Rationalized evaluation subgroups of mapping spaces between complex Grassmannians

Paul Antony Otieno¹ · Jean Baptiste Gatsinzi² · Vitalis Onyango-Otieno¹

Received: 22 December 2018 / Accepted: 14 October 2019 / Published online: 31 October 2019
 © The Author(s) 2019

Abstract

We determine evaluation subgroups of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + 1)$ between complex Grassmannians.

Keywords Evaluation subgroups · Gottlieb group · G -sequence

Mathematics Subject Classification Primary 55P62 · 54C35

1 Introduction

Given a pointed topological space (X, x_0) , the n th Gottlieb group of X , also called the evaluation subgroup of $\pi_n(X)$ and denoted by $G_n(X)$, consists of those $\alpha \in \pi_n(X)$ for which there is a map $F : X \times S^n \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{F} & X \\ J \uparrow & & \nabla \uparrow \\ X \vee S^n & \xrightarrow{1_X \vee f} & X \vee X, \end{array}$$

where $f : S^n \rightarrow X$ is a representative of α and ∇ is the folding map. Thus for every $\alpha \in G_n(X, x_0)$, there exists at least one map $F : X \times S^n \rightarrow X$ such that $F(x_0, s) = f(s)$. We say that F is an affiliated map to α [3]. If X has a base point x_0 and $aut X$ denotes the monoid of self homotopy equivalences of X with $ev : aut X \rightarrow X$ the evaluation map at

✉ Paul Antony Otieno
 paotieno@strathmore.edu
 Jean Baptiste Gatsinzi
 gatsinzi@biust.ac.bw
 Vitalis Onyango-Otieno
 vonyango@strathmore.edu

¹ Strathmore Institute of Mathematical Sciences, Strathmore University, Box 59857, Nairobi, Kenya

² Department of Mathematics, Botswana International University of Science and Technology, Private Bag 16, Palapye, Botswana

x_0 , then it follows from the definition that

$$G_n(X) = \text{im} \left(\text{ev}_{\sharp} : \pi_n(\text{aut } X, 1_X) \longrightarrow \pi_n(X, x_0) \right).$$

Similarly, if $f : X \longrightarrow Y$ is a based map between simply connected CW complexes and $\text{map}(X, Y; f)$, the space of maps from X to Y which are homotopic to f , then $G_n(Y, X; f) = \text{im}(\text{ev}_{\sharp} : \text{map}(X, Y; f) \longrightarrow \pi_n(Y))$ is the n th evaluation subgroup of f [9]. In [10], Woo and Lee defined relative evaluation subgroups $G_n^{\text{rel}}(X, Y; f)$ and showed that they fit in a sequence

$$\cdots \longrightarrow G_{n+1}^{\text{rel}}(Y, X; f) \longrightarrow G_n(X) \longrightarrow G_n(Y, X; f) \longrightarrow \cdots$$

called the G -sequence of f .

We use Sullivan models to compute rational relative Gottlieb groups of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$. We refer to [4] for details and work over a field of characteristic zero in this case $\mathbb{Q}$.

Definition 1 A differential graded algebra (dga) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ together with a derivation d , $d = d_n : A^n \longrightarrow A^{n+1}$ such that $d \circ d = 0$. Then (A, d) is called a cochain algebra. A graded algebra A is commutative if $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ for $a, b \in A$ [2, Chap. 3].

Definition 2 A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$ where $V = \{V^p\}_{p \geq 2}$ and $\wedge V$ denotes the graded free commutative algebra on V . A Sullivan model for a commutative cochain algebra (A, d) is a quasi-isomorphism $m : (\wedge V, d) \longrightarrow (A, d)$ from a Sullivan algebra $(\wedge V, d)$. A Sullivan algebra is said to be minimal if the differential is decomposable, that is, $\text{Im } d \subset \wedge^+ V \cdot \wedge^+ V$. Moreover, if $H^0(A) = \mathbb{Q}$ then (A, d) has a minimal model which is unique up to isomorphism. If X is a nilpotent space and $A_{PL}(X)$ the commutative differential graded algebra ($cdga$) of piecewise linear forms on X , then a Sullivan model of X is a Sullivan model of $A_{PL}(X)$ [2, Chap.12].

2 Derivation spaces and the rationalized G -sequence

Given commutative differential graded algebras (A, d_A) and (B, d_B) and a map $\phi : A \longrightarrow B$, define a ϕ -derivation of degree n to be a linear map $\theta : A^* \longrightarrow B^{*-n}$ which satisfies $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y)$. We only consider derivations of positive degree. Let $\text{Der}_n(A, B; \phi)$ denote the vector space of all ϕ -derivations of degree n for $n > 0$. Define a linear map $D : \text{Der}_n(A, B; \phi) \longrightarrow \text{Der}_{n-1}(A, B; \phi)$ by $D(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A$. Then, $(\text{Der}_*(A, B; \phi), D)$ is a chain complex. In case $A = B$ and $\phi = 1_B$, the chain complex of derivations $\text{Der}_*(B, B; 1)$ is just the usual complex of derivations on the commutative differential graded algebra B [6]. If $\phi : (\wedge V, d) \longrightarrow (\wedge W, d)$ is a Sullivan minimal model of $f : X \longrightarrow Y$, then $H_n(\text{Der}(\wedge V, \wedge W; \phi), D) \cong \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$; $n \geq 2$ [6], [1]. We note that $\text{Der}(\wedge V, B; \phi) \cong \text{Hom}(V, B)$. If $\{v_i\}$ is a basis of V , we denote by (v_i, b) , the unique ϕ -derivation θ such that

$$\begin{cases} \theta(v_i) = b_i & b_i \in B, \\ \theta(v_j) = 0 & i \neq j. \end{cases}$$

Pre-composition with ϕ , respectively post-composition with the augmentation $\varepsilon : B \longrightarrow \mathbb{Q}$, gives a map of chain complexes

$$\phi^* : \text{Der}_*(B, B; 1) \longrightarrow \text{Der}_*(A, B; \phi),$$

respectively

$$\varepsilon_* : \text{Der}_*(A, B; \phi) \longrightarrow \text{Der}_*(A, \mathbb{Q}; \varepsilon).$$

Definition 3 Let $\phi : V \longrightarrow W$ be a map of differential graded vector spaces. Define a differential graded vector space, $\text{Rel}_*(\phi)$, called the mapping cone as follows. $\text{Rel}_n(\phi) = sV_{n-1} \oplus W_n$ with differential δ of degree -1 given by $\delta(sv, w) = (-sd_V(v), \phi(v) + d_W(w))$ [7]. There are chain maps $J : W_n \longrightarrow \text{Rel}_n(\phi)$ and $P : \text{Rel}_n(\phi) \longrightarrow V_{n-1}$ defined by $J(w) = (0, w)$ and $P(sv, w) = v$. These give a short exact sequence of chain complexes

$$0 \longrightarrow W_* \xrightarrow{J} \text{Rel}_*(\phi) \xrightarrow{P} V_{*-1} \longrightarrow 0,$$

which leads to a long exact sequence in homology

$$\cdots \rightarrow H_{n+1}(\text{Rel}(\phi)) \xrightarrow{H(P)} H_n(V) \xrightarrow{H(\phi)} H_n(W) \xrightarrow{H(J)} H_n(\text{Rel}(\phi)) \rightarrow \cdots,$$

whose connecting homomorphism is $H(\phi)$. We refer to this sequence as the long exact homology sequence of ϕ .

We use Theorem 3.3 of [6]

Theorem Let $f : X \longrightarrow Y$ be a map between simply connected CW-complexes and $\phi : (\wedge V, d) \longrightarrow (\wedge W, d)$ its Sullivan model. The long exact sequence induced by $f_* : \text{map}(X, X; 1) \longrightarrow \text{map}(X, Y; f)$ on rational homotopy groups is equivalent to the long exact homology sequence of the map

$$\phi^* : \text{Der}_*(\wedge W, \wedge W; 1) \longrightarrow \text{Der}_*(\wedge V, \wedge W; \phi)$$

induced by the minimal model $\phi : (\wedge V, d) \longrightarrow (\wedge W, d)$ of the map $f : X \longrightarrow Y$.

Definition 4 Given a commutative differential graded algebra map $\phi : A \longrightarrow B$, we have the following commutative diagram of differential graded vector spaces;

$$\begin{array}{ccc} \text{Der}_*(B, B; 1) & \xrightarrow{\phi^*} & \text{Der}_*(A, B; \phi) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* \\ \text{Der}_*(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & \text{Der}_*(A, \mathbb{Q}; \varepsilon). \end{array}$$

Here, ε denotes the augmentation of either A or B . On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder ($n \geq 2$),

$$\begin{array}{ccccccc} \cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_n(\text{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_n(\text{Der}(A, B; \phi)) \cdots \\ & & \downarrow H(\varepsilon_* \varepsilon_*) & & \downarrow H(\varepsilon_*) & & \downarrow H(\varepsilon_*) \\ \cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\widehat{\phi^*})) & \xrightarrow{H(\widehat{P})} & H_n(\text{Der}(B, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{\phi^*})} & H_n(\text{Der}(A, \mathbb{Q}; \varepsilon)) \cdots \end{array}$$

Definition 5 Suppose $\phi : A \longrightarrow B$ is a map of commutative differential graded algebras, we define the evaluation subgroups of ϕ by

$$G_n(A, B; \phi) = \text{im}\{H(\varepsilon_*) : H_n(\text{Der}(A, B; \phi)) \longrightarrow H_n(\text{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In the special case $A = B$ and $\phi = 1_B$, we refer to the Gottlieb group of B , and use the notation $G_n(B)$. If $B = (\wedge V, d)$ is a model of a simply connected space X , then an element

$\alpha \in G_n(B)$ is represented by a linear mapping $f : V^n \rightarrow \mathbb{Q}$ that extends into a derivation θ of $\wedge V$ such that $\delta\theta = 0$. Moreover, $G_n(B) \cong G_n(X_{\mathbb{Q}})$ [2, Proposition 29.8]. The n th relative evaluation subgroup of ϕ is defined by;

$$G_n^{rel}(A, B; \phi) = im\{H(\varepsilon_*, \varepsilon_*) : H_n(Rel(\phi^*)) \rightarrow H_n(Rel(\widehat{\phi^*}))\}.$$

Then the image of the upper long sequence in the lower, of the ladder above, gives a sequence

$$\dots \xrightarrow{H(\widehat{J})} G_{n+1}^{rel}(A, B; \phi) \xrightarrow{H(\widehat{P})} G_n(B) \xrightarrow{H(\widehat{\phi^*})} G_n(A, B; \phi) \xrightarrow{H(\widehat{J})} \dots \quad (1)$$

that terminates in $G_2(A, B; \phi)$.

We refer to this sequence as the G -sequence of the map $\phi : A \rightarrow B$. This can be applied to the minimal model $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ of the map $f : X \rightarrow Y$ as stated and proved in [6, Theorem 3.5].

3 The inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$

Let $Gr(k, n)$ be the Grassmann manifold of k -dimensional subspaces of $\mathbb{C}^n$. The cohomology ring $H^*(Gr(k, n), \mathbb{Q})$ is generated by the Chern classes $c_i \in H^{2i}(Gr(k, n), \mathbb{Q})$, for $1 \leq i \leq k$. Further, the cohomology ring has a presentation

$$H^*(Gr(k, n), \mathbb{Q}) = \wedge(c_1, c_2, \dots, c_k) / (h_{n-k+1}, \dots, h_n),$$

as the quotient of the polynomial ring generated by $c_1, c_2, \dots, c_k$, $|c_i| = 2i$, modulo the ideal generated by the elements h_j , $n-k+1 \leq j \leq n$. Here, h_j is defined as the $2j$ th degree term in the Taylor's expansion of $(1+c_1+c_2+c_3+\dots+c_k)^{-1}$ where $(1+c_1+c_2+c_3+\dots+c_k)$ is the total Chern class [4].

In particular, the cohomology rings of $Gr(2, n)$ and $Gr(2, n+1)$ are:

$$\begin{aligned} H^*(Gr(2, n), \mathbb{Q}) &= \wedge(y_2, y_4) / (h_{n-1}, h_n) \text{ and} \\ H^*(Gr(2, n+1), \mathbb{Q}) &= \wedge(x_2, x_4) / (h_n, h_{n+1}) \text{ respectively.} \end{aligned}$$

The minimal model of $Gr(2, n)$ is $(\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d)$ with $d(y_2) = d(y_4) = 0$, $d(y_{2n-3}) = h_{n-1}$, $d(y_{2n-1}) = h_n$. In the same way, a model of $Gr(2, n+1)$ is given by $(\wedge(x_2, x_4, x_{2n-1}, x_{2n+1}), d)$ with $dx_2 = dx_4 = 0$, $dx_{2n-1} = h_n$ and $dx_{2n+1} = h_{n+1}$.

Lemma 1 $h_{n+1} = -x_2h_n - x_4h_{n-1}$.

Proof Write the Taylor series $(1+x_2+x_4)^{-1} = 1+h_1+h_2+\dots$ where, $|h_i| = 2i$. From $(1+x_2+x_4)(1+x_2+x_4)^{-1} = 1$, one gets the relation $h_{n+1} = -x_2h_n - x_4h_{n-1}$ $\square$

In particular, h_{n+1} is co-boundary in $(\wedge(x_2, x_4, x_{2n-3}, x_{2n-1}), d)$, that is, there exists α of degree $2n+1$ such that $d\alpha = h_{n+1}$.

Theorem 1 Let $B = (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d)$. Then $G_n(B) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$.

Proof Let $\alpha_{2n-1} = (y_{2n-1}, 1)$ and $\alpha_{2n-3} = (y_{2n-3}, 1)$. Then $\delta\alpha_{n-1} = \delta\alpha_{2n-3} = 0$. Moreover, α_{2n-3} and α_{2n-1} can not be boundaries for degree reason. Therefore, $[\alpha_{2n-3}]$ and $[\alpha_{2n-1}]$ are non zero homology classes in $H_*(Der(B, B; 1))$. Further, $\varepsilon_*(\alpha_{2n-3}) = y_{2n-3}^*$ and $\varepsilon_*(\alpha_{2n-1}) = y_{2n-1}^*$.

As $Gr(2, n)$ is a finite CW-complex then $G_{even}(B) = 0$ [2, Pg.379]. Hence, $G_n(B) = \langle [y_{2n-3}^*], [y_{2n-1}^*] \rangle$. $\square$

The inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ has a model of the form

$$\phi : \wedge V = \left(\wedge (x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \longrightarrow \left(\wedge (y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) = B$$

where $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_{2n-1}) = y_{2n-1}$ and $\phi(x_{2n+1}) = \alpha$ where $d\alpha = h_{n+1}$ by Lemma 1.

Theorem 2 Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ and $\phi : (\wedge V, d) \longrightarrow (B, d)$ its Sullivan model, then $G_*(\wedge V, B; \phi) \cong \langle [x_{2n-1}^*], [x_{2n+1}^*] \rangle$.

Proof As $Gr(2, n)$ is formal, $Der(\wedge V, B; \phi) \xrightarrow{\sim} Der(\wedge V, H(B); f \circ \phi)$ where $f : B \xrightarrow{\sim} H(B)$ is a quasi isomorphism. Similarly, since B is formal $Der(B, B; 1) \simeq Der(B, H(B); f)$. Define $\theta_{2n-1} = (x_{2n-1}, 1)$, $\theta_{2n+1} = (x_{2n+1}, 1)$ in $Der(\wedge V, H(B); f \circ \phi)$. Then $\delta\theta_{2n-1} = \delta\theta_{2n+1} = 0$. Moreover, $[\theta_{2n-1}]$ and $[\theta_{2n+1}]$ are nonzero cohomology classes in $H_*(Der(\wedge V, H^*(B); f \circ \phi))$.

We note that, $\theta_2 = (x_2, 1)$ and $\theta_4 = (x_4, 1)$ are not cycles in $Der(\wedge V, H(B); f \circ \phi)$ [8]. Further, $H(\varepsilon_*)([\theta_{2n-1}]) = [x_{2n-1}^*] \in G_{2n-1}(\wedge V, B; f \circ \phi)$. In a similar way, $H(\varepsilon_*)([\theta_{2n+1}]) = [x_{2n+1}^*] \in G_{2n+1}(\wedge V, B; f \circ \phi)$. It then follows that $G_*(\wedge V, B; f \circ \phi) = \langle [x_{2n-1}^*], [x_{2n+1}^*] \rangle$. $\square$

Theorem 3 Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n+1)$ and

$$\begin{aligned} \phi : (\wedge V, d) &= \left(\wedge (x_2, x_4, x_{2n-1}, x_{2n+1}), d \right) \\ &\longrightarrow \left(\wedge V(y_2, y_4, y_{2n-3}, y_{2n-1}), d \right) = B \end{aligned}$$

its Sullivan model, then $G_*^{rel}(\wedge V, B; \phi) = \langle [(x_{2n-3}^*, 0)], [(0, y_{2n+1}^*)] \rangle$.

Proof Consider the diagram below [6].

$$\begin{array}{ccccc} Der(B, H(B); f) & \xrightarrow{\phi^*} & Der(\wedge V, H(B); f \circ \phi) & \xrightarrow{J} & Rel(\phi^*) \\ \downarrow \varepsilon_* & & \downarrow \varepsilon_* & & \downarrow (\varepsilon_*, \varepsilon_*) \\ Der(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & Der(\wedge V, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & Rel(\widehat{\phi^*}). \end{array}$$

Let $\alpha_{2n-1} = (y_{2n-1}, 1)$, $\alpha_{2n-3} = (y_{2n-3}, 1) \in Der(B, H(B); f)$ and θ_{2n-1} , $\theta_{2n+1} \in Der(\wedge V, H^*(B); \phi)$ as defined above. Then $\phi^*(\alpha_{2n-1}) = \theta_{2n-1}$ and $\phi^*(\alpha_{2n-3}) = 0$. Further, $D(\alpha_{2n-1}, 0) = (0, \theta_{2n-1})$, $D(\alpha_{2n-3}, 0) = (0, 0)$ and $D(0, \theta_{2n-1}, 0) = 0 = D(0, \theta_{2n+1})$. Therefore, $[(\alpha_{2n-3}, 0)]$ and $[(0, \theta_{2n+1})]$ are non zero homology classes in $H_*(Rel(\phi^*))$. Moreover, $H(\varepsilon_*, \varepsilon_*)([\alpha_{2n-3}, 0]) = [(x_{2n-3}^*, 0)]$ and $H(\varepsilon_*, \varepsilon_*)([(0, \theta_{2n+1})]) = [(0, y_{2n+1}^*)]$. A straightforward computation shows that $[(x_{2n-3}^*, 0)]$ and $[(0, y_{2n+1}^*)]$ span $H(\varepsilon_*, \varepsilon_*)$. $\square$

The G -sequence reduces to

$$\begin{aligned} 0 &\longrightarrow G_{2n+1}(\wedge V, B; \phi) \xrightarrow[\simeq]{H(J)} G_{2n+1}^{rel}(\wedge V, B; \phi) \longrightarrow 0 \cdots \\ \cdots 0 &\longrightarrow G_{2n-1}(B) \xrightarrow[\simeq]{H(\phi^*)} G_{2n-1}(\wedge V, B; \phi) \longrightarrow 0 \cdots \\ \cdots 0 &\longrightarrow G_{2n-2}^{rel}(\wedge V, B; \phi) \xrightarrow[\simeq]{H(P)} G_{2n-3}(B) \longrightarrow 0. \end{aligned}$$

and is exact.

Example 1 Consider $Gr(2, 4) \hookrightarrow Gr(2, 5)$. A model of the inclusion is given by

$$\phi : \wedge V = \left(\wedge (x_2, x_4, x_7, x_9), d \right) \longrightarrow \left(\wedge (y_2, y_4, y_5, y_7), d \right) = B,$$

where $dx_2 = dx_4 = 0$, $dx_7 = x_4^2 - 3x_2^2x_4 + x_2^4$, $dx_9 = 4x_2^3x_4 - 3x_2x_4^2 - x_2^5$, $dy_2 = dy_4 = 0$, $dy_5 = 2y_2y_4 - y_2^3$ and $dy_7 = y_4^2 - 3y_2^2y_4 + y_2^4$.

Moreover, $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_7) = y_7$ and $\phi(x_9) = -y_2y_7 - y_4y_4$.

We compute $G_*^{rel}(\wedge V, B; \phi)$. Let $\alpha_7 = (y_7, 1)$, $\alpha_5 = (y_5, 1) \in Der_7(B, H(B); f)$ where $f : B \xrightarrow{\simeq} H(B)$ and $\theta_7 = (x_7, 1)$, $\theta_9 = (x_9, 1) \in Der(\wedge V, H^*(B); f \circ \phi)$ then $\phi^*(\alpha_7) = \theta_7$ and $\phi^*(\alpha_5) = 0$. Moreover, $D(s\alpha_7, 0) = (0, \theta_7)$, $D(s\alpha_5, 0) = (0, 0)$ and $D(0, \theta_7) = D(0, \theta_9) = (0, 0)$. Hence $[(s\alpha_5, 0)]$ and $[(0, \theta_9)]$ are non zero homology classes. Moreover, $(\epsilon_*, \epsilon_*)(s\alpha_5, 0) = (sy_5^*, 0)$, $(\epsilon_*, \epsilon_*)(0, \theta_9) = (0, y_9^*)$. Therefore

$$G_*^{rel}(\wedge V, B; \phi) = \langle [(0, x_9^*)], [(sy_5^*, 0)] \rangle.$$

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Buijs, U., Murillo, A.: The rational homotopy Lie algebra of function spaces. *Comment. Math. Helv.* **83**(4), 723–739 (2008)
2. Félix, Y., Halperin, S., Thomas, J.-C.: *Rational Homotopy theory*, Graduate Texts in Mathematics, vol. 205. Springer, New York (2001)
3. Gottlieb, D.H.: Evaluation subgroups of homotopy groups. *Am. J. Math.* **91**, 729–756 (1969)
4. Hoffman, M.: Endomorphisms of the cohomology of complex Grassmannians. *Trans. Am. Math. Soc.* **281**(2), 745–760 (1984)
5. Lee, K.-Y., Mimura, M., Woo, M.H.: Gottlieb groups of homogeneous spaces. *Topol. Appl.* **145**(1–3), 147–155 (2004)
6. Lupton, G., Smith, S.B.: Rationalized evaluation subgroups of a map I: Sullivan models, derivations and G -sequences. *J. Pure Appl. Algebra* **209**(1), 159–171 (2007)
7. MacLane, S.: *Homology*. Springer, New York (2012)
8. Otieno, P.A., Gatsinzi, J.-B., Otieno, V.O.: Rational homotopy type of mapping spaces between complex Grassmannian. *Quaest. Math.*
9. Woo, M.H., Kim, J.-R.: Certain subgroups of homotopy groups. *J. Korean Math. Soc.* **21**(2), 109–120 (1984)

10. Woo, H.H., Lee, K.Y.: On the relative evaluation subgroups of a CW-pair. *J. Korean Math. Soc.* **25**, 149–160 (1988)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.