Stronger Lie derivations on MA -semirings

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Abstract

On an additively inverse MA -semiring S we consider the additive mapping $f: S \rightarrow S$ satisfying the identity $f(xy) = [x, f(y)]$, where $[a, b]$ is a commutator of a and b . We investigate the properties of such a mapping and determine the commutativity of S using this mapping.

Keywords Additively inverse semirings · MA -semirings · Semiprime semiring · Centralizing mappings

Mathematics Subject Classification 16Y60 · 16N60

1 Introduction

The notion of a semiring was introduced by Vandiver in 1934 [21]. Semirings which provide a common generalization of rings and distributive lattices appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (cf. [8]). Hemirings, as semirings with commutative addition and zero element, have also proved to be an important algebraic tool in theoretical computer science (cf. [9, 10, 20]).

Nowadays such semirings are studied by many authors in various directions regardless of the derivations in rings where other methods are used (cf. [2, 15, 18]). One of the main directions of such studies is investigation of properties of ideals (cf. [1, 7, 24]). The second important direction is the study of commutativity. For this purpose are used commutators and derivations (cf. [12, 19]).

In this paper, we will describe basic properties of the additive mapping $f: S \rightarrow S$ satisfying the identity $f(xy) = [x, f(y)]$ and we will use this mapping to the characterization of commutativity of certain types of MA -semirings.

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2 Basic definitions and facts

By a *semiring* $(S, +, \cdot)$ we mean a nonempty set S equipped with two binary operations $+$ and $\cdot$ (called addition and multiplication) such that the multiplication is distributive with respect to the addition, $(S, +)$ is a semigroup with neutral element 0, and $(S, \cdot)$ is a semigroup with zero 0, i.e., $0a = a0 = 0$ for all $a \in S$. If a semigroup $(S, \cdot)$ is commutative, then we say that a semiring S is commutative.

A semiring S is an *additively inverse*, if for every $a \in S$ there exists a uniquely determined element $a' \in S$ such that

$$a + a' + a = a \quad \text{and} \quad a' + a + a' = a'. \quad (1)$$

Then, according to [14], for all $a, b \in S$ we have

$$(ab)' = a'b = ab', \quad (a + b)' = b' + a', \quad a'b' = ab, \quad (a')' = a, \quad 0' = 0. \quad (2)$$

Also the following implication is valid

$$a + b = 0 \quad \text{implies} \quad b = a' \quad \text{and} \quad a + a' = 0. \quad (3)$$

An additively inverse semiring S with commutative addition satisfying for all $a, b \in S$ the Bandelt and Petrich condition A_2 (cf. [4]):

$$(a + a')b = b(a + a'), \quad (4)$$

is called an *MA-semiring* (cf. [13]).

An important role in investigations of such semirings plays the *commutator* $[x, y] = xy + y'x$ and *derivations* defined as additive mappings $d: S \rightarrow S$ (i.e., endomorphisms of an additive semigroup of S) such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in S$. If the last identity is valid only for $y = x$, then d is called the *Jordan derivation* (cf. [11]). The *Lie derivation* on S is an additive mapping $L: S \rightarrow S$ satisfying the identity $L([x, y]) = [L(x), y] + [x, L(y)]$. Lie derivations are used to the description of algebras of different types (cf. [3, 17, 19, 25]) and have an important application in differential geometry (cf. [23]).

Let A be a nonempty subset of S . An additive mapping is called *centralizing* on A if $[[f(x), x], y] = 0$ for all $x \in A$ and $y \in S$. In a special case where $[f(x), x] = 0$ (or equivalently, $[x, f(x)] = 0$) for all $x \in A$, the mapping f is said to be *commuting* on A (cf. [22]), or *skew-commuting* on A if $f(x)x + xf(x) = 0$ for all $x \in A$ (cf. [5] or [6]).

Let's also remind that a semiring S is *semiprime* if $xSx = 0$ implies $x = 0$, and *prime* if $xSy = 0$ implies $x = 0$ or $y = 0$. A semiring S is *2-torsion free* if $2x = 0$ is possible only for $x = 0$.

We end this section with three simple lemmas that will be useful later.

Lemma 2.1 *If $f: S \rightarrow S$ is an additive mapping on an additively inverse semiring S , then $f(x + x') \in Z(S)$ for any $x + x' \in Z(S)$, where $Z(S)$ is the center of S .*

Lemma 2.2 *In any MA-semiring*

- (1) $[x, y] = xy + (yx)' = xy + yx'$,
- (2) $[x, y]' = [x, y'] = [x', y] = [y, x]$,
- (3) $[x', y'] = [x, y]$,
- (4) $[x, y] = 0$ implies $xy = yx$.

Lemma 2.3 *In any MA -semiring the following identities*

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z, \quad (5)$$

$$[z, z](x^2 + (x^2)') = [x, z]x(x + x') \quad (6)$$

are valid.

Proof The Jacobi identities (5) are proved in [13]. The proof of (6) is straightforward. $\square$

3 Stronger Lie derivations

Throughout this paper, S denotes an MA -semiring with zero 0.

Consider the additive mapping $f: S \rightarrow S$ satisfying the identity

$$f(xy) = [x, f(y)].$$

Then, as it is not difficult to see,

$$f([x, y]) = f(xy) + f(y'x) = [f(x), y] + [x, f(y)].$$

Hence such mapping is a Lie derivation on an MA -semiring S (in an inverse semiring too). Since the converse statement is not true in general, we will call f the *stronger Lie derivation* (*sl-derivation*, in short).

Example 3.1 Using the above two lemmas it is not difficult to see that the mapping $f(x) = [a, x]$ defined on a non-commutative MA -semiring S is an example of a Lie derivation which is not a *sl-derivation*. It is a *sl-derivation* in a commutative MA -semiring.

Example 3.2 The real vector space $\mathbb{R}^2$ with the multiplication of vectors defined by $(a, b) \cdot (c, d) = (ac, ad)$ is an MA -semiring. The mapping $f(a, b) = (0, a + b)$ is a *sl-derivation* on this MA -semiring. This MA -semiring is isomorphic to an MA -semiring $\mathcal{M}_2(S)$ of matrices $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ over an MA -semiring S described in [13]. The mapping $f: \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & x + y \\ 0 & 0 \end{bmatrix}$ is a *sl-derivation* on $\mathcal{M}_2(S)$.

Example 3.3 The real vector space $\mathbb{R}^3$ with the multiplication of vectors defined by $(a, b, c) \cdot (x, y, z) = (0, az, 0)$ is an MA -semiring. The mapping $f(a, b, c) = (0, b, 0)$ is a *sl-derivation* on this MA -semiring.

Let's we start with the following simple observations.

Lemma 3.4 *Let f be a *sl-derivation* on an MA -semiring S . Then*

- (1) $f(x'y) = f(xy') = f(xy)'$ for all $x, y \in S$.
- (2) If $f(x^2) = 0$ for all $x \in S$, then f is commuting.
- (3) If S is with unity, then $f(x) = f(x) + f(x)'$ for all $x \in S$.
- (4) If S is semiprime and $f(x)f(y) = 0$ for all $x, y \in S$, then $f = 0$.
- (5) If f is commuting, then $f(x^n) = 0$ for all $n \geq 2$.
- (6) If $f(ya) = 0$ for all $y \in S$, then $f(a) \in Z(S)$.

Proposition 3.5 *If a *sl-derivation* on a semiprime MA -semiring S satisfies the identity $f(a)x + x'f(b) = 0$, where $a, b \in S$ are fixed, $f(a) = f(b) \in Z(S)$ and $f(yb) = 0$ for all $y \in S$.*

Proof By (3), from $f(a)x + x'f(b) = 0$ it follows $f(a)x = xf(b)$. Therefore, according to the assumption, for all $x, y \in S$ we have

$$0 = f(a)xy + (xy)'f(b) = xf(b)y + x'yf(b) = x[f(b), y].$$

Thus $0 = [f(b), y]x[f(b), y]$, which, by semiprimeness and Lemma 2.2, implies $[f(b), y] = 0 = [f(b), y]' = [y, f(b)] = f(yb)$. This, by Lemma 2.2(4), gives $f(b) \in Z(S)$. Therefore, $0 = f(a)x + x'f(b) = (f(a) + f(b)')x$, and consequently, $(f(a) + f'(b))x(f(a) + f'(b)) = 0$. Whence, by semiprimeness, we get $f(a) + f(b)' = 0$. So, $f(a) = f(b)$. $\square$

Theorem 3.6 *If a sl-derivation f defined on an MA-semiring S satisfies the identity $2f(x)x = 0$, then it is a Jordan derivation. Moreover, if S is 2-torsion free, then f is a derivation.*

Proof By (3), from $2f(x)x = 0$ it follows $f(x)x = f(x)'x = f(x)x'$. Hence $f(x^2) = [x, f(x)] = xf(x) + f(x)'x = xf(x) + f(x)x$. So, f is a Jordan derivation.

If S is 2-torsion free, then $2f(x)x = 0$ implies $f(x)x = 0$. Thus $f(x)'x = 0$, and consequently $f(x^2) = xf(x)$. By adding $x'f(x)$ to the last equation and using (4), we get

$$f(x^2) + x'f(x) = xf(x) + x'f(x) = (x + x')f(x) = f(x)(x + x') = 0.$$

Hence

$$f(x^2) + x'f(x) = 0. \quad (7)$$

By linearization of (7) we get $f((x + y)^2) + (x + y)'f(x + y) = 0$, whence after reduction, we obtain

$$f(xy + yx) + x'f(y) + y'f(x) = 0.$$

Therefore

$$\begin{aligned} 0 &= f(xy) + f(yx) + x'f(y) + y'f(x) \\ &= [x, f(y)] + [y, f(x)] + x'f(y) + y'f(x) \\ &= (x + x')f(y) + f(y)'x + (y + y')f(x) + f(x)'y \\ &\stackrel{(4)}{=} f(y)(x + x' + x') + f(x)(y + y' + y') \\ &= f(y)x' + f(x)y', \end{aligned}$$

which means that

$$f(y)x + f(x)y = 0. \quad (8)$$

Consequently,

$$\begin{aligned} f(xy) + f(x)y' + x'f(y) &\stackrel{(2)}{=} xf(y) + f(y)x' + f(x)y' + x'f(y) \\ &= (x + x')f(y) + f(y)x' + f(x)y' \\ &\stackrel{(4)}{=} f(y)(x + x' + x') + f(x)y' \\ &= f(y)x' + f(x)y' \stackrel{(8)}{=} 0, \end{aligned}$$

which, by (2) and (3), gives $f(xy) = f(x)y + xf(y)$. So, f is a derivation on S . $\square$

Proposition 3.7 *If a sl-derivation f on an MA-semiring S satisfies the identity $f(x^2) + 2x'f(x) = 0$, then it is skew-commuting.*

Proof Indeed, by (2), we have $0 = f(x^2) + 2x'f(x) = x'f(x) + (xf(x)' + xf(x) + f(x)'x) = x'f(x) + f(x)'x$, which implies $xf(x) + f(x)x = 0$. So, f is skew-commuting. $\square$

Comparing Proposition 3.7 with Theorem 3.2 in [16] we obtain

Corollary 3.8 *In a 2-torsion free semisimple MA-semiring S a sl-derivation f satisfies the identity $f(x^2) + 2x'f(x) = 0$ if and only if $f(x) = 0$ for all $x \in S$.*

Proposition 3.9 *If a sl-derivation f on an MA-semiring S satisfies the identity $f(x^2) + 2f(x)x' = 0$, then also*

$$2f(x^2) = xf(x) + f(x)x, \quad (9)$$

$$xf(x) = 3f(x)x \quad (10)$$

for all $x \in S$.

Proof Indeed, $2f(x^2) = [x, f(x)] + f(x^2) = xf(x) + (f(x)'x + 2f(x)x) = xf(x) + f(x)x$. This proves (9).

Also $0 = f(x^2) + 2f(x)x' = xf(x) + 3f(x)x'$. This, by (3), implies (10). $\square$

Theorem 3.10 *If a sl-derivation f defined on an MA-semiring S satisfies the identity $2f(x)x = 0$, then it also satisfies the identities:*

$$2f(xy + yx) = 2xf(y) + 2yf(x), \quad (11)$$

$$f(xy + x'y) = 2f(x)y + 2f(y)x, \quad (12)$$

$$f(x)(xy + yx) = xf(x)y + yf(x)x. \quad (13)$$

Proof (11) From $2f(x)x = 0$ it follows $f(x)x = f(x)x'$ and $f(x)x + f(x)x' = 0$, which together with (4) gives

$$f(x)(x + x') = 0 = (x + x')f(x). \quad (14)$$

Consider the equation $f((x+y)^2) = f(x^2) + f(y^2) + f(xy + yx)$. Since, by Theorem 3.6, f is a Jordan derivation, the last equation can be rewritten in the form

$$(x + y)f(x + y) + (x + y)f(x + y) = xf(x) + f(x)x + yf(y) + f(y)y + f(xy + yx).$$

By adding $x'f(x) + f(x)x' + y'f(y) + f(y)y'$ to both sides of the last equation and using (14) we get

$$f(xy + yx) = xf(y) + yf(x) + f(x)y + f(y)x. \quad (15)$$

Thus, $f(xy + yx) = [x, f(y)] + [y, f(x)]$, or in the other words

$$f(xy + yx) = xf(y) + f(y)x' + yf(x) + f(x)y'. \quad (16)$$

Adding (16) to (15) we obtain (11).

Further we have

$$\begin{aligned} f(xy + x'y) &= f((x + x')y + xy') = f((x + x')y + x(y + y')) \\ &\stackrel{(4)}{=} f(xy + yx + x'y + y'x) = f(xy + yx) + f(xy + yx)' \\ &\stackrel{(15), (16)}{=} xf(y) + yf(x) + f(x)y + f(y)x + x'f(y) + f(y)x + y'f(x) + f(x)y \\ &= (x + x')f(y) + 2f(y)x + (y + y')f(x) + 2f(x)y \\ &\stackrel{(4)}{=} f(y)(x + x') + 2f(y)x + f(x)(y + y') + 2f(x)y \\ &= f(y)(x + x' + 2x) + f(x)(y + y' + 2y) \\ &= 2f(y)x + 2f(x)y. \end{aligned}$$

This proves (12).

Now from $f(xyx) = f(x(yx)) = [x, f(yx)] = x[y, f(x)] + [y, f(x)]x'$, we calculate

$$f(xyx) = xyf(x) + xf(x)y' + yf(x)x' + f(x)yx.$$

Analogously

$$f(xyx) = f((xy)x) = [xy, f(x)] = xyf(x) + f(x)xy'.$$

Using these two identities we obtain

$$\begin{aligned} f((x+x')yx) &= f(xyx) + f(xyx)' \\ &= xyf(x) + xf(x)y' + yf(x)x' + f(x)yx + x'yf(x) + f(x)xy \\ &= (x+x')yf(x) + xf(x)y' + yf(x)x' + f(x)yx + f(x)xy \\ &\stackrel{(4),(14)}{=} xf(x)y' + yf(x)x' + f(x)yx + f(x)xy \\ &= (xf(x)y + yf(x)x)' + f(x)(yx + xy). \end{aligned}$$

But on the other hand, in view of (4) and (14), we also have

$$f((x+x')yx) = [(x+x')y, f(x)] = 0.$$

Thus $(xf(x)y + yf(x)x)' + f(x)yx + f(x)xy = 0$, which by (3) implies (13). This completes the proof. $\square$

4 Commutativity of MA -semirings

In this section we characterize the commutativity of MA -semirings by the properties of some their ideals and a sl -derivation.

Let's we start with the following three simple lemmas.

Lemma 4.1 *Let S be a semiprime MA -semiring. Then any element $a \in S$ satisfying the identity $a[a, x] = 0$ is in the center of S .*

Proof By hypothesis, $a[a, xy] = 0$ for all $x, y \in S$. Hence, applying Lemma 2.3, we obtain $ax[a, y] + a[a, x]y = 0$. Thus $ax[a, y] = 0$. In particular, for $x = yx$ we have $ayx[a, y] = 0$. Multiplying the previous identity by y' and adding to the last we obtain $[a, y]x[a, y] = 0$. So, $[a, y] = 0$ because S is semiprime. Lemma 2.2 completes the proof. $\square$

Lemma 4.2 *Let U be a non-zero right ideal of a semiprime MA -semiring S and $a \in U$. If $[a, x] = 0$ for all $x \in U$, then $a \in Z(S)$.*

Proof Indeed, $ar \in U$ for each $r \in S$. Thus, according to the assumption, $0 = [a, ar] = a[a, r]$. So, by Lemma 4.1, $a \in Z(S)$. $\square$

Lemma 4.3 *The center of a semiprime semiring does not contain any non-zero nilpotent elements.*

Proof Let $a \in Z(S)$ be a nilpotent element of index n . If $n = 2$, then $0 = a^2s = asa$ for each $s \in S$. This implies $a = 0$. If $n > 2$, then $a^m = 0$ for each $m \geq n$. Thus $0 = a^{2n-2} = a^{2n-2}s = a^{n-1}sa^{n-1}$ for each $s \in S$. Hence $a^{n-1} = 0$. This is possible only for $a = 0$. $\square$

Theorem 4.4 *Let f be a non-zero sl-derivation on a prime MA-semiring S . If U is a non-zero right ideal of S such that $f(U) = 0$, then $f(S) \subseteq Z(S)$ and f is commuting on S .*

Proof For all $a \in U$ and $s \in S$ we have $0 = f(as) = [a, f(s)]$. Thus also $f(s)[f(s), a] = 0$, whence, by Lemma 4.1, $f(S) \subseteq Z(S)$. Moreover, by Lemma 2.3, for all $s \in S$ we also have $0 = f((as)s) = [as, f(s)] = a[s, f(s)] + [a, f(s)]s = a[s, f(s)]$. Because $xS \subseteq U$ for each $x \in U$, the last implies $xS[s, f(s)] = 0$, whence $[s, f(s)] = 0$ for all $s \in S$. So, f is commuting on S . $\square$

Let's remind that the Lie ideal of a semiring S is defined as an additive subsemigroup U of S such that $[u, s] = us + s'u \in U$ for all $u \in U$ and $s \in S$.

Proposition 4.5 *Let f be a non-zero sl-derivation on a prime MA-semiring S . If U is a non-zero Lie ideal of S such that $f(U) = 0$, then $f(S) \subseteq Z(U)$. Moreover, if U is not a ring, then f is centralizing on S .*

Proof Since $f(u) = 0$ for $u \in U$, for all $u \in U$ and $x \in S$ we have $0 = f(ux + x'u) = [u, f(x)] + [x', f(u)] = [u, f(x)]$. Hence, by Lemma 4.2, $f(S) \subseteq Z(S)$.

From $0 = [u, f(x)]$, for $u := ux + x'u$, we obtain $0 = [ux + x'u, f(x)] = u[x, f(x)] + [x, f(x)]u' = [u, [x, f(x)]] = [u, f(x^2)]$. This for $u := ur + r'u$, where $r \in S$, gives $0 = u[r, f(x^2)] + [r, f(x^2)]u' = [u, [r, f(x^2)]]$, whence by Lemma 4.2, $[r, f(x^2)] \in Z(S)$. Thus $0 = [u, [r, f(x^2)]] = (u + u')[r, f(x^2)]$. So, $u + u' = 0$ or $[r, f(x^2)] = [r, [x, f(x)]] = 0$. As U is not a ring, must be $[r, [x, f(x)]] = 0$. This completes the proof. $\square$

Proposition 4.6 *Let f be a sl-derivation of an MA-semiring S such that $f(u) + u' = 0$ for all elements of some subset $U \subseteq S$.*

(a) *If U is a left ideal of S , then*

$$U \subseteq l(S) = \{y \in S : yx = 0 \text{ for all } x \in S\}.$$

Moreover, if S is semiprime, then $U = \{0\}$.

(b) *If U is a right ideal of S , then $u^2 \in Z(S)$ for all $u \in U$. Moreover, if S is prime, then $U = \{0\}$ or S is a ring.*

Proof Let's notice first that $f(u) + u' = 0$ implies $f(u) = u$ and $u + u' = 0$.

(a). If U is a left ideal, then for $u \in U$ and $s \in S$ we have $su \in U$ and $0 = f(su) + su' = [s, f(u)] + su' = [s, u] + su' = us'$, because $u + u' \in Z(S)$. Thus $uS = \{0\}$. The rest is obvious.

(b). Let U be a right ideal. Then for all $u \in U$ and $s \in S$ we have $0 = f(u(su)) + (u(su))' = usu + u^2s' + s'u^2$ and $0 = f(u(us)) + (u(us))' = u^2s + usu' + u^2s'$. Adding up these two identities we obtain $u^2s + s'u^2 = 0$. So, by (3), $u^2 \in Z(S)$.

Moreover, in this case also $0 = u^2(s + s') = u(us + us')$. But S is a MA-semiring, hence $us + us' \in Z(S)$, and consequently, $0 = uS(us + us')$. Thus, if S is prime, $u = 0$ or $u(s + s') = 0$ for all $s \in S$. So, $0 = u(s + s')S = uS(s + s')$ again implies $u = 0$ or $s + s' = 0$. Therefore $U = \{0\}$ or S is a ring. $\square$

Proposition 4.7 *If a sl-derivation f on a prime MA-semiring S is injective on a non-zero left ideal U of S , then it is injective on whole S .*

Proof Indeed, $(\ker f)SU \subseteq (\ker f)U \subseteq (\ker f) \cap U = \{0\}$, because U is a non-zero left ideal and f is injective on U . Thus $\ker f = \{0\}$. $\square$

Theorem 4.8 *If a sl-derivation f on a semiprime MA -semiring S is centralizing on an ideal U of S , then it is commuting on U .*

Proof As f is centralizing on U , therefore

$$[[f(x), x], z] = 0 \quad (17)$$

for all $x \in U$ and $z \in S$. This, by Lemma 2.2, means that $[f(x), x] \in Z(S)$. Now, replacing in (17) an element x by $x + y$, where $y \in U$, we get

$$[[f(x), y] + [f(y), x], z] = 0. \quad (18)$$

From this, for $y = x^2$ and $z = f(x)$, we obtain

$$\begin{aligned} 0 &= [[f(x), x^2] + [f(x^2), x], f(x)] \\ &= [x[f(x), x] + [f(x), x]x + [[x, f(x)], x], f(x)] \\ &\stackrel{(17)}{=} [2x[f(x), x], f(x)] = 2\left(x[[f(x), x], f(x)] + [x, f(x)][f(x), x]\right) \\ &= 2[x, f(x)][f(x), x] = 2[f(x), x]^2. \end{aligned}$$

Consequently, $0 = 2[f(x), x]^2 + 2[f(x), x]^2 = 4[f(x), x]^2 = (2[f(x), x])^2$. This, by Lemma 4.3, means that

$$2[f(x), x] = 0. \quad (19)$$

Replacing in this equation x by $x + y$, $x, y \in U$, we obtain

$$2([y, f(x)] + [x, f(y)]) = 0. \quad (20)$$

Now using Lemma 2.3 we have

$$\begin{aligned} [xy + yx, f(x)] + [x^2, f(y)] &= [xy, f(x)] + [yx, f(x)] + x[x, f(y)] + [x, f(y)]x \\ &= x[y, f(x)] + [x, f(x)]y + y[x, f(x)] + [y, f(x)]x + x[x, f(y)] + [x, f(y)]x \\ &\stackrel{(18)}{=} 2y[x, f(x)] + 2x([y, f(x)] + [x, f(y)]) \stackrel{(19), (20)}{=} 0. \end{aligned}$$

Therefore,

$$[xy + yx, f(x)] + [x^2, f(y)] = 0 \quad (21)$$

for $x, y \in U$.

Since for $x \in U$ elements $[f(x), x], [x, f(x)], [f(x), f(x)] = [f(x), f(x)]'$ are in the center of S , replacing in the above equation y by $f(x)x$ and using Lemma 2.2 we obtain

$$\begin{aligned} 0 &= [xf(x)x + f(x)xx, f(x)] + [x^2, f(f(x)x)] \\ &\stackrel{(1)}{=} \left[\left(xf(x) + f'(x)x + f(x)x + f(x)x \right) x, f(x) \right] + [x^2, [f(x), f(x)]] \\ &= \left[\left([x, f(x)] + 2f(x)x \right) x, f(x) \right] + [x^2, [f(x), f(x)]] \\ &= [[x, f(x)]x, f(x)] + [2f(x)x^2, f(x)] + x^2[f(x), f(x)] + [f(x), f(x)]x^2 \\ &\stackrel{(18)}{=} [x, f(x)]^2 + 2f(x)[x^2, f(x)] + 2[f(x), f(x)]x^2 + [[x, f(x)], f(x)]x \\ &\quad + 2[f(x), f(x)]x^2 \\ &\stackrel{(5)}{=} [x, f(x)]^2 + 2f(x)x[x, f(x)] + 2f(x)[x, f(x)] + [x, f(x)]f(x)x \end{aligned}$$

$$\begin{aligned}
& + f(x)'[x, f(x)]x + 2[f(x), f(x)]x^2 + 2[f(x), f(x)](x^2)' \\
& \stackrel{(18)}{=} [x, f(x)]^2 + [x, f(x)]f(x)x + f(x)'[x, f(x)]x \\
& \quad + [f(x), f(x)](x^2 + x^2 + (x^2)' + (x^2)') \\
& \stackrel{(1)}{=} [x, f(x)]^2 + [x, f(x)]f(x)(x + x') + [f(x), f(x)](x^2 + (x^2)') \\
& \stackrel{(6)}{=} [x, f(x)]^2 + 2[x, f(x)]f(x)(x + x') \stackrel{(19)}{=} [x, f(x)]^2.
\end{aligned}$$

Thus $0 = [x, f(x)]^2 = [x, f(x)]S[x, f(x)]$. As S is semiprime, the last implies $[x, f(x)] = 0$ and hence f is commuting on U . $\square$

Corollary 4.9 *Any sl-derivation centralizing on whole semiprime MA-semiring is also commuting on this semiring.*

By Proposition 4.5 and Corollary 4.9, we get the following result.

Corollary 4.10 *Let f be a sl-derivation of a prime MA-semiring S . If S has a non-zero Lie ideal that is not a ring and $f(x) = 0$ for all $x \in U$, then f is commuting on S .*

Proposition 4.11 *Let U be a non-zero right ideal of a prime MA-semiring S . If a sl-derivation f is surjective and $f(uv) = 0$ for all $u, v \in U$, then U is a ring or S is commutative.*

Proof Since f is surjective for every $z \in S$ there is $s \in S$ such that $z = f(s)$ and $us \in U$ for all $u \in U$. Thus $0 = f(u(us)) = [u, [u, f(s)]] = [u, [u, z]] = (u + u')[u, z]$. Hence $(u + u')S[u, z] = 0$, which implies $u + u' = 0$ or $[u, z] = 0$ for all $z \in S$ and $u \in U$. In the first case U is a ring; in the second S is commutative by Lemma 4.2. $\square$

Theorem 4.12 *If a semiprime MA-semiring has a centralizing surjective sl-derivation, then it is commutative.*

Proof Let f be a centralizing sl-derivation on S . By Theorem 4.8, it is commuting on S . Therefore $[f(x), x] = 0$ holds for all $x \in S$. From this, putting $x = y + z$, we can see that

$$[f(x), y] + [x, f(y)]' = 0, \quad (22)$$

and consequently, by (3),

$$[f(y), z] = [y, f(z)] \quad (23)$$

for all $y, z \in S$. Replacing in this identity y by zx , we obtain

$$[f(zx), z] = [zx, f(z)] = z[x, f(z)] + [z, f(z)]x = z[x, f(z)].$$

But by definition, $[f(zx), z] = [[z, f(x)], z]$. Therefore, by Lemma 2.2,

$$\begin{aligned}
& [f(zx), z] + [f(zx), z]' = z[x, f(z)] + [[z, f(x)], z]' \\
& = z[x, f(z)] + [z, [z, f(x)]] = z[x, f(z)] + z[z, f(x)] + [z, f(x)]'z \\
& \stackrel{(23)}{=} z[f(x), z] + z[z, f(x)] + [z, f(x)]z' \stackrel{(1),(4)}{=} [z, f(x)]z' = 0,
\end{aligned}$$

because

$$\begin{aligned}
& [f(zx), z] + [f(zx), z]' \stackrel{(23)}{=} [zx, f(z)] + [zx, f(z)]' \\
& \stackrel{(5)}{=} z[x, f(z)] + [z, f(z)]x + z[x, f(z)]' + [z, f(z)]'x
\end{aligned}$$

$$= z[x, f(z)] + z[x, f(z)]' \stackrel{(23)}{=} z([f(x), z] + [x, f(z)]') \stackrel{(22)}{=} 0.$$

So, $0 = [z, f(x)]z'$, i.e., $0 = [z, w]z$ for all $w, z \in S$ since f is surjective. From this, by putting $w = su$ and using (5), we obtain $0 = [z, s]uz$ for all $s, u, z \in S$. Consequently, $0 = [z, s]uzs$ and $0 = [s, z]usz = [z, s]uz's$. Thus, $[s, z]S[s, z] = 0$, which implies $[s, z] = 0$. This by last statement of Lemma 2.2 means that S is a commutative semiring. $\square$

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