vertex $[U]_2$ in $X_2$.

**Proposition 15.3** *Let $\mathcal{C}_2$ be the full subcategory of $\mathcal{C}$ whose objects are all the objects of $\mathcal{C}$ that are modules of type 2. Then the graph $G(\mathcal{C}_2)$ and the subgraph of $B(F_1, F_2)$ corresponding to $\mathcal{C}_2$ are canonically isomorphic.*

*Remark 15.4* We want to stress the difference between the graph $G(\mathcal{C})$ introduced in Section 14 and the graph $B(F_1, F_2) = (V, E)$ we have introduced now here. The starting point in both cases is a class $\mathcal{C}$ of indecomposable modules of type either 1 or 2. In the graph $G(\mathcal{C})$, the vertices are the ideals in the category $\mathcal{C}$ that are associated to maximal ideals of the endomorphism rings of the objects of $\mathcal{C}$. The edges are the objects of the skeleton $V(\mathcal{C})$ of $\mathcal{C}$ that are modules of type 2. In $B(F_1, F_2)$, the vertices are the classes $[U]_i$, where $U$ is any object of $\mathcal{C}$ $(i = 1, 2)$. The edges are the objects in $V(\mathcal{C})$, including all those that are modules of type 1. The graph $G(\mathcal{C})$ is not necessarily a bipartite graph (by Proposition 14.7, $G(\mathcal{C})$ is bipartite if and only if there exist two suitable functors $F_1, F_2$), whereas the graph $B(F_1, F_2)$ is always bipartite (its construction depends on the two functors $F_1$ and $F_2$). In a sense, the bipartite graph $B(F_1, F_2)$ is a refinement of the graph $G(\mathcal{C})$($G(\mathcal{C})$ is constructed only from the category $\mathcal{C}$, whereas $B(F_1, F_2)$ is constructed from more data, that is, also from the functors $F_1$ and $F_2$. In the refinement from the graph $G(\mathcal{C})$ to the graph $B(F_1, F_2)$, we essentially do not touch the structure relative to the indecomposable modules of type 2, but in $B(F_1, F_2)$ we substitute the isolated points of $G(\mathcal{C})$, which correspond to modules of type 1, with an edge.)

More precisely, the reason of this difference between $G(\mathcal{C})$ and $B(F_1, F_2)$ as far as modules of type 1 are concerned and similarity between the two graphs as far as modules of type 2 are concerned depends on the fact that the graph $B(F_1, F_2)$ is essentially constructed from the completely prime ideals $P_{1,U}$ and $P_{2,U}$ of $\mathrm{End}_R(U)$ for every $U \in \mathrm{Ob}(\mathcal{C})$, where $P_{i,U} = \{f \in \mathrm{End}_R(U) \mid F_i(f) = 0\}$. If $U$ is an indecomposable module of type 2, then $P_{1,U}$ and $P_{2,U}$ are exactly the two maximal ideals of $\mathrm{End}_R(U)$ and the ideal $\mathcal{P}_{i,U}$ associated to $P_{i,U}$ corresponds to the $F_i$-class $[U]_i$ of $U$. If $U$ is a module of type 1, then one of the $P_{i,U}$'s is the maximal ideal of $\mathrm{End}_R(U)$, but the other is not necessarily the maximal ideal, though $U$ still has the two $F_i$-classes $[U]_1$ and $[U]_2$. Cf. the case of a uniserial module $U$.

Proposition 10.1 can be adapted from maximal ideals to completely prime ideals as in the following lemma.

**Lemma 15.5** *Let $\mathcal{C}$ be a full subcategory of* Mod-$R$ *and $M_i (i = 1, 2)$ be two object of $\mathcal{C}$. Let $P_i$ be a fixed completely prime ideal of* $\mathrm{End}_R(M_i)$ *and $\mathcal{A}_{P_i}$ be the associated ideal in the category $\mathcal{C}$. The following are equivalent:*

(a) $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$.
(b) *There exist morphisms $f : M_1 \to M_2$, $g : M_2 \to M_1$ such that $gf \notin P_1$, $fg \notin P_2$, $gP_2 f \subseteq P_1$ and $fP_1 g \subseteq P_2$.*

Let us go back to the pair of functors $F_1, F_2$. Recall that $[A]_i = [B]_i$ indicates that there are two morphisms $f: A \to B$ and $g: B \to A$ such that $F_i(f)$ and $F_i(g)$ are two isomorphisms. If $U$ and $V$ are objects of $\mathcal{C}$, then $[U]_i = [V]_i$ if and only if $\mathcal{A}_{P_{i,U}} = \mathcal{A}_{P_{i,V}}$ [3, Proposition 5.7]. If $U$ and $V$ are objects of $\mathcal{C}$ and the edges $\langle U \rangle$ and $\langle V \rangle$ are incident in $B(F_1, F_2)$, then $U$ and $V$ have the same type. It follows that modules in the same connected component of $B(F_1, F_2)$ have the same type [3, Corollary 5.8].

The connected components of $B(F_1, F_2)$ corresponding to modules of type 1 are "star graphs", that is, isomorphic to $K_{1,\beta}$ for some non-empty class $\beta$.

We now adapt a definition we had given immediately before the statement of Lemma 14.4 for classes $\mathcal{C}$ of indecomposable modules of type 2 to the present setting of a full subcategory $\mathcal{C}$ of Mod-$R$ with two functors $F_i: \mathrm{add}(\mathcal{C}) \to \mathcal{A}_i$. We say that the full subcategory $\mathcal{C}$ of Mod-$R$ satisfies *weak* (*DSP*) if for every $U, V, W \in \mathrm{Ob}(\mathcal{C})$ such that the edges $\langle U \rangle$ and $\langle V \rangle$ are not incident in $B(F_1, F_2)$ and for every module $X$, $U \oplus V \cong W \oplus X$ implies $X \in \mathcal{C}$ [43, Section 5].

**Proposition 15.6** *The category $\mathcal{C}$ satisfies weak (DSP) if and only if all connected components of $B(F_1, F_2)$ are complete bipartite graphs.*

**Theorem 15.7** *If the graph $B(F_1, F_2)$ is a complete bipartite graph, every object of* $\mathrm{add}(\mathcal{C})$ *is a direct sum of finitely many objects of $\mathcal{C}$.*

This theorem can also be stated in the language of commutative monoids. Let $V(\mathrm{add}(\mathcal{C}))$ denote the class of objects of a skeleton of $\mathrm{add}(\mathcal{C})$, so that $V(\mathrm{add}(\mathcal{C}))$ is a commutative monoid with respect to the operation induced by coproduct. Assume that the hypotheses of Theorem 15.7 hold, so that in particular the graph $B(F_1, F_2)$ is a complete bipartite graph. Since every object of $\mathrm{add}(\mathcal{C})$ is isomorphic to a direct sum of objects of $\mathcal{C}$, we can extend the position $\langle U \rangle \mapsto ([U]_1, [U]_2)$ to a monoid morphism $\Gamma: V(\mathrm{add}(\mathcal{C})) \to \mathbb{N}_0^{(X_1)} \oplus \mathbb{N}_0^{(X_2)}$. By the Weak Krull–Schmidt Theorem 5.14, $\Gamma$ is a well defined injective divisor homomorphism into $\mathbb{N}_0^{(X_1)} \oplus \mathbb{N}_0^{(X_2)}$, whose image consists of all the elements $((n_{x_1})_{x_1 \in X_1}, (m_{x_2})_{x_2 \in X_2})$ with $\sum_{x_1 \in X_1} n_{x_1} = \sum_{x_2 \in X_2} m_{x_2}$.

## 16 Quasismall modules and weak Krull–Schmidt for infinite direct sums of uniserials

Let $U$ be a right module over an arbitrary ring $R$. Recall that $U$ is *small* if for any family $\{M_\lambda \mid \lambda \in \Lambda\}$ of right $R$-modules and any monomorphism $f: U \to \oplus_{\lambda \in \Lambda} M_\lambda$, one has that $\pi_\lambda \circ f = 0$ for all but finitely many indices $\lambda$. Here $\oplus_{\lambda \in \Lambda} M_\lambda$ denotes the direct sum of the modules in the family and $\pi_\lambda$ denotes the canonical projection of the direct sum onto $M_\lambda$ (see [48] or [83]).

We say that the $R$-module $U$ is *quasismall* if whenever $U$ is isomorphic to a direct summand of a direct sum $\oplus_{\lambda \in \Lambda} M_\lambda$ of right $R$-modules $M_\lambda$, there is a finite subset $F$ of $\Lambda$ such that $U$ is isomorphic to a direct summand of $\oplus_{\lambda \in F} M_\lambda$. Clearly, every small module is quasismall. To see it, let $U$ be a small module and assume that $U$ is isomorphic to a direct summand of a direct sum $\oplus_{\lambda \in \Lambda} M_\lambda$. Then there are a monomorphism $f: U \to \oplus_{\lambda \in \Lambda} M_\lambda$ and an epimorphism $g: \oplus_{\lambda \in \Lambda} M_\lambda \to U$ with $gf = 1_U$. As $U$ is small, there is a finite subset $F$ of $\lambda$ with $\pi_\lambda \circ f = 0$ for every $\lambda \in \Lambda \setminus F$. Now it is easy to conclude.

Since finitely generated modules are small, finitely generated modules are quasismall. Every uniserial module is either countably generated or small ([18, Lemma 24], [22, Lemma 4.2]). In particular, every uniserial module that is not countably generated is quasismall. Every module with a local endomorphism ring is quasismall [22, pp. 111–112].

If $A$ and $B$ are modules, a family $\{f_\lambda \mid \lambda \in \Lambda\}$ of morphisms of $A$ into $B$ is a *summable family* if for every element $x \in A$ there exists a finite subset $F_x \subseteq \Lambda$ such that $f_\lambda(x) = 0$ for every $\lambda \in \Lambda \setminus F_x$. Thus if $\{f_\lambda \mid \lambda \in \Lambda\}$ is a summable family of morphisms of $A$ into $B$, it is possible to define its sum $\sum_{\lambda \in \Lambda} f_\lambda$, which turns out to be a morphism of $A$ into $B$.

**Lemma 16.1** *Let $U$ be a uniserial module. The following conditions are equivalent:*

(a) *$U$ is quasismall.*
(b) *For every summable family $\{f_\lambda \mid \lambda \in \Lambda\}$ of endomorphisms of $U$ with $\sum_{\lambda \in \Lambda} f_\lambda = 1_U$, there exists an index $\mu \in \Lambda$ for which $f_\mu$ is an epimorphism.*

Here is a characterization of uniserial modules that are not quasismall. Recall that, as we have seen above, any uniserial module that is not quasismall must be countably generated.

**Lemma 16.2** *The following conditions are equivalent for a countably generated uniserial module $U$:*

(a) *$U$ is non-quasismall.*
(b) *For every element $x \in U$, there exists an endomorphism $f_x$ of $U$ that is not an automorphism and such that $f_x(x) = x$.*

*Moreover, if $U$ is non-quasismall, then any nonzero homomorphic image of $U$ is non-quasismall.*

In particular, if $U$ and $V$ are uniserial modules with $[U]_e = [V]_e$, then $U$ is quasismall if and only if $V$ is quasismall.

The importance of the dichotomy of quasismall/non-quasismall uniserial modules is given by the following theorem proved by Příhoda [71, Theorem 2.6]. It extends Theorem 5.2 to arbitrary, possibly infinite, families of uniserial modules.

**Theorem 16.3** *Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be two families of non-zero uniserial modules. Set $I' := \{i \in I \mid U_i$ is quasismall$\}$ and $J' = \{j \in J \mid V_j$ is quasismall$\}$. Then $\oplus_{i \in I} U_i \cong \oplus_{j \in J} V_j$ is and only if there exist a bijection $\sigma : I \to J$ and a bijection $\tau : I' \to J'$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ for every $i \in I$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I'$.*

An example of a uniserial non-quasismall module was given by Puninski in [76]. To present it, recall that a ring $R$ such that both the modules $R_R$ and ${}_RR$ are uniserial is called a *chain ring*. A *chain domain* is a (non-necessarily commutative) integral domain that is a chain ring. A chain domain is said to be *nearly simple* if it has exactly three two-sided ideals. Clearly, if a chain domain has exactly three ideals, they must necessarily be the ideals 0, $R$ and $J(R)$ (every chain domain $R$ is local, hence $J(R)$ is its maximal ideal). Notice that nearly simple chain domains are necessarily non-commutative. If $R$ is a nearly simple chain domain and $a$ and $b$ are any two non-zero non-invertible elements of $R$, that is, $a, b \in J(R)\setminus\{0\}$, then the right $R$-modules $R/aR$ and $R/bR$ are always isomorphic [76, Corollary 4.3]. Puninski shows in [76, Proof of Lemma 8.3] that if $R$ is a nearly simple chain domain, then there exists a non-invertible element $s \in R$ such that $\bigcap_{n\geq 1} Rs^n \neq 0$. Then he defines a uniserial right $R$-module $U_R$ via generators and relations taking as set of generators a countable set $\{x_1, x_2, x_3, \dots\}$ and set of relations $\{x_1 s = 0, x_{n+1}s = x_n \mid n \geq 1\}$. It is easily seen that $U_R$ is a uniserial module. Now consider the uniserial finitely presented module $V_R := R/sR$. Then

$$U_R \oplus V_R^{(\aleph_0)} \cong V_R^{(\aleph_0)}$$

[76, Proposition 8.1]. Thus $U_R$ is not quasismall, otherwise it would be isomorphic to a direct summand of a module $V_R^n$, hence it would be a finitely generated module, which is not.

The description of the behavior of direct-sum decompositions of infinite direct sums of modules with a semilocal endomorphism ring is a problem that has not been solved yet, even for the class of projective modules over semilocal rings. Very interesting results in this direction have been obtained by Puninski [77], Příhoda and Puninski [74], Příhoda [72,73], Herbera and Příhoda [57,58].

## 17 Open problems

We conclude this survey mentioning some of the many problems that still remain open in this setting.

(1) It would be interesting to explicitly determine the maximal ideals of the semilocal categories $\mathcal{C}$ we have met. For instance, if $R$ is any ring with identity and $\mathcal{C}$ is the full subcategory of Mod-$R$ whose objects are all artinian right $R$-modules, which are the maximal ideals of the semilocal category $\mathcal{C}$? Similarly, determine the valuations $V(\mathcal{C}) \to \mathbb{N}_0$. Determine the essential valuations $V(\mathcal{C}) \to \mathbb{N}_0$. Answers would be particularly interesting also restricting the attention to the case where the ring $R$ is local.

(2) Generalize Theorem 5.10 to the case of infinite direct sums of kernels of morphisms between indecomposable injective modules. That is, find the analog of Theorem 16.3 for kernels of morphisms between indecomposable injective modules instead of uniserial modules. Does the phenomenon of non-quasismall modules appears in this setting?

(3) Determine further classes of modules with a semilocal endomorphism ring and for which some form of the weak Krull–Schmidt Theorem holds.

(4) (Goodearl) Characterize the monoids $V$ that are isomorphic to $V(R)$ for the rings $R$ in some important classes $\mathcal{C}$ of rings. For instance, if $\mathcal{C}$ is the class of all Von Neumann regular rings with identity, which are the monoids $V$ that are isomorphic

to $V(R)$ for some $R \in \mathcal{C}$, that is, some Von Neumann regular ring $R$? This is a classical problem. The best result in this direction has been obtained by Wehrung [86], who constructs a reduced monoid $V$ with order-unit, with the refinement property (that is, such that, for every $x_1, x_2, y_1, y_2 \in V$ with $x_1 + x_2 = y_1 + y_2$, there exist $z_{ij} \in V$, $i, j = 1, 2$, such that $x_i = z_{i1} + z_{i2}$ for $i = 1, 2$ and $y_j = z_{1j} + z_{2j}$ for $j = 1, 2$), of cardinality $\aleph_2$ that is not isomorphic to $V(R)$ for any Von Neumann regular ring $R$. See the review MR2563739 by Enrique Pardo in Mathematical Reviews.

(5) Does some form of the weak Krull–Schmidt Theorem appear also for other algebraic structures, like groups, lattices, Lie algebras, etc.?



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