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On fully regular $\mathcal{AG}$-groupoids

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Abstract One of the best approaches to study one type of algebraic structure is to connect it with other type of algebraic structure which is better explored. In this paper we have accomplished this aim by connecting $\mathcal{AG}$-groupoids with some useful associative and commutative algebraic structures. We have also introduced a fully regular class of an $\mathcal{AG}$-groupoid and shown that an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is fully regular if and only if $\mathcal{L} = \mathcal{L}^{i+1}$, for any left ideal $\mathcal{L}$ of $\mathcal{S}$, where $i = 1, \ldots, n$.

Keywords $\mathcal{AG}$-groupoids, Left invertive law, Medial law and Paramedial law

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1 Introduction

The concept of an Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid) was first given by Kazim and Naseeruddin in 1972 [3] and they have called it a left almost semigroup (LA-semigroup). In [2], the same structure is called a left invertive groupoid. Several examples and interesting properties of LA-semigroups can be found in [4,7].

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The most generalized algebraic structure is a groupoid which is a set together with a binary operation. A groupoid satisfying the following left invertive law is known as an $\mathcal{AG}$-groupoid [9].

$$(ab)c = (cb)a, \quad \text{for all } \; a, b, c \in \mathcal{S}.$$

This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc = cba$.

In an $\mathcal{AG}$-groupoid $\mathcal{S}$, the medial law holds [3]

$$(ab)(cd) = (ac)(bd), \quad \text{for all } \; a, b, c, d \in \mathcal{S}.$$

Since $\mathcal{AG}$-groupoids satisfy medial law, they belong to the class of entropic groupoids which are also called abelian quasigroups [10].

An $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of an $\mathcal{AG}$-groupoid allow us to introduce the inverses of elements in an $\mathcal{AG}$-groupoid. If an $\mathcal{AG}$-groupoid contains a left identity, then it is unique [7].

In an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the paramedial law holds [7]

$$(ab)(cd) = (dc)(ba), \quad \text{for all } \; a, b, c, d \in \mathcal{S}.$$

Further if an $\mathcal{AG}$-groupoid $\mathcal{S}$ contains a left identity, the following law holds [7]

$$a(bc) = b(ac), \quad \text{for all } \; a, b, c \in \mathcal{S}.$$

If an $\mathcal{AG}$-groupoid $\mathcal{S}$ satisfies the above law without left identity, then it is called an $\mathcal{AG}^{**}$-groupoid. An $\mathcal{AG}^{**}$-groupoid also satisfies a paramedial law without left identity. An $\mathcal{AG}^{**}$-groupoid is the generalization of an $\mathcal{AG}$-groupoid with left identity. Every $\mathcal{AG}$-groupoid with left identity is an $\mathcal{AG}^{**}$-groupoid but the converse is not true in general [1].

An $\mathcal{AG}$-groupoid is a useful algebraic structure, midway between a groupoid and a commutative semigroup with wide applications in theory of flocks [8]. An $\mathcal{AG}$-groupoid is non-associative and non-commutative in general, however, there is a close relation with semigroup as well as with commutative structures. It has been investigated in [7] that if an $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative monoid. An $\mathcal{AG}$-groupoid is the generalization of a semigroup theory [7] and has vast applications in collaboration with semigroup like other branches of mathematics. The connections of an $\mathcal{AG}$-groupoid with the vector spaces over finite fields have been investigated in [4].

From the above discussion, we see that $\mathcal{AG}$-groupoids have very closed links with semigroups and vector spaces which make an $\mathcal{AG}$ -groupoid to be among the most interesting non-associative algebraic structure.

2 $\mathcal{AG}$-groupoids generated by other algebraic structures

Example 1 Define a binary operation "$\diamond$" on a commutative inverse semigroup $(\mathcal{S}, .)$ as follows:

$$a \diamond b = ba^{-1} \prod_{i=1}^{n} r_i^{-1}, \quad \text{where } \; a, b, r_i \in \mathcal{S} = \mathbb{R}\backslash\{0\}, \; \text{for } \; i = 1, 2, \ldots, n.$$

Then $(\mathcal{S}, \diamond)$ becomes an $\mathcal{AG}$-groupoid. Indeed

$$
\begin{aligned}
(a \diamond b) \diamond c &= ba^{-1} \prod_{i=1}^{n} r_i^{-1} \diamond c \\
&= c \left(ba^{-1} \prod_{i=1}^{n} r_i^{-1} \right)^{-1} \prod_{i=1}^{n} r_i^{-1} \\
&= c \left(\prod_{i=1}^{n} r_i^{-1} \right)^{-1} ab^{-1} \prod_{i=1}^{n} r_i^{-1} \\
&= a \left(\prod_{i=1}^{n} r_i^{-1} \right)^{-1} cb^{-1} \prod_{i=1}^{n} r_i^{-1}, \quad \text{as } (\mathcal{S}, .) \text{ is commutative} \\
&= a \left(bc^{-1} \prod_{i=1}^{n} r_i^{-1} \right)^{-1} \prod_{i=1}^{n} r_i^{-1} \\
&= bc^{-1} \prod_{i=1}^{n} r_i^{-1} \diamond a \\
&= (c \diamond b) \diamond a.
\end{aligned}
$$

It is not hard to see that $(\mathcal{S}, \diamond)$ is non-commutative and non-associative.

Example 2 Let us consider an abelian group $(\mathbb{R}, +)$ of all real numbers under the binary operation of addition. If we define

$$
a \odot b = b - a - \left(\sum_{i=1}^{n} r_i \right), \quad \text{where } a, b, r_i \in \mathbb{R}, \text{ for } i = 1, 2, \ldots, n.
$$

Then $(\mathbb{R}, \odot)$ becomes an $\mathcal{AG}$-groupoid. Indeed

$$
\begin{aligned}
(a \odot b) \odot c &= c - (a \odot b) - \left(\sum_{i=1}^{n} r_i \right) \\
&= c - \left\{ b - a - \left(\sum_{i=1}^{n} r_i \right) \right\} - \left(\sum_{i=1}^{n} r_i \right) \\
&= c - b + a + \left(\sum_{i=1}^{n} r_i \right) - \left(\sum_{i=1}^{n} r_i \right) \\
&= a - b + c + \left(\sum_{i=1}^{n} r_i \right) - \left(\sum_{i=1}^{n} r_i \right), \quad \text{as } (\mathbb{R}, +) \text{ is an abelian group} \\
&= a - \left\{ b - c - \left(\sum_{i=1}^{n} r_i \right) \right\} - \left(\sum_{i=1}^{n} r_i \right) \\
&= a - (c \odot b) - \left(\sum_{i=1}^{n} r_i \right) \\
&= (c \odot b) \odot a.
\end{aligned}
$$

One can easily verify that $(\mathbb{R}, \odot)$ is non-commutative and non-associative.

3 Some algebraic structures generated by $\mathcal{AG}$-groupoids

In [10], it has been shown that an $\mathcal{AG}$*-groupoid $(\mathcal{S}, .)$ under the binary operation "$\circ$" (sandwich operation) defined as follows:

$$x \circ y = (xa)y, \quad \text{for all } x, y \in \mathcal{S} \text{ and } a \in \mathcal{S} \text{ is fixed,} \tag{1}$$

becomes a commutative semigroup $(\mathcal{S}, \circ)$.

Example 3 An $\mathcal{AG}$**-groupoid $(\mathcal{S}, .)$ under the sandwich operation "$\circ$" [defined in (1)] also becomes a commutative semigroup $(\mathcal{S}, \circ)$.

Example 4 Let $(\mathcal{S}, .)$ be an $\mathcal{AG}$-groupoid with left identity e . Define a binary operation "$\circ_e$" (e-sandwich operation) as follows:

$$a \circ_e b = (ae)b, \quad \text{for all } a, b \in \mathcal{S}.$$

Then $(\mathcal{S}, \circ_e)$ becomes a commutative monoid. Indeed

$$a \circ_e b = (ae)b = (be)a = b \circ_e a,$$

and

$$\begin{aligned}(a \circ_e b) \circ_e c &= ((ae)b) \circ_e c = (((ae)b)e)c \\ &= (((be)a)e)c = (ce)((be)a) \\ &= (a(be))(ec) = (ae)((be)c) \\ &= a \circ_e ((be)c) = a \circ_e (b \circ_e c),\end{aligned}$$

also

$$a \circ_e e = (ae)e = (ee)a = a = ea = (ee)a = e \circ_e a.$$

4 Some studies in fully regular$\mathcal{AG}$-groupoids

Definition 1 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a regular element of $\mathcal{S}$ if there exists some $x \in \mathcal{S}$ such that $a = (ax)a$ and $\mathcal{S}$ is called regular $\mathcal{AG}$-groupoid if all elements of $\mathcal{S}$ are regular.

Definition 2 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an intra-regular element of $\mathcal{S}$ if there exist some $u, v, x, y \in \mathcal{S}$ such that $a = (ua)(av) = (xa^2)y$ and $\mathcal{S}$ is called intra-regular if all elements of $\mathcal{S}$ are intra-regular.

Definition 3 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a weakly regular element of $\mathcal{S}$ if there exist some $x, y \in \mathcal{S}$ such that $a = (ax)(ay)$ and $\mathcal{S}$ is called weakly regular if all elements of $\mathcal{S}$ are weakly regular.

Definition 4 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a right (left) regular element of $\mathcal{S}$ if there exists some $x \in \mathcal{S}$ such that $a = a^2x = (aa)x$ $(a = xa^2 = x(aa))$ and $\mathcal{S}$ is called right (left) regular if all elements of $\mathcal{S}$ are right (left) regular.

Definition 5 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a left quasi regular element of $\mathcal{S}$ if there exist some $x, y \in \mathcal{S}$ such that $a = (xa)(ya)$ and $\mathcal{S}$ is called left quasi regular if all elements of $\mathcal{S}$ are left quasi regular.

Definition 6 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a (2, 2)-regular element of $\mathcal{S}$ if there exists some $x \in \mathcal{S}$ such that $a = (a^2x)a^2$ and $\mathcal{S}$ is called (2, 2)-regular $\mathcal{AG}$-groupoid if all elements of $\mathcal{S}$ are (2, 2)-regular.

Definition 7 [5] An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a completely regular element of $\mathcal{S}$ if a is regular, right regular and left regular. An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called completely regular if it is regular, right and left regular.

In [5], it has been shown that all of these classes coincide in an $\mathcal{AG}$-groupoid with left identity except a regular class. It is important to note that none of any two mention classes coincide in a semigroup.

Here we introduce a new class of an $\mathcal{AG}$-groupoid as follows:

An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a fully regular element of $\mathcal{S}$ if there exist some $p, q, r, s, t, u, v, w, x, y, z \in \mathcal{S}$ ($p, q, \ldots, z$ may be repeated) such that

$$a = (pa^2)q = (ra)(as) = (at)(au) = (aa)v = w(aa) = (xa)(ya) = (a^2z)a^2.$$

An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called fully regular if all elements of $\mathcal{S}$ are fully regular.

Example 5 Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\}$ be an $\mathcal{AG}$-groupoid with the following multiplication table.

.	1	2	3	4	5	6	7
1	2	4	6	1	3	5	7
2	5	7	2	4	6	1	3
3	1	3	5	7	2	4	6
4	4	6	1	3	5	7	2
5	7	2	4	6	1	3	5
6	3	5	7	2	4	6	1
7	6	1	3	5	7	2	4

It is easy to verify that $\mathcal{S}$ is fully regular $\mathcal{AG}$-groupoid.

Remark 1 An $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($\mathcal{AG}^{**}$-groupoid) is fully regular if and only if $\mathcal{S}$ is intra-regular [weakly regular, right regular, left regular, left quasi regular, (2,2)-regular and completely regular].

Every intra-regular [weakly regular, right regular, left regular, left quasi regular, (2,2)-regular and completely regular] class of an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($\mathcal{AG}^{**}$-groupoid) is regular but the converse is not true in general [5].

Example 6 If we consider an $\mathcal{AG}$-groupoid $\mathcal{S} = \{1, 2, 3, 4\}$ with left identity 3 in the following Cayley's table.

.	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

Then by routine calculation, it is easy to see that $\mathcal{S}$ is regular. Note that $\mathcal{S}$ is not fully regular, because $1 \in \mathcal{S}$ is not a fully regular element of $\mathcal{S}$.

Remark 2 An $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($\mathcal{AG}$ **-groupoid) is regular if $\mathcal{S}$ is fully regular but the converse is not valid in general which can be followed from Example 6.

If $\mathcal{S}$ is an intra-regular [weakly regular, right regular, left regular, left quasi regular, (2,2)-regular or completely regular] $\mathcal{AG}$-groupoid,then $\mathcal{S} = \mathcal{S}^2$ holds but the converse is not true in general [5].

Remark 3 If an $\mathcal{AG}$-groupoid $\mathcal{S}$ is fully regular, then $\mathcal{S} = \mathcal{S}^2$ holds but the converse is not valid in general which can be followed from Example 6.

Definition 8 A non-empty subset $\mathcal{A}$ of an $\mathcal{AG}$-groupoid $\mathcal{S}$ called left (right) ideal of $\mathcal{S}$ if and only if $\mathcal{SA} \subseteq \mathcal{A}$ ($\mathcal{AS} \subseteq \mathcal{A}$) and is called two-sided ideal or ideal of $\mathcal{S}$ if and only if it is both left and right ideal of $\mathcal{S}$.

Definition 9 A non-empty subset $\mathcal{A}$ of an $\mathcal{AG}$-groupoid $\mathcal{S}$ called semiprime if and only if $a^2 \in \mathcal{A} \Longrightarrow a \in \mathcal{A}$.

Lemma 1 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is fully regular.*
(ii) *$\mathcal{R} \cap \mathcal{L} = \mathcal{RL}$, where $\mathcal{R}$ and $\mathcal{L}$ are any right and left ideals of $\mathcal{S}$ respectively such that $\mathcal{R}$ is semiprime.*

Proof (i) $\Longrightarrow$ (ii): Suppose that an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is fully regular. Now by using Remark 1, $\mathcal{S}$ is an intra-regular. Let $\mathcal{R}$ and $\mathcal{L}$ be any right and left ideals of $\mathcal{S}$ respectively and let $a \in \mathcal{S}$, then there exist $x, y \in \mathcal{S}$ such that $a = (xa^2)y$. Now let $a^2 \in \mathcal{R}$, then

$$\begin{aligned} a &= (xa^2)y = ((ex)(aa))y = ((aa)(xe))y \\ &= (y(xe))(aa) = (aa)((xe)y) \in \mathcal{RS} \subseteq \mathcal{R}. \end{aligned}$$

Thus $\mathcal{R}$ is semiprime. It is easy to see that $\mathcal{RL} \subseteq \mathcal{R} \cap \mathcal{L}$. Clearly $a^2 \in \mathcal{S}a^2$ and also $\mathcal{S}a^2$ right ideal of $\mathcal{S}$, therefore $a \in a^2\mathcal{S}$. Let $a \in \mathcal{R} \cap \mathcal{L}$, then $a \in \mathcal{R}$ and $a \in \mathcal{L}$. Thus

$$\begin{aligned} a \in a^2\mathcal{S} &= (aa)(\mathcal{SS}) = (a\mathcal{S})(a\mathcal{S}) = (\mathcal{S}a)(\mathcal{S}a) \\ &= ((\mathcal{SS})a)(\mathcal{S}a) = ((a\mathcal{S})\mathcal{S})(\mathcal{S}a) \\ &\subseteq ((\mathcal{RS})\mathcal{S})(\mathcal{SL}) \subseteq \mathcal{RL}, \end{aligned}$$

which shows that $\mathcal{R} \cap \mathcal{L} = \mathcal{RL}$, where $\mathcal{R}$ and $\mathcal{L}$ are any right and left ideals of $\mathcal{S}$ respectively such that $\mathcal{R}$ is semiprime. $(ii) \Longrightarrow (i)$: Let every right ideal $\mathcal{R}$ be semiprime and $\mathcal{L}$ be any left ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity such that $\mathcal{R} \cap \mathcal{L} = \mathcal{RL}$. Since $\mathcal{S}a^2$ and $\mathcal{S}a$ are any right and left ideals of $\mathcal{S}$ respectively, then it is easy to see that $a \in a^2\mathcal{S}$ and $a \in \mathcal{S}a$, therefore

$$\begin{aligned} a \in (a^2\mathcal{S}) \cap (\mathcal{S}a) &= (a^2\mathcal{S})(\mathcal{S}a) = ((aa)(\mathcal{SS}))(\mathcal{S}a) \\ &= ((\mathcal{SS})(aa))(\mathcal{S}a) \subseteq (\mathcal{S}a^2)\mathcal{S}, \end{aligned}$$

that is $a = (xa^2)y$ for some $x, y \in \mathcal{S}$. Thus by using Remark 1, $\mathcal{S}$ is fully regular. □

Lemma 2 *A non-empty subset $\mathcal{A}$ of a fully regular $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is a left ideal of $\mathcal{S}$ if and only if it is a right ideal of $\mathcal{S}$.*

Proof It is simple. □

Lemma 3 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is fully regular.*
(ii) *$\mathcal{L} \cap \mathcal{R} = \mathcal{LR}$, where $\mathcal{L}$ and $\mathcal{R}$ are any left and right ideals of $\mathcal{S}$ respectively such that $\mathcal{R}$ is semiprime.*

Proof $(i) \Longrightarrow (ii)$ can be followed by using Lemmas 1 and 2.
$(ii) \Longrightarrow (i)$ is straightforward. □

Definition 10 A non-empty subset $\mathcal{A}$ of an $\mathcal{AG}$-groupoid $\mathcal{S}$ called idempotent if and only if $\mathcal{A} = \mathcal{A}^2$.

Corollary 1 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is fully regular.*
(ii) *Every left ideal of $\mathcal{S}$ is idempotent.*

Corollary 2 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is fully regular.*
(ii) *Every right ideal of $\mathcal{S}$ is idempotent and semiprime.*

Theorem 1 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is fully regular.*
(ii) *$\mathcal{L} = \mathcal{L}^3$, where $\mathcal{L}$ is any left ideal of $\mathcal{S}$.*

Proof $(i) \Longrightarrow (ii)$: Let $\mathcal{S}$ be a fully regular $\mathcal{AG}$-groupoid with left identity and $\mathcal{L}$ be any left ideal of $\mathcal{S}$. Then by using Corollory 1, we have

$$\mathcal{L}^3 = \mathcal{L}^2\mathcal{L} = \mathcal{LL} \subseteq \mathcal{SL} \subseteq \mathcal{L}.$$

Now let $a \in \mathcal{L}$, then by using Lemmas 1 and 2, $\mathcal{S}a^2$ is left ideal of $\mathcal{S}$ such that $a \in a^2\mathcal{S}$, therefore

$$\begin{aligned} a \in a^2\mathcal{S} = (aa)\mathcal{S} &= (\mathcal{S}a)a \subseteq (\mathcal{S}(a^2\mathcal{S}))a \\ &= (a^2(\mathcal{SS}))a = ((aa)\mathcal{S})a = ((\mathcal{S}a)a)a \\ &\subseteq ((\mathcal{SL})\mathcal{L})\mathcal{L} \subseteq (\mathcal{LL})\mathcal{L} \subseteq \mathcal{L}^3, \end{aligned}$$

which is what we set out to prove. $(ii) \Longrightarrow (i)$: Let $\mathcal{L}$ be any left ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity such that $\mathcal{L} = \mathcal{L}^3$. Since $\mathcal{S}a$ is left ideals of $\mathcal{S}$ and $a \in \mathcal{S}a$, therefore

$$a \in \mathcal{S}a = ((\mathcal{S}a)(\mathcal{S}a))(\mathcal{S}a) = ((\mathcal{SS})(aa))(\mathcal{S}a) \subseteq (\mathcal{S}a^2)\mathcal{S},$$

that is $a = (xa^2)y$ for some $x, y \in \mathcal{S}$. Thus by using Remark 1, $\mathcal{S}$ is fully regular. □

Theorem 2 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is fully regular.*
(ii) *$\mathcal{L} = \mathcal{L}^{i+1}$, for any left ideal $\mathcal{L}$ of $\mathcal{S}$, where $i = 1, \ldots, n$.*

Proof It can be easily followed by generalizing the proof of Theorem 1.
From the left–right dual of Theorem 2, we have the following theorem. □

Theorem 3 *For an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent.*

(i) $\mathcal{S}$ *is fully regular.*
(ii) $\mathcal{R} = \mathcal{R}^{i+1}$, *for any right ideal* $\mathcal{R}$ *of* $\mathcal{S}$ *such that* $\mathcal{R}$ *is semiprime, where* $i = 1, \dots, n$.

Definition 11 An $\mathcal{AG}$-groupoid is called left (right) simple if and only if it has no proper left (right) ideal and is called simple if and only if it has no proper two-sided ideal.

Note that if an $\mathcal{AG}$-groupoid $\mathcal{S}$ contains a left identity, then $\mathcal{S} = \mathcal{S}^2$.

Theorem 4 *The following conditions are equivalent for an* $\mathcal{AG}$ *-groupoid* $\mathcal{S}$ *with left identity.*

(i) $a\mathcal{S} = \mathcal{S}$, *some* $a \in \mathcal{S}$.
(ii) $\mathcal{S}a = \mathcal{S}$, *for some* $a \in \mathcal{S}$.
(iii) $\mathcal{S}$ *is simple.*
(iv) $\mathcal{AS} = \mathcal{S} = \mathcal{SA}$, *where* $\mathcal{A}$ *two-sided ideal of* $\mathcal{S}$.
(v) $\mathcal{S}$ *is fully regular.*

Proof $(i) \Longrightarrow (ii)$: Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity and assume that $a\mathcal{S} = \mathcal{S}$ holds for some $a \in \mathcal{S}$, then

$$\mathcal{S} = \mathcal{SS} = (a\mathcal{S})\mathcal{S} = (\mathcal{SS})a = \mathcal{S}a.$$

$(ii) \Longrightarrow (iii)$: Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity such that $a\mathcal{S} = \mathcal{S}$ holds for some $a \in \mathcal{S}$. Suppose that $\mathcal{S}$ is not left simple and let $\mathcal{L}$ be a proper left ideal of $\mathcal{S}$, then

$$\begin{aligned} \mathcal{SL} \subseteq \mathcal{L} \subseteq \mathcal{S} = \mathcal{SS} &= (\mathcal{S}a)\mathcal{S} = ((\mathcal{SS})(ea))\mathcal{S} \\ &= ((ae)(\mathcal{SS}))\mathcal{S} = ((ae)\mathcal{S})(\mathcal{SS}) \\ &= ((\mathcal{S}e)a)(\mathcal{SS}) = (\mathcal{SS})(a(\mathcal{S}e)) \\ &= a((\mathcal{SS})(\mathcal{S}e)) \subseteq a\mathcal{S}, \end{aligned}$$

implies that $sl = at$, for some $a, s, t \in \mathcal{S}$ and $l \in \mathcal{L}$. Since $sl \in \mathcal{L}$, therefore $at \in \mathcal{L}$, but $at \in a\mathcal{S}$. Thus $a\mathcal{S} \subseteq \mathcal{L}$ and therefore, we haves

$$\mathcal{S} = a\mathcal{S} \subseteq \mathcal{L},$$

implies that $\mathcal{S} = \mathcal{L}$, which contradicts the given assumption. Thus $\mathcal{S}$ is left simple and similarly we can show that $\mathcal{S}$ is right simple, which shows that $\mathcal{S}$ is simple.

$(iii) \Longrightarrow (iv)$: Let $\mathcal{S}$ be a simple $\mathcal{AG}$-groupoid with left identity and let $\mathcal{A}$ be any two-sided ideal of $\mathcal{S}$, then $\mathcal{A} = \mathcal{S}$. Therefore, we have

$$\mathcal{AS} = \mathcal{SS} = \mathcal{SA}.$$

$(iv) \Longrightarrow (v)$: Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity such that $\mathcal{AS} = \mathcal{S} = \mathcal{SA}$ holds for any two-sided ideal $\mathcal{A}$ of $\mathcal{S}$. Since $\mathcal{S}a^2$ is a right ideal of $\mathcal{S}$ and we know that every right ideal of $\mathcal{S}$ with left identity is two-sided ideal of $\mathcal{S}$. Thus $\mathcal{S}a^2$ is two-sided ideal of $\mathcal{S}$ such that $(a^2\mathcal{S})\mathcal{S} = \mathcal{S} = \mathcal{S}(a^2\mathcal{S})$. Let $a \in \mathcal{S}$, then

$$\begin{aligned} a \in \mathcal{S} = (a^2\mathcal{S})\mathcal{S} &= ((aa)(\mathcal{SS}))\mathcal{S} \\ &= ((\mathcal{SS})(aa))\mathcal{S} = (\mathcal{S}a^2)\mathcal{S}, \end{aligned}$$

that is $a = (xa^2)y$ for some $x, y \in \mathcal{S}$. Thus by using Remark 1, $\mathcal{S}$ is fully regular.
$(v) \Longrightarrow (i)$: Let $\mathcal{S}$ be a fully regular $\mathcal{AG}$ -groupoid with left identity, then by using remark 1, $\mathcal{S}$ is an intra-regular. Let $a \in \mathcal{S}$, then there exist $x, y \in \mathcal{S}$ such that $a = (xa^2)y$. Thus

$$a = (xa^2)y = ((ex)(aa))y = ((aa)(ex))y$$
$$= (y(ex))(aa) = a((y(ex))a) \in a\mathcal{S},$$

which shows that $\mathcal{S} \subseteq \mathcal{S}a$ and $\mathcal{S}a \subseteq \mathcal{S}$ is obvious. Thus $\mathcal{S}a = \mathcal{S}$ holds for some $a \in \mathcal{S}$. □

Corollary 3 *The following conditions are equivalent for an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity.*

(i) $a\mathcal{S} = \mathcal{S}$*, for some* $a \in \mathcal{S}$.
(ii) $\mathcal{S}a = \mathcal{S}$*, for some* $a \in \mathcal{S}$.
(iii) $\mathcal{S}$ *is right simple.*
(iv) $\mathcal{AS} = \mathcal{S} = \mathcal{SA}$*, where* $\mathcal{A}$ *is any right ideal of* $\mathcal{S}$.
(v) $\mathcal{S}$ *is fully regular.*

Corollary 4 *If $\mathcal{S}$ is an $\mathcal{AG}$-groupoid with left identity, then the following conditions are equivalent.*

(i) $\mathcal{S}a = \mathcal{S}$*, for some* $a \in \mathcal{S}$.
(ii) $a\mathcal{S} = \mathcal{S}$*, for some* $a \in \mathcal{S}$.

Example 7 Let $\mathcal{S} = \{a, b, c, d, e\}$ be an $\mathcal{AG}$-groupoid with left identity b in the following multiplication table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

It is clear to see that $\mathcal{S}a = \mathcal{S}$ or $a\mathcal{S} = \mathcal{S}$ does not holds for all $a \in \mathcal{S}$.

Corollary 5 *If $\mathcal{S}$ is an $\mathcal{AG}$-groupoid, then $e\mathcal{S} = \mathcal{S} = \mathcal{S}e$ holds for $e \in \mathcal{S}$, where e is a left identity of $\mathcal{S}$. Note that e does not act as a right identity of $\mathcal{S}$. For example, an element $b \in \mathcal{S}$ considered in Example 7 is a left identity of $\mathcal{S}$ such that $\mathcal{S}b = \mathcal{S}$ holds for $b \in \mathcal{S}$ but $cb = e \neq c = eb \neq e$ for $c, e \in \mathcal{S}$.*

Theorem 5 *The following conditions are equivalent for an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity.*

(i) $\mathcal{S}$ *is fully regular.*
(ii) $\mathcal{S}a = \mathcal{S} = a\mathcal{S}$*, for some* $a \in \mathcal{S}$.

Proof It can be easily followed by using Remark 1 and Theorem 4. □

5 On some algebraic structures in terms of $\mathcal{AG}$-groupoids

Lemma 4 *A groupoid $\mathcal{S}$ with left identity is an $\mathcal{AG}$-groupoid if and only if $\mathcal{S}$ satisfy medial and paramedial laws.*

Proof Let a groupoid $\mathcal{S}$ with left identity be medial and paramedial and $a, b, c \in \mathcal{S}$, then

$$(ab)c = (ab)(ec) = (ae)(bc) = (cb)(ea) = (cb)a.$$

The converse is obvious. □

Lemma 5 *An $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($\mathcal{AG}$**-groupoid) is a semigroup if and only if $a(bc) = (ca)b = (cb)a$ holds for all $a, b, c \in \mathcal{S}$.*

Proof Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity such that $a(bc) = (ca)b = (cb)a$ holds for all $a, b, c \in \mathcal{S}$. Then

$$a(bc) = (ca)b = (cb)a = (ab)c.$$

Conversely suppose that an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($\mathcal{AG}$**-groupoid) is semigroup, then

$$\begin{aligned} a(bc) &= b(ac) = (ba)c = (ca)b \\ &= (ba)c = b(ac) = a(bc) \\ &= (ab)c = (cb)a. \end{aligned}$$

Which is what we set out to prove. □

Definition 12 An element a of an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity e is called left (right) invertive if and only if there exists $a' \in S$ such that $a'a = e$ $(aa' = e)$ and a is called invertive if and only if it is both left and right invertive. An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called left (right) invertible if and only if every element of $\mathcal{S}$ is left (right) invertive and $\mathcal{S}$ is called invertible if and only if it is both left and right invertible.

Theorem 6 *A left (right) invertible $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is an abelian group if and only if $a(bc) = c(ba)$, for all $a, b, c \in \mathcal{S}$.*

Proof Assume that $\mathcal{S}$ be a left invertible $\mathcal{AG}$-groupoid with left identity e, then

$$ae = a(ee) = e(ea) = ea = a,$$

implies that e is an identity of $\mathcal{S}$. Since $\mathcal{S}$ is left invertible, then there exists $a' \in \mathcal{S}$ such that $a'a = e$, then

$$aa' = (ea)a' = (a'a)e = ee = e.$$

Thus $\mathcal{S}$ have inverses for each $a \in \mathcal{S}$. Now

$$\begin{aligned} (ab)c &= (ab)(ec) = (ce)(ba) = a(b(ce)) = a(e(cb)) \\ &= a(cb) = c(ab) = (ec)(ab) = (ba)(ce) \\ &= e(c(ba)) = c(ba) = a(bc), \end{aligned}$$

implies that $\mathcal{S}$ is associative, also

$$ab = a(eb) = b(ea) = ba,$$

which shows that $\mathcal{S}$ is commutative and therefore $\mathcal{S}$ is an abelian group.
The converse is simple. □

Corollary 6 *An invertible $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is an abelian group if and only if $a(bc) = c(ba)$, for all $a, b, c \in \mathcal{S}$.*

Definition 13 [6] An $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $\mathcal{AG}$*-groupoid if the following holds:

$$(ab)c = b(ac), \quad \text{for all } a, b, c \in \mathcal{S}.$$

In an $\mathcal{AG}$*-groupoid $\mathcal{S}$, the following law holds [9]:

$$(x_1x_2)(x_3x_4) = (x_{\pi(1)}x_{\pi(2)})(x_{\pi(3)}x_{\pi(4)}), \tag{2}$$

where $x_1, x_2, x_3, x_4 \in \mathcal{S}$ and $\{\pi(1), \pi(2), \pi(3), \pi(4)\}$ means any permutation on the set $\{1, 2, 3, 4\}$. It is an easy consequence that if $\mathcal{S} = \mathcal{S}^2$, then $\mathcal{S}$ becomes a commutative semigroup.

Many characteristics of a non-associative AG*-groupoid are similar to a commutative semigroup.

Theorem 7 *A fully regular $\mathcal{AG}$*-groupoid becomes a semigroup.*

Proof It can be followed from Remark 3 and Eq. (2). □

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