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1-absorbing and weakly 1-absorbing prime submodules of a module over a noncommutative ring

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Abstract
In this study, we aim to introduce the concepts of 1-absorbing prime submodules and weakly 1-absorbing prime submodules of a unital module over a noncommutative ring with nonzero identity. This is a new class of submodules between prime submodules (weakly prime submodules) and 2-absorbing submodules (weakly 2-absorbing submodules). Let R be a noncommutative ring with a nonzero identity $1 \neq 0$ and M an R-module. A proper submdule P of M is said to be a 1-absorbing prime submodule (weakly 1-absorbing prime submodule) if for all nonunits $x, y \in R$ and $m \in M$ with $xRyRm \subseteq P$ ($\{0\} \neq xRyRm \subseteq P$), then $xy \in (M :_R P)$ or $m \in P$. Various properties and characterizations of these classes of submodules are considered.

Keywords 1-absorbing prime submodule · Weakly 1-absorbing prime submodule

Mathematics Subject Classification Primary 16D10 · 16D25; Secondary 16D80 · 16L30 · 16N60

1 Introduction

In this article, we focus only on noncommutative rings with nonzero identity and nonzero unital left modules. Let R always denote such a ring and let M denote such an R-module. The concept of prime ideals and its generalizations have a significant place in noncommutative algebra since they are used in understanding the structure of rings. Recall that in a commutative ring a proper ideal I of R is said to be a prime ideal if whenever $xy \in I$ then $x \in I$ or $y \in I$. In [1], Anderson and Smith introduced a notion of weakly prime ideal which is a generalization of prime ideals. A proper ideal I of R is called weakly prime ideal if $0 \neq xy \in I$ for some elements $x, y \in R$ implies that $x \in I$ or $y \in I$. It is clear that every prime ideal is weakly prime but the converse is not true in general. Afterwards, Badawi, in his celebrated paper [2], introduced the notion of 2-absorbing ideals and used them to characterize Dedekind domains. Recall from [2], that a nonzero proper ideal I of

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R is called 2-absorbing ideal if $xyz \in I$ for some $x, y, z \in R$ implies either $xy \in I$ or $xz \in I$ or $yz \in I$. Note that every prime ideal is also a 2-absorbing ideal. After this, over the past decades, 2-absorbing version of ideals and many generalizations of 2-absorbing ideals attracted considerable attention by many researchers. Badawi and Darani in [3] defined and studied the notion of weakly 2-absorbing ideals which is a generalization of weakly prime ideals. A proper ideal I of R is called a weakly 2-absorbing ideal if for each $x, y, z \in R$ with $0 \neq xyz \in I$, then either $xy \in I$ or $xz \in I$ or $yz \in I$.

In 2010, Hirano et al. extended the notion of weakly prime ideals in rings, not necessarily commutative or with identity. According their celebrated paper [11], a proper ideal P of R is called a weakly prime ideal of a ring R if whenever $a, b \in R$ such that $\{0\} \neq aRb \subseteq P$, then $a \in P$ or $b \in P$. They also verified that P is weakly prime ideal if and only if whenever J, K are right ideals of R such that $\{0\} \neq JK \subseteq P$, then $J \subseteq P$ or $K \subseteq P$. An ideal I of R is said to be proper if $I \neq R$. Recall that a proper ideal I of R is called 2-absorbing as in [6] if whenever $aRbRc \subseteq I$ for some $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. Let I be a proper ideal of R. Recall from [7] that a proper ideal I of R is said to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ with $\{0\} \neq aRbRc \subseteq I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Note that a 2-absorbing ideal is a weakly 2-absorbing ideal. However, these are different concepts.

In 2011, Darani and Soheilnia [5] introduced the concept of 2-absorbing and weakly 2-absorbing submodules of modules over commutative rings with identities. A proper submodule P of a module M over a commutative ring R with identity is said to be a 2-absorbing (weakly 2-absorbing) submodule of M if whenever $a, b \in R$ and $m \in M$ with $abm \in P$ ($0 \neq abm \in P$), then $abM \subseteq P$ or $am \in P$ or $bm \in P$. One can see that 2-absorbing submodules are generalization of prime submodules. Moreover, it is obvious that 2-absorbing ideals are special cases of 2-absorbing submodules.

In [8] and [9] the notions of 2-absorbing and weakly 2-absorbing submodules of a module over a noncommutative ring were introduced. A proper submodule P of a module M over a noncommutative ring R with identity is said to be a 2-absorbing (weakly 2-absorbing) submodule of M if whenever $a, b \in R$ and $m \in M$ with $aRbRm \subseteq P$ ($\{0\} \neq aRbRm \subseteq P$), then $ab \in (P : M) \subseteq P$ or $am \in P$ or $bm \in P$.

Recently, in [16], Yassine et al. introduced a 1-absorbing prime ideal. This type of ideal which is a generalization of prime ideals of a commutative ring with identity. A proper ideal I of R is called 1–absorbing prime ideal if whenever $xyz \in I$ for some nonunits $x, y, z \in R$, then either $xy \in I$ or $z \in I$. Note that every prime ideal is 1-absorbing prime and every 1-absorbing prime ideal is 2-absorbing. The converses are not true. More currently in [12] Koç et al. defined weakly 1-absorbing prime ideals which is a generalization of 1-absorbing prime ideal. A proper ideal I of R is called weakly 1-absorbing prime ideal if $0 \neq xyz \in I$ for some nonunits $x, y, z \in R$ implies that $xy \in I$ or $z \in I$. Following Yassine et al. [15] and Koç [12] in [10] we introduced 1-absorbing prime ideals and weakly 1-absorbing prime ideals in noncommutative rings. For a noncommutative ring R, whenever $xRyRz \subseteq I$ ($\{0\} \neq xRyRz \subseteq I$) for some nonunits $x, y, z \in R$, then $xy \in I$ or $z \in I$, then I is a 1-absorbing prime ideal (weakly 1 absorbing prime ideal). In [14] Ugurlu introduced the concept of a 1-absorbing prime submodule of a unital module over a commutative ring with a non-zero identity. Also in [4] Celikel introduced the notion of 1-absorbing primary submodules of a unital module over a commutative ring with a non-zero identity. In this paper, after introducing the notion of 1-absorbing and weakly 1-absorbing prime submodules of a unital left module over a noncommutative ring with nonzero identity, we examine the properties of the new classes. We show that many of the results of Ugurlu in [14] for 1-absorbing prime submodules of a unital module over a commutative ring with a non-zero identity are also valid for 1-absorbing prime submodules of a unital left module over a noncommutative ring with nonzero identity. For all nonunit elements $a, b \in R$ and $m \in M$, if $aRbRm \subseteq N$, $(\{0\} \neq aRbRm \subseteq N)$ either $ab \in (N :_R M)$ or $m \in N$, then N is called a 1-absorbing prime submodule (weakly 1-absorbing prime submodule) of M. Recall that a proper submodule N of the R-module M is a prime (weakly prime) submodule if $aRm \subseteq N(\{0\} \neq aRm \subseteq N)$ for $a \in R$ and $m \in M$ then $a \in (N :_R M)$ or $m \in N$.

Among many results in this paper, it is shown in Proposition 2.4 if N is a 1-absorbing prime submodule of M then $(N :_R M)$ is a 1-absorbing prime ideal of R. It is also proved in Corollary 2.7 that if M is an R-module and N_1, N_2 submodules of M with $N_2 \subseteq N_1$, then N_1 is a 1-absorbing prime submodule of M if and only if N_1/N_2 is a 1-absorbing prime submodule of M/N_2. We also have the following charaterization of 1-absorbing prime submodules in Theorem 2.9. A proper submodule of an R-module M is a 1-absorbing prime submodule of M if $I_1 I_2 K \subseteq N$ for some proper ideals I_1, I_2 of R and some submodule K of M, then either $I_1 I_2 \subseteq (N :_R M)$ or $K \subseteq N$. If there exists a weakly 1-absorbing prime submodule N in the R-module M that is not a prime submodule, then we show in Theorem 2.10 that R is a local ring. If R is a local ring and N is a weakly 1-absorbing prime submodule that is not 1-absorbing prime, then we show in Proposition 3.10 that $(N :_R M)^2 N = \{0\}$, and in particular, $(N :_R M)^3 \subseteq \mathrm{Ann}(M)$. If R is a local ring and M is a multiplication module and N is a weakly 1-absorbing prime submodule of M that is not a 1-absorbing prime submodule, then $N^3 = \{0\}$ (Proposition 3.11). It is shown in Theorem 3.14 that if N is a proper submodule of the R module M, then N is a weakly 1-absorbing prime submodule of M if for any proper ideals I_1, I_2 of R and a submodule K of M such that $\{0\} \neq I_1 I_2 K \subseteq N$ and N is free triple-zero with respect to I_1, I_2, K, we have either $I_1 I_2 \subseteq (N :_R M)$ or $K \subseteq N$.

We have the following diagram which clarifies the place of 1-absorbing prime submodules and weakly 1-absorbing prime submodules. Here, the arrows in the diagram are irreversible.

$$\begin{array}{ccccc} \text{prime submodule} & \Rightarrow & \text{1-absorbing prime} & \Rightarrow & \text{2-absorbing} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{weakly prime} & \Rightarrow & \text{weakly 1-absorbing prime} & \Rightarrow & \text{weakly 2-absorbing} \end{array}$$

2 1-absorbing prime submodules

Definition 2.1 Let M be an R-module and N be a proper submodule of M. For all nonunit elements $a, b \in R$ and $m \in M$ if $aRbRm \subseteq N$ either $ab \in (N :_R M)$ or $m \in N$, then N is called 1-absorbing prime submodule of M.

Proposition 2.2 *Prime submodules* $\Rightarrow$ 1-*absorbing prime submodules* $\Rightarrow$ 2-*absorbing submodules.*

Proof Let N be a prime submodule of M. Take nonunit elements $a, b \in R$ and $m \in M$ such that $aRbRm \subseteq N$. Now $abRm \subseteq N$ and since N is a prime submodule, $ab \in (N :_R M)$ or $m \in N$, as desired.

Suppose N is a 1-absorbing prime submodule of M. Take any $a, b \in R$ and $m \in M$ such that $aRbRm \subseteq N$. If a and b are nonunits, we have $ab \in (N :_R M)$ or $m \in N$ and we are done. If a is a unit element, then $abm \in N$ implies $bm \in N$. If b is a unit element, then there exists $b' \in R$ such that $b'b = 1$ and we have $am = ab'bm \in N$, as desired. □

Example 2.3 For field K the ring $R = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{bmatrix} : a, b, c \in K \right\}$ is a local ring whose unique maximal ideal M has square zero. Consider the R-module M. Then every proper submodule is a 1-absorbing prime submodule of M. To see this, choose nonunits $x, y \in R$ and $m \in M$ such that $xRyRm \subseteq N$. Since $xRy \subseteq M^2 = \{0\}$, we have $xy \in (N :_R M)$ which implies N is a 1-absorbing prime submodule of M.

Proposition 2.4 *If N is a 1-absorbing prime submodule of M then we have the following:*

1. $(N :_R M)$ *is a* 1*-absorbing prime ideal of R.*
2. $(N : Rm)$ *is a* 1*-absorbing prime ideal of R for every $m \in M \backslash N$.*

Proof Let N be a 1-absorbing prime submodule of M.

1. Choose nonunits $a, b, c \in R$ such that $aRbRc \subseteq (N :_R M)$. For all $m \in M$, then $aRbRcm \subseteq N$. By our hypothesis, $ab \in (N :_R M)$ or $cm \in N$. If $ab \in (N :_R M)$, then we done. So suppose $ab \notin (N :_R M)$. Hence $cm \in N$ for all $m \in M$. This implies that $c \in (N :_R M)$. Consequently, $(N :_R M)$ is 1-absorbing prime ideal of R.

2. Choose nonunits $a, b, c \in R$ such that $aRbRc \subseteq (N : Rm)$. Hence $aRbRcRm \subseteq N$ and therefore $aRbRcrm \subseteq N$ for all $r \in R$. By our hypothesis or $ab \in (N :_R M)$ or $crm \in N$ for all $r \in R$. Thus $ab \in (N :_R M) \subseteq (N : Rm)$ or $c \in (N : Rm)$. Consequently, $(N : Rm)$ is 1-absorbing prime ideal of R. □

The converse of the above proposition is not true in general.

Example 2.5 Let p be a fixed prime integer. Then $\mathbb{Z}(p^\infty) = \{a \in Q/\mathbb{Z} : a = r/p^n + \mathbb{Z}$ for some $r \in \mathbb{Z}$ and $n \geq 0\}$ is a nonzero submodule of $Q/\mathbb{Z}$. Let $G_t = \{a \in Q/\mathbb{Z} : a = r/p^t + \mathbb{Z}$ for some $r \in \mathbb{Z}\}$ for all $t \geq 0$. It is well known that each proper submodule of $\mathbb{Z}(p^\infty)$ is equal to G_t for some $t \geq 0$. G_t is not a 1-absorbing prime submodule of $\mathbb{Z}(p^\infty)$ since for $p^2(1/p^{t+2} + \mathbb{Z}) \in G_t$ we have $(1/p^{t+2} + Z) \notin G_t$ and $p^2 \notin (G_t :_\mathbb{Z} \mathbb{Z}(p^\infty)) = \{0\}$. We can see that $(G_t :_\mathbb{Z} \mathbb{Z}(p^\infty)) = \{0\}$ is a 1-absorbing prime ideal of $\mathbb{Z}$ for all $t \geq 0$.

Note that from the above remark we have that some modules do not have any 1-absorbing prime submodules. Since each proper submodule of $\mathbb{Z}(p^\infty)$ is equal to G_t for some $t \geq 0$, so $\mathbb{Z}(p^\infty)$ does not have any 1-absorbing prime submodule.

Proposition 2.6 *Let M_1 and M_2 be R-modules and $f : M_1 \to M_2$ be a module homomorphism. Then the following statements hold:*

1. *If N_2 is a* 1*-absorbing prime submodule of M_2, then $f^{-1}(N_2)$ is a* 1*-absorbing prime submodule of M_1.*
2. *Let f be an epimorphism. If N_1 is a* 1*-absorbing prime submodule of M_1 containing* $\ker(f)$*, then $f(N_1)$ is a* 1*-absorbing prime submodule of M_2.*

Proof 1. Suppose that a, b are nonunit elements of R, $m_1 \in M_1$ and $aRbRm_1 \subseteq f^{-1}(N_2)$. Then $aRbRf(m_1) \subseteq N_2$. Since N_2 is a 1-absorbing prime submodule, we have either $ab \in (N_2 :_R M_2)$ or $f(m_1) \in N_2$. Here, we show that $(N_2 :_R M_2) \subseteq (f^{-1}(N_2) : M_1)$. Let $r \in (N_2 : M_2)$. Then $rM_2 \subseteq N_2$ which implies that $rf^{-1}(M_2) \subseteq f^{-1}(N_2)$, i.e., $rM_1 \subseteq f^{-1}(N_2)$. Thus $r \in (f^{-1}(N_2) : M_1)$. Hence $ab \in (f^{-1}(N_2) : M_1)$ or $m_1 \in f^{-1}(N_2)$. Hence $f^{-1}(N_2)$ is a 1-absorbing prime submodule of M_1.

2. Suppose that are nonunit elements a and b of R, $m_2 \in M_2$ and $aRbRm_2 \in f(N_1)$. Since f is an epimorphism, there exists $m_1 \in M_1$ such that $f(m_1) = m_2$. Since $\ker(f) \subseteq N_1$,

$aRbRm_1 \subseteq N_1$. Hence $ab \in (N_1 : M_1)$ or $m_1 \in N_1$. Here, we show that $(N_1 : M_1) \subseteq (f(N_1) : M_2)$. Let $r \in (N_1 : M_1)$. Then $rM_1 \subseteq N_1$ which implies that $rf(M_1) \subseteq f(N_1)$. Since f is onto, we conclude that $rM_2 \subseteq f(N_1)$, that is, $r \in (f(N_1) : M_2)$. Thus $ab \in (f(N_1) : M_2)$ or $m_2 \in f(N_1)$, as desired □

As a consequence of Proposition 2.6, we have the following result.

Corollary 2.7 *Let M be an R-module and N_1, N_2 be submodules of M with $N_2 \subseteq N_1$. Then N_1 is a 1-absorbing prime submodule of M if and only if N_1/N_2 is a 1-absorbing prime submodule of M/N_2.*

Proof Suppose that N_1 is a 1-absorbing primary submodule of M. Consider the canonical epimorphism $f : M \to M/N_2$ in Proposition 2.6. Then N_1/N_2 is a 1-absorbing prime submodule of M/N_2. Conversely, let a and b are nonunit elements of R, $m \in M$ such that $aRbRm \subseteq N_1$. Hence $aRbR(m + N_2) \subseteq N_1/N_2$. Since N_1/N_2 is a 1-absorbing prime submodule of M/N_2, it implies either $ab \in (N_1/N_2 :_R M/N_2)$ or $m + N_2 \in M/N_2$. Therefore $ab \in (N_1 :_R M)$ or $m \in N_1$. Thus N_1 is a 1-absorbing prime submodule of M. □

Let M_1 be R_1-module and M_2 be R_2-module where R_1 and R_2 are noncommutative rings with identity. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M is an R-module and every submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1, N_2 of M_1, M_2, respectively.

Proposition 2.8 *Let M_1 be R_1-module and M_2 be R_2-module where R_1 and R_2 are non-commutative rings with identity. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Suppose that N_1 is a proper submodule of M_1. If $N = N_1 \times M_2$ is a 1-absorbing prime submodule of the R-module M, then N_1 is a 1-absorbing prime submodule of M.*

Proof Suppose that $N = N_1 \times M_2$ is a 1-absorbing prime submodule of M. Put $M' = M/(\{0\} \times M_2)$ and $N' = N/(\{0\} \times N_2)$. From Corollary 2.7, N' is a 1-absorbing prime submodule of M'. Since $M' \cong M_1$ and $N' \cong N_1$, we conclude the result. □

Next we give several characterizations of 1-absorbing prime submodules of an R-module.

Theorem 2.9 *Let N be a proper submodule of an R-module M. Then the following statements are equivalent:*

(1) *N is a 1-absorbing prime submodule of M.*
(2) *If a, b are nonunit elements of R such that $ab \notin (N :_R M)$, then $(N :_M aRbR) \subseteq N$.*
(3) *If a, b are nonunit elements of R, and K is a submodule of M with $aRbK \subseteq N$, then $ab \in (N :_R M)$ or $K \subseteq N$.*
(4) *If $I_1 I_2 K \subseteq N$ for some proper ideals I_1, I_2 of R and some submodule K of M, then either $I_1 I_2 \subseteq (N :_R M)$ or $K \subseteq N$.*

Proof (1) ⇒ (2) Suppose that a, b are nonunit elements of R such that $ab \notin (N :_R M)$. Let $m \in (N :_M aRbR)$. Hence $aRbRm \subseteq N$. Since N is 1-absorbing prime submodule and $ab \notin (N :_R M)$, we have $m \in N$, and so $(N :_M aRbR) \subseteq N$.

(2) ⇒ (3) Suppose that $ab \notin (N :_R M)$. Since $aRbRK \subseteq aRbK \subseteq N$, we have $K \subseteq (N :_M aRbR) \subseteq N$ by (2). (3) ⇒ (4) Suppose $I_1 I_2 K \subseteq N$ for some proper ideals I_1, I_2 of R and some submodule K of M. Assume on the contrary that neither $I_1 I_2 \subseteq (N :_R M)$ nor $K \subseteq N$. Then there exist nonunit elements $a \in I_1, b \in I_2$ with $ab \notin (N :_R M)$. Thus $aRbK \subseteq N$, which contradicts (3).

(4) $\Rightarrow$ (1) Let $a, b \in R$ be nonunit elements, $m \in M$ and $aRbRm \subseteq N$. Put $I_1 = RaR$, $I_2 = RbR$, $K = Rm$. Now $RaRRbRRm \subseteq RaRbRm \subseteq N$. Thus $ab \in RaRRbR \subseteq (N :_R M)$ or $m \in Rm \subseteq N$ and we are done. □

Theorem 2.10 *Let M be an R-module. If N is a 1-absorbing prime submodule of M that is not a prime submodule, then R is a local ring.*

Proof Suppose that N is a 1-absorbing prime submodule of M that is not a prime submodule. Then there exist a nonunit $r \in R$ and $m \in M$ such that $rRm \subseteq N$ but $r \notin (N :_R M)$ and $m \notin N$. Choose a nonunit element $s \in R$. Hence we have that $rRsRm \subseteq rRm \subseteq N$ and $m \notin N$. Since N is 1-absorbing prime, $rs \in (N :_R M)$. Let us take a unit element $u \in R$. We claim that $s + u$ is a unit element of R. To see this, assume $s + u$ is a nonunit. Then $rR(s + u)Rm \subseteq rRm \subseteq N$. As N is 1-absorbing prime, $r(s + u) \in (N :_R M)$. This means that $ru \in (N :_R M)$, i.e., $r \in (N :_R M)$, which is a contradiction. Thus for any nonunit element s and unit element u in R, we have $s + u$ is a unit element. From [9, Lemma 4.1], we have that R is a local ring. □

Corollary 2.11 *Let M be an R-module where R is not a local ring. Then a proper submodule N of M is a 1-absorbing prime submodule if and only if N is a prime submodule of M.*

Proposition 2.12 *Let $\{N_i : i \in \Delta\}$ be a chain of 1-absorbing prime submodules of the R-module M. Then $\bigcap_{i\in\Delta} N_i$ is a 1-absorbing prime submodule of M.*

Proof Let $\{N_i : i \in \Delta\}$ be a chain of 1-absorbing prime submodules of M. Take nonunit elements $a, b \in R$ and $m \in M$ such that $aRbRm \subseteq \bigcap_{i\in\Delta} N_i$. Assume that $m \notin \bigcap_{i\in\Delta} N_i$, so there exists $i \in \Delta$ such that $m \notin N_i$. Since N_i is 1-absorbing prime, we conclude $ab \in (N_i : M)$. For any $j \in \Delta$, we have $N_i \subseteq N_j$ or $N_j \subseteq N_i$. Without loss of generality, if $N_i \subseteq N_j$ then $(N_i : M) \subseteq (N_j : M)$, that is, $ab \in (N_j : M)$. If $N_j \subseteq N_i$, then $ab \in (N_j : M)$ since $m \notin N_j$ and N_j is 1-absorbing prime. Hence we have $ab \in \bigcap_{i\in\Delta} \{(N_i : M) : i \in \Delta\} = ((\bigcap_{i\in\Delta} N_i : i \in \Delta) : M)$. □

Definition 2.13 Let M be an R-module and N be a proper submodule of M. Let P be a 1-absorbing prime submodule of M such that $N \subseteq P$. If there does not exist a 1-absorbing prime submodule P' such that $N \subseteq P' \subset P$, then P is called a minimal 1-absorbing prime submodule over N.

Proposition 2.14 *Let M be an R-module and N be a proper submodule of M. If P is a 1-absorbing prime submodule of M such that $N \subseteq P$, then there exists a minimal 1-absorbing prime submodule over N that is contained in P.*

Proof Let $\Lambda = \{P_i : P_i$ is a submodule of M such that $N \subseteq P_i \subseteq P\}$. Since $N \subseteq P$, we have $\Lambda \neq \emptyset$. Consider $(\Lambda, \supseteq)$. Let us take a chain $\{N_i : i \in \Delta\}$ in Λ. Since by Proposition 2.12, $\cap_{i\in\Delta} N_i$ is a 1-absorbing prime submodule of M, there exists a maximal element $K \in \Lambda$ by applying Zorn's Lemma. Then K is 1-absorbing prime and $N \subseteq K \subseteq P$. Now we will show that K is a minimal 1-absorbing prime submodule over N. On the contrary, assume that there exists a 1-absorbing prime submodule K' such that $N \subseteq K' \subseteq K$. Then $K' \in \Lambda$ and $K \subseteq K'$. This implies $K = K'$. Consequently, K is a minimal 1-absorbing prime submodule of N. □

Corollary 2.15 *Let M be an R-module. Every 1-absorbing prime submodule of M contains at least one minimal 1-absorbing prime submodule of M.*

3 Weakly 1-absorbing prime submodules

Definition 3.1 Let R be a ring and N be a proper submodule of an R-module M. Then N is a weakly 1-absorbing prime submodule of M if $\{0\} \neq aRbRm \subseteq N$ implies $abM \subseteq N$ i.e. $ab \in (N :_R M)$ or $m \in N$ for nonunits $a, b \in R$ and $m \in M$.

Remark 3.2 1. Every 1–absorbing prime submodule is weakly 1-absorbing prime but the converse does not necessarily hold. For example consider the case where $R = \mathbb{Z}$, $M = \mathbb{Z}/30\mathbb{Z}$ and $N = \{0\}$. Then $2 \cdot 3 \cdot (5 + 30\mathbb{Z}) = 0 \in N$ while $2 \cdot 3 \notin (N :_R M)$, $(5 + 30Z) \notin N$. Therefore N is not 1-absorbing prime while it is weakly 1-absorbing prime.

2. Every weakly prime submodule is weakly 1 absorbing prime but the converse does not necessarily hold. Let $M = \mathbb{Z}_{12}$ be a module over $\mathbb{Z}$ and $W = \{\overline{0}, \overline{4}, \overline{8}\}$ be a proper submodule of M. Let $r, s \in \mathbb{Z}$ and $m \in M$. Now $(W : M) = 4\mathbb{Z}$. Therfore, for $0 \neq rsm \in W$, we get $m = \overline{4}$ or $m = \overline{8}$ which are elements in W or $rs \in 4\mathbb{Z}$. So W is a weakly 1 absorbing prime submodule. W is not weakly prime since $\overline{0} \neq 2 \cdot \overline{2} \in W$ with $\overline{2} \notin W$ and $2 \notin (W : M)$.

Question. Suppose that L is a weakly 1-absorbing prime submodule of an R-module M and $\{0\} \neq IJK \subseteq L$ for some ideals I, J of R and a submodule K of M. Does it imply that $IJ \subseteq (M :_R L)$ or $K \subseteq L$?

This section is devoted to studying the above question for modules over noncommutative rings.

Proposition 3.3 *Let $x \in M$ and $a \in R$. Then if $ann_l(x) \subseteq (Rx : M)$, the submodule Rx is 1-absorbing prime if and only if Rx is weakly 1-absorbing prime.*

Proof If Rx is 1-absorbing prime then it is clear that Rx is weakly 1- absorbing prime. Let Rx be a weakly 1-absorbing prime submodule of M and suppose $r, s \in R$ are nonunits and $m \in M$ with $rRsRm \subseteq Rx$. Since Rx is a weakly 1-absorbing prime submodule, we may assume $rRsRm = \{0\}$, otherwise Rx is 1-absorbing prime. Now $rRsR(x + m) \subseteq Rx$. If $rRsR(x + m) \neq \{0\}$ then we have $rs \in (Rx : M)$ or $(x + m) \in Rx$, as Rx is a weakly 1-absorbing prime submodule. Hence $rs \in (Rx : M)$ or $m \in Rx$. Now let $rRsR(x+m) = \{0\}$. Then $rRsRm = \{0\}$ implies $rRsRx = \{0\}$. Hence $rs \in ann_l(x) \subseteq (Rx : M)$. Thus Rx is 1-absorbing prime. □

Proposition 3.4 *Let R be a ring and N be a proper submodule of an R-module M.*

1. *If N is weakly prime, then it is weakly 1-absorbing prime.*
2. *If N is a weakly 1-absorbing prime submodule of M, then it is a weakly 2-absorbing submodule.*

Proof 1. Assume N is a weakly prime submodule of the R-module M and $\{0\} \neq aRbRm \subseteq N$ for nonunits $a, b \in R$ and $m \in M$. Suppose $m \notin N$. Since $abRm \subseteq N$ and $m \notin N$, we have $ab \in (N :_R M)$. Hence N is weakly 1-absorbing prime. 2. Assume N is a weakly 1-absorbing prime submodule of the R-module M and $\{0\} \neq aRbRm \subseteq N$ for $a, b \in R$ and $m \in M$. If a is a unit, then it is easy to see that $bm \in N$. If b is a unit, then there exists $b' \in R$ such that $ab'bm = am \in N$. If both a and b are nonunits, then $ab \in (N :_R M)$ since N is a weakly 1-absorbing prime submodule of the R-module M. Hence N is weakly 2-absorbing. □

Proposition 3.5 *Let N be a weakly 1-absorbing prime submodule of an R-module M. Assume that K is a submodule of M with $N \subsetneq K$. Then N is a weakly 1-absorbing prime submodule of K.*

Proof Let $a, b \in R$ be nonunits and $k \in K$ with $\{0\} \neq aRbRk \subseteq N$. Then $ab \in (N :_R M)$ or $k \in N$ as N is a weakly 1-absorbing prime submodule of M. Thus $ab \in (N :_R K)$ or $k \in K$ since $(N :_R M) \subseteq (N :_R K)$ and $N \subseteq K$. □

Proposition 3.6 *Let N, K be submodules of an R-module M with $K \subseteq N$. If N is a weakly 1-absorbing prime submodule of M, then N/K is a weakly 1-absorbing prime submodule of M/K. The converse is true when K is a weakly 1-absorbing prime submodule.*

Proof Assume that N is a weakly 1-absorbing prime submodule of M. Let $a, b \in R$ be nonunits and $m + K \in M/K$ where $\{0_{M/K}\} \neq aRbR(m + K) \subseteq N/K$. Since $aRbR(m + K) \neq \{0_{M/K}\}$, we get $aRbRm \subseteq N$ and $aRbPm \nsubseteq K$. If $aRbRm = \{0\}$, we obtain $aRbRm + K = \{0_{M/K}\}$. So $aRbRm \neq \{0\}$. Thus $ab \in (N :_R M)$ or $m \in N$ as N is weakly 1-absorbing prime. Consequently, we get $ab \in (N/K :_R M/K)$ or $m + K \in N/K$. Conversely, let K be a weakly 1-absorbing prime submodule. Assume that N/K is a weakly 1-absorbing prime submodule of M/K. Let $a, b \in R$ be nonunits and $m \in M$ where $\{0\} \neq aRbRm \subseteq N$. Then we have $aRbRm + K \subseteq N/K$. If $aRbRm + K = \{0_{M/K}\}$, then $aRbRm \subseteq K$. Thus $ab \in (K :_R M)$ or $m \in K$, since K is weakly 1-absorbing prime. Therefore, $ab \in (N :_R M)$ or $m \in N$, since $K \subseteq N$. Let $aRbRm + K = aRbR(m + K) \neq \{0_{M/K}\}$. Then $ab \in (N/K :_R M/K)$ or $m + K \in N/K$. Thus $ab \in (N :_R M)$ or $m \in N$. □

Definition 3.7 Let N be a weakly 1-absorbing prime submodule of an R-module M. For nonunits $a, b \in R$ and $m \in M$, (a, b, m) is called a triple-zero of N if $aRbRm = 0$, $ab \notin (N :_R M)$ and $m \notin N$.

Note that if N is a weakly 1-absorbing prime submodule of M and there is no triple-zero of N, then N is a 1-absorbing prime submodule of M.

Proposition 3.8 *Let N be a weakly 1-absorbing prime submodule of M and K be a proper submodule of M with $K \subseteq N$. Then for any nonunits $a, b \in R$ and $m \in M$, (a, b, m) is a triple-zero of N if and only if $(a, b, m + K)$ is a triple-zero of N/K.*

Proof Let (a, b, m) be a triple-zero of N for some nonunits $a, b \in R$ and $m \in M$. Then $aRbRm = \{0\}, ab \notin (N :_R M)$ and $m \notin N$. By Proposition 3.6, we get that N/K is a weakly 1-absorbing prime submodule of M/K. Thus $aRbR(m + K) = K, ab \notin (N/K : M/K)$ and $(m + K) \notin N/K$. Hence $(a, b, m + K)$ is a triple-zero of N/K. Conversely, assume that $(a, b, m + K)$ is a triple-zero of N/K. Suppose that $aRbRm \neq \{0\}$. Then $aRbRm \subseteq N$ since $aRbR(m + K) = K$. Thus $ab \in (N :_R M)$ or $m \in N$ as N is weakly 1-absorbing prime, a contradiction. So it must be $aRbRm = \{0\}$. Consequently, (a, b, m) is a triple-zero of N. □

Theorem 3.9 *Let R be a local ring and let N be weakly 1-absorbing prime submodule of M and (a, b, m) be a triple-zero of N for some nonunits $a, b \in R$ and $m \in M$. Then the following hold.*

1. $aRbN = a(N :_R M)m = b(N :_R M)m = \{0\}$.
2. $a(N :_R M)N = b(N :_R M)N = (N :_R M)bN = (N :_R M)bm = (N :_R M)^2 m = \{0\}$.

Proof Suppose that (a, b, m) is a triple-zero of N for some nonunits $a, b \in R$ and $m \in M$. 1. Assume that $aRbN \neq \{0\}$. Then there is an element $n \in N$ such that $aRbRn \neq \{0\}$. Now $aRbR(m+n) = aRbRm + aRbRn = aRbRn \neq \{0\}$ since $aRbRm = \{0\}$. Since $\{0\} \neq aRbR(m+n) \subseteq N$ and N a weakly 1-absorbing prime submodule, we have $ab \in (N :_R M)$ or $(m+n) \in N$. Hence $ab \in (N :_R M)$ or $m \in N$. This is a contradiction since (a, b, m) is a triple zero of N. Hence $aRbN = \{0\}$. Now, we suppose that $a(N :_R M)m \neq \{0\}$. Thus there exists an element $r \in (N :_R M)$ such that $arm \neq 0$. Hence $aR(r+b)Rm = aRrRm + aRbRm = aRrRm$. Hence $\{0\} \neq aR(r+b)Rm \subseteq N$. Since R is local, the set of nonunit elements of R is an ideal of R. Therefore $(r+b)$ is a nonunit. Since N is a weakly 1-absorbing prime submodule, we have $a(r+b) \in (N :_R M)$ or $m \in N$. Consequently, $ab \in (N :_R M)$ or $m \in N$ a contradiction since (a, b, m) is a triple-zero of N. Hence $a(N :_R M)m = \{0\}$. Similarly, we can proof that $b(N :_R M)m = \{0\}$. 2. Assume that $a(N :_R M)N \neq \{0\}$. Then there are $r \in (N :_R M)$, $n \in N$ such that $arn \neq 0$. By (1), we get $a(b+r)(m+n) = abm + abn + arm + arn = arn \neq 0$. Now $\{0\} \neq aR(b+r)R(m+n) \subseteq N$. Again, since R is a local ring $(b+r)$ is a nonunit and since N is 1-absorbing prime, we have $a(b+r) \in (N :_R M)$ or $(m+n) \in N$. Hence we obtain $ab \in (N :_R M)$ or $m \in N$ a contradiction. Hence $a(N :_R M)N = \{0\}$. In a similar way we get $b(N :_R M)N = \{0\}$. Now, we suppose that $(N :_R M)bN \neq \{0\}$. Then there are $r \in (N :_R M)$, $n \in N$ such that $rbn \neq 0$. Now, from above $(a+r)b(n+m) = abn + abm + rbn + rbm = rbn \neq 0$. Hence $\{0\} \neq (a+r)RbR(n+m) \subseteq N$ and since N is weakly 1-absorbing and $(a+r)$ is a nonunit, we have $(a+r)b \in (N :_R M)$ or $(n+m) \in N$. Hence $ab \in (N :_R M)$ or $m \in N$ a contradiction since (a, b, m) is a triple-zero of N. Now, we suppose that $(N :_R M)bm \neq \{0\}$. Then there is $r \in (N :_R M)$ such that $rbm \neq 0$. Hence $0 \neq rbm = (a+r)bm \in (a+r)RbRm = rRbRm \subseteq N$. Since N is a weakly 1-absorbing prime submodule, we have $(a+r)b \in (N :_R M)$ or $m \in N$. Therefore $ab \in (N :_R M)$ or $m \in N$ a contradiction since (a, b, m) is a triple-zero of N. Hence $(N :_R M)bm = \{0\}$. Lastly, we show that $(N :_R M)^2 m = \{0\}$. Let $(N :_R M)^2 m \neq \{0\}$. Thus there exist $r, s \in (N :_R M)$ where $rsm \neq 0$. By (1), we get $(a+r)(b+s)m = rsm \neq 0$. Thus we have $\{0\} \neq (a+r)R(b+s)Rm \subseteq N$. Since R is a local ring, $(a+r)$ and $(b+s)$ are nonunits. Hence $(a+r)(b+s) \in (N :_R M)$ or $m \in N$. Consequently $ab \in (N :_R M)$ or $m \in N$ a contradiction since (a, b, m) is a triple-zero of N. Therefore $(N :_R M)^2 m = \{0\}$. □

Proposition 3.10 *Let R be a local ring. Assume that N is a weakly 1-absorbing prime submodule of an R-module M that is not 1-absorbing prime. Then $(N :_R M)^2 N = \{0\}$. In particular, $(N :_R M)^3 \subseteq Ann(M)$.*

Proof Suppose that N is a weakly 1-absorbing prime submodule of an R-module M that is not 1-absorbing prime. Then there is a triple-zero (a, b, m) of N for some nonunits $a, b \in R$ and $m \in M$. Assume that $(N :_R M)^2 N \neq \{0\}$. Thus there exist $r, s \in (N :_R M)$ and $n \in N$ with $rsn \neq 0$. By Theorem 3.9, we get $(a+r)(b+s)(n+m) = rsn \neq 0$. Then we have $\{0\} \neq (a+r)R(b+s)R(n+m) \subseteq N$. Since N is weakly 1-absorbing prime, we have $(a+r)(b+s) \in (N :_R M)$ or $(n+m) \in N$ and so $ab \in (N :_R M)$ or $m \in N$ which is a contradiction. Thus $(N :_R M)^2 N = \{0\}$. We get $(N :_R M)^3 \subseteq ((N :_R M)^2 N : M) = (\{0\} : M) = \text{Ann}(M)$. □

Proposition 3.11 *Let R be a local ring and let M be a multiplication R-module and N be a weakly 1-absorbing prime submodule of M that is not a 1-absorbing prime submodule. Then $N^3 = \{0\}$.*

Proof We have that $(N :_R M)M = N$ since M is a multiplication module. Then $N^3 = (N :_R M)^3 M = (N :_R M)^2 N = \{0\}$. Consequently, $N^3 = \{0\}$. □

Definition 3.12 Let N be a weakly 1-absorbing prime submodule of an R-module M and let $a, b \in R$ be nonunits. Let $\{0\} \neq I_1 I_2 K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M. N is called free triple-zero in regard to I_1, I_2, K if (a, b, m) is not a triple-zero of N for every $a \in I_1, b \in I_2$ and $m \in K$.

Lemma 3.13 *Let N be a weakly 1-absorbing prime submodule of M. Assume that $aRbK \subseteq N$ for some nonunits $a, b \in R$ and some submodule K of M where (a, b, m) is not a triple-zero of N for every $m \in K$. If $ab \notin (N :_R M)$, then $K \subseteq N$.*

Proof Suppose that $aRbK \subseteq N$, but $ab \notin (N :_R M)$ and $K \nsubseteq N$. Then there exists an element $k \in K \backslash N$. But (a, b, k) is not a triple-zero of N and $aRbRk \subseteq N$ and $ab \notin (N :_R M)$ and $k \notin N$, a contradiction. Hence $K \subseteq N$. □

Let N be a weakly 1-absorbing prime submodule of an R-module M and $I_1 I_2 K \subseteq N$ for some for some ideals I_1, I_2 of R and some submodule K of M where N is free triple-zero in regard to I_1, I_2, K. Note that if $a \in I_1, b \in I_2$ and $m \in K$, then $ab \in (N :_R M)$ or $m \in N$.

Theorem 3.14 *Suppose that N is a proper submodule of the R module M. Then the following statements are equivalent.*

1. *N is a weakly 1-absorbing prime submodule of M.*
2. *For any proper ideals I_1, I_2 of R and a submodule K of M such that $\{0\} \neq I_1 I_2 K \subseteq N$ and N is free triple-zero with respect to I_1, I_2, K, we have either $I_1 I_2 \subseteq (N :_R M)$ or $K \subseteq N$.*

Proof (1) $\Rightarrow$ (2) Suppose that N is a weakly 1-absorbing prime submodule of M and $\{0\} \neq I_1 I_2 K \subseteq N$ for proper ideals I_1, I_2 of R and a submodule K of M such that N is free triple-zero with respect to I_1, I_2, K. Then there are nonunit elements $a \in I_1$ and $b \in I_2$ such that $ab \notin (N :_R M)$. Since $aRbK \subseteq N$, $ab \notin (N :_R M)$ and (a, b, k) is not a triple-zero of N for every $k \in K$, it follows from Lemma 3.13 that $K \subseteq N$. (2) $\Rightarrow$ (1) Suppose that $\{0\} \neq aRbRm \subseteq N$ for some nonunit elements $a, b \in R$ and $m \in M$. Suppose $ab \notin (N :_R M)$. Let $I_1 = RaR$, $I_2 = RbR$ and $K = Rm$. Now, $\{0\} \neq RaRRbRRm \subseteq RN \subseteq N$. Hence $\{0\} \neq I_1 I_2 K \subseteq N$ and $I_1 I_2 \nsubseteq (N :_R M)$. From (2), it follows that $m \in Rm = K \subseteq N$ and we are done. □

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