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Re-embeddings of affine algebras via Gröbner fans of linear ideals

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Abstract
Given an affine algebra $R = K[x_1, \dots, x_n]/I$ over a field K, where I is an ideal in the polynomial ring $P = K[x_1, \dots, x_n]$, we examine the task of effectively calculating re-embeddings of I, i.e., of presentations $R = P'/I'$ such that $P' = K[y_1, \dots, y_m]$ has fewer indeterminates. For cases when the number of indeterminates n is large and Gröbner basis computations are infeasible, we have introduced the method of Z-separating re-embeddings in Kreuzer et al. (J Algebra Appl 21, 2022) and Kreuzer, et al. (São Paulo J Math Sci, 2022). This method tries to detect polynomials of a special shape in I which allow us to eliminate the indeterminates in the tuple Z by a simple substitution process. Here we improve this approach by showing that suitable candidate tuples Z can be found using the Gröbner fan of the linear part of I. Then we describe a method to compute the Gröbner fan of a linear ideal, and we improve this computation in the case of binomial linear ideals using a cotangent equivalence relation. Finally, we apply the improved technique in the case of the defining ideals of border basis schemes.

Keywords Re-embedding · Optimal embedding · Gröbner fan · Cotangent space · Border basis scheme

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1 Introduction

A finitely generated algebra R over a field K is also called an *affine K-algebra*. In order to analyse an affine K-algebra and to perform basic operations effectively, we usually assume that the algebra is given by generators and relations, i.e., that $R = P/I$ where $P = K[x_1, \dots, x_n]$ is a polynomial ring over K and I is an explicitly given ideal in P. The question of whether we can actually perform the intended calculations, many of which are based on the theory of Gröbner bases, depends chiefly on the number n of indeterminates involved in the presentation $R = P/I$. If the number n is too large, essentially all computations, with the possible exception of those which require solely linear algebra, become prohibitively expensive. Thus it is an important task to find better presentations $R \cong P'/I'$ where $P' = K[y_1, \dots, y_m]$ is a polynomial ring in fewer indeterminates and I' is an ideal in P'.

In the language of Algebraic Geometry, we are given an affine scheme $\mathrm{Spec}(R)$ embedded into an affine space $\mathbb{A}^n_K$, and we are looking for a re-embedding of it into a lower dimensional affine space $\mathbb{A}^m_K$. In this setting, the topic has a long history, beginning with the classical result that a smooth variety of dimension d can be embedded into $\mathbb{A}^{2d+1}$ if the field K is infinite. In the fundamental paper (Srinivas 1991), this bound is generalized to a bound for arbitrary affine schemes over infinite fields. The usual way to achieve the desired improvements of an embedding in algebraic geometry is by using *generic projections*. The obstruction is then mainly given by the secant variety of the scheme (see Holme 1975, Sect. 7). If we intend to perform actual computer calculations, computing projections corresponds to calculating elimination ideals, possibly after a change of coordinates. Herein lies the heart of the problem: finding suitable projections and calculating the resulting presentations $R \cong P'/I'$ is traditionally done by computing Gröbner bases with respect to elimination term orderings, and this is usually infeasible when the number of indeterminates is large.

In two previous papers (Kreuzer et al. 2022a) and (Kreuzer et al. 2022b), the authors introduced and developed a new method for re-embedding affine algebras based on tuples of *separating indeterminates*. This method can be applied to any finitely generated K-algebra $R = P/I$ if I is an ideal contained in the maximal ideal $\mathfrak{M} = \langle x_1, \dots, x_n \rangle$ of P. If this hypothesis is not satisfied from the outset, it may be necessary to know a K-rational point of $\mathrm{Spec}(R)$ and to perform a suitable change of coordinates. Then, for a tuple $Z = (z_1, \dots, z_s)$ of indeterminates in P, we say that I is *Z-separating* if there exist a tuple of non-zero polynomials $(f_1, \dots, f_s)$ of I and a term ordering σ such that $\mathrm{LT}_\sigma(f_i) = z_i$ for $i = 1, \dots, s$. Given a tuple Z such that I is Z-separating, we can eliminate the indeterminates in Z and get a *Z-separating re-embedding* $\Phi : P/I \longrightarrow K[Y]/(I \cap K[Y])$ where $Y = \{x_1, \dots, x_n\} \setminus \{z_1, \dots, z_s\}$. Here the actual elimination can be carried out by interreducing $(f_1, \dots, f_s)$ such that they become *coherently separating* and performing substitutions in the remaining generators of I. Of course, in this way we will usually end up with a system of generators of the elimination ideal, but not a Gröbner basis with respect to any term ordering.

Notice that the question of whether an ideal I in P is Z-separating for some tuple of indeterminates Z is related to the famous *Epimorphism Problem* in Affine Geometry (see for instance van den Essen 2004; Drensky 2005, or Gupta 2022). However, we are looking for even more special generators of I than required in this problem, and therefore we generally cannot expect I to be Z-separating for some tuple Z. Our main focus is to try to find such tuples Z computationally, even if we cannot know whether they exist for a given ideal I, because for larger numbers of indeterminates this is our best chance to effectively calculate good, and possibly optimal, re-embeddings.

Using the method of Z-separating re-embeddings, the task of finding good re-embeddings is split into two steps:

(1) Find a suitable candidate tuple of indeterminates Z.
(2) Check if this tuple Z really works.

For the second step we can use a method based on Linear Programming Feasibility (LPF), if we have a candidate tuple $(f_1, \dots, f_s)$ of polynomials $f_i \in I$ available (see Kreuzer et al. 2022b, Section 3). Other algorithms for this task, which are efficient but not guaranteed to succeed, are in development. The current paper is chiefly concerned with the first step of finding suitable candidate tuples Z. A first idea of how to restrict the number of tuples one has to consider was introduced in Kreuzer et al. (2022b, Section 5), namely the idea is to use the *Z-restricted Gröbner fan* of I. However, this approach may still be computationally heavy, and this is where the first new result of the current paper comes in: it turns out that it is enough to consider the Gröbner fan of the *linear part* $\mathrm{Lin}_{\mathfrak{M}}(I)$ of I (see Propositions 2.6 and 3.1). The linear part of I is the linear ideal generated by the linear parts of an arbitrary system of generators of I, and thus easy to compute.

In Sect. 2 we begin by recalling and extending the basic theory of Z-separating re-embeddings. For the later applications, it is necessary to pay close attention to the questions of when the newly found re-embeddings are *optimal*, i.e., when the number m of indeterminates of P' is the minimal possible one (see Corollary 2.8), and when the re-embedded ring is actually a polynomial ring, i.e., when $I' = \langle 0 \rangle$ and $\mathrm{Spec}(R)$ is isomorphic to an affine space (see Corollary 2.9).

Thus we are led to examine the task of calculating the Gröbner fan of a linear ideal I_L in Sect. 3. First we reduce the problem to check whether a given tuple of indeterminates Z consists of the leading terms of a reduced Gröbner basis of I_L to a linear algebra computation (see Proposition 3.1). Then we reduce the calculation of the Gröbner fan of I_L to the task of finding the maximal minors of a matrix (see Theorem 3.5). We note that this step can also be tackled via any method for computing the bases of a linear matroid (see Remark 3.7.b). Combining all steps, we present an algorithm for finding Z-separating re-embeddings using the Gröbner fan of $\mathrm{Lin}_{\mathfrak{M}}(I)$ in Sect. 4. Notice that Algorithm 4.1 also allows us to find optimal Z-separating re-embeddings, if they exist.

As the calculation of the maximal minors of a matrix required by Theorem 3.5 could still be quite demanding, we look at a practically relevant special case in Sect. 5, namely the case of binomial linear ideals. In this case we define an equivalence relation on the indeterminates in $X = (x_1, \dots, x_n)$, called *cotangent equivalence* and based on the equality of $K \cdot \bar{x}_i$ and $K \cdot \bar{x}_j$ in the cotangent space $\mathrm{Cot}_{\mathfrak{m}}(R) = \mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$ is the maximal ideal of R generated by the residue classes of the indeterminates. Using this equivalence relation, the indeterminates in X can be divided into three types: *basic* indeterminates which can never by a part of any separating tuple Z, *trivial* indeterminates which can be part of such a tuple Z, and *proper* equivalence classes of indeterminates for which a proper subset can be put into Z. With the help of this equivalence relation we describe the leading term Gröbner fan of a linear ideal explicitly (see Theorem 5.5), apply it to the classification of possible separating tuples Z (see Theorem 5.6), and spell out an explicit efficient algorithm to compute those tuples (see Algorithm 5.7).

In the final section we apply the new theory to the case which prompted its development in the first place, namely to the coordinate rings of border basis schemes. These schemes are important moduli spaces in Algebraic Geometry which parametrize 0-dimensional polynomial ideals. Their vanishing ideals have exactly the correct structure to make Z-separating re-embeddings work: they are contained in the maximal ideal generated by the indeterminates, their linear part is a binomial linear ideal, and the cotangent equivalence classes can be calculated quickly (using the algorithm given in Kreuzer et al. 2020). In this setting the indeterminates of the underlying polynomial ring can be classified further into *interior* indeterminates and *rim* indeterminates, and we are able to provide additional information about their distribution in the cotangent equivalence classes (see Theorem 6.7). Our final Example 6.8 shows the machinery of Z-separating re-embeddings at work and verifies some claims in Huibregtse (2002, Remark 7.5.3).

All algorithms mentioned in this paper were implemented in the computer algebra system CoCoA (see [1]) and are available as a package for ApCoCoA (see [20]) on the first author's web page[1] This package can be applied to perform the calculations underlying most examples. Their use was essential in the discovery of properties and features which eventually evolved into theorems or disproved previous conjectures. The general notation and definitions in this paper follow (Kreuzer and Robbiano 2000) and (Kreuzer and Robbiano 2005).

2 Z-Separating Re-embeddings

In this paper we let K be an arbitrary field, let $P = K[x_1, \ldots, x_n]$, and let $\mathfrak{M} = \langle x_1, \ldots, x_n \rangle$. The tuple formed by the indeterminates of P is denoted by $X = (x_1, \ldots, x_n)$. Moreover, let $1 \leq s \leq n$, let $z_1, \ldots, z_s$ be pairwise distinct indeterminates in X, and let $Z = (z_1, \ldots, z_s)$. Denote the remaining indeterminates by $\{y_1, \ldots, y_{n-s}\} = \{x_1, \ldots, x_n\} \setminus \{z_1, \ldots, z_s\}$, and let $Y = (y_1, \ldots, y_{n-s})$. Committing a slight abuse of notation, we shall also write $Y = X \setminus Z$. The monoid of terms in P is denoted by $\mathbb{T}^n = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0\}$. Given a term ordering σ on $\mathbb{T}^n$, its restriction to $\mathbb{T}(y_1, \ldots, y_{n-s})$ is denoted by σ_Y.

Recall that an algebra of type $R = P/I$ where $P = K[x_1, \ldots, x_n]$ is a polynomial ring over a field K and I is a proper ideal in P is called an **affine K-algebra**. In this setting, re-embeddings are defined as follows.

Definition 2.1 Let $P = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, let I be a proper ideal in P, and let $R = P/I$.

(a) A K-algebra isomorphism $\Psi : R \longrightarrow P'/I'$, where P' is a polynomial ring over K and I' is an ideal in P', is called a **re-embedding** of I.

[1] https://symbcomp.fim.uni-passau.de/en/symbolic-computation/projects.

(b) A re-embedding $\Psi : R \longrightarrow P'/I'$ of I is called **optimal** if every K-algebra isomorphism $R \longrightarrow P''/I''$ with a polynomial ring P'' over K and an ideal I'' in P'' satisfies the inequality $\dim(P'') \geq \dim(P')$.

In Kreuzer et al. (2022a) and Kreuzer et al. (2022b), the authors examined re-embeddings of affine K-algebras, i.e., isomorphisms with presentations requiring fewer K-algebra generators. For these techniques to work, we need to assume that the given ideal I is contained in a linear maximal ideal of P. As explained in Kreuzer et al. (2022a, Sect. 1), we can then perform a linear change of coordinates and assume that I is contained in $\mathfrak{M} = \langle x_1, \dots, x_n \rangle$. In particular, it was shown that the following situation leads to such re-embeddings.

Definition 2.2 Let I be an ideal in P with $I \subseteq \mathfrak{M}$, and let $Z = (z_1, \dots, z_s)$ be a tuple of distinct indeterminates in X.

(a) We say that the ideal I is Z-**separating** if there exist a term ordering σ on $\mathbb{T}^n$ and $f_1, \dots, f_s \in I \setminus \{0\}$ such that $\mathrm{LT}_\sigma(f_i) = z_i$ for $i = 1, \dots, s$. In this situation σ is called a Z-**separating term ordering** for I, and the tuple $(f_1, \dots, f_s)$ is called a Z-**separating tuple**.
(b) The ideal I is called **coherently** Z-**separating** if it contains a Z-separating tuple $(f_1, \dots, f_s)$ such that for $i \neq j$ the indeterminate z_i does not divide any term in the support of f_j.

Given a Z-separating term ordering σ for I, the reduced σ-Gröbner basis of I is of the form $G = \{z_1 - h_1, \dots, z_s - h_s, g_1, \dots, g_r\}$ with $h_i, g_j \in K[Y]$. In this case the K-algebra homomorphism $\Phi : P/I \longrightarrow K[Y]/(I \cap K[Y])$ given by $\Phi(\bar{x}_i) = \bar{x}_i$ for $x_i \in Y$ and $\Phi(\bar{x}_i) = \bar{h}_j$ for $x_i = z_j \in Z$ is an isomorphism of K-algebras. It is called the Z-**separating re-embedding** of I (see Kreuzer et al. 2022a, Theorem 2.13). Notice that the map Φ is a re-embedding of I such that the new polynomial ring $K[Y]$ involves fewer indeterminates, and the size of Z measures the improvement $\#Z = \#X - \#Y$ we achieved. Geometrically, the original variety can be viewed as the graph of the functions $h_1, \dots, h_s$ over the re-embedded variety.

For the choice of a Z-separating term ordering, we have the following observation.

Proposition 2.3 *Let I be an ideal in P which is contained in $\mathfrak{M}$, and let $Z = (z_1, \dots, z_s)$ be a tuple of distinct indeterminates in X. Then the following conditions are equivalent.*

(a) *The ideal I is Z-separating.*
(b) *For every elimination ordering σ for Z, we have $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(I)$.*
(c) *There exists an elimination ordering σ for Z such that $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(I)$.*

Proof To show that (a) implies (b), we note that, by definition, there exists a Z-separating term ordering σ for I. By Kreuzer et al. (2022b, Remark 4.3), it follows that any elimination ordering for Z is then also a Z-separating term ordering for I. Now the claim follows from Kreuzer et al. (2022b, Proposition 4.2).

Condition (b) obviously implies (c), and the remaining implication (c)⇒(a) follows from the definition. □

The following example illustrates this proposition.

Example 2.4 Consider the ring $P = \mathbb{Q}[x, y, z]$, the tuple $Z = (x)$, and the ideal $I = \langle f_1, \ldots, f_{10}\rangle$, where

$$\begin{aligned}
f_1 &= xy^2 + \tfrac{1}{2}y^3 - \tfrac{1}{2}y^2z - x^2 - \tfrac{1}{2}xy - y^2 + \tfrac{1}{2}xz + x,\\
f_2 &= y^2z^2 + 3y^3 - 4y^2z - xz^2 - 3xy + 4xz,\\
f_3 &= y^3z - xyz - y^2z + xz,\\
f_4 &= y^4 - xy^2 - y^3 + xy,\\
f_5 &= x^2y^2 - x^3,\\
f_6 &= x^3 + \tfrac{1}{2}x^2y + xy^2 + \tfrac{1}{2}y^3 - \tfrac{1}{2}x^2z - \tfrac{1}{2}y^2z - x^2 - y^2,\\
f_7 &= x^2z^2 + y^2z^2 + 3x^2y + 3y^3 - 4x^2z - 4y^2z,\\
f_8 &= x^2yz + y^3z - x^2z - y^2z,\\
f_9 &= x^2y^2 + y^4 - x^2y - y^3,\\
f_{10} &= x^4 + x^2y^2.
\end{aligned}$$

At first glance, the ideal I does not appear to be Z-separating, even if we use linear combinations of the generators. However, for any elimination ordering σ for Z, the reduced σ-Gröbner basis of the ideal I is $\{x - y^2,\ y^4 + y^2\}$, and this proves that I is indeed Z-separating.

The above proposition provides one way to solve the following task.

Remark 2.5 (*Checking Z-Separating Tuples*) Given a tuple of indeterminates Z, there are several methods for checking whether the ideal I is Z-separating.

(a) Condition (b) of the last proposition says that we can check $\langle Z\rangle \subseteq \mathrm{LT}_\sigma(I)$ for any elimination ordering σ for Z. However, the required Gröbner basis computation may be too costly, in particular if the given ideal I is not Z-separating.

(b) If we have a tuple of polynomials $(f_1, \ldots, f_s)$ with $f_i \in I$ and want to check whether it is Z-separating, we can use the methods explained in Kreuzer et al. (2022b, Sect. 4). They use Linear Programming Feasibility solvers and are usually very fast.

In the following, our main focus is the possibility to weed out many candidate tuples Z beforehand. In Kreuzer et al. (2022b) it was suggested to use the Gröbner fan of I for this purpose. Actually, as we shall see later, it suffices to use the Gröbner fan of the linear part of I which we introduce now. Recall that $\mathrm{Lin}_{\mathfrak{M}}(f)$ denotes the homogeneous component of standard degree 1 of a polynomial $f \in \mathfrak{M}$. It is called the **linear part** of f. Given an ideal I contained in $\mathfrak{M}$, the K-vector space $\mathrm{Lin}_{\mathfrak{M}}(I) = \langle \mathrm{Lin}_{\mathfrak{M}}(f) \mid f \in I\rangle_K$ is called the **linear part** of I.

In Kreuzer et al. (2022a, Proposition 1.6), we showed that $\mathrm{Lin}_{\mathfrak{M}}(I)$ is easy to compute, since it is equal to $\langle \mathrm{Lin}_{\mathfrak{M}}(f_1), \ldots, \mathrm{Lin}_{\mathfrak{M}}(f_s)\rangle_K$, where $\{f_1, \ldots, f_s\}$ is any set of generators of I. In this setting we make the following useful observation.

Proposition 2.6 *Let I be an ideal in P which is contained in $\mathfrak{M}$. Suppose that I is Z-separating for some tuple of indeterminates Z in X, let σ be a Z-separating term ordering for I, and let $Y = X \setminus Z$.*

(a) *We have* $\langle Z\rangle \subseteq \mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$,
(b) *Let* S_σ *be the set of indeterminates which generate* $\mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$. *Then we have* $Y \supseteq X\setminus S_\sigma$.
(c) *We have* $Z \subseteq \mathrm{LT}_\tau(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$ *for every elimination ordering* τ *for* Z.

Proof First we show (a). By assumption, there exists a tuple $(f_1,\dots,f_s)$ of polynomials in I which is the reduced σ-Gröbner basis of $\langle f_1,\dots,f_s\rangle$ and such that $z_i = \mathrm{LT}_\sigma(f_i)$ for $i=1,\dots,s$. Then we have $z_i = \mathrm{LT}_\sigma(\mathrm{Lin}_{\mathfrak{M}}(f_i))$ for $i=1,\dots,s$, and thus $\langle Z\rangle \subseteq \mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$.

Claim (b) follows from (a) and the definition of Y, and claim (c) is a consequence of (a) and Proposition 2.3. □

Let us denote the image of $\mathfrak{M}$ in P/I by $\mathfrak{m}$. Recall that the K-vector space $\mathrm{Cot}_{\mathfrak{m}}(R) = \mathfrak{m}/\mathfrak{m}^2$ is called the **cotangent space** of P/I at the origin.

Remark 2.7 As shown in Kreuzer et al. (2022a, Proposition 1.8.b), the canonical map $P_1 \longrightarrow \mathfrak{m}/\mathfrak{m}^2$ induces an isomorphism of K-vector spaces $P_1/\mathrm{Lin}_{\mathfrak{M}}(I) \cong \mathfrak{m}/\mathfrak{m}^2$. In the setting of the proposition, the set of the residue classes of the elements in $X\setminus S_\sigma$ is a K-basis of $P_1/\mathrm{Lin}_{\mathfrak{M}}(I)$. Therefore the residue classes of the indeterminates in $Y = X\setminus S_\sigma$ generate the cotangent space $\mathrm{Cot}_{\mathfrak{m}}(R) \cong \mathfrak{m}/\mathfrak{m}^2$ of P/I at the origin.

When we are only looking for Z-separating re-embeddings of I which are optimal, the above proposition yields the following characterization.

Corollary 2.8 *In the setting of the proposition, assume that* $s = \#Z$ *is equal to the* K*-vector space dimension of* $\mathrm{Lin}_{\mathfrak{M}}(I)$. *Then the following claims hold.*

(a) *The map* $\Phi : P/I \longrightarrow K[Y]/(I\cap K[Y])$ *is an optimal re-embedding of* I.
(b) *We have* $\langle Z\rangle = \mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$, *and hence* $Y = X\setminus S_\sigma$.
(c) *The residue classes of the indeterminates in* $Y = X\setminus Z$ *form a* K*-vector space basis of the cotangent space* $\mathrm{Cot}_{\mathfrak{m}}(R) \cong \mathfrak{m}/\mathfrak{m}^2$ *of* P/I *at the origin.*

Proof Claim (a) is a consequence of Kreuzer et al. (2022a, Corollary 4.2). Let us prove (b). By claim (a) of the proposition, we have $\langle Z\rangle \subseteq \mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$. Since we have the equality $s = \dim_K(\langle Z\rangle_K) = \dim_K(\mathrm{Lin}_{\mathfrak{M}}(I))$, we deduce that Z minimally generates $\mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$.

Finally, let us prove (c). By (b), we have $Z = S_\sigma$. As mentioned in the preceding remark, the set of the residue classes of the elements in $X\setminus S_\sigma$ is a K-basis of $P_1/\mathrm{Lin}_{\mathfrak{M}}(I)$. This implies the claim. □

An important situation in which we obtain an optimal re-embedding is described by the following corollary.

Corollary 2.9 *In the setting of the preceding corollary, the following claims hold.*

(a) *The localization* $P_{\mathfrak{M}}/I_{\mathfrak{M}}$ *is a regular local ring if and only if* $I\cap K[Y] = \{0\}$.

Now assume that these conditions are satisfied.

(b) *The map* $\Phi : P/I \longrightarrow K[Y]$ *is an isomorphism with a polynomial ring and the scheme* $\mathrm{Spec}(P/I)$ *is isomorphic to an affine space.*

(c) *The reduced σ-Gröbner basis of I is of the form $G = \{z_1 - h_1, \ldots, z_s - h_s\}$ with $h_i \in K[Y]$ and it is a minimal set of generators of I.*
(d) *Every Z-separating tuple $F = (f_1, \ldots, f_s)$ is a Gröbner basis of I with respect to any elimination ordering for Z. It is a minimal system of generators of I and a permutable regular sequence in P.*
(e) *The reduced Gröbner bases of I for all elimination orderings for Z coincide.*

Proof Claim (a) follows from Kreuzer et al. (2022b, Proposition 6.7). Parts (b) and (c) are immediate consequences of (a).

Now we show (d). Since F is separating, we have $\langle Z\rangle = \mathrm{LT}_\sigma(\langle F_Z\rangle)$ for every elimination ordering σ for Z. The theorem on the computation of elimination modules (cf. Kreuzer and Robbiano 2000, Theorem 3.4.5) implies that every σ-Gröbner basis of I consists of polynomials with leading terms in $\langle Z\rangle$ and polynomials in $K[Y]$ generating $I \cap K[Y]$. By (a), we have $I \cap K[Y] = \{0\}$, whence it follows that F is in fact a minimal σ-Gröbner basis of I. Consequently, F is a system of generators of I. Moreover, since the leading terms of $f_1, \ldots, f_s$ form a regular sequence, also $F = (f_1, \ldots, f_s)$ is a regular sequence.

Finally, to prove (e), we use the definitions and results of Kreuzer et al. (2022b, Section 5). By Thm. 5.5, the map $\Gamma_Z : \mathrm{GFan}_Z(I) \longrightarrow \mathrm{GFan}(I \cap K[Y])$ is bijective. Since we have $I \cap K[Y] = \{0\}$, we obtain $\mathrm{GFan}(I \cap K[Y]) = \{\emptyset\}$. Therefore $\mathrm{GFan}_Z(I)$, which is not empty, has cardinality one. $\square$

Let us apply this corollary in a concrete case.

Example 2.10 Let $P = \mathbb{Q}[x, y, z, w]$, let $f_1 = w^2 + x - y + 3z$, $f_2 = zw^2 + w^3 + y$, $f_3 = w^3 - xz + yz - 3z^2 + y$, and let $I = \langle f_1, f_2, f_3\rangle$. By substituting y with $-f_2 + y = -zw^2 - w^3$ in f_1, we get $f_1' = zw^2 + w^3 + w^2 + x + 3z$. Then, by substituting y with $-zw^2 - w^3$ and x with $-f_1' + x = -zw^2 - w^3 - w^2 - 3z$ in f_3 we get 0.

Consequently, we have $I = \langle f_1', f_2\rangle$ and (f_1', f_2) is clearly the reduced Gröbner basis of I with respect to every elimination ordering for (x, y). According to the above corollary, we obtain an isomorphism $P/I \cong \mathbb{Q}[z, w]$, and (f_1', f_2) is a permutable regular sequence.

The final example in this section shows that not all optimal embeddings fall into the area of application of the above corollary.

Example 2.11 In $\mathbb{Q}[x, y]$, consider the polynomial $F = 2x^8 + 8x^6y + 12x^4y^2 + 8x^2y^3 + 2y^4 + x$ Then the $\mathbb{Q}$-algebra homomorphism $\alpha : \mathbb{Q}[x, y] \longrightarrow \mathbb{Q}[x]$ defined by $\alpha(x) = -2x^4$and $\alpha(y) = x - 4x^8$ satisfies $\alpha(x^2 + y) = x$ as well as $\mathrm{Ker}(\alpha) = \langle F\rangle$. It follows that $\bar{\alpha} : \mathbb{Q}[x, y]/\langle F\rangle \longrightarrow \mathbb{Q}[x]$ is an optimal re-embedding of $\langle F\rangle$, although F is not x-separating. Unlike the case of Example 2.4, there is no separating term ordering for the ideal $\langle F\rangle$ here at all.

Another example of this type is the famous Koras-Russel cubic threefold whose coordinate ring is $R = K[x, y, z, t]/\langle x + x^2y + z^2 + t^3\rangle$. Using completely different techniques, it was shown that R is not isomorphic to a polynomial ring in three indeterminates (see Makar-Limanov 2005 and Crachiola 2005).

3 Gröbner fans of linear ideals

As before, let K be a field, let $P = K[x_1, \ldots, x_n]$, and let I be an ideal of P which is contained in $\mathfrak{M} = \langle x_1, \ldots, x_n \rangle$. In the preceding section we saw that for the existence of a Z-separating re-embedding of I it is necessary that we have $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I) \rangle)$ for some term ordering σ. Here $\langle \mathrm{Lin}_{\mathfrak{M}}(I) \rangle$ is a **linear ideal**, i.e., an ideal generated by linear polynomials. The possible ideals $\mathrm{LT}_\sigma(\langle \mathrm{Lin}_{\mathfrak{M}}(I) \rangle)$ are classified by the **Gröbner fan** (see Mora and Robbiano 1988) of $\langle \mathrm{Lin}_{\mathfrak{M}}(I) \rangle$. Therefore we study Gröbner fans of linear ideals in this section with a special emphasis on the task of computing them efficiently.

In the following we let $L = (\ell_1, \ldots, \ell_r)$ be a tuple of linear forms in P, and we let $I_L = \langle L \rangle$ be the linear ideal generated by L. For $i = 1, \ldots, r$, we write $\ell_i = a_{i1}x_1 + \cdots + a_{in}x_n$ with $a_{ij} \in K$. Then the matrix $A = (a_{ij}) \in \mathrm{Mat}_{r,n}(K)$ is called the **coefficient matrix** of L. In view of Proposition 2.6, we are interested in the condition $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(I_L)$. It can be rephrased as follows.

Proposition 3.1 *Let $L = (\ell_1, \ldots, \ell_r)$ be a tuple of K-linearly independent linear forms in P, let $I_L = \langle L \rangle$, and let $A = (a_{ij})$ be the coefficient matrix of L. Moreover, let $s \le r$, let $Z = (z_1, \ldots, z_s)$ be a tuple of distinct indeterminates in $X = (x_1, \ldots, x_n)$, and let $Y = X \setminus Z$. Then the following conditions are equivalent.*

(a) *There exists a term ordering σ such that $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(I_L)$.*
(b) *The residue classes of the elements of Y generate the K-vector space $P_1/\langle L \rangle_K$.*
(c) *Let $i_1, \ldots, i_s \in \{1, \ldots, n\}$ be the indices such that $z_j = x_{i_j}$ for $j = 1, \ldots, s$. Then the columns $i_1, \ldots, i_s$ of A are linearly independent.*

Proof To show (a)⇒(b), we first note that $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(I_L)$ implies that the canonical map $K[Y] \cong P/\langle Z \rangle \longrightarrow P/\mathrm{LT}_\sigma(I_L)$ is surjective. Hence the residue classes of the elements of Y generate the K-algebra $P/\mathrm{LT}_\sigma(I_L)$. By Macaulay's Basis Theorem (see Kreuzer and Robbiano 2000, Theorem 1.5.7), it follows that the residue classes of the elements of Y generate the K-algebra P/I_L. We observe that that I_L is generated by linear forms, and therefore P/I_L is isomorphic to a polynomial ring. Thus the residue classes of a tuple of indeterminates Y are a K-algebra system of generators of the ring P/I_L if and only if they are a system of generators of the K-vector space given by its homogeneous component $P_1/(I_L)_1$ of degree one, where $(I_L)_1 = \langle L \rangle_K$.

The assumption in (b) implies that the indeterminates in Z can be expressed as linear combinations of the indeterminates in Y. Consequently, any elimination ordering σ for Z satisfies $\langle Z \rangle \subseteq \mathrm{LT}_\sigma(I_L)$. This proves (b)⇒(a).

Finally, we show that (b) and (c) are equivalent. Notice that both conditions imply $s \le r$. The tuple $Y = X \setminus Z$ is a system of generators of the vector space $P_1/\langle L \rangle_K$ if and only if $\{\ell_1, \ldots, \ell_r\}$ together with Y is a system of generators of P_1. This means that, if we extend A with $n - s$ rows that are unit vectors having their non-zero entries at the positions of the indeterminates in Y, the resulting matrix $\overline{A}$ of size $(r+n-s) \times n$ has the maximal rank n. Now we consider the matrix $A_{(i_1,\ldots,i_s)}$ consisting of columns $i_1, \ldots, i_s$ of A. By renumbering the indeterminates, we may assume that $i_1 = 1, \ldots, i_s = s$, and hence that the extended matrix is upper block triangular of the form

$$\overline{A} \;=\; \begin{pmatrix} A_{(i_1,\ldots,i_s)} & * \\ 0 & I_{n-s} \end{pmatrix}$$

where I_{n-s} is the identity matrix of size $n-s$. Now it is clear that the rows of $\overline{A}$ generate K^n if and only if the rows of $A_{(i_1,\ldots,i_s)}$ generate K^s, and this is equivalent to $A_{(i_1,\ldots,i_s)}$ having maximal rank s. This concludes the proof of the proposition. □

As a special case of the proposition, we get the following characterization of tuples Z which are leading term tuples of a marked reduced Gröbner basis in $\mathrm{GFan}(I_L)$. Recall that a **marked Gröbner basis** of an ideal J is a set of pairs

$$\overline{G} \;=\; \{\,(\mathrm{LT}_\sigma(g_1),\; g_1), \ldots, (\mathrm{LT}_\sigma(g_k),\; g_k)\,\}$$

where σ is a term ordering and $G = \{g_1, \ldots, g_k\}$ is the reduced σ-Gröbner basis of J. The set of all marked reduced Gröbner bases of J is the **Gröbner fan** $\mathrm{GFan}(J)$ of J.

Corollary 3.2 *In the setting of the proposition, assume that $s = r$. Then the following conditions are equivalent.*

(a) *There exists a term ordering σ such that $\langle Z\rangle = \mathrm{LT}_\sigma(I_L)$.*
(b) *The residue classes of the elements of Y are a K-basis of $P_1/\langle L\rangle_K$.*
(c) *Let $i_1, \ldots, i_s \in \{1, \ldots, n\}$ be the indices such that $z_j = x_{i_j}$ for $j = 1, \ldots, s$. Then the columns $i_1, \ldots, i_s$ of A form an invertible matrix of size $s \times s$.*

Our next goal is to construct a bijection between the Gröbner fan of I_L and the non-zero maximal minors of A. The following terminology will prove useful.

Definition 3.3 Let J be an ideal in P.

(a) For a marked reduced Gröbner basis $G = \{(\mathrm{LT}_\sigma(g_1), g_1), \ldots, (\mathrm{LT}_\sigma(g_k), g_k)\}$ of J, we call $\mathrm{LT}_\sigma(G) = \{\mathrm{LT}_\sigma(g_k), \ldots, \mathrm{LT}_\sigma(g_k)\}$ the **leading term set** of G.
(b) The set $\mathrm{LTGFan}(J)$ of all leading term sets of marked reduced Gröbner bases in $\mathrm{GFan}(J)$ is called the **leading term Gröbner fan** of J.

The following lemma provides some information about changing the basis of I_L. As above, by $A_{(i_1,\ldots,i_s)}$ we denote the matrix consisting of columns $i_1, \ldots, i_s$ of a matrix A.

Lemma 3.4 *Let $L = (\ell_1, \ldots, \ell_s)$ be a tuple of K-linearly independent linear forms in P, let $I_L = \langle L\rangle$, and let $A = (a_{ij})$ be the coefficient matrix of L. Moreover, let $L' = (\ell'_1, \ldots, \ell'_r)$ be a further tuple of linear forms in I_L, and let $A' \in \mathrm{Mat}_{r,n}(K)$ be its coefficient matrix.*

(a) *The tuple L' is a minimal system of generators of I_L if and only if $r = s$ and there exists a matrix $U \in \mathrm{GL}_s(K)$ such that $A' = U \cdot A$.*
(b) *A set of pairs $\{(x_{i_1}, \ell'_1) \ldots, (x_{i_r}, \ell'_r)\}$, where $1 \le i_1 < \cdots < i_r \le n$, is a marked reduced Gröbner basis of I_L, if and only if $r = s$, the matrix $A_{(i_1,\ldots,i_s)}$ is invertible, and $A' = (A_{(i_1,\ldots,i_s)})^{-1} \cdot A$.*

Proof Claim (a) follows from the fact that every tuple of minimal generators of I_L is also a basis of the K-vector space $(I_L)_1$.

To prove (b) we observe that a minimal Gröbner basis of a linear ideal is also a minimal set of generators of I_L. This yields $r = s$. Moreover, it is reduced if and only if the submatrix $A'_{(i_1,\dots,i_s)}$ of A' is the identity matrix, and hence the conclusion follows from (a). $\square$

This lemma can also be interpreted in terms of the Plücker embedding of the Graßmannian $\mathrm{Gr}(s, n)$. Now we are ready to present the key result for computing $\mathrm{GFan}(I_L)$.

Theorem 3.5 *(The Gröbner Fan of a Linear Ideal) Let $L = (\ell_1, \dots, \ell_s)$ be a tuple of K-linearly independent linear forms in P, let $I_L = \langle L \rangle$, and let $A = (a_{ij})$ be the coefficient matrix of L. Furthermore, let M be the set of tuples $(i_1, \dots, i_s)$ such that $1 \le i_1 < \cdots < i_s \le n$ and such that the corresponding maximal minor of the matrix $A = (a_{ij})$ is non-zero.*

(a) *The map $\varphi : \mathrm{LTGFan}(I_L) \longrightarrow M$ given by $\varphi(Z) = (i_1, \dots, i_s)$ for a tuple $Z = (x_{i_1}, \dots, x_{i_s}) \in \mathrm{LTGFan}(I_L)$ with $1 \le i_1 < \cdots < i_s \le n$ is well-defined and bijective.*

(b) *The map $\psi : \mathrm{GFan}(I_L) \longrightarrow M$ given by $\psi(G) = \varphi(\mathrm{LT}_\sigma(G))$ for every $G \in \mathrm{GFan}(I_L)$ is well-defined and bijective.*

Proof First we prove (a). To begin with, let us check that φ is well-defined. For an element $G \in \mathrm{GFan}(I_L)$, the tuple $Z = \mathrm{LT}_\sigma(G)$ satisfies $\langle Z \rangle = \mathrm{LT}_\sigma(I_L)$. Let $Z = (x_{i_1}, \dots, x_{i_s})$ with $1 \le i_1 < \cdots < i_s \le n$. Then Corollary 3.2.c shows $\det(A_{(i_1,\dots,i_s)}) \ne 0$, and therefore φ is well-defined.

Since the map φ is clearly injective, we still need to show that it is surjective. Given $(i_1, \dots, i_s) \in M$, part (b) of the lemma implies that $(L')^{\mathrm{tr}} = A^{-1}_{(i_1,\dots,i_s)} \cdot L^{\mathrm{tr}}$ is a reduced Gröbner basis of I_L with leading term tuple $Z = (x_{i_1}, \dots, x_{i_s})$. Therefore we have $\varphi(Z) = (i_1, \dots, i_s)$, and this proves the desired surjectivity.

To show (b), we note that the map ψ is clearly well-defined and injective. By the definition of $\mathrm{LTGFan}(I_L)$, it is also surjective. $\square$

Based on this theorem, we can compute the Gröbner fan of a linear ideal as follows.

Corollary 3.6 *(Computing the Gröbner Fan of a Linear Ideal) Let $I_L = \langle \ell_1, \dots, \ell_s \rangle$ be an ideal in P generated by linearly independent linear forms $\ell_1, \dots, \ell_s \in P_1$. Then we can compute $\mathrm{GFan}(I_L)$ as follows.*

(1) *Let A be the coefficient matrix of L, and let $S = \emptyset$.*
(2) *For every tuple $(i_1, \dots, i_s) \in M$, compute the maximal minor $\det(A_{(i_1,\dots,i_s)})$ of A.*
(3) *If $\det(A_{(i_1,\dots,i_s)}) \ne 0$, compute the vector $(L')^{\mathrm{tr}} = (A_{(i_1,\dots,i_s)})^{-1} \cdot L^{\mathrm{tr}}$ whose tuple of leading terms is $Z = (x_{i_1}, \dots, x_{i_s})$. Append the corresponding marked reduced Gröbner basis of I_L to S. Continue with the next tuple in Step (2).*
(4) *Return the set $S = \mathrm{GFan}(I_L)$.*

Of course, depending on the numbers n and s, computing $\binom{n}{s}$ determinants could be quite costly. The following remark may help us out.

Remark 3.7 Let $I_L = \langle \ell_1, \ldots, \ell_s \rangle$ be an ideal in P generated by linearly independent linear forms $\ell_1, \ldots, \ell_s \in P_1$ as above.

An alternative way of viewing the task to compute $\mathrm{GFan}(I_L)$ is obtained by applying Corollary 3.2. The complements of the leading term sets of reduced Gröbner bases of I_L correspond uniquely to sets of indeterminates whose residue classes form a K-basis of $P_1/(I_L)_1$. All sets of indeterminates whose residue classes are linearly independent in $P_1/(I_L)_1$ are the independent sets of a linear matroid, and maximal such sets are the bases of the matroid. The task of computing the bases of a linear matroid has been studied intensively, and many algorithms are known, see for instance the reverse search technique of D. Avis and K. Fukuda (cf. Avis and Fukuda 1996).

Let us calculate the Gröbner fan of an explicit linear ideal.

Example 3.8 Let $P = \mathbb{Q}[x, y, z, w]$, let $\ell_1 = x + y - z + 4w$, $\ell_2 = x - y - z$, and let $I_L = \langle \ell_1, \ell_2 \rangle$. The set $\{\ell_1, \ell_2\}$ is a set of minimal generators of I_L and its coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & -1 & -1 & 0 \end{pmatrix}.$$

One 2×2-submatrix is singular. The others are

$$A_{12} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ A_{14} = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix},\ A_{23} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix},\ A_{24} = \begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix},\ A_{34} = \begin{pmatrix} -1 & 4 \\ -1 & 0 \end{pmatrix}.$$

Multiplying their inverses by A we get the matrices

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1/2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1/2 & 0 & -1/2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{pmatrix}.$$

They correspond to the following marked reduced Gröbner bases of I_L which form the Gröbner fan of I_L:

$$\begin{aligned} &\{(x, x - z + 2w), (y, y + 2w)\}, \quad \{(x, x - y - z), (w, w + \tfrac{1}{2}y)\}, \\ &\{(y, y + 2w), (z, z - x - 2w)\}, \quad \{(y, y - x + z), (w, w + \tfrac{1}{2}x - \tfrac{1}{2}z\}, \\ &\{(z, z - x + y), (w, w + \tfrac{1}{2}y)\}. \end{aligned}$$

4 Finding Z-separating re-embeddings

In this section we show how to apply the Gröbner fan of the linear part of an ideal to find tuples Z which are good candidates for providing Z-separating re-embeddings of the ideal. In Kreuzer et al. (2022a) we gave some answers to this question which use the computation of the Gröbner fan of the ideal I itself. Unfortunately, the computation of $\mathrm{GFan}(I)$ may be infeasible for large examples. The Gröbner fan of the ideal generated by the linear part of I is in general much smaller and thus provides a better set of candidate tuples Z.

Using the definitions and notation introduced in the preceding sections, the following algorithm uses the Gröbner fan of the linear part of I in order to find Z-separating re-embeddings.

Algorithm 4.1 *(Z-Separating Re-embeddings via GFan ($\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle$)) Let $I \subseteq \mathfrak{M}$ be an ideal of P, and let $s \le \dim_K(\mathrm{Lin}_{\mathfrak{M}}(I))$. Consider the following sequence of instructions.*

(1) *Using Corollary 3.6, compute* $\mathrm{GFan}(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$.
(2) *Form the set S of all tuples $Z = (z_1, \dots, z_s)$ such that there is marked reduced Gröbner basis $\overline{G}$ in* $\mathrm{GFan}(\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle)$ *for which $z_1, \dots, z_s$ are among the marked terms.*
(3) *If $S = \emptyset$, return* `"No re-embedding found"`. *While $S \neq \emptyset$, perform the following steps.*
(4) *Choose a tuple $Z = (z_1, \dots, z_s) \in S$ and remove it from S.*
(5) *Using Remark 2.5, check whether the ideal I is Z-separating. If it is, return Z and stop. Otherwise, continue with Step (3).*

This is an algorithm which, if successful, finds a tuple of distinct indeterminates $Z = (z_1, \dots, z_s)$ in X such that I is Z-separating.

Moreover, if $s = \dim_K(\mathrm{Lin}_{\mathfrak{M}}(I))$ and the algorithm is successful then the output tuple Z defines an optimal re-embedding of I.

Proof Every tuple Z such that there exists a Z-separating re-embedding of I is contained in the tuple of leading terms of a marked reduced Gröbner basis of $\mathrm{Lin}_{\mathfrak{M}}(I)$ by Proposition 2.6.c. The set of all possible such tuples Z is computed in Steps (1) and (2). If the loop in Steps (3)-(5) finds a tuple Z such that I is Z-separating, we are done.

In addition, if $s = \dim_K(\mathrm{Lin}_{\mathfrak{M}}(I))$ and the algorithm is successful, then Kreuzer et al. (2022a, Corollary 4.2), shows that the Z-separating re-embedding of I is optimal. $\square$

If we are looking for optimal re-embeddings and use the method of Remark 2.5.a to perform Step (5), then Algorithm 4.1 is able to certify that no optimal Z-separating re-embedding of I exists. However, we may have to compute some huge Gröbner bases. Moreover, the next remark points out some further limitations.

Remark 4.2 Notice that Algorithm 4.1 provides only a sufficient condition for detecting optimal re-embeddings of I. On one side, it can happen that an optimal re-embedding is obtained using a subset of generators of a leading term ideal of $\mathrm{Lin}_{\mathfrak{M}}(I)$ (see Kreuzer et al. 2022b, Example 6.6). On the other side, it can happen that an optimal re-embedding cannot be achieved by a separating re-embedding, as shown in Example 2.11.

The following example illustrates Algorithm 4.1 at work.

Example 4.3 Let $P = \mathbb{Q}[x, y, z, w]$, let $F = (f_1, f_2, f_3)$, where $f_1 = x - y - w^2$, $f_2 = x + y - z^2$, and $f_3 = z + w + z^3$, and let $I = \langle f_1, f_2, f_3\rangle$.

(1) We obtain $\mathrm{Lin}_{\mathfrak{M}}(I) = \langle z + w, x, y\rangle_K$ and the methods explained below return the two marked reduced Gröbner bases $\{(x, x), (y, y), (z, z + w)\}$ and $\{(x, x), (y, y), (w, w + z)\}$.
(2) We get $S = \{(x, y, z), (x, y, w)\}$.
(4) We pick $Z = (x, y, z)$ and delete it from S.
(5) We construct an elimination ordering σ for Z and find that the minimal set of generators of $\mathrm{Lin}_{\mathfrak{M}}(\mathrm{LT}_\sigma(I))$ is $L = \{x\}$. Therefore $L \neq \{x, y, z\}$ and continue with the next iteration.
(4) Next we let $Z = (x, y, w)$ and let $S = \emptyset$.
(5) We construct an elimination ordering σ for Z and compute the minimal set of generators L of $\mathrm{Lin}_{\mathfrak{M}}(\mathrm{LT}_\sigma(I))$. Since $L = \{x, y, w\}$, we return $Z = (x, y, w)$ and stop.

To get Z-separating polynomials, it suffices to replace f_2 with $f_2' = f_2 - f_1$.

To find the actual polynomials defining the optimal re-embedding, we compute the reduced σ-Gröbner basis of I. It is

$$\left(x - \tfrac{1}{2}z^6 - z^4 - z^2,\ y + \tfrac{1}{2}z^6 + z^4,\ w + z^3 + z\right).$$

This tuple gives rise to a $\mathbb{Q}$-algebra isomorphism $P/I \cong \mathbb{Q}[z]$ via $x \mapsto \frac{1}{2}z^6 + z^4 + z^2$, $y \mapsto -\frac{1}{2}z^6 - z^4$, and $w \mapsto -z^3 - z$.

5 Cotangent equivalence classes

The task to compute the Gröbner fan of $\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle$ in Algorithm 4.1 can be simplified when $\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle$ is a binomial linear ideal. Recall that an ideal J in $P = K[x_1, \dots, x_n]$ is called a **binomial ideal** if it is generated by polynomials containing at most two terms in their support.

As before, we let $I = \langle f_1, \dots, f_r\rangle$ be an ideal in P with $f_i \in \mathfrak{M}$. Letting $\ell_i = \mathrm{Lin}_{\mathfrak{M}}(f_i)$ for $i = 1, \dots, r$ and $L = (\ell_1, \dots, \ell_r)$, the linear part of I is $\mathrm{Lin}_{\mathfrak{M}}(I) = \langle L\rangle_K$ and it generates the ideal $I_L = \langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle$.

In the following we assume that the linear forms ℓ_i are **binomials**, i.e., for $i = 1, \dots, r$, we have $\ell_i = a_i\, x_{i_1} + b_i\, x_{i_2}$ with $a_i, b_i \in K$ and $i_1, i_2 \in \{1, \dots, n\}$. In this case the ideal I_L is called a **binomial linear ideal**.

Recall that, by Remark 2.7 and Corollary 2.8.c, we can detect whether an ideal I is Z-separating by looking at the residue classes of the entries of $Y = X \setminus Z$ in the cotangent space $\mathrm{Cot}_{\mathfrak{m}}(R) = \mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the image of $\mathfrak{M} = \langle x_1, \dots, x_n\rangle$ in $R = P/I$. Moreover, we have $\mathrm{Cot}_{\mathfrak{m}}(R) \cong P_1/\langle L\rangle_K$. This point of view leads us to the following definition.

Definition 5.1 For every indeterminate $x_i \in X$, let $\bar{x}_i$ denote its residue class in the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ of $R = P/I$ at the origin.

(a) The relation $\sim$ on X defined by $x_i \sim x_j \Leftrightarrow \langle \bar{x}_i\rangle_K = \langle \bar{x}_j\rangle_K$ is an equivalence relation called **cotangent equivalence**.

(b) An indeterminate $x_i \in X$ is called **trivial** if $\bar{x}_i = 0$. The trivial indeterminates form the **trivial cotangent equivalence class** in X.
(c) A non-trivial indeterminate $x_i \in X$ is called **basic** if its cotangent equivalence class consists only of x_i. In this case, the cotangent equivalence class $\{x_i\}$ is also called **basic**.
(d) A non-trivial indeterminate $x_i \in X$ is called **proper** if its cotangent equivalence class contains at least two elements. In this case, the cotangent equivalence class of x_i is also called **proper**.

The meaning of these notions will become clear in the next theorems. First we need a lemma which provides further information about the above definition.

Lemma 5.2 *Let us assume to be in the above setting.*

(a) *The union U of the supports of the elements in a minimal set of generators of the ideal I_L does not depend on the choice of a minimal set of generators.*
(b) *The set of basic indeterminates is $X \setminus U$.*
(c) *The union of the sets of trivial and proper indeterminates is U.*

Proof To prove (a) we note that any minimal set of generators of the ideal $\langle \mathrm{Lin}_{\mathfrak{M}}(I)\rangle$ is also a minimal set of generators of the K-vector space $\mathrm{Lin}_{\mathfrak{M}}(I)$. Let A and B be two such sets. Since every linear form ℓ in A is a linear combination of linear forms in B, each indeterminate in $\mathrm{Supp}(\ell)$ is in the support of some linear form in B. By interchanging the roles of A and B, the conclusion follows.

Since $\sim$ is an equivalence relation on X, to prove claims (b) and (c) it suffices to show that basic indeterminates are not in U, while trivial and proper indeterminates are in U. Firstly, let x_i be a basic indeterminate. For a contradiction, assume that $x_i \in U$. From $x_i \in U$ and the fact that x_i is the only element in its equivalence class, we deduce that there is a polynomial in I of the form $x_i + q$ with $q \in \mathfrak{M}^2$. Hence we get $\bar{x}_i = 0$, a contradiction to the fact that x_i is basic. Secondly, let x_i be trivial. Then there is a polynomial in I of the form $x_i + q$ with $q \in \mathfrak{M}^2$, and hence we get $x_i \in U$. Thirdly, let x_i be proper. Then there exist another indeterminate x_j and a polynomial in I of the form $ax_i + bx_j + q$ with $a, b \in K\setminus\{0\}$ and $q \in \mathfrak{M}^2$. Hence we get $x_i \in U$. $\square$

Our next step is to order the indeterminates in the cotangent equivalence classes using a term ordering. The following definition will come in handy.

Definition 5.3 Let $E = \{x_{i_1}, \ldots, x_{i_p}\}$ be a proper equivalence class in X, and let σ be a term ordering on $\mathbb{T}^n$ with $x_{i_1} >_\sigma \cdots >_\sigma x_{i_p}$. Then the set $E \setminus \{x_{i_p}\}$ is called the σ**-leading set** of E and denoted by E^σ.

Notation 5.4 Let σ be a term ordering on $\mathbb{T}^n$. In accordance with the notation introduced in Proposition 2.6, the unique minimal set of indeterminates generating the ideal $\mathrm{LT}_\sigma(I_L)$ will be denoted by S_σ.

In the following theorem we give an explicit representation of S_σ for every term ordering σ on $\mathbb{T}^n$ and show the importance of σ-leading sets. This implies a description of $\mathrm{LTGFan}(I_L)$ which has several advantages compared to the general description given in Theorem 3.5. For instance, it does not have to deal with huge matrices.

Theorem 5.5 *Let E_0 be the trivial equivalence class, and let $E_1, \dots, E_q$ be the proper equivalence classes in X. Let $I_L = \langle \mathrm{Lin}_{\mathfrak{M}}(I) \rangle$ be the ideal generated by the linear parts of the polynomials in I.*

(a) *Let σ be a term ordering on $\mathbb{T}^n$. Then we have $S_\sigma = E_0 \cup E_1^\sigma \cup \cdots \cup E_q^\sigma$, and hence $\#S_\sigma = \#E_0 + \sum_{i=1}^q \#E_i - q$.*
(b) *For $i = 1, \dots, q$, let E_i^* be a set obtained from E_i by deleting one of its elements. Then there exists a term ordering σ on $\mathbb{T}^n$ such that we have $S_\sigma = E_0 \cup E_1^* \cup \cdots \cup E_q^*$.*
(c) *Let Σ be the set of all sets of the form $E_0 \cup E_1^* \cup \cdots \cup E_q^*$, where E_i^* is obtained from the set E_i by deleting one of its elements. Then the map $\varphi : \mathrm{LTGFan}(I_L) \longrightarrow \Sigma$ given by $\varphi(S_\sigma) = E_0 \cup E_1^\sigma \cup \cdots \cup E_q^\sigma$ is well-defined and bijective.*
(d) *We have $\#\,\mathrm{LTGFan}(I_L) = \#\,\mathrm{GFan}(I_L) = \prod_{i=1}^{q} \#E_i$.*

Proof To prove claim (a) we observe that the inclusion $E_0 \subseteq S_\sigma$ follows from $E_0 \subseteq \mathrm{Lin}_{\mathfrak{M}}(I)$, and that the inclusion $S_\sigma \subseteq E_0 \cup E_1 \cup \cdots \cup E_q$ follows from Lemma 5.2.c.

For $k \in \{1, \dots, q\}$, we write the proper equivalence class $E_k = \{x_{i_1}, \dots, x_{i_p}\}$ such that $x_{i_1} >_\sigma \cdots >_\sigma x_{i_p}$. Using the definition of a proper equivalence class, it follows that $x_{i_m} - x_{i_p} \in \mathrm{Lin}_{\mathfrak{M}}(I)$, so that $\mathrm{LT}_\sigma(x_{i_m} - x_{i_p}) = x_{i_m}$ for every $m \in \{1, \dots, p-1\}$. Hence we have proved

$$E_0 \cup E_1^\sigma \cup \cdots \cup E_q^\sigma \;\subseteq\; S_\sigma \;\subseteq\; E_0 \cup E_1 \cup \cdots \cup E_q.$$

As the σ-smallest element in each proper equivalence class does not belong to S_σ, claim (a) follows.

Claim (b) follows from (a) if we show that there exists a term ordering σ such that $E_i^\sigma = E_i^*$ for $i = 1, \dots, q$. By definition, the sets E_i are pairwise disjoint. Consequently, a term ordering which solves the problem can be chosen as a block term ordering, and hence it suffices to consider the case $q = 1$. So, let $E_1 = \{x_{i_1}, \dots, x_{i_p}\}$. W.l.o.g. assume that $E_1^* = \{x_{i_2}, \dots, x_{i_p}\}$. As we observed before, we have $x_{i_k} - x_{i_1} \in \mathrm{Lin}_{\mathfrak{M}}(I)$ for $k = 2, \dots, p$. To finish the proof it suffices to take a term ordering σ such that $x_{i_k} >_\sigma x_{i_1}$ for $k = 2, \dots, p$.

Claim (c) follows immediately from (b). To prove claim (d) we note that the first equality is obvious. To show $\#\,\mathrm{LTGFan}(I_L) = \prod_{i=1}^q \#E_i$, it suffices to deduce from (b) that the number of the leading term ideals of I_L equals the number of q-tuples of indeterminates x_i, exactly one chosen in each proper equivalence class. $\square$

Now we are ready to classify the indeterminates in X which can be used for a Z-separating re-embedding of I as follows.

Theorem 5.6 *Let Z be a tuple of indeterminates from X such that there exists a Z-separating re-embedding of I, and let $Y = X \setminus Z$.*

(a) *The basic indeterminates of X are contained in Y.*
(b) *Each proper equivalence class in X contains at least one element of Y.*
(c) *If $\#Z = \dim_K(\mathrm{Lin}_{\mathfrak{M}}(I))$, then the Z-separating re-embedding of I is optimal, the trivial indeterminates of X are contained in Z, and each proper equivalence class in X contains exactly one element of Y.*

Proof To prove (a), let σ be a Z-separating term ordering for I, let S_σ be the minimal set of indeterminates generating $\mathrm{LT}_\sigma(I_L)$, and let U be the union of the supports of the elements in a minimal set of generators of I_L. Note that Proposition 2.6.c implies $Y \supseteq X \setminus S_\sigma$. From $S_\sigma \subseteq X$ we deduce the inclusion $Y \supseteq X \setminus U$, and thus the claim follows from Lemma 5.2.b.

Claim (b) follows from the mentioned relation $Y \supseteq X \setminus S_\sigma$ and Theorem 5.5.a.

Finally, we prove (c). If $\#(Z) = \dim_K(\mathrm{Lin}_{\mathfrak{M}}(I))$ then Corollary 2.8.a implies that the Z-separating re-embedding of I is optimal. Moreover, Corollary 2.8.b implies that $Z = S_\sigma$, and hence the claim follows from Theorem 5.5.a. □

Based on the preceding results and on Algorithm 3.8 in Kreuzer et al. (2020) for computing the cotangent equivalence classes, we can now check effectively whether a given ideal I admits a Z-separating embedding. Notice that we are excluding some trivial cases (namely $n = 1$ and $I = \mathfrak{M}$) in order to be able to apply Kreuzer et al. (2020, Algorithm 3.8), but these cases can be dealt with easily by a direct computation.

Algorithm 5.7 *(Z-sep. Re-embeddings Via Cotangent Equivalence) Let $I \subsetneq \mathfrak{M}$ be an ideal in $P = K[x_1, \ldots, x_n]$, where $n \geq 2$, and let $X = (x_1, \ldots, x_n)$. Consider the following sequence of instructions.*

(1) *Compute the trivial cotangent equivalence class E_0 and also the proper cotangent equivalence classes $E_1, \ldots, E_q$.*
(2) *Let $S = \emptyset$.*
(3) *Turn each set $Z_0 \cup Z_1 \cup \cdots \cup Z_q$ such that $Z_0 \subseteq E_0$ and $Z_i \subsetneq E_i$ for $i = 1, \ldots, q$ into a tuple Z and perform the following steps.*
(4) *Using Remark 2.5, check whether the ideal I is Z-separating. If it is, append Z to S.*
(5) *Continue with Step (3) using the next tuple Z until all tuples have been dealt with. Then return S and stop.*

Then the following two claims hold.

(a) *This is an algorithm which computes the set S of all tuples Z of distinct indeterminates in X such that there exists a Z-separating re-embedding of I.*
(b) *Assume that Step (3) is replaced by the following step.*

    (3') *Turn each set $E_0 \cup E_1^* \cup \cdots \cup E_q^*$, where E_i^* is obtained from E_i by deleting one element, into a tuple Z and perform the following steps.*

    Then the result is an algorithm which computes the set S of all tuples Z of distinct indeterminates in X such that there exists an optimal Z-separating re-embedding of I.

Proof Both claims follow from Theorem 5.5 and Theorem 5.6. □

For an example to illustrate this algorithm, we refer the reader to the next section.

6 Application to border basis schemes

In this section we apply the methods developed above to the ideals defining border basis schemes. These affine schemes are moduli spaces of 0-dimensional ideals which are canonically embedded into very high dimensional affine spaces. To study them carefully, it is imperative to re-embed them into lower dimensional affine spaces. For instance, one important question is whether a given border basis scheme is an **affine cell**, i.e., isomorphic to an affine space. As we shall recall below, the natural generators of the defining ideals of border basis schemes have binomial linear parts, so that the theory developed in the preceding section is perfectly suited to re-embed border basis schemes. In fact, we also provide some information complementing Theorem 5.6 in this situation.

In the following we assume that the reader has a basic knowledge of border basis theory, e.g., to the extent it is covered in Kreuzer and Robbiano (2005, Section 6.4). Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal of terms in $\mathbb{T}^n$, and let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ be its border. By replacing the coefficients of an $\mathcal{O}$-border prebasis with indeterminates c_{ij}, we obtain the **generic $\mathcal{O}$-border prebasis** $G = \{g_1, \ldots, g_\nu\}$, where $g_j = b_j - \sum_{i=1}^{\mu} c_{ij}\, t_i$ for $i = 1, \ldots, \mu$ and $j = 1, \ldots, \nu$. Furthermore, let C be the set of indeterminates $C = \{c_{ij} \mid i = 1, \ldots, \mu;\ \ j = 1, \ldots, \nu\}$. The ideal $I(\mathbb{B}_\mathcal{O})$ in $K[C]$ defining the border basis scheme $\mathbb{B}_\mathcal{O}$ can be constructed in several ways (see Kreuzer and Robbiano 2008, 2011; Kreuzer et al. 2020):

(1) As in Kreuzer and Robbiano (2008, Definition 3.1), construct the **generic multiplication matrices** $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathrm{Mat}_\mu(K[C])$ and let $I(\mathbb{B}_\mathcal{O})$ be the ideal in $K[C]$ generated by all entries of all commutators $\mathcal{A}_i\, \mathcal{A}_j - \mathcal{A}_j\, \mathcal{A}_i$ with $1 \le i < j \le n$.
(2) Construct the set of next-door generators $\mathrm{ND}_\mathcal{O}$ and the set of across-the-rim generators $\mathrm{AR}_\mathcal{O}$ of $I(\mathbb{B}_\mathcal{O})$ and take the union $\mathrm{ND}_\mathcal{O} \cup \mathrm{AR}_\mathcal{O}$ (see below).

For us, the most important properties of these sets of polynomials $\{f_1, \ldots, f_r\}$ are that f_i consists for $i = 1, \ldots, r$ of a linear and a quadratic part. In the following we describe these homogeneous components in more detail. We begin by making the construction in (2) explicit.

Definition 6.1 (*Neighbour Generators*) Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be the generic multiplication matrices, and for $j = 1, \ldots, \mu$ let $c_j = (c_{1j}, \ldots, c_{\mu j})^{\mathrm{tr}}$ be the j-th column of (c_{ij}).

(a) Let $j, j' \in \{1, \ldots, \nu\}$ be such that $b_j = x_\ell b_{j'}$ for some $\ell \in \{1, \ldots, n\}$. Then $b_j, b_{j'}$ are called **next-door neighbours** and the tuple of polynomials $(c_j - \mathcal{A}_\ell c_{j'})^{\mathrm{tr}}$ is denoted by $\mathrm{ND}(j, j')$.
(b) The union of all entries of the tuples $\mathrm{ND}(j, j')$ is called the set of **next-door generators** of $I(\mathbb{B}_\mathcal{O})$ and is denoted by $\mathrm{ND}_\mathcal{O}$.
(c) Let $j, j' \in \{1, \ldots, \nu\}$ be such that $b_j = x_\ell b_m$ and $b_{j'} = x_k b_m$ for some $m \in \{1, \ldots, \nu\}$. Then $b_j, b_{j'}$ are called **across-the-corner neighbours**.
(d) Let $j, j' \in \{1, \ldots, \nu\}$ be such that $b_j = x_\ell t_m$ and $b_{j'} = x_k t_m$ for some $m \in \{1, \ldots, \mu\}$. Then $b_j, b_{j'}$ are called **across-the-rim neighbours** and the tuple of polynomials $(\mathcal{A}_k c_j - \mathcal{A}_\ell c_{j'})^{\mathrm{tr}}$ is denoted by $\mathrm{AR}(j, j')$.
(e) The union of all entries of the tuples $\mathrm{AR}(j, j')$ is called the set of **across-the-rim generators** of $I(\mathbb{B}_\mathcal{O})$ and is denoted by $\mathrm{AR}_\mathcal{O}$.
(f) The polynomials in $\mathrm{ND}_\mathcal{O} \cup \mathrm{AR}_\mathcal{O}$ are called the **neighbour generators** of $I(\mathbb{B}_\mathcal{O})$.

In Kreuzer and Robbiano (2008, Proposition 4.1), it is shown that the polynomials corresponding to across-the-corner neighbours are not necessary to generate $I(\mathbb{B}_\mathcal{O})$

and that the neighbour generators are precisely the non-trivial entries of the commutators $\mathcal{A}_i\,\mathcal{A}_j - \mathcal{A}_j\,\mathcal{A}_i$.

An important property of the neighbour generators is that they are homogeneous with respect to the following multigrading. Recall that the logarithm of a term $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is defined by $\log(t) = (\alpha_1, \dots, \alpha_n)$.

Definition 6.2 The $\mathbb{Z}^n$-grading on $K[C]$ defined by $\deg_W(c_{ij}) = \log(b_j) - \log(t_i)$ for $i = 1, \dots, \mu$ and $j = 1, \dots, \nu$ is called the **arrow grading**.

As mentioned above, the neighbour generators of $I(\mathbb{B}_{\mathcal{O}})$ have (standard) degree two and no constant term. Their linear parts can be described in detail as follows.

Proposition 6.3 *(Linear Parts of Neighbour Polynomials) Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$.*

(a) *Let $j, j' \in \{1, \dots, \nu\}$ be such that $b_j, b_{j'}$ are next-door neighbours, i.e., such that $b_j = x_\ell b_{j'}$, and let $i \in \{1, \dots, \mu\}$. Then the linear part of the corresponding next-door generator $ND(j, j')_i$ is (up to sign) given by*

$$\begin{cases} c_{ij} - c_{i'j'} & \text{if } x_\ell \text{ divides } t_i, \\ c_{ij} & \text{otherwise.} \end{cases}$$

The polynomial $f = \mathrm{ND}(j, j')_i$ is homogeneous with respect to the arrow degree with $\deg_W(f) = \deg_W(c_{ij})$.

(b) *Let $j, j' \in \{1, \dots, \nu\}$ be such that $b_j, b_{j'}$ are across-the-rim neighbours, i.e., such that $b_j = x_\ell t_{m'}$ and $b_{j'} = x_k t_{m'}$ for some $m' \in \{1, \dots, \mu\}$. For $m \in \{1, \dots, \mu\}$, the non-zero linear part of the corresponding across-the-rim generator $\mathrm{AR}(j, j')_m$ is (up to sign) given by*

$$\begin{cases} c_{ij} - c_{i'j'} & \text{if } t_m = x_k t_i = x_\ell t_{i'}, \\ c_{ij} & \text{if } t_m = x_k t_i, \text{ but } x_\ell \text{ does not divide } t_m, \\ c_{i'j'} & \text{if } t_m = x_\ell t_{i'}, \text{ but } x_k \text{ does not divide } t_m. \end{cases}$$

The polynomial $g = \mathrm{AR}(j, j')_m$ is homogeneous with respect to the arrow degree with $\deg_W(g) = \deg_W(c_{mj}) + e_k = \deg(c_{mj'}) + e_\ell$.

Proof The claims for the linear parts are shown in Kreuzer et al. (2020, Corollary 2.8). The claims for the arrow degree follow from Kreuzer et al. (2020, Lemma 3.4), and the description of the linear parts. □

To describe the quadratic parts of the neighbour generators in detail, the following concepts are convenient.

Definition 6.4 Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$.

(a) The **rim** $\mathcal{O}^\nabla$ of $\mathcal{O}$ consists of all terms t_i such that $x_k t_i \in \partial\mathcal{O}$ for some $k \in \{1, \dots, n\}$. The indeterminate c_{ij} is called a **rim indeterminate** if $t_i \in \mathcal{O}^\nabla$, and the set of all rim indeterminates is denoted by C^∇.

(b) The **interior** of $\mathcal{O}$ is $\mathcal{O}^\circ = \mathcal{O} \setminus \mathcal{O}^\nabla$. The indeterminate c_{ij} is called an **interior indeterminate** if $t_i \in \mathcal{O}^\circ$, and the set of all interior indeterminates is denoted by C°.

Clearly, we have a disjoint union $C = C^\nabla \mathbin{\dot\cup} C^\circ$. One more definition, and we are ready to go. The following notion was introduced in Huibregtse (2002, Section 4.1).

Definition 6.5 Let $j \in \{1, \dots, \nu\}$, and let $\ell \in \{1, \dots, n\}$. Then the border term b_j is called x_ℓ-**exposed** if it is of the form $b_j = x_\ell\, t_i$ with $i \in \{1, \dots, \mu\}$. In this case we also say that t_i x_ℓ-**exposes** the border term b_j.

Finally we are ready to describe the homogeneous components of (standard) degree two of the neighbour generators of $I(\mathbb{B}_\mathcal{O})$.

Proposition 6.6 *(Quadratic Parts of Neighbour Generators) Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$.*

(a) *Let $j, j' \in \{1, \dots, \nu\}$ be such that $b_j, b_{j'}$ are next-door neighbours with $b_j = x_\ell b_{j'}$, and let $i \in \{1, \dots, \mu\}$. Assume that $b_{\lambda_1}, \dots, b_{\lambda_s} \in \partial\mathcal{O}$ are the x_ℓ-exposed border terms, and let $b_{\lambda_p} = x_\ell\, t_{\varrho_p}$ for $p = 1, \dots, s$. Then the quadratic terms in the support of $\mathrm{ND}(j, j')_i$ are the products $c_{i\lambda_p}\, c_{\varrho_p j'}$ with $p = 1, \dots, s$.*
In particular, all terms in the quadratic part are of the form $c_{i\lambda} c_{\varrho j'}$ with $\lambda \in \{1, \dots, \nu\}$ and a rim indeterminate $c_{\varrho j'}$.
(b) *Let $j, j' \in \{1, \dots, \nu\}$ be such that $b_j, b_{j'}$ are across-the-rim neighbours with $b_j = x_\ell t_{m'}$ and $b_{j'} = x_k t_{m'}$ for some $m' \in \{1, \dots, \mu\}$. For every $m \in \{1, \dots, \mu\}$, the quadratic terms in the support of $\mathrm{AR}(j, j')_m$ are of the following types.*

  (1) *Let $b_{\kappa_1}, \dots, b_{\kappa_s} \in \partial\mathcal{O}$ be the x_k-exposed border terms, and let us write $b_{\kappa_p} = x_\ell\, t_{\varrho_p}$ for $p = 1, \dots, s$. Then the terms $c_{m\kappa_p}\, c_{\varrho_p j}$ may appear in the support of $\mathrm{AR}(j, j')_m$.*
  (2) *Let $b_{\lambda_1}, \dots, b_{\lambda_u} \in \partial\mathcal{O}$ be the x_ℓ-exposed border terms, and let us write $b_{\lambda_q} = x_\ell t_{\sigma_q}$ for $q = 1, \dots, u$. Then the terms $c_{m\lambda_q}\, c_{\sigma_q j'}$ may appear in the support of $\mathrm{AR}(j, j')_m$.*

  In particular, all terms in the quadratic part are of the form $c_{m\lambda} c_{\varrho j}$ or $c_{m\lambda} c_{\varrho j'}$ with $\lambda \in \{1, \dots, \nu\}$ and rim indeterminates $c_{\varrho j}$ and $c_{\varrho j'}$, respectively.

Proof Let $G = \{g_1, \dots, g_\nu\}$ be the generic $\mathcal{O}$-border prebasis. We shall use the construction of the neighbour generators using the lifting of neighbour syzygies (see Kehrein and Kreuzer 2005, Sect. 5 and Kreuzer and Robbiano 2005, Proposition 6.4.34).

First we prove (a). The polynomials in $\mathrm{ND}(j, j')$ are the coefficients of $t_1, \dots, t_\mu$ in the reduction of $x_\ell g_{j'} - g_j$, viewed as a polynomial in $K[C][x_1, \dots, x_n]$. The leading terms $x_\ell b_{j'}$ and b_j cancel by definition. Hence we only have to reduce the x_ℓ-exposed border terms in $x_\ell g_{j'}$. The coefficient of b_{λ_p} in $x_\ell g_{j'}$ is $c_{\varrho_p j'}$. The coefficient of t_i in g_{λ_p} is $c_{i\lambda_p}$. Therefore the coefficient of t_i in $c_{\varrho_p j'}\, g_{\lambda_p}$ is $c_{\varrho_p j'} c_{i\lambda_p}$ and it will be a part of the coefficient of t_i in the final result of the reduction, i.e., in $\mathrm{ND}(j, j')_i$.

Next we prove (b). Again the entries of $\mathrm{AR}(j, j')$ are obtained as the coefficients of $t_1, \dots, t_\mu$ in the reduction of $x_k g_j - x_\ell g_{j'}$, and again the highest terms $x_k b_j$ and $x_\ell b_{j'}$

cancel. To reduce $x_k g_j$, we have to reduce the x_k-exposed border terms $b_{\kappa_1}, \dots, b_{\kappa_s}$. The coefficient of b_{κ_p} in $x_k g_j$ is $c_{\varrho_p j}$. The polynomial $\mathrm{AR}(j, j')_m$ has arrow degree $\deg_W(c_{mj}) + e_k$. It is therefore the coefficient of t_m in the reduction of $x_k g_j - x_\ell g_{j'}$. The coefficient of t_m in g_{κ_p} is $c_{m\kappa_p}$. Hence the coefficient of t_m in $c_{\varrho_p j}\, g_{\kappa_p}$ is $c_{\varrho_p j} c_{m\kappa_p}$, and it may appear in $\mathrm{AR}(j, j')_m$. These quadratic terms are the ones listed in case (1).

The analysis of the terms in the reduction of $x_\ell g_{j'}$ is completely analogous and leads to the quadratic terms in case (2). □

Based on this detailed study of the support of the neighbour generators, we are now able to provide some additional information on the distribution of the rim indeterminates in the cotangent equivalence classes.

Theorem 6.7 *Let Z be a tuple of indeterminates from C such that there exists a Z-separating re-embedding of $I(\mathbb{B}_\mathcal{O})$ and let $Y = C \setminus Z$.*

(a) *All basic indeterminates of C are rim indeterminates.*
(b) *Each proper cotangent equivalence class in C contains a rim indeterminate.*

Proof For the proof of (a), we assume that $c_{ij} \in C$ is a basic indeterminate and that $t_i \in \mathcal{O}^\circ$. If there exists an indeterminate x_ℓ such that $x_\ell b_j \in \partial\mathcal{O}$ then we let $j' \in \{1, \dots, \nu\}$ such that $b_{j'} = x_\ell b_j$. As we have $t_i \in \mathcal{O}^\circ$, there exists an index $i' \in \{1, \dots, \mu\}$ such that $t_{i'} = x_\ell t_i \in \mathcal{O}$. Hence we have a next-door neighbour pair $(b_j, b_{j'})$ and Proposition 6.3 implies $c_{ij} \sim c_{i'j'}$. This contradicts the hypothesis that c_{ij} is a basic indeterminate. Thus no next-door neighbour pair $(b_j, b_{j'})$ exists.

Since the border $\partial\mathcal{O}$ is connected with respect to the neighbour relations (see Kehrein and Kreuzer 2005, Proposition 19), there exists an across-the-rim neighbour pair $(b_j, b_{j'})$. Thus we may assume that there are $m' \in \{1, \dots, \mu\}$ and $k, \ell \in \{1, \dots, n\}$ such that $b_j = x_\ell t_{m'}$ and $b_{j'} = x_k t_{m'} \in \partial\mathcal{O}$ for some $j' \in \{1, \dots, \nu\}$. Now, if t_i is not divisible by x_ℓ, then Proposition 6.6 implies that c_{ij} is a trivial indeterminate, in contradiction to the hypothesis. Hence t_i has to be divisible by x_ℓ. The term $t_i x_k / x_\ell$ cannot be in the border of $\mathcal{O}$, because then also $t_i x_k$ would be outside $\mathcal{O}$, in contradiction to the hypothesis $t_i \in \mathcal{O}^\circ$. Therefore there exists an index $i' \in \{1, \dots, \mu\}$ such that $x_\ell t_{i'} = x_k t_i$. By Proposition 6.3, we get $c_{ij} \sim c_{i'j'}$, in contradiction to the hypothesis that c_{ij} is basic.

Now we show (b). For a contradiction, assume that $c_{ij} \in C$ is proper and that every element in the cotangent equivalence class of c_{ij} is an interior indeterminate. Since $\mathcal{O}$ is an order ideal, the term t_i is not a multiple of $b_j \notin \mathcal{O}$. Thus there exists an index $\ell \in \{1, \dots, n\}$ such that the arrow degree $\deg_W(c_{ij}) = \log(b_j) - \log(t_i)$ of c_{ij} has a positive ℓ-th component. Let $j' \in \{1, \dots, \nu\}$ be such that the ℓ-th component of $\log(b_{j'})$ is zero. (For instance, let $k \neq \ell$ and consider the unique term of the form $b_{j'} = x_k^N \in \partial\mathcal{O}$ with $N \geq 1$.) Since the border is connected, we can find a sequence of border terms $b_j = b_{j_1} \sim \cdots \sim b_{j_q} = b_{j'}$ such that b_{j_p} and $b_{j_{p+1}}$ are next-door or across-the-rim neighbours for $p = 1, \dots, q - 1$.

Notice that it is not possible to find indeterminates $c_{i_1} \sim \cdots \sim c_{i_q j_q}$ because the equality of arrow degrees $\deg_W(c_{i_q j_q}) = \deg_W(c_{i_1 j_1})$ would imply that the x_ℓ-degree of t_{i_q} is negative. Let us try to construct such a sequence of cotangent equivalences inductively. When we have found terms $t_{i_1}, \dots, t_{i_p} \in \mathcal{O}$ with the property that $c_{i_1 j_1} \sim$

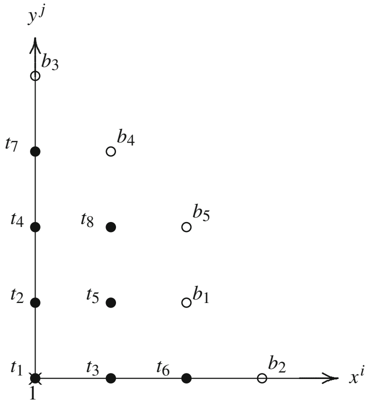
$\cdots \sim c_{i_p j_p}$ and try to find a term $t_{i_{p+1}} \in \mathcal{O}$ such that $c_{i_p j_p} \sim c_{i_{p+1} j_{p+1}}$, three things can happen:

(1) A term $t_{i_{p+1}}$ of the desired kind exists in $\mathcal{O}^\circ$.
(2) A term $t_{i_{p+1}}$ of the desired kind exists in $\mathcal{O}^\nabla$.
(3) No term $t_{i_{p+1}}$ of the desired kind exists in $\mathcal{O}$ because one of the components of $\log(t_{i_{p+1}})$ would be negative.

In case (1), we can continue our inductive construction for one further step. By the hypothesis that c_{ij} is not equivalent to a rim indeterminate, case (2) never occurs. Hence case (3) has to happen for some $p \in \{1, \ldots, q-1\}$. In this case we have $\bar{c}_{i_p j_p} = 0$ by Proposition 6.3, and thus $\bar{c}_{ij} = 0$. Hence we have arrived at a contradiction to the hypothesis that c_{ij} is proper, and the proof is complete. □

The following example illustrates the results of this section and the preceding one. The readers may also check that it verifies some claims in Huibregtse (2002, Remark 7.5.3).

Example 6.8 In $P = \mathbb{Q}[x, y]$, consider the order ideal $\mathcal{O} = \{t_1, \ldots, t_8\}$ given by $t_1 = 1$, $t_2 = y$, $t_3 = x$, $t_4 = y^2$, $t_5 = xy$, $t_6 = x^2$, $t_7 = y^3$, and $t_8 = xy^2$. Then we have $\partial\mathcal{O} = \{b_1, \ldots, b_5\}$ with $b_1 = x^2y$, $b_2 = x^3$, $b_3 = y^4$, $b_4 = xy^3$, and $b_5 = x^2y^2$.



Thus $\mathbb{Q}[C] = \mathbb{Q}[c_{11}, \ldots, c_{85}]$ is a polynomial ring in 40 indeterminates. Notice that the dimension of $\mathbb{B}_\mathcal{O}$ is $\dim(\mathbb{B}_\mathcal{O}) = \mu n = 16$, and that there are 32 neighbour generators of the ideal $I(\mathbb{B}_\mathcal{O})$. The linear parts of these generators are

$$c_{65},\ c_{51} - c_{85},\ c_{45},\ c_{44},\ c_{55},\ c_{43} - c_{54},\ c_{42},\ c_{41} - c_{75},\ c_{52} - c_{75},\ c_{35},$$
$$c_{34},\ c_{33},\ c_{31},\ c_{25},\ c_{24},\ c_{23},\ c_{22},\ c_{21},\ c_{32},\ c_{15},\ c_{14},\ c_{13},\ c_{12},\ c_{11}.$$

If we let U be the union of the supports of these elements, we get

$$C \setminus U = \{c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}\}$$

which is exactly the set of basic indeterminates by Lemma 5.2.b. Moreover, note that $C \setminus U$ is contained in the set of rim indeterminates (see Theorem 6.7.a).

For the trivial cotangent equivalence class E_0 and the proper cotangent equivalence classes $E_1, \ldots, E_q$, we get

$$\begin{aligned} E_0 =& \{c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, \\ & c_{42}, c_{44}, c_{45}, c_{55}, c_{65}\} \\ E_1 =& \{c_{51}, c_{85}\}, \qquad E_2 = \{c_{43}, c_{54}\}, \qquad E_3 = \{c_{41}, c_{52}, c_{75}\}. \end{aligned}$$

Using $\sigma = \texttt{DegRevLex}$, we obtain $E_1^\sigma = \{c_{51}\}$, $E_2^\sigma = \{c_{43}\}$, and $E_3^\sigma = \{c_{41}, c_{52}\}$. Next we compute the set S_σ of the minimal generators of the leading term ideal of $I_L = \langle \mathrm{Lin}_{\mathfrak{M}}(I(\mathbb{B}_{\mathcal{O}})) \rangle$ and get

$$\begin{aligned} S_\sigma = \{& c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, \\ & c_{41}, c_{42}, c_{43}, c_{44}, c_{45}, c_{51}, c_{52}, c_{55}, c_{65}\} \end{aligned}$$

which coincides with $E_0 \cup E_1^\sigma \cup E_2^\sigma \cup E_3^\sigma$ (see the first claim of Theorem 5.5.a). Here we have $\#S_\sigma = 24$, $\#E_0 = 20$, $\#E_1 = 2$, $\#E_2 = 2$, $\#E_3 = 3$, and therefore

$$\#S_\sigma = \#E_0 + \#E_1 + \#E_2 + \#E_3 - 3$$

in accordance with the second claim of Theorem 5.5.a.

The minimal sets of terms generating the sets in $\mathrm{LTGFan}(I_L)$ are

$$\begin{aligned} Z_1 =& E_0 \cup \{c_{51}\} \cup \{c_{43}\} \cup \{c_{41},\ c_{52}\} \\ Z_2 =& E_0 \cup \{c_{51}\} \cup \{c_{43}\} \cup \{c_{41},\ c_{75}\} \\ Z_3 =& E_0 \cup \{c_{85}\} \cup \{c_{43}\} \cup \{c_{41},\ c_{52}\} \\ Z_4 =& E_0 \cup \{c_{85}\} \cup \{c_{43}\} \cup \{c_{41},\ c_{75}\} \\ Z_5 =& E_0 \cup \{c_{51}\} \cup \{c_{54}\} \cup \{c_{41},\ c_{52}\} \\ Z_6 =& E_0 \cup \{c_{51}\} \cup \{c_{54}\} \cup \{c_{41},\ c_{75}\} \\ Z_7 =& E_0 \cup \{c_{85}\} \cup \{c_{54}\} \cup \{c_{41},\ c_{52}\} \\ Z_8 =& E_0 \cup \{c_{85}\} \cup \{c_{54}\} \cup \{c_{41},\ c_{75}\} \\ Z_9 =& E_0 \cup \{c_{51}\} \cup \{c_{43}\} \cup \{c_{52},\ c_{75}\} \\ Z_{10} =& E_0 \cup \{c_{85}\} \cup \{c_{43}\} \cup \{c_{52},\ c_{75}\} \\ Z_{11} =& E_0 \cup \{c_{51}\} \cup \{c_{54}\} \cup \{c_{52},\ c_{75}\} \\ Z_{12} =& E_0 \cup \{c_{85}\} \cup \{c_{54}\} \cup \{c_{52},\ c_{75}\}. \end{aligned}$$

Thus we see that $\prod_{i=1}^3 \#E_i = 12 = \#\,\mathrm{LTGFan}(I_L)$ (see Theorem 5.5.d). As remarked before, $C \backslash U$ is the set of basic indeterminates, and we notice that $Z_i \cap (C \backslash U) = \emptyset$ for $i = 1, \ldots, 12$ (see Theorem 6.7.a).

The complements of the sets Z_i in C are

$$
\begin{aligned}
Y_1 &= \{c_{53}, c_{54}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{75}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}\}\\
Y_2 &= \{c_{52}, c_{53}, c_{54}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}\}\\
Y_3 &= \{c_{51}, c_{53}, c_{54}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{75}, c_{81}, c_{82}, c_{83}, c_{84}\}\\
Y_4 &= \{c_{51}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}\}\\
Y_5 &= \{c_{43}, c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{75}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}\}\\
Y_6 &= \{c_{43}, c_{52}, c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}\}\\
Y_7 &= \{c_{43}, c_{51}, c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{75}, c_{81}, c_{82}, c_{83}, c_{84}\}\\
Y_8 &= \{c_{43}, c_{51}, c_{52}, c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}\}\\
Y_9 &= \{c_{41}, c_{53}, c_{54}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}\}\\
Y_{10} &= \{c_{41}, c_{51}, c_{53}, c_{54}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}\}\\
Y_{11} &= \{c_{41}, c_{43}, c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}, c_{85}\}\\
Y_{12} &= \{c_{41}, c_{43}, c_{51}, c_{53}, c_{61}, c_{62}, c_{63}, c_{64}, c_{71}, c_{72}, c_{73}, c_{74}, c_{81}, c_{82}, c_{83}, c_{84}\}.
\end{aligned}
$$

It is straightforward to verify that $C \setminus U$, the set of basic indeterminates, is contained in each set Y_i (see Theorem 5.6.a). Since we have $\#Z_i = \dim_{\mathbb{Q}}(\mathrm{Lin}_{\mathfrak{M}}(I(\mathbb{B}_{\mathcal{O}})))$ for every $i = 1, \ldots 12$, we are in the situation considered in Theorem 5.6.c. Using Algorithm 5.7, we check that for each Z_i there exists an optimal Z_i-separating re-embedding of $I(\mathbb{B}_{\mathcal{O}})$. Finally, for $i \in \{1, \ldots, 12\}$, we use $\#Z_i = 24$ and conclude that the set Z_i defines an isomorphism $\mathbb{B}_{\mathcal{O}} \cong \mathbb{Q}[Y_i]$, where $\mathbb{Q}[Y_i]$ is a polynomial ring having 16 indeterminates.

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