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Integration over facet-simple polytopes

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Abstract

We present an elementary approach for the computation of integrals of the form $\int_{\mathcal{P}} f^{(n)}(\mathbf{s}\cdot\mathbf{x})\,\mathbf{dx}$ over polytopes $\mathcal{P}$, where $f:\mathbb{C}\to\mathbb{C}$ is analytic. The proof is based on an independence theorem on exponential functions over the field of rational functions and needs only simple facts from the theory of polyhedra. In particular we present an explicit formula for generalized facet-simple polytopes. Here a convex polytope is called facet-simple if each of its facets is simple and a set of points is called a generalized facet-simple polytope if it is a finite union of n-dimensional facet-simple convex polytopes such that any two distinct members are either disjoint or intersect in a common facet.

Keywords Integration · Polytope · Facet-simple polytope · Rational function

1 Introduction

It is well-known that by the fundamental theorem of calculus for all $a, b \in \mathbb{R}$ with $a \le b$

$$\int_a^b f'(x)\,dx = f(b) - f(a) \tag{1}$$

if $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Moreover, if in addition $s \in \mathbb{R}$, then

$$\int_a^b f'(s\cdot x)\,dx = \frac{1}{s}(f(s\cdot b) - f(s\cdot a)), \tag{2}$$

where the case $s = 0$ has to be interpreted as the limit $s \to 0$. The equalities (1) and (2) remain true if $f:\mathbb{C}\to\mathbb{C}$ is an analytic (holomorphic) function and $s \in \mathbb{C}$. An

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important example is the case $f(z) = e^z$. Then (2) reads

$$\int_a^b e^{s \cdot x}\, dx = \frac{1}{s}(e^{s \cdot b} - e^{s \cdot a}). \tag{3}$$

This is (up to a replacement of $-s$ by s) the *Laplace transform* of the function

$$g(x) = \begin{cases} 1, & \text{if } a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

In the special case $s = -i\omega$, where $\omega \in \mathbb{R}$, we have the *Fourier transform*. With $I = \{x \in \mathbb{R} : a \le x \le b\}$ we can write the left side of (3) in the form $\int_I e^{s \cdot x}\, dx$. Clearly, I is a polytope in $\mathbb{R}$ and it arises the question how (3) can be generalized to polytopes $\mathcal{P}$ in $\mathbb{R}^n$, i.e., how the integral

$$\int_{\mathcal{P}} e^{\mathbf{s} \cdot \mathbf{x}}\, \mathbf{dx} \tag{4}$$

can be computed. Here bold symbols denote real or complex tuples, written as column vectors, $\mathbf{s} \cdot \mathbf{x} = \sum_j s_j x_j$ and $\mathbf{dx}$ is an abbreviation for $dx_1 \ldots dx_n$. In this context the integral (4) is called the *Fourier-Laplace transform of the polytope* $\mathcal{P}$. It has many applications, see e.g. Barvinok (2008), Beck and Robins (2015), Engel and Laasch (2022), Engel (2023).

More generally, instead of (4) we study the integral

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx}, \tag{5}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an analytic function and $\mathbf{s} \in \mathbb{C}^n$, which corresponds to the left side of (2).

We point out that in the case $f(z) = z^n/n!$ the integral (5) is equal to the volume of $\mathcal{P}$ for all $\mathbf{s} \in \mathbb{C}^n$.

An analytical approach to compute such integrals is the application of Stoke's formula like in Komrska (1982), Baldoni et al. (2011), Wuttke (2017). The algebraic approach is part of the theory on the exponential valuation of polytopes built by Brion (1988), Lawrence (1991), Pukhlikov and Khovanskii (1992), Barvinok (1994) which has origins in results of Motzkin and Schoenberg (mentioned by Davis 1964) as well of Grunsky (1955). We recommend Barvinok (2008), Beck and Robins (2015) for studying this theory.

The aim of this paper is to present an elementary way for the computation of the integral (5) and to derive an explicit formula under some additional condition such that all (non-self-intersecting) 2- and 3-dimensional polytopes are included. We do not need much more than a suitable identity theorem, the computation for simplices using Lagrange polynomials and a little bit polytope theory (triangulation and supporting hyperplanes).

2 The main theorem

In order to formulate the main theorem we need some definitions and notations. For a polytope $\mathcal{P}$ let $V_{\mathcal{P}}$ be the set of its vertices and $\mathcal{F}_{\mathcal{P}}$ be the set of its facets. We consider the elements of $V_{\mathcal{P}}$ as points in $\mathbb{R}^n$. For a vertex $\mathbf{v} \in V_{\mathcal{P}}$ let $N_{\mathcal{P},\mathbf{v}}$ be the set of its *neighbors*, i.e., those vertices $\mathbf{v}'$ that are connected with $\mathbf{v}$ by a 1-dimensional face, and let $E_{\mathcal{P},\mathbf{v}} = \{\mathbf{v} - \mathbf{v}' : \mathbf{v}' \in N_{\mathcal{P},\mathbf{v}}\}$ be the set of edges with end point $\mathbf{v}$. Let $\mathcal{F}_{\mathcal{P},\mathbf{v}}$ be the set of facets containing the vertex $\mathbf{v}$.

A vertex $\mathbf{v}$ of an n-dimensional polytope $\mathcal{P}$ is called *simple* if the set $E_{\mathcal{P},\mathbf{v}}$ consists of n linearly independent vectors, i.e., if the points from $N_{\mathcal{P},\mathbf{v}} \cup \{\mathbf{v}\}$ form a simplex. An n-dimensional convex polytope is called *simple* if all of its vertices are simple.

We call an n-dimensional convex polytope in $\mathbb{R}^n$ *facet-simple* if each of its facets is an $(n-1)$-dimensional simple polytope. But we admit also non-convex polytopes. We define a set of points in $\mathbb{R}^n$ to be an n-dimensional *generalized facet-simple polytope* if it is a finite union of n-dimensional facet-simple convex polytopes such that any two distinct members are either disjoint or intersect in a common facet. We define the vertex set $V_{\mathcal{P}}$ of such a generalized facet-simple polytope $\mathcal{P}$ to be the union of the vertex sets of its members. Note that 2- and 3-dimensional non-self-intersecting polytopes are facet-simple and hence indeed the facet-simple polytopes are important generalizations of polytopes from the real world.

For a simple vertex $\mathbf{v}$ of an n-dimensional convex polytope $\mathcal{P}$ in $\mathbb{R}^n$ let $D_{\mathcal{P},\mathbf{v}}$ be the determinant of the matrix whose columns are formed by the n linearly independent elements of $E_{\mathcal{P},\mathbf{v}}$ in some fixed order. Note that $|D_{\mathcal{P},\mathbf{v}}|$ is $n!$-times the volume of the simplex with vertex set $N_{\mathcal{P},\mathbf{v}} \cup \{\mathbf{v}\}$. Let

$$\Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = \frac{|D_{\mathcal{P},\mathbf{v}}|}{\prod_{\mathbf{e}\in E_{\mathcal{P},\mathbf{v}}} \mathbf{e}\cdot\mathbf{s}}. \tag{6}$$

Here we have terms of the form $\mathbf{e}\cdot\mathbf{s}$ in the denominator. Hence we must exclude that $\mathbf{s}$ is contained in the hyperplane given by $\mathbf{e}\cdot\mathbf{s} = 0$, which is indeed a hyperplane because $\mathbf{e} \neq \mathbf{0}$ for all $\mathbf{e} \in E_{\mathcal{P},\mathbf{v}}$. Therefore we use the following notation: $\forall_h \mathbf{s} \in \mathbb{C}^n$ means that we consider all $\mathbf{s} \in \mathbb{C}^n$ up to those $\mathbf{s}$ that are contained in a finite union of hyperplanes.

We mention that we may use (6) also if $\mathcal{P}$ is degenerated in the sense that it is only $(n-1)$-dimensional, but $E_{\mathcal{P},\mathbf{v}}$ still consists of n elements different from the zero vector, which are in this case linearly dependent. Then $D_{\mathcal{P},\mathbf{v}}$ as well as $\Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s})$ are equal to zero.

For a convex $(k-1)$-dimensional polytope F in $\mathbb{R}^n$, where $k \in [n] = \{1, \dots, n\}$, and a point $\mathbf{p}$ let $F + \mathbf{p}$ be the convex hull of $F \cup \{\mathbf{p}\}$. Note that $F + \mathbf{p}$ is a k-dimensional pyramid if $\mathbf{p}$ is affinely independent of the vertices of F, otherwise $F + \mathbf{p}$ is a $(k-1)$-dimensional polytope.

Let $\mathcal{P}$ be a convex n-dimensional polytope in $\mathbb{R}^n$ and F a facet of $\mathcal{P}$. Let $H(F)$ be the hyperplane defined by F. We put

$$\operatorname*{sgn}_{\mathcal{P}}(F,\mathbf{p})=\begin{cases}1, & \text{if } \mathbf{p} \text{ and } \mathcal{P} \text{ are on the same side of F},\\ -1, & \text{if } \mathbf{p} \text{ and } \mathcal{P} \text{ are on different sides of F},\\ 0, & \text{if } \mathbf{p}\in H(F).\end{cases}$$

For generalized facet-simple polytopes $\mathcal{P}$ we can proceed as follows: Recall that $\mathcal{P}$ is a union of facet-simple convex polytopes $\mathcal{P}_i$, where i belongs to some finite index set I, such that any two distinct members are either disjoint or intersect in a common facet. We say that a facet F of the union of the facets of all $\mathcal{P}_i$, $i \in I$, is *visible* if it is the facet of only one of the $\mathcal{P}_i$ and it is *invisible* if it is the intersection of two of the polytopes $\mathcal{P}_i$. We define $\mathcal{F}_{\mathcal{P}}$ to be the set of all visible facets. Let $\mathcal{F}_{\mathcal{P},\mathbf{v}}$ be the set of all visible facets containg $\mathbf{v}$. Now let F be visible, i.e., there is a unique $\mathcal{P}_i$ having F as facet. Then we set

$$\operatorname*{sgn}_{\mathcal{P}}(F,\mathbf{p})=\begin{cases}1, & \text{if } \mathbf{p} \text{ and } \mathcal{P}_i \text{ are on the same side of F},\\ -1, & \text{if } \mathbf{p} \text{ and } \mathcal{P}_i \text{ are on different sides of F},\\ 0, & \text{if } \mathbf{p}\in H(F).\end{cases}$$

Let $\mathcal{P}$ be an n-dimensional generalized facet-simple polytope in $\mathbb{R}^n$, $\mathbf{v}$ a vertex, F a visible facet containing $\mathbf{v}$, i.e., $F \in \mathcal{F}_{\mathcal{P},\mathbf{v}}$, and $\mathbf{p} \notin H(F)$. Note that then $\mathbf{v}$ is a simple vertex of $F + \mathbf{p}$ and hence the function $\Phi_{F+\mathbf{p},\mathbf{v}}(\mathbf{s})$ is defined by (6). In fact we may even allow that $\mathbf{p} \in H(F)$, but $\mathbf{p} \neq \mathbf{v}$, because in this case $D_{F+\mathbf{p},\mathbf{v}}$ and hence $\Phi_{F+\mathbf{p},\mathbf{v}}(\mathbf{s})$ is equal to zero. We emphasize that an equation defining the hyperplane $H(F)$ is given by $D_{F+\mathbf{p},\mathbf{v}} = 0$. This can be used to determine $\operatorname{sgn}_{\mathcal{P}}(F, \mathbf{p})$: Let $\mathbf{v}'$ be a vertex of $\mathcal{P}_i$ (see above) that is not contained in F. Then $\operatorname{sgn}_{\mathcal{P}}(F, \mathbf{p}) = 1$ resp. -1 if $D_{F+\mathbf{p},\mathbf{v}}$ and $D_{F+\mathbf{v}',\mathbf{v}}$ have the same sign resp. different signs.

Theorem 2.1 *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $\mathcal{P}$ be an n-dimensional generalized facet-simple polytope in $\mathbb{R}^n$. Moreover let $\mathbf{p}$ be any point in $\mathbb{R}^n$ different from any vertex of $\mathcal{P}$. Then*

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s}\cdot\mathbf{x})\,\mathbf{dx}=\sum_{\mathbf{v}\in V_{\mathcal{P}}}\left(\sum_{F\in\mathcal{F}_{\mathcal{P},\mathbf{v}}}\operatorname*{sgn}_{\mathcal{P}}(F,\mathbf{p})\Phi_{F+\mathbf{p},\mathbf{v}}(\mathbf{s})\right)f(\mathbf{v}\cdot\mathbf{s})\quad\forall_h\mathbf{s}\in\mathbb{C}^n.$$

We mention that this theorem can also be considered as true for all $\mathbf{s} \in \mathbb{C}^n$ if one interprets the right side as a limit in those cases where a zero appears in one of the denominators. This follows from the continuity of the integral.

Since $D_{F+\mathbf{p_1},\mathbf{v}}$ and $D_{F+\mathbf{p_2},\mathbf{v}}$ as well as $\operatorname{sgn}_{\mathcal{P}}(F, \mathbf{p}_1)$ and $\operatorname{sgn}_{\mathcal{P}}(F, \mathbf{p_2})$ have different signs iff $\mathbf{p}_1$ and $\mathbf{p}_2$ are on different sides of F, the product $|D_{F+\mathbf{p},\mathbf{v}}| \operatorname{sgn}_{\mathcal{P}}(F, \mathbf{p})$ equals $+D_{F+\mathbf{p},\mathbf{v}}$ for all $\mathbf{p} \in \mathbb{R}^n$ or $-D_{F+\mathbf{p},\mathbf{v}}$ for all $\mathbf{p} \in \mathbb{R}^n$. The sign depends on the order of the columns in the determinant. Thus in concrete examples we do not need the absolute value function or the sign function.

In order to become acquainted with the formula in Theorem 2.1 we choose $f(z) = e^z$ and study the integration over a triangle in $\mathbb{R}^2$ which is illustrated with all necessary informations in Fig. 1.

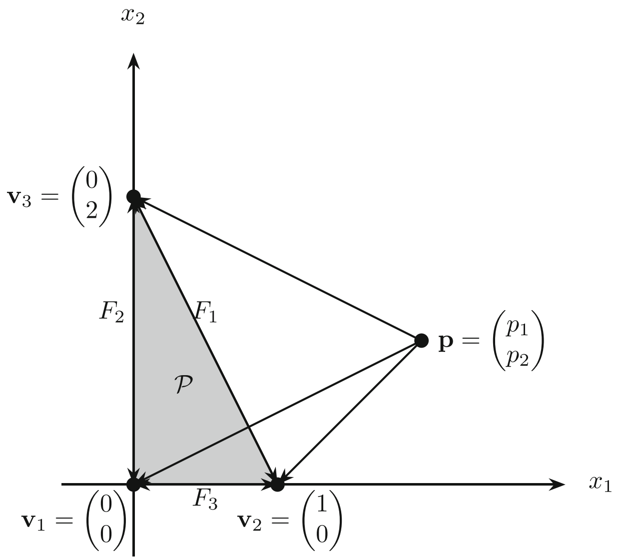

Fig. 1 Example for the computation of the integral using Theorem 2.1

We have

$$\begin{aligned}
\underset{\mathcal{P}}{\operatorname{sgn}}(F_2,\mathbf{p})\Phi_{F_2+\mathbf{p},\mathbf{v_1}}(\mathbf{s}) &= \frac{2p_1}{(-p_1s_1-p_2s_2)(-2s_2)},\\
\underset{\mathcal{P}}{\operatorname{sgn}}(F_3,\mathbf{p})\Phi_{F_3+\mathbf{p},\mathbf{v_1}}(\mathbf{s}) &= \frac{p_2}{(-p_1s_1-p_2s_2)(-s_1)},\\
\underset{\mathcal{P}}{\operatorname{sgn}}(F_3,\mathbf{p})\Phi_{F_3+\mathbf{p},\mathbf{v_2}}(\mathbf{s}) &= \frac{p_2}{((1-p_1)s_1-p_2s_2)(s_1)},\\
\underset{\mathcal{P}}{\operatorname{sgn}}(F_1,\mathbf{p})\Phi_{F_1+\mathbf{p},\mathbf{v_2}}(\mathbf{s}) &= \frac{-(2p_1+p_2-2)}{((1-p_1)s_1-p_2s_2)(s_1-2s_2)},\\
\underset{\mathcal{P}}{\operatorname{sgn}}(F_1,\mathbf{p})\Phi_{F_1+\mathbf{p},\mathbf{v_3}}(\mathbf{s}) &= \frac{-(2p_1+p_2-2)}{(-p_1s_1+(2-p_2)s_2)(-s_1+2s_2)},\\
\underset{\mathcal{P}}{\operatorname{sgn}}(F_2,\mathbf{p})\Phi_{F_2+\mathbf{p},\mathbf{v_3}}(\mathbf{s}) &= \frac{2p_1}{(-p_1s_1+(2-p_2)s_2)(2s_2)}.
\end{aligned}$$

Consequently,

$$\int_{\mathcal{P}} e^{s_1x_1+s_2x_2}\,dx_1dx_2 = \frac{1}{s_1s_2}e^0 + \frac{2}{s_1(s_1-2s_2)}e^{s_1} + \frac{1}{s_2(-s_1+2s_2)}e^{2s_2}.$$

In this special case, this result can be obtained also easily by iterated integration.

Note that in the final summation the terms with **p** cancel each other. This must be the case since the result does not depend on **p**. But the use of **p** enables a short closed formula and an easy implementation if one has the necessary informations for $\mathcal{P}$.

3 An independence theorem on exponential functions over the field of rational functions

Let, as usual, $\mathbb{C}(x)$ (resp. $\mathbb{C}(\mathbf{x})$) be the set of rational functions in the variable x (resp. in the variables that are the components of $\mathbf{x}$) with complex coefficients. In the following let $\mathcal{O}$ be the zero function, i.e., the function that is everywhere 0 on a certain domain which is given by functions that are included in the concrete context. The following lemma is a special case of Theorem 3.1 of Engel (2023), but can be proved in this situation much more easily.

Lemma 3.1 *Let $a_1, \ldots, a_m$ be distinct real numbers, $b_1, \ldots, b_m$ be real numbers and $p_1, \ldots, p_m$ be rational functions. Let I be an open interval in $\mathbb{R}$. If*

$$\sum_{k=1}^{m} p_k(x) e^{a_k x + b_k} = 0 \quad \forall x \in I, \tag{7}$$

then

$$p_k = \mathcal{O} \quad \forall k \in [m]. \tag{8}$$

Proof We may assume that the p_k are polynomials (if necessary, multiply by the common denominator) and that (7) holds for all $x \in \mathbb{R}$ (use the identity theorem for analytic functions). We proceed by induction on m. The base case $m = 1$ is trivial. For the induction step $m - 1 \to m$ let, without loss of generality, $a_1 < \cdots < a_m$.

If we divide (7) by $e^{a_m x + b_m}$, we obtain after a rearrangement

$$p_m(x) = -\sum_{k=1}^{m-1} p_k(x) e^{(a_k - a_m)x + (b_k - b_m)} \quad \forall x \in \mathbb{R}. \tag{9}$$

Since $a_k - a_m < 0$ for all $k \in [m-1]$, the right side tends to 0 for $x \to \infty$. Hence

$$\lim_{x \to \infty} p_m(x) = 0.$$

The only polynomial with this property is the zero polynomial, and hence $p_m = \mathcal{O}$. By the induction hypothesis and (9) also $p_k = \mathcal{O}$ for all $k \in [m-1]$. □

The following lemma is well-known and follows easily from the fundamental theorem of algebra, see e.g. Lemma 3.2 in Engel (2023).

Lemma 3.2 *Let $P \in \mathbb{C}(\mathbf{x})$. If there is an open subset O of $\mathbb{R}^n$ such that*

$$P(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in O,$$

then $P = \mathcal{O}$.

Our main auxiliary theorem is the following:

Theorem 3.1 *Let* $\mathbf{v}_k$, $k \in [m]$, *be distinct points in* $\mathbb{R}^n$ *and let* $P_k \in \mathbb{C}(\mathbf{x})$ *for all* $k \in [m]$. *Let* O *be an open subset of* $\mathbb{R}^n$. *If*

$$\sum_{k=1}^{m} P_k(\mathbf{x}) e^{\mathbf{v}_k \cdot \mathbf{x}} = 0 \quad \forall \mathbf{x} \in O,$$

then

$$P_k = \mathcal{O} \quad \forall k \in [m].$$

Proof Let $\mathbf{x} \in O$. Obviously, we can choose some $\mathbf{z} \in \mathbb{R}^n$ in such a way that $(\mathbf{v}_k - \mathbf{v}'_k) \cdot \mathbf{z} \neq 0$ whenever $k \neq k'$. Since $\mathbf{x}$ is an inner point of O there is some open interval $I \subseteq \mathbb{R}$ such that $0 \in I$ and $\mathbf{x} + \lambda \mathbf{z} \in O$ for all $\lambda \in I$. Consequently,

$$\sum_{k=1}^{m} P_k(\mathbf{x} + \lambda \mathbf{z}) e^{a_k \lambda + b_k} = 0 \quad \forall \lambda \in I,$$

where $a_k = \mathbf{v}_k \cdot \mathbf{z}$ and $b_k = \mathbf{v}_k \cdot \mathbf{x}$. Clearly, $p_k(\lambda) = P_k(\mathbf{x} + \lambda \mathbf{z}) \in \mathbb{C}(\lambda)$ for all $k \in [m]$. From Lemma 3.1 we obtain $p_k(0) = P_k(\mathbf{x}) = 0$ for all $k \in [m]$. Now Lemma 3.2 implies $P_k = \mathcal{O}$ for all $k \in [m]$. □

4 Proof of Theorem 2.1

First we need some preparation for the case of simplices. We apply iterated integration as in Baldoni et al. (2011), but use Lagrange polynomials.

For distinct numbers $s_0, s_1, \ldots, s_n$ let

$$l_{j,n}(x; s_0, \ldots, s_n) = \prod_{k=0, k \neq j}^{n} (x - s_k).$$

Recall that the Lagrange polynomials are given by

$$L_{j,n}(x; s_0, \ldots, s_n) = \frac{l_{j,n}(x; s_0, \ldots, s_n)}{l_{j,n}(s_j; s_0, \ldots, s_n)}.$$

Interpolating the function that is constant 1 leads to

$$\sum_{j=0}^{n} \frac{l_{j,n}(x; s_0, \ldots, s_n)}{l_{j,n}(s_j; s_0, \ldots, s_n)} = 1 \quad \forall x \in \mathbb{C}$$

and comparing the coefficient of x^n yields

$$\sum_{j=0}^{n} \frac{1}{l_{j,n}(s_j; s_0, \ldots, s_n)} = 0. \tag{10}$$

Let $\triangle_n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{1} \cdot \mathbf{x} \leq 1 \text{ and } \mathbf{x} \geq \mathbf{0}\}$ and let $\lambda \triangle_n = \{\lambda \mathbf{x} : \mathbf{x} \in \triangle_n\}$. Note that the vertices of the simplex $\triangle_n$ are given by the zero vector and standard basis vectors.

Lemma 4.1 *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic, let $s_0, s_1, \ldots, s_n$ be distinct complex numbers with $s_0 = 0$, let $\mathbf{s} = (s_1, \ldots, s_n)^{\mathbf{T}}$ and let c be a complex number. Then*

$$\int_{\triangle_n} f^{(n)}(\mathbf{s} \cdot \mathbf{x} + c)\, \mathbf{dx} = \sum_{j=0}^{n} \frac{f(s_j + c)}{l_{j,n}(s_j; s_0, \ldots, s_n)} \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

Proof We proceed by induction on n. The base of induction $n = 1$ is trivial. For the induction step $n - 1 \to n$ we use the notation $\overline{\mathbf{x}} = (x_1, \ldots, x_{n-1})^{\mathbf{T}}$ and analogously $\overline{\mathbf{s}}$, $\overline{\mathbf{y}}$. Using the change of variables $\overline{\mathbf{x}} = \lambda \overline{\mathbf{y}}$ with $0 < \lambda \leq 1$ and the induction hypothesis we obtain

$$\begin{aligned}
\int_{\lambda \triangle_{n-1}} f^{(n)}(\overline{\mathbf{s}} \cdot \overline{\mathbf{x}} + c)\, \mathbf{d}\overline{\mathbf{x}} &= \lambda^{n-1} \int_{\triangle_{n-1}} f^{(n)}(\lambda \overline{\mathbf{s}} \cdot \overline{\mathbf{y}} + c)\, \mathbf{d}\overline{\mathbf{y}} \\
&= \lambda^{n-1} \sum_{j=0}^{n-1} \frac{f'(\lambda s_j + c)}{l_{j,n-1}(\lambda s_j; \lambda s_0, \ldots, \lambda s_{n-1})} \\
&= \sum_{j=0}^{n-1} \frac{f'(\lambda s_j + c)}{l_{j,n-1}(s_j; s_0, \ldots, s_{n-1})}.
\end{aligned}$$

In view of (10) this equality is also true for $\lambda = 0$. Iterated integration gives

$$\begin{aligned}
\int_{\triangle_n} f^{(n)}(\mathbf{s} \cdot \mathbf{x} + c)\, \mathbf{dx} &= \int_0^1 \left(\int_{(1-x_n)\triangle_{n-1}} f^{(n)}(\overline{\mathbf{s}} \cdot \overline{\mathbf{x}} + s_n x_n + c)\, \mathbf{d}\overline{\mathbf{x}} \right) dx_n \\
&= \sum_{j=0}^{n-1} \int_0^1 \frac{f'((1 - x_n)s_j + s_n x_n + c)}{l_{j,n-1}(s_j; s_0, \ldots, s_{n-1})}\, dx_n \\
&= \sum_{j=0}^{n-1} \frac{f(s_j + c) - f(s_n + c)}{(s_j - s_n) l_{j,n-1}(s_j; s_0, \ldots, s_{n-1})} \\
&= \sum_{j=0}^{n-1} \frac{f(s_j + c)}{l_{j,n}(s_j; s_0, \ldots, s_n)} - \sum_{j=0}^{n-1} \frac{f(s_n + c)}{l_{j,n}(s_j; s_0, \ldots, s_n)} \\
&= \sum_{j=0}^{n} \frac{f(s_j + c)}{l_{j,n}(s_j; s_0, \ldots, s_n)}.
\end{aligned}$$

Here the last equality follows from (10). □

Lemma 4.2 *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $\mathcal{P}$ be an n-dimensional simplex in $\mathbb{R}^n$. Then*

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} = \sum_{\mathbf{v} \in V_{\mathcal{P}}} \Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s}) f(\mathbf{v} \cdot \mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

Proof Let $V_{\mathcal{P}} = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$. Let T be the matrix whose j-th column is $\mathbf{v}_j - \mathbf{v}_0$, $j \in [n]$. Then the affine transformation

$$\mathbf{x} = T\mathbf{y} + \mathbf{v}_0$$

maps $\triangle_n$ onto $\mathcal{P}$. Consequently, using Lemma 4.1

$$\begin{aligned}
\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} &= |\det(T)| \int_{\triangle_n} f^{(n)}(\mathbf{s} \cdot (T\mathbf{y} + \mathbf{v}_0))\, \mathbf{dy} \\
&= |\det(T)| \int_{\triangle_n} f^{(n)}(T^{\mathbf{T}}\mathbf{s} \cdot \mathbf{y} + \mathbf{v}_0 \cdot \mathbf{s})\, \mathbf{dy} \\
&= |\det(T)| \sum_{j=0}^{n} \frac{f((\mathbf{v}_j - \mathbf{v}_0) \cdot \mathbf{s} + \mathbf{v}_0 \cdot \mathbf{s})}{l_{j,n}((\mathbf{v}_j - \mathbf{v}_0) \cdot \mathbf{s}; (\mathbf{v}_0 - \mathbf{v}_0) \cdot \mathbf{s}, \ldots, (\mathbf{v}_n - \mathbf{v}_0) \cdot \mathbf{s})} \\
&= \sum_{\mathbf{v} \in V_{\mathcal{P}}} \frac{|\det(T)|}{\prod_{\mathbf{v}' \in V \setminus \{\mathbf{v}\}} (\mathbf{v} - \mathbf{v}') \cdot \mathbf{s}} f(\mathbf{v} \cdot \mathbf{s}) \\
&= \sum_{\mathbf{v} \in V_{\mathcal{P}}} \Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s}) f(\mathbf{v} \cdot \mathbf{s})\,.
\end{aligned}$$

□

Note that the coefficients $\Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s})$ in Lemma 4.2 do not depend on f.

Corollary 4.1 *Let $\mathcal{P}$ be an n-dimensional convex polytope in $\mathbb{R}^n$. Then there exist unique rational functions $Q_{\mathcal{P},\mathbf{v}}(\mathbf{s})$, $\mathbf{v} \in V_{\mathcal{P}}$, such that for all analytic functions $f : \mathbb{C} \to \mathbb{C}$*

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} = \sum_{\mathbf{v} \in V_{\mathcal{P}}} Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) f(\mathbf{v} \cdot \mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

Proof The assertion on the existence follows immediately from Lemma 4.2 and a triangulation of $\mathcal{P}$ because the functions $\Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s})$ are rational. But triangulations are not unique. Hence we have to prove that each triangulation leads to the same coefficients. Assume that we have two representations of this form with functions $Q_{\mathcal{P},\mathbf{v}}(\mathbf{s})$ and $Q'_{\mathcal{P},\mathbf{v}}(\mathbf{s})$, $\mathbf{v} \in V_{\mathcal{P}}$. Then for all analytic functions $f : \mathbb{C} \to \mathbb{C}$

$$\sum_{\mathbf{v} \in V_{\mathcal{P}}} Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) f(\mathbf{v} \cdot \mathbf{s}) = \sum_{\mathbf{v} \in V_{\mathcal{P}}} Q'_{\mathcal{P},\mathbf{v}}(\mathbf{s}) f(\mathbf{v} \cdot \mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

We use the special function $f(z) = e^z$ and obtain

$$\sum_{\mathbf{v}\in V_{\mathcal{P}}} Q_{\mathcal{P},\mathbf{v}}(\mathbf{s})e^{\mathbf{v}\cdot\mathbf{s}} = \sum_{\mathbf{v}\in V_{\mathcal{P}}} Q'_{\mathcal{P},\mathbf{v}}(\mathbf{s})e^{\mathbf{v}\cdot\mathbf{s}} \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

Now $Q_{\mathcal{P},\mathbf{v}} - Q'_{\mathcal{P},\mathbf{v}} = \mathcal{O}$ for all $\mathbf{v} \in V_{\mathcal{P}}$ follows from Theorem 3.1. □

For a fixed vertex $\mathbf{v}$ of the n-dimensional convex polytope $\mathcal{P}$ let $C_{\mathcal{P},\mathbf{v}}$ be a hyperplane with the following properties:

- It is parallel to a supporting hyperplane meeting $\mathcal{P}$ only in $\mathbf{v}$.
- All other vertices of $\mathcal{P}$ are on the other side of $C_{\mathcal{P},\mathbf{v}}$ as $\mathbf{v}$.

Obviously, $C_{\mathcal{P},\mathbf{v}}$ indeed exists. Then $C_{\mathcal{P},\mathbf{v}}$ cuts $\mathcal{P}$ into a pyramid $\mathcal{P}_\mathbf{v}$ containing the vertex $\mathbf{v}$ and into a remainder polytope $\overline{\mathcal{P}}_\mathbf{v}$ not containing $\mathbf{v}$. Here $\mathcal{P}_\mathbf{v}$ and $\overline{\mathcal{P}}_\mathbf{v}$ share only a facet. We call $C_{\mathcal{P},\mathbf{v}}$ a *truncating hyperplane* and $\mathcal{P}_\mathbf{v}$ the *separated pyramid*.

Lemma 4.3 *Let* $\mathbf{v}$ *be a vertex of the n-dimensional convex polytope* $\mathcal{P}$ *in* $\mathbb{R}^n$ *and* $\mathcal{P}_\mathbf{v}$ *be the separated pyramid by a truncating hyperplane. Then*

$$Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = Q_{\mathcal{P}_\mathbf{v},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n \ \forall \mathbf{v} \in V_{\mathcal{P}}.$$

Proof Let $\mathbf{v} \in V_{\mathcal{P}}$ be fixed. Clearly, for all analytic functions $f : \mathbb{C} \to \mathbb{C}$

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s}\cdot\mathbf{x})\,\mathbf{dx} = \int_{\mathcal{P}_\mathbf{v}} f^{(n)}(\mathbf{s}\cdot\mathbf{x})\,\mathbf{dx} + \int_{\overline{\mathcal{P}}_\mathbf{v}} f^{(n)}(\mathbf{s}\cdot\mathbf{x})\,\mathbf{dx} \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

We use again the special function $f(z) = e^z$ and expand both sides according to Corollary 4.1. Then the coefficient of $e^{\mathbf{v}\cdot\mathbf{s}}$ on the left side is $Q_{\mathcal{P},\mathbf{v}}(\mathbf{s})$ and the coefficient of $e^{\mathbf{v}\cdot\mathbf{s}}$ on the right side is $Q_{\mathcal{P}_\mathbf{v},\mathbf{v}}(\mathbf{s})$. Now the assertion follows immediately from Theorem 3.1. □

Lemma 4.4 *If* $\mathbf{v}$ *is a simple vertex of the convex polytope* $\mathcal{P}$*, then*

$$Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = \Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

Proof Since $\mathcal{P}_\mathbf{v}$ is a simplex we obtain from Lemmas 4.2 and 4.3 that

$$Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = \Phi_{\mathcal{P}_\mathbf{v},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

But we have also

$$\Phi_{\mathcal{P}_\mathbf{v},\mathbf{v}}(\mathbf{s}) = \Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n$$

because the corresponding edges in $E_{\mathcal{P}_\mathbf{v},\mathbf{v}}$ and in $E_{\mathcal{P},\mathbf{v}}$ differ only by a scalar and these scalars cancel each other in (6). □

Now Corollary 4.1 and Lemma 4.4 immediately yield the following known result for simple polytopes. An accessible sketch of proof using cones can be found in Gravin et al. (2012).

Theorem 4.1 *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $\mathcal{P}$ be an n-dimensional convex simple polytope in $\mathbb{R}^n$. Then*

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} = \sum_{\mathbf{v} \in V_{\mathcal{P}}} \Phi_{\mathcal{P},\mathbf{v}}(\mathbf{s}) f(\mathbf{v} \cdot \mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

We need some further definitions for n-dimensional convex polytopes $\mathcal{P}$. The *characteristic function* $\chi_{\mathcal{P}} : \mathbb{R}^n \to \mathbb{R}$ of $\mathcal{P}$ is defined by

$$\chi_{\mathcal{P}}(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}'_{\mathcal{P}}$ be the (finite) set of all faces of $\mathcal{P}$ of dimension at most $n-2$. For a point $\mathbf{p}$ we set

$$S_{\mathbf{p}} = \bigcup_{F \in \mathcal{F}'_{\mathcal{P}}} (F + \mathbf{p}).$$

Lemma 4.5 *Let $\mathcal{P}$ be an n-dimensional convex polytope in $\mathbb{R}^n$ and $\mathbf{p}$ be a point. Then*

$$\chi_{\mathcal{P}}(\mathbf{x}) = \sum_{F \in \mathcal{F}_{\mathcal{P}}} \underset{\mathcal{P}}{\mathrm{sgn}}(F, \mathbf{p}) \chi_{F+\mathbf{p}}(\mathbf{x}) \quad \forall \mathbf{x} \in (\mathcal{P} + \mathbf{p}) \setminus S_{\mathbf{p}}. \tag{11}$$

Proof First let $\mathbf{p} \in \mathcal{P}$. Then $\mathcal{P} + \mathbf{p} = \mathcal{P}$.

If $\mathbf{x} \in \mathcal{P} \setminus S_{\mathbf{p}}$, then the open ray $\overrightarrow{\mathbf{px}}$ meets exactly one facet $F_{\mathbf{x}}$ (using that $\mathbf{x} \notin S_{\mathbf{p}}$) and $F_{\mathbf{x}}$ is the only facet F of $\mathcal{P}$ such that $\mathbf{x} \in F + \mathbf{p}$. Hence both sides of (11) are equal to 1.

Now let $\mathbf{p} \notin \mathcal{P}$ and $\mathbf{x} \in (\mathcal{P} + \mathbf{p}) \setminus S_{\mathbf{p}}$. Then the ray $\overrightarrow{\mathbf{px}}$ meets exactly two facets $F_{\mathbf{x}}^{in}$ (where $\mathcal{P}$ is entered) and $F_{\mathbf{x}}^{out}$ (where $\mathcal{P}$ is left). If $\mathbf{x} \in \mathcal{P}$, then $F_{\mathbf{x}}^{out}$ is the only facet F of $\mathcal{P}$ such that $\mathbf{x} \in F + \mathbf{p}$. Hence both sides of (11) are equal to 1. If $\mathbf{x} \notin \mathcal{P}$, then $F_{\mathbf{x}}^{in}$ and $F_{\mathbf{x}}^{out}$ are the only facets F of $\mathcal{P}$ such that $\mathbf{x} \in F + \mathbf{p}$, but $-\operatorname{sgn}_{\mathcal{P}}(F_{\mathbf{x}}^{in}, \mathbf{p}) = +\operatorname{sgn}_{\mathcal{P}}(F_{\mathbf{x}}^{out}, \mathbf{p}) = 1$. Hence both sides of (11) are equal to 0. $\square$

In the example given in Fig. 1 (with the illustrated position of $\mathbf{p}$) we have

$$\chi_{\mathcal{P}}(\mathbf{x}) = -\chi_{F_1+\mathbf{p}}(\mathbf{x}) + \chi_{F_2+\mathbf{p}}(\mathbf{x}) + \chi_{F_3+\mathbf{p}}(\mathbf{x}) \quad \forall \mathbf{x} \in (\mathcal{P} + \mathbf{p}) \setminus S_{\mathbf{p}}.$$

Proof of Theorem 2.1 First we assume that $\mathcal{P}$ is a *convex* facet-simple polytope. By Corollary 4.1 it is sufficient to compute the rational function $Q_{\mathcal{P},\mathbf{v}}(\mathbf{s})$ for each fixed vertex $\mathbf{v}$ of $\mathcal{P}$.

Let $\mathbf{p}$ be any point in $\mathbb{R}^n$ different from any vertex of $\mathcal{P}$. First we assume that $\mathbf{p} \notin H(F)$ for all facets F of $\mathcal{P}$. Note that $S_{\mathbf{p}}$ is a set of measure zero since it is a finite union of at most $(n-1)$-dimensional polytopes.

Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. If we integrate (11), multiplied by $f^{(n)}(\mathbf{s}\cdot\mathbf{x})$, over $\mathcal{P} + \mathbf{p}$, we obtain

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} = \sum_{F \in \mathcal{F}_{\mathcal{P}}} \underset{\mathcal{P}}{\mathrm{sgn}}(F, \mathbf{p}) \int_{F+\mathbf{p}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} \quad \forall_h \mathbf{s} \in \mathbb{C}^n. \tag{12}$$

Expanding these integrals according to Corollary 4.1 yields for a fixed vertex $\mathbf{v}$ of $\mathcal{P}$

$$Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = \sum_{F \in \mathcal{F}_{\mathcal{P},\mathbf{v}}} \underset{\mathcal{P}}{\mathrm{sgn}}(F, \mathbf{p}) Q_{F+\mathbf{p},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n.$$

But since all facets F are simple and $\mathbf{p} \notin H(F)$, the vertex $\mathbf{v}$ is a simple vertex of $F + \mathbf{p}$ for all facets $F \in \mathcal{F}_{\mathcal{P},\mathbf{v}}$. Lemma 4.4 gives

$$Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = \sum_{F \in \mathcal{F}_{\mathcal{P},\mathbf{v}}} \underset{\mathcal{P}}{\mathrm{sgn}}(F, \mathbf{p}) \Phi_{F+\mathbf{p},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n. \tag{13}$$

If there are facets F such that $\mathbf{p} \in H(F)$, the result remains true because the contribution of the corresponding items in (12) is zero since the integral over the $(n-1)$-dimensional polytope $F + \mathbf{p}$ is zero and also the contribution of the corresponding items in (13) is zero. Thus the assertion of Theorem 2.1 is proved for convex facet-simple polytopes.

Now let $\mathcal{P}$ be a not necessarily convex facet-simple polytope. Recall that $\mathcal{P}$ is a union of facet-simple convex polytopes $\mathcal{P}_i$, where i belongs to some finite index set I, such that any two distinct members are either disjoint or intersect in a common facet. Let $\overline{\mathcal{F}}$ be the set of the facets of all $\mathcal{P}_i$, $i \in I$. Recall that $\mathcal{F}_{\mathcal{P}}$ is the set of all visible facets from $\overline{\mathcal{F}}$ which implies that $\mathcal{F}'_{\mathcal{P}} = \overline{\mathcal{F}} \setminus \mathcal{F}_{\mathcal{P}}$ is the set of all invisible facets. Let as before $\mathcal{F}_{\mathcal{P},\mathbf{v}}$ and $\mathcal{F}'_{\mathcal{P},\mathbf{v}}$ be the corresponding subsets of facets containing the vertex $\mathbf{v}$.

Since

$$\int_{\mathcal{P}} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} = \sum_{i \in I} \int_{\mathcal{P}_i} f^{(n)}(\mathbf{s} \cdot \mathbf{x})\, \mathbf{dx} \quad \forall_h \mathbf{s} \in \mathbb{C}^n$$

we have by (13)

$$Q_{\mathcal{P},\mathbf{v}}(\mathbf{s}) = \sum_{i \in I} \sum_{F \in \mathcal{F}_{\mathcal{P}_i,\mathbf{v}}} \underset{\mathcal{P}_i}{\mathrm{sgn}}(F, \mathbf{p}) \Phi_{F+\mathbf{p},\mathbf{v}}(\mathbf{s}) \quad \forall_h \mathbf{s} \in \mathbb{C}^n. \tag{14}$$

If F is invisible and is a facet of $\mathcal{P}_j$ and $\mathcal{P}_k$, then obviously

$$\underset{\mathcal{P}_j}{\mathrm{sgn}}(F, \mathbf{p}) \Phi_{F+\mathbf{p},\mathbf{v}} = - \underset{\mathcal{P}_k}{\mathrm{sgn}}(F, \mathbf{p}) \Phi_{F+\mathbf{p},\mathbf{v}}. \tag{15}$$

Hence invisible facets cancel each other in (14) and thus the summation is only over visible facets.

If $F \in \mathcal{F}_{\mathcal{P}_i,\mathbf{v}}$ is visible, then by definion

$$\operatorname*{sgn}_{\mathcal{P}_i}(F,\mathbf{p}) = \operatorname*{sgn}_{\mathcal{P}}(F,\mathbf{p}). \tag{16}$$

Now (13) follows from (14)–(16) and the whole theorem is proved. □

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