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The Jacobson property in rings and Banach algebras

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Abstract
In a Banach algebra A it is well known that the usual spectrum has the following property:

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$$

for elements $a, b \in A$. In this note we are interested in subsets of A that have the Jacobson Property, i.e. $X \subset A$ such that for $a, b \in A$:

$$1 - ab \in X \implies 1 - ba \in X.$$

We are interested in sets with this property in the more general setting of a ring. We also look at the consequences of ideals having this property. We show that there are rings for which the Jacobson radical has this property.

Keywords Banach algebra · Ring · Topological ring · Dedekind finite · Spectral theory

Mathematics Subject Classification 13A10 · 13J99 · 46H10

1 Preliminaries

In this article we continue the investigation into sets that satisfy the Jacobson property started in [9].

In a commutative ring with identity every set satisfies trivially the Jacobson property, so in this article an arbitrary ring or algebra will be unital and not commutative. For a ring R, we let R_l^{-1} (R_r^{-1}) denote the left (right) invertibles in R. By an ideal I in a ring R we will mean a two sided ideal. When we wish to speak of a one sided ideal, we will call the ideal a *right* (*left*) ideal. If R is a ring and I an ideal in R then R/I will denote the factor ring of R modulo I, i.e. $R/I = \{a + I : a \in R\}$. In this case, if $a \in R$, the symbol $\overline{a}$ will denote the equivalence class of a, i.e. $\overline{a} = a + I \in R/I$. If R is a ring and $a, b \in R$ then $ab - ba$ will be called the *commutator* of a and b, denoted by $[a, b]$. For R a ring, we will denote by R_D the ideal generated by all commutators in R.

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We are interested in subsets of a ring that have the *Jacobson property*. We are interested in the case where these sets are ideals and specifically, we want to investigate sufficient conditions for the Jacobson radical (Rad R) of the ring to have the Jacobson property.

Definition 1.1 Let R be a ring and $S \subseteq R$. We will say that S satisfies the Jacobson property if $a, b \in R$ and $1 - ab \in S \implies 1 - ba \in S$.

Definition 1.2 Let R be a ring. We will say R is *Dedekind finite* if

$$ab = 1 \implies ba = 1 \text{ for all } a, b \in R$$

Definition 1.3 [1] If R is a ring and $S \subseteq R$, we will say that S is *commutatively closed* if for any $a, b \in R$ we have

$$ab \in S \implies ba \in S.$$

Definition 1.4 [4] A ring R is called *reversible* if

$$a, b \in R \text{ and } ab = 0 \implies ba = 0.$$

We will use the following definition of a topological ring:

Definition 1.5 [10, Definition 1.1] A topology $\mathcal{T}$ on a ring R is a *ring topology* and R, furnished with $\mathcal{T}$, is a *topological ring* if the following conditions hold:

1. $(x, y) \to x + y$ is continuous from $R \times R$ to R
2. $x \to -x$ is continuous from R to R
3. $(x, y) \to xy$ is continuous from $R \times R$ to R

where R is given topology $\mathcal{T}$ and $R \times R$ the cartesian product topology determined by $\mathcal{T}$.

2 Ideals with the Jacobson property and commutatively closed ideals

Proposition 2.1 *Let R be a ring, and I an ideal such that $R_D \subseteq I$. Then I has the Jacobson property and I is commutatively closed.*

Proof If $1 - ab \in I$ then $1 - ba = 1 - ab + ab - ba = 1 - ab + [a, b] \in I$.
Also, if $ab \in I$ then $ba = ab + ba - ab = ab + [b, a] \in I$. □

For ease of reference we state the following proposition.

Proposition 2.2 [1, Proposition 3.11] *Let R be a ring and I a one sided ideal in R that is commutatively closed. Then I is a two-sided ideal.*

Proposition 2.3 *Let R be a ring, and I an ideal which contains R_D. Then R/I partitions R into equivalence classes, each of which satisfies the Jacobson property and is commutatively closed.*

Proof The fact that R/I partitions the ring is clear. So we prove that each equivalence class satisfies the Jacobson property. For $c \in R$, let $1 - ab \in \overline{c}$. Then $1 - ab - c \in I$, and

$$1 - ba - c = 1 - ab + ab - ba - c = 1 - ab - c + ab - ba = 1 - ab - c + [a, b] \in I.$$

Hence $1 - ba \in \overline{c}$ as required. Similarly, suppose $ab \in \overline{c}$. Then $ab - c \in I$. Hence

$$ba - c = ab + ba - ab - c = ab - c + ba - ab = ab - c + [b, a] \in I.$$

Hence $ba \in \overline{c}$, as required. □

Definition 2.1 Let R be a ring and I be an ideal in R. We say I *preserves right invertibility* if $a \in R_r^{-1} \iff \overline{a} \in (R/I)_r^{-1}$. I *preserves left invertibility* if $a \in R_l^{-1} \iff \overline{a} \in (R/I)_l^{-1}$. I *preserves invertibility* if it preserves left and right invertibility.

Proposition 2.4 *Let R be ring and I an ideal that preserves invertibility. Then R is Dedekind finite $\iff$ R/I is Dedekind finite.*

Proof Suppose that R is Dedekind finite and suppose that $\overline{a}\overline{b} = \overline{1}$. Then $\overline{a} \in (R/I)_r^{-1}$, and so by assumption we have $a \in R_r^{-1}$. So there exists $c \in R$ such that $ac = 1$. Since R is Dedekind finite, we have that $ca = 1$ also. This means that $a \in R^{-1}$, and by preservation of invertibility we have that $\overline{a}$ is left and right invertible in R/I. Hence $\overline{b}\overline{a} = \overline{1}$ and R/I is Dedekind finite.
Conversely, suppose that R/I is Dedekind finite, and let $ab = 1$. Then $a \in R_r^{-1}$, so there exists $c \in R$ such that $\overline{ac} = \overline{1}$. Since R/I is Dedekind finite, we also have that $\overline{ca} = \overline{1}$. By preservation of invertibility we have that a is left and right invertible. Since inverses in a ring are unique, we have that $ba = 1$, and R is Dedekind finite. □

Proposition 2.5 *Let R be a ring, and I an ideal in R. Then I satisfies the Jacobson property $\iff$ R/I is Dedekind finite.*

Proof Suppose I satisfies the Jacobson property and let $\overline{a}\overline{b} = \overline{1}$. Then $\overline{ab} = \overline{1}$, so that $1 - ab \in I$. Since I satisfies the Jacobson property, we also have that $1 - ba \in I$. Hence $\overline{b}\overline{a} = \overline{1}$, and R/I is Dedekind finite.

Conversely, suppose that R/I is Dedekind finite, and suppose that $1 - ab \in I$. Then $\overline{ab} = \overline{1}$, hence $\overline{a}\overline{b} = \overline{1}$. Since R/I is Dedekind finite, we have $\overline{ba} = \overline{b}\overline{a} = \overline{1}$ also. Hence $1 - ba \in I$, and so I has the Jacobson property. □

We can now combine Proposition 2.4 and Theorem 2.5 to yield:

Theorem 2.1 *Let R be a ring and I an ideal that preserves invertibility. Then R is Dedekind finite $\iff$ R/I is Dedekind finite.*

Remark If R is a ring, and I is an ideal in R, it is easy to see that R/I is commutative if and only if $R_D \subseteq I$.

Theorem 2.2 *Let R be a ring and I an ideal that contains R_D. Then R/I is a field $\iff$ I is maximal.*

Proof Suppose R/I is a field and let J be an ideal such that $I \subseteq J$. Let $a \in J$, $a \notin I$. Then $\overline{a}$ is a nonzero element of R/I, and since R/I is a field there exists $b \in R$ such that $\overline{a}\overline{b} = \overline{1}$. Hence $1 - ab \in J$. Since $a \in J$, we also have $ab \in J$. Hence $1 = 1 - ab + ab \in J$. This means that $J = R$, and hence I is a maximal ideal.

Conversely, suppose I is a maximal ideal in R that contains R_D. Then R/I is commutative. We will show that every nonzero element of R/I is a unit. So, suppose that $\overline{a}$ is a nonzero element of R/I. Then $a \notin I$. Consider the right ideal defined as $J = \{ar + b : r \in R, b \in I\}$. Since $R_D \subset J$, J must be a two sided ideal by Proposition 2.2. Next, $a \in J$, and $a \notin I$, and since I is maximal we have that $J = R$, so that there exists $b_0 \in I$, $r_0 \in R$ such that $1 = ar_0 + b_0$. This means that $1 - ar_0 \in I$, so that $\overline{a}$ is invertible in R/I. □

3 Rings in which the radical has the Jacobson property

Example 1 Let R be a division ring. Then $\operatorname{Rad} R = \{0\}$, and using Remark 2.1 from [9], together with the fact that R^{-1} has the Jacobson property, we can conclude that $\operatorname{Rad} R$ has the Jacobson property.

Example 2 Let R be a local ring. Then $R = R^{-1} \cup \operatorname{Rad} R$, and again, Remark 2.1 from [9] allows us to conclude that $\operatorname{Rad} R$ has the Jacobson property.

Example 3 Let R be a ring in which $R_D \subset \operatorname{Rad} R$. By Proposition 2.1 $\operatorname{Rad} R$ has the Jacobson property.

From Theorem 3.1.5 in [2] we have that $\operatorname{Rad} R$ preserves invertibility, and so we have the following corollary to Theorem 2.1.

Corollary 3.1 *Let R be a ring. Then R is Dedekind finite $\iff$ $R/\operatorname{Rad} R$ is Dedekind finite.*

We also have the following theorem.

Theorem 3.1 *Let R be a ring. Then $\operatorname{Rad} R$ has the Jacobson Property $\iff$ $R/\operatorname{Rad} R$ is Dedekind finite.*

Proof Suppose $\operatorname{Rad} R$ has the Jacobson Property, and let $\overline{a}\overline{b} = \overline{1}$ in $R/\operatorname{Rad} R$. Then $\overline{ab} = \overline{1}$, hence $1 - ab \in \operatorname{Rad} R$. Then $1 - ba \in \operatorname{Rad} R$, since $\operatorname{Rad} R$ has the Jacobson Property. That means $\overline{b}\overline{a} = \overline{1}$ and $R/\operatorname{Rad} R$ is Dedekind finite.
Conversely, suppose $R/\operatorname{Rad} R$ is Dedekind finite and let $1 - ab \in \operatorname{Rad} R$. Then $\overline{a}\overline{b} = \overline{1}$ and since $R/\operatorname{Rad} R$ is Dedekind finite we have $\overline{b}\overline{a} = \overline{1}$ as well. This means $\overline{ba} = \overline{1}$ or $1 - ba \in \operatorname{Rad} R$ as required. □

Now we can combine Corollary 3.1 and Theorem 3.1 to get our main result:

Theorem 3.2 *Let R be a ring. Then $\operatorname{Rad} R$ has the Jacobson property $\iff$ R is Dedekind finite.* □

Next, we look at some examples of Dedekind finite rings.

Example 4 From Exercise 12, page 25 of [8], we know that every Noetherian ring is Dedekind finite.

Example 5 It is easy to see that any finite ring must be Noetherian, hence Dedekind finite.

The examples above illustrate that there are many examples of rings R for which $\operatorname{Rad} R$ has the Jacobson Property. A simple application of Lemma 2.1 from [9] proves the following theorem:

Theorem 3.3 *Let R be a ring in which $\operatorname{Rad} R$ has the Jacobson property. Then $R \setminus (R^{-1} \cup \operatorname{Rad} R)$ has the Jacobson property.* □

Consider the theorem above. If R is a local ring then $R \backslash (R^{-1} \cup \operatorname{Rad} R) = \emptyset$. To show that this is not always the case, in the next example we construct a ring R, for which $\operatorname{Rad} R$ has the Jacobson property and such that $R \backslash (R^{-1} \cup \operatorname{Rad} R) \neq \emptyset$. Also, this ring will be noncommutative and not semisimple. For ease of reference, we state some results from [3]. Let R be a commutative ring, and denote by $M_n(R)$ the ring of all n by n matrices with entries from R.

Corollary 3.2 [3, Corollary 2.21] *Let $A \in M_n(R)$. Then $A \in (M_n(R))^{-1} \iff \det A \in R^{-1}$.* □

Theorem 3.4 [3, Theorem 3.20] *For any commutative ring R,* $\text{Rad}\, M_n(R) = M_n(\text{Rad}\, R)$. □

Example 6 Let $R = M_2(\mathbb{Z}_{12})$. Since $\mathbb{Z}_{12}$ is finite, R is a finite ring, hence it is Dedekind finite. Also, R is not commutative, since

$$\begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{bmatrix} \begin{bmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix} = \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{bmatrix} \neq \begin{bmatrix} \overline{1} & \overline{3} \\ \overline{0} & \overline{0} \end{bmatrix} = \begin{bmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{bmatrix} \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{bmatrix}$$

The prime divisors of 12 are 2 and 3, so that the maximal ideals in the ring $\mathbb{Z}_{12}$ are $\langle \overline{2} \rangle$ and $\langle \overline{3} \rangle$. Hence $\text{Rad}\, \mathbb{Z}_{12} = \{\overline{0}; \overline{6}\}$. Then, from Theorem 3.4 above

$$\text{Rad}\, R = \left\{ \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} \middle|\, a, b, c, d \in \{0, 6\} \right\}.$$

Clearly then R is not semisimple.

In the ring $\mathbb{Z}_{12}$, an element $\overline{a}$ is invertible if and only if a is relatively prime to 12, hence $\left(\mathbb{Z}_{12}\right)^{-1} = \{\overline{1}; \overline{5}; \overline{7}; \overline{11}\}$. Consider $A = \begin{bmatrix} \overline{1} & \overline{0} \\ 0 & 2 \end{bmatrix}$. We have $\det A = \overline{2} \notin \left(\mathbb{Z}_{12}\right)^{-1}$, hence by Corollary 3.2 we have that $A \notin R^{-1}$. Clearly $A \notin \text{Rad}\, R$ as well. Hence $R \backslash (R^{-1} \cup \text{Rad}\, R) \neq \emptyset$.

4 Multiplicative linear functionals

Part of the first result below is from the original article that showed us how to build a multiplicative linear functional on a noncommutative Banach algebra ([6, p 214,]). We prove it here for ease of reference, since we need it in the theorem that follows.

Lemma 4.1 *Let A be a noncommutative Banach algebra and $f : A \to \mathbb{C}$ be a nonzero multiplicative linear functional. Then*

a) $A_D \subseteq \ker(f)$, *and*
b) If $a \in A$ is nilpotent, then $a \in \ker(f)$

Proof a) For $a, b \in A$, we have that $f(ab - ba) = 0$, so that $\ker(f)$ contains every commutator in A. Since $\ker(f)$ is an ideal, and since it contains all commutators, it must contain the smallest ideal containing all commutators. Hence $R_D \subseteq \ker(f)$.

b) Suppose $a \in A$ is nilpotent and that $a^n = 0$ for some $n \in \mathbb{N}$. Then

$$0 = f(0) = f(a^n) = [f(a)]^n$$

This can only be if $f(a) = 0$.

□

Theorem 4.2 *Let A be a noncommutative Banach algebra and $f : A \to \mathbb{C}$ a nonzero multiplicative linear functional. Then* $\ker(f)$ *is a maximal ideal.*

Proof We have that $\ker(f)$ contains all commutators, so that $A/\ker(f)$ is commutative. Suppose that $b \notin \ker(f)$. Then $0 \neq f(b) \in \mathbb{C}$, hence $[f(b)]^{-1} \in \mathbb{C}$. Since f is onto $\mathbb{C}$ there exists $a \in A$ such that $f(a) = [f(b)]^{-1}$. Then $f(1 - ab) = 0$, hence $1 - ab \in \ker(f)$, so that $\overline{a}\overline{b} = \overline{1}$. This shows that $A/\ker(f)$ is a field. By Lemma 4.1 and Theorem 2.2 we have that $\ker(f)$ is a maximal ideal. □

Theorem 4.3 *Let A be a noncommutative Banach algebra and $f : A \to \mathbb{C}$ a nonzero multiplicative linear functional. Then $P = \{f^{-1}(\lambda) : \lambda \in \mathbb{C}\}$ is a partition of A into subsets, each of which is multiplicatively closed.*

Proof Since f is a multiplicative linear functional, it is onto $\mathbb{C}$. Hence we have that $\emptyset \notin P$. Next, f is defined for each $a \in A$, hence for each a there is a part of P that contains it. Suppose that $a \in f^{-1}(\lambda_1) \cap f^{-1}(\lambda_2)$ where $\lambda_1 \neq \lambda_2$. Then $f(a) = \lambda_1$ and $f(a) = \lambda_2$ which is impossible since f is a function, hence well-defined. We have shown that P is a partition of A.

To see that each part of P is multiplicatively closed, let $ab \in f^{-1}(\lambda)$, where $\lambda \in \mathbb{C}$. Then $f(ab) = \lambda$ hence $f(a)f(b) = \lambda \implies f(b)f(a) = \lambda$ hence $f(ba) = \lambda$, hence $ba \in f^{-1}(\lambda)$. □

We can generalize the above result as follows:

Theorem 4.4 *Let $B \subset \mathbb{C}$, $B \neq \emptyset$. Then $f^{-1}(B)$ is multiplicatively closed.*

Proof The proof is similar to the proof of the previous theorem. □

5 A topological result

This section has a single result, below.

Theorem 5.1 *Let R be a ring. Define $\mathcal{T}$ as:*

$$\mathcal{T} = \{A \subseteq R : A \text{ has the Jacobson property}\}.$$

Then $\mathcal{T}$ is a topology.

Proof It is clear that $R \in \mathcal{T}$. $\emptyset \in \mathcal{T}$ is vacuously true. Let $A_i \in \mathcal{T}$ for each $i \in I$, and suppose that $1 - ab \in \bigcup_{i \in I} A_i$. Then there exists $i_0 \in I$ such that $1 - ab \in A_{i_0}$, which means that $1 - ba \in A_{i_0} \subseteq \bigcup_{i \in I} A_i$. Hence $\bigcup_{i \in I} A_i \in \mathcal{T}$, as required. Finally, suppose $A_i \in \mathcal{T}$ for all $i \leqslant n, n \in \mathbb{N}$. Then, let $1 - ab \in \bigcap_{i=1}^{n} A_i$. That means $1 - ab \in A_i$ for all i, and since $A_i \in \mathcal{T}$, we have $1 - ba \in A_i$ for each i. Hence $1 - ba \in \bigcap_{i=1}^{n} A_i$. This means that $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$, and so $\mathcal{T}$ is a topology. □

So the Jacobson Property gives us a means of defining a topology on any ring. Notice that from Remark 2.1 in [9] we have that every set that is open in $\mathcal{T}$ is also closed in $\mathcal{T}$. Also, if $A \in \mathcal{T}$ then $R \backslash A \in \mathcal{T}$. In the next section we look at a topological ring in which every open set has the Jacobson property.

6 The spectral topology in rings

In [5](p 268) the authors define a closure operation on the subsets of a ring. This operation turns out to be a Kuratowski closure operation, and so defines a topology on the ring, which the authors call the *spectral* topology. The operation has a number of interesting properties. A key property for us is that the spectral topology makes the ring into a topological ring.

In this section we will discuss some of the relevant properties of the spectral topology. Specifically, we will prove explicitly that in the spectral topology on a ring R the Jacobson radical of the ring is an open set. This will be key in proving the main result of this section, which is an application of the theory developed in Sect. 2 above.

We start with the definition of the spectral closure of a set.

Definition 6.1 [5, Definition 1] Let R be a ring, and $K \subseteq R$. The *spectral closure* of K is the set

$$\mathrm{Cl}(K) = \{r \in R : \forall \text{ finite } J \subseteq R\ \exists\, r' \in K, 1 - J(r - r') \subseteq R^{-1}\}.$$

Theorem 2 in [5] proves that the closure operator above defines a topology giving jointly continuous addition and multiplication. That theorem also establishes the fact that if R is a ring and $a \in R$, then we have

$$\mathrm{Cl}(\{a\}) = a + \mathrm{Rad}\, R. \tag{1}$$

It is also easy to see that if R is a ring, for any $K \subseteq R$, we have

$$K \subseteq \mathrm{Cl}(K) \tag{2}$$

In this topology the closure operator defines the closed sets, and a set is open if and only if its complement is closed, so that we have the following. Suppose R is a ring and τ is the spectral topology generated by the operator in Definition 6.1. Then, for $K \subseteq R$

$$K \text{ is closed in } \tau \iff \mathrm{Cl}(K) = K. \tag{3}$$

This means that we can define τ as:

$$\tau = \{K \subseteq R : \mathrm{Cl}(R \setminus K) = R \setminus K\}.$$

Lemma 6.1 *Let R be a ring, $a \in R$. Then*

$$\mathrm{Cl}(\{a\}) = \mathrm{Rad}\, R \iff a \in \mathrm{Rad}\, R.$$

Proof Suppose that $\mathrm{Cl}(\{a\}) = \mathrm{Rad}\, R$. Then, using (2), $a \in \mathrm{Cl}(\{a\}) = \mathrm{Rad}\, R$. This proves the forward implication. Conversely, suppose that $a \in \mathrm{Rad}\, R$. Let $b \in \mathrm{Cl}(\{a\})$. Then $b \in a + \mathrm{Rad}\, R$, so there exists $c \in \mathrm{Rad}\, R$ such that $b = a + c$. Since $\mathrm{Rad}\, R$ is an ideal, we have $a + c \in \mathrm{Rad}\, R$, hence $b \in \mathrm{Rad}\, R$. This means $\mathrm{Cl}(\{a\}) \subseteq \mathrm{Rad}\, R$. To see that $\mathrm{Rad}\, R \subseteq \mathrm{Cl}(\{a\})$, suppose that $b \in \mathrm{Rad}\, R$. Since $a \in \mathrm{Rad}\, R$, we have that $b - a \in \mathrm{Rad}\, R$. Then $b = a + (b - a) \in a + \mathrm{Rad}\, R = \mathrm{Cl}(\{a\})$. This shows that $\mathrm{Rad}\, R \subseteq \mathrm{Cl}(\{a\})$ and the proof is complete. □

Corollary 6.1 *Let R be a ring, $a \in R$. Then $a \in R \setminus \mathrm{Rad}\, R \iff \mathrm{Cl}(\{a\}) \subseteq R \setminus \mathrm{Rad}\, R$.*

Proof Suppose that $a \in R \setminus \mathrm{Rad}\, R$. We show that $\mathrm{Cl}(\{a\}) \cap \mathrm{Rad}\, R = \emptyset$. Suppose not, so that there exists $b \in \mathrm{Cl}(\{a\}) \cap \mathrm{Rad}\, R$. Then $b \in a + \mathrm{Rad}\, R$ and $b \in \mathrm{Rad}\, R$. Since $b \in a + \mathrm{Rad}\, R$, there exists $c \in \mathrm{Rad}\, R$ such that $b = a + c$. Then $a = b - c$. Since $b \in \mathrm{Rad}\, R$ and $a = b - c$, we get $a \in \mathrm{Rad}\, R$, contradicting $a \in R \setminus \mathrm{Rad}\, R$. Conversely, suppose that $\mathrm{Cl}(\{a\}) \subseteq R \setminus \mathrm{Rad}\, R$. Then $a \in \{a\} \subseteq \mathrm{Cl}(\{a\})$, hence $a \in R \setminus \mathrm{Rad}\, R$. □

Proposition 6.1 *Let R be a ring, $a \in R$ and $R' = R \setminus \mathrm{Rad}\, R$. Then*

$$\bigcup_{a \in R'} \mathrm{Cl}(\{a\}) = R'.$$

Proof Let $b \in R'$. Then $b \in \mathrm{Cl}(\{b\}) \subseteq \bigcup_{a \in R'} \mathrm{Cl}(\{a\})$. This shows that $R' \subseteq \bigcup_{a \in R'} \mathrm{Cl}(\{a\})$. Next, suppose that $a \in R'$. Then, by Corollary 6.1 we have that $\mathrm{Cl}(\{a\}) \subseteq R'$. Hence $\bigcup_{a \in R'} \mathrm{Cl}(\{a\}) \subseteq R'$. The result follows. □

Theorem 6.2 *Let R be a ring, $R' = R \setminus \mathsf{Rad}\, R$. Then*

$$\mathsf{Cl}\left(\bigcup_{a\in R'} \mathsf{Cl}(\{a\})\right) = \bigcup_{a\in R'} \mathsf{Cl}\left(\{a\}\right)$$

Proof From (2), it is clear that $\bigcup_{a\in R'} \mathsf{Cl}\left(\{a\}\right) \subseteq \mathsf{Cl}\left(\bigcup_{a\in R'} \mathsf{Cl}(\{a\})\right)$. We show that $\mathsf{Cl}\left(\bigcup_{a\in R'} \mathsf{Cl}(\{a\})\right) \subseteq \bigcup_{a\in R'} \mathsf{Cl}\left(\{a\}\right)$. Suppose that $b \in \mathsf{Cl}\left(\bigcup_{a\in R'} \mathsf{Cl}(\{a\})\right)$. We show that there exists $a_0 \in R'$ such that $b \in \mathsf{Cl}(\{a_0\})$. Let $J \subseteq R$, J finite. Then there exists $c \in \bigcup_{a\in R'} \mathsf{Cl}(\{a\})$ such that $1 - J(b - c) \subseteq R^{-1}$. Since $c \in \bigcup_{a\in R'} \mathsf{Cl}(\{a\})$, there exists $a_0 \in R'$ with the property that $c \in \mathsf{Cl}(\{a_0\})$. Suppose $j \in J$. Then

$$\begin{aligned} 1 - j(b - a_0) &= 1 - j(b - c + c - a_0) \\ &= 1 - j(b - c) - j(c - a_0) \end{aligned}$$

Let $G = 1 - J(b - c)$. Then $G \subseteq R^{-1}$ and $G^{-1}J$ is a finite set. Then

$$\begin{aligned} 1 - j(b - a_0) &= 1 - j(b - c) - j(c - a_0) \\ &\in G - J(c - a_0) \\ &\subseteq G(1 - G^{-1}J(c - a_0)) \end{aligned}$$

Since $c \in \mathsf{Cl}(\{a_0\})$ and $G^{-1}J$ is a finite set, we have that $1 - G^{-1}J(c - a_0) \subseteq R^{-1}$. Hence $G(1-G^{-1}J(c-a_0)) \subseteq R^{-1}$. Hence $1-J(b-a_0) \subseteq R^{-1}$ and so $b \in \mathsf{Cl}(\{a_0\})$ as required. □

Theorem 6.3 *Let R be a ring. Then $\mathsf{Rad}\, R$ is an open set in the spectral topology.*

Proof From Proposition 6.1 we have that $\bigcup_{a\in R'} \mathsf{Cl}(\{a\}) = R\setminus \mathsf{Rad}\, R$. From Theorem 6.2 we have that $\bigcup_{a\in R'} \mathsf{Cl}(\{a\})$ is a closed set. Hence $R \setminus \mathsf{Rad}\, R$ is closed, which means that $\mathsf{Rad}\, R$ is an open set. □

Proposition 6.2 *Let R be a ring. Then, in the spectral topology, $\mathsf{Rad}\, R$ is the smallest open set containing 0.*

Proof Suppose this is not the case. Then there exists $U \subseteq \mathsf{Rad}\, R$ such that U is open, $0 \in U$ and there exists $a \in \mathsf{Rad}\, R$ such that $a \notin U$. Since $a \in R\setminus U$, we have that $\mathsf{Cl}(\{a\}) \subseteq \mathsf{Cl}(R\setminus U) = R\setminus U$. Since $0 \in U$, we have that $0 \notin R\setminus U$. Hence $0 \notin \mathsf{Cl}(\{a\})$. However, by Lemma 6.1 we also have $0 \in \mathsf{Rad}\, R = \mathsf{Cl}(\{a\})$, which of course, is impossible. This contradiction proves the point. □

In Theorem 2.7 of [10] the author describes the fact that in a topological ring the open sets are translations of the open sets containing 0. This fact, together with Proposition 2.3 can be used to prove the following theorem.

Theorem 6.4 *Let R be a ring in which $\mathsf{Rad}\, R$ has the Jacobson property and $R_D \subseteq \mathsf{Rad}\, R$. Then the spectral topology on R is a ring topology in which every open (closed) set satisfies the Jacobson Property.*

In [11](p 257) the author of that article proves that if the spectral radius of a Banach algebra is subadditive or submultiplicative then the Banach algebra is commutative modulo its radical. In [7], the authors state a condition on the joint spectra that characterizes a Banach algebra being commutative modulo its radical. That article also contains an example (Example 2) of an algebra which is noncommutative but commutative modulo its radical.

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