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ORIGINAL PAPER

On Crofton's type formulas and the solid angle of convex sets

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Abstract
Here we analyze three dimensional analogues of the classic Crofton formula for planar compact convex sets. In this formula a fundamental role is played by the visual angle of the convex set from an exterior point. A generalization of the visual angle to convex sets in the Euclidean space is the visual solid angle. This solid angle, being an spherically convex set in the unit sphere, has perimeter, area and other geometric quantities to be considered. The main goal of this note is to express invariant quantities of the original convex set depending on volume, surface area and mean curvature integral by means of integrals of functions related to the solid angle.

Keywords Invariant measures · Convex set · Dihedral angle · Solid angle · Constant width

Mathematics Subject Classification Primary 52A15; Secondary 53C65

1 Introduction and statement of results

The purpose of this note is to give three-dimensional analogues of the classic Crofton formula for a planar compact convex set K,

$$\int_{P \notin K} 2(\omega - \sin \omega)\, dP = L^2 - 2\pi F. \tag{1.1}$$

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Here L is the length of the boundary of K, F the area of K and $\omega = \omega(P)$ is the *visual angle* of K as seen from P, and dP is the Lebesgue measure on the Euclidean plane.

A generalization of the visual angle in the plane to compact convex sets in the Euclidean space $\mathbb{E}^3$ is the visual solid angle. The *solid angle* $\Omega(P)$ of a compact convex set K from a point $P \notin K$ is the set of unit directions u such that the ray $P + tu, t \geq 0$ meets K. Instead of an arc in the unit circle S^1 and its length ω we have a (spherically convex) set in the unit sphere S^2 with a richer geometry, having area, perimeter, and other geometric quantities that might be considered.

In our analysis we will be lead to two set functions $\alpha(\Omega), \beta(\Omega)$ defined for spherically convex sets $\Omega \subset S^2$, both replacing $\omega - \sin\omega$ in different senses. To introduce them we first recall some classical notions about spherical geometry, the main reference being (Santaló 2004). The radii perpendicular to the support planes of the cone spanned by Ω form another cone whose intersection with S^2 is the so-called *dual curve* of $\partial\Omega$. The region in S^2 bounded by this curve and its symmetral with respect to the origin, which we denote by $\tilde{\Omega}$, consists in the unit directions v such that the plane $v^\perp$ meets Ω. We may identify it with the set of great circles in S^2 meeting Ω. Its complementary $\tilde{\Omega}^c$ in S^2 consists of two symmetrical components. Since the scalar product $\langle v, u\rangle$ does not vanish for $v \in \tilde{\Omega}^c, u \in \Omega$, this product has constant sign in each component. The one with positive sign is called the *dual solid cone* and denoted by Ω^*.

The set functions α, β are respectively defined as

$$\alpha(\Omega) = \frac{1}{2}\int_{u\in\Omega,\, v\in\tilde{\Omega}} |\langle u, v\rangle|\, du\, dv, \tag{1.2}$$

$$\beta(\Omega) = \frac{1}{8}\int_{v_i\in\tilde{\Omega}} |\det(v_1, v_2, v_3)|\, dv_1\, dv_2\, dv_3, \tag{1.3}$$

where du, dv and dv_i denote the Lebesgue measure in S^2.

The factors in front of the integrals are explained by the fact that the set $\tilde{\Omega}$ doubly parametrizes the planes through the origin meeting Ω, or equivalently the great circles meeting Ω.

It is immediate to check that both set functions are invariant by rigid motions T of the sphere, that is $\alpha(T(\Omega)) = \alpha(\Omega), \beta(T(\Omega)) = \beta(\Omega)$. We point out that a straightforward computation shows that the formal analogues in the plane

$$\alpha(I) = \frac{1}{2}\int_{u\in I,\, v\in\tilde{I}} |\langle u, v\rangle|\, du\, dv, \quad \beta(I) = \frac{1}{4}\int_{v_i\in\tilde{I}} |\det(v_1, v_2)|\, dv_1\, dv_2,$$

where now I is a convex subset of S^1, are both equal to $\omega - \sin\omega$ up to a constant, ω being the length of I.

The Crofton type formulas we obtain are then:

Theorem 1.1 *For a compact convex set $K \subset \mathbb{E}^3$ with mean curvature integral M, volume V and surface area F one has*

$$\frac{1}{2}\pi MF - 2\pi^2 V = \int_{P \notin K} \alpha(\Omega(P))\, dP, \tag{1.4}$$

with the set function α given by (1.2).

Recall that the *mean curvature integral M* is given by

$$M = \int_{\partial K} \frac{k_1 + k_2}{2}\, dS$$

where k_1, k_2 are the principal curvatures of ∂K and dS is the surface element.

Theorem 1.2 *For a compact convex set K in $\mathbb{E}^3$ with mean curvature integral M and volume V one has*

$$M^3 - \pi^4 V = \int_{P \notin K} \beta(\Omega(P))\, dP, \tag{1.5}$$

with the set function β given by (1.3).

Theorem 1.3 *For a compact convex set K in $\mathbb{E}^3$ with volume V one has*

$$\int L(K \cap E)^2\, dE = \int_{P \notin K} |\Omega(P)|^2\, dP + 4\pi^2 V, \tag{1.6}$$

where dE is the invariant measure for affine planes E in the Euclidean space, L denotes the perimeter and $|\Omega|$ the surface measure.[1]

As a consequence of Theorem 1.1 and Minkowski's inequality $12\pi V \leq MF$, which follows from the inequalities $4\pi F \leq M^2$ and $3MV \leq F^2$ (see Minkowski 1901, p. 120 or Schneider 2013, p. 387), one has

$$\int_{P \notin K} \alpha(\Omega(P))\, dP \geq 4\pi^2 V,$$

with equality only when K is a ball.

Theorem 1.3 has as a consequence:

Theorem 1.4 *For a compact convex set K in $\mathbb{E}^3$ with volume V one has*

$$\int_{P \notin K} |\Omega(P)|^2\, dP \geq 4\pi^2 V, \tag{1.7}$$

and equality holds if and only if K is a ball.

[1] Formula (1.6) was previously considered in private conversations with Teufel in 2003.

Part of our analysis will consist in understanding the set functions α, β in terms of metric properties of Ω. For the set function α a satisfactory description is easily obtained, namely

$$\alpha(\Omega) = \pi|\Omega| - \langle c(\Omega), c(\Omega^*)\rangle, \tag{1.8}$$

where $c(\Omega) = \int_\Omega u\,du$ and the similarly defined $c(\Omega^*)$ are the (unweighted) centroids of Ω and Ω^* respectively. Again, the analogue of this expression for an arc I in S^1 (I^* being in this case the concentric arc with length $\pi - \omega$) equals $\omega - \sin\omega$ up to constants. The explicit computation of $\alpha(\Omega)$ is possible just for spherical caps and other simple cases.

For the set function β the description is not so neat. Nevertheless we show that the difference $\beta(\Omega) - \frac{\pi^2}{2}\alpha(\Omega)$ can be expressed in terms of the *dihedral visual angles* $\mathcal{D}(\Omega, u)$ of Ω from points $u \in S^2$ not in Ω, the analogue in spherical geometry of the visual angle. More precisely $\mathcal{D}(\Omega, u)$ is the angle between two planes through the origin and through u, tangent to Ω. From the above linear combination between $\alpha(\Omega)$ and $\beta(\Omega)$ and the classical Crofton–Herglotz formula we will deduce that Theorems 1.1 and 1.2 can be seen as equivalent statements (see Sect. 2.3).

Regarding Theorem 1.3, it would be interesting to understand the left-hand side of (1.6) in terms of the geometry of K.

2 Proofs of the theorems

2.1 Proof of Theorems 1.1, 1.2, 1.3 and 1.4

All of them are obtained by mimicking the integral geometry proof of the plane Crofton formula (1.1). We denote by E, G affine planes and lines in the Euclidean space, respectively, and by dE, dG the corresponding canonical invariant measures as used for instance in Santaló (2004).

For a compact convex set K with mean curvature integral M, surface area F and volume V, Crofton's classic formulas for intrinsic volumes give (see Santaló (2004))

$$\int_{G\cap K\neq\emptyset} dG = \frac{\pi}{2}F, \quad \int L(K\cap G)dG = 2\pi V, \tag{2.1}$$

and

$$\int_{E\cap K\neq\emptyset} dE = M, \int L(K\cap E)dE = \frac{\pi^2}{2}F, \int F(K\cap E)dE = 2\pi V. \tag{2.2}$$

We have used the notation $L(K\cap G)$ for the length of the segment $K\cap G$ and $L(K\cap E)$ and $F(K\cap E)$ for the perimeter and the area, respectively, of the planar convex set $K\cap E$.

We shall consider pairs and triples of linear varieties provided with the product measure. It will be useful to express these measures using the parametrizations given in the next lemma, whose proof can be deduced from Section 12 of Santaló (2004):

Lemma 2.1 *Let dG and dE be the canonical invariant measures of affine lines and planes respectively. Then*

a) *For pairs of planes E and lines G we have*

$$dG\, dE = |\langle u, v\rangle| dG_P\, dE_P\, dP, \tag{2.3}$$

where v is a unit normal direction to the plane E, u is a unit direction of the straight line G, dP is the Lebesgue measure on $\mathbb{E}^3$, dG_P is the measure of lines in $\mathbb{E}^3$ through P and dE_P the measure of planes in $\mathbb{E}^3$ through P.

b) *For triples of planes E^1, E^2, E^3 we have*

$$dE^1\, dE^2\, dE^3 = |\det(v_1, v_2, v_3)| dE_P^1\, dE_P^2\, dE_P^3\, dP, \tag{2.4}$$

where v_i are the unit normal directions to the planes E^i, $P \in E^1 \cap E^2 \cap E^3$, and dE_P^i is the measure of planes in $\mathbb{E}^3$ through P.

c) *For pairs of straight lines G^1, G^2 intersecting at a point $P \in \mathbb{E}^3$ we have*

$$dG_P^1\, dG_P^2\, dP = dG_E^1\, dG_E^2\, dE \tag{2.5}$$

where dG_P^i is the measure of lines through P and dG_E^i is the measure of lines within E.

In the above items, the cases in which two linear varieties are parallel have zero measure.

Proof a) This is a particular case of formula (12.47) in Santaló (2004) when $n = 3$, $r = 1$ and $s = 2$.

b) It is known that $dE = ds \wedge du$ where s is the distance from the plane E to the origin O and $u \in S^2$ is the normal vector to E. If $P \in E$ then $\langle \overrightarrow{OP}, u\rangle = s$. Hence the equations of the planes E_i are $\langle \overrightarrow{OP}, u_i\rangle = s_i$. It follows that

$$ds_1 \wedge ds_2 \wedge ds_3 = |\det(u_1, u_2, u_3)|\, dx \wedge dy \wedge dz$$

with $P = (x, y, z)$. From this (2.4) follows immediately.

c) We consider orthonormal moving frames $\{P; e_1, e_2, e_3\}$ and $\{P; e_1^*, e_2^*, e_3^* = e_3\}$ such that $P = E \cap G^1 \cap G^2$, e_3 is normal to E, e_1 is the unit direction of G^1 and e_1^* is the unit direction of G^2. Considering the Maurer-Cartan forms of these moving frames, by (12.48) in Santaló (2004) we have

$$dG_E^1\, dG_E^2\, dE = (\omega_{12} \wedge \omega_2) \wedge (\omega_{12}^* \wedge \omega_2^*) \wedge (\omega_{13} \wedge \omega_{23} \wedge \omega_3) \tag{2.6}$$

and

$$dG_P^1\, dG_P^2\, dP = (\omega_{21} \wedge \omega_{31}) \wedge (\omega_{21}^* \wedge \omega_{31}^*) \wedge (\omega_1 \wedge \omega_2 \wedge \omega_3), \qquad (2.7)$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of $\{e_1, e_2, e_3\}$, the forms ω_{ij} are defined by $de_i = \omega_{i1}e_1 + \omega_{i2}e_2 + \omega_{i3}e_3$ and the forms $\omega_i^*, \omega_{ij}^*$ are defined analogously. Then we get

$$\omega_2^* = \langle e_1, e_2^* \rangle \omega_1 + \langle e_2, e_2^* \rangle \omega_2$$

and

$$\omega_{31}^* = \langle de_3, e_1^* \rangle = \langle e_1^*, e_1 \rangle \omega_{31} + \langle e_1^*, e_2 \rangle \omega_{32}.$$

Taking into account that $\langle e_1, e_2^* \rangle = -\langle e_1^*, e_2 \rangle$ it comes from (2.6) and (2.7) that

$$dG_P^1\, dG_P^2\, dP = dG_E^1\, dG_E^2\, dE$$

as announced (we always consider the absolute value for densities).

□

We can now proceed to prove the announced theorems.

Proof of Theorem 1.1 We consider pairs (E, G) of planes and lines both meeting K. From (2.1), (2.2) and (2.3) and denoting by G_P a line through P and E_P a plane through P one has

$$\frac{\pi}{2} MF = \int_{G \cap K \neq \emptyset, E \cap K \neq \emptyset} dG\, dE = \int_{\mathbb{E}^3} \int_{G_P \cap K \neq \emptyset, E_P \cap K \neq \emptyset} |\langle u, v \rangle| dG_P\, dE_P\, dP.$$

If $P \in K$ the integral in the right-hand side is extended to all lines G_P and to all planes E_P and since these lines and planes are doubly parametrized by u, v respectively, it is

$$\int |\langle u, v \rangle| dG_P\, dE_P = \frac{1}{4} \int_{u, v \in S^2} |\langle u, v \rangle| du\, dv.$$

Obviously the integral with respect to v does not depend on u and so the above integrals are equal to

$$\pi \int |\langle u, v \rangle|\, dv.$$

Choosing $u = (0, 0, 1)$ and computing in spherical coordinates one obtains the value $2\pi^2$.

If $P \notin K$ then the line G_P is parametrized by $u \in \Omega(P)$ and the plane E_P is doubly parametrized by $v \in \tilde{\Omega}(P)$ whence

$$\int_{G_P \cap K \neq \emptyset, E_P \cap K \neq \emptyset} |\langle u, v\rangle| dG_P\, dE_P = \frac{1}{2} \int_{u \in \Omega(P), v \in \tilde{\Omega}(P)} |\langle u, v\rangle|\, du\, dv = \alpha(\Omega(P)),$$

thus proving (1.4). □

Proof of Theorem 1.2 Here we use triples of planes meeting K and proceed analogously to the above proof. Using (2.2) and (2.4) it follows that

$$M^3 = \int_{\mathbb{E}^3} \int_{E_P^i \cap K \neq \emptyset} |\det(v_1, v_2, v_3)| dE_P^1\, dE_P^2\, dE_P^3\, dP.$$

Again, if P is within K, there is no restriction on the planes E_P^i and one has

$$\int |\det(v_1, v_2, v_3)| dE_P^1\, dE_P^2\, dE_P^3 = \frac{1}{8} \int_{v_i \in S^2} |\det(v_1, v_2, v_3)|\, dv_1\, dv_2\, dv_3.$$

The integral in v_1, v_2 is independent of v_3 and so the above integrals are equal to

$$\frac{\pi}{2} \int |\det(v_1, v_2, v_3)|\, dv_1\, dv_2.$$

Choosing $v_3 = (0, 0, 1)$ and computing in spherical coordinates we get the value π^4.

If $P \notin K$ then the planes E_P^i are doubly parametrized by $v_i \in \tilde{\Omega}(P)$, so that

$$\int_{E_P^i \cap K \neq \emptyset} |\det(v_1, v_2, v_3)| dE_P^1\, dE_P^2\, dE_P^3 = \beta(\Omega(P)),$$

and (1.5) is proved. □

Proof of Theorem 1.3 We use now pairs of intersecting lines, both meeting K. The measure of this set of lines is

$$\int_{P \in \mathbb{E}^3, G_P^i \cap K \neq \emptyset} dG_P^1\, dG_P^2\, dP = \int_{\mathbb{E}^3} \left(\int_{G_P^i \cap K \neq \emptyset} dG_P^1\, dG_P^2 \right) dP.$$

On the one hand, the contribution of K in the dP integral is $(2\pi)^2 V$, while that of the complementary of K is $\int_{P \notin K} |\Omega(P)|^2\, dP$. On the other hand, by (2.5), the above integrals are equal to

$$\int \left(\int_{G_1, G_2 \subset E, G_i \cap K \neq \emptyset} dG_E^1\, dG_E^2 \right) dE.$$

Since the Cauchy-Crofton formula in the plane E states that $\int_{G \subset E, G \cap K \neq \emptyset} dG = L(K \cap E)$, we are done. □

We point out that another proof of Theorem 1.3 can be obtained integrating the planar Crofton formula (1.1) applied to the convex set $E \cap K$ over all the planes E.

Proof of Theorem 1.4 Using the isoperimetric inequality in the plane E the left-hand side of (1.6) is bigger than

$$4\pi \int F(K \cap E)\, dE,$$

which by the last equality in (2.2) is equal to $8\pi^2 V$, thus proving (1.7). If equality holds, then $L(K \cap E)^2 = 4\pi F(K \cap E)$ for all E, and all convex sets $K \cap E$ are discs, and this implies easily that K is a ball (see for instance Corollary 7.1.4 in Gardner (2006)). □

2.2 On the set function α

To prove the relation (1.8) just notice that

$$\int_{v\in\tilde{\Omega}} |\langle u, v\rangle|\, dv = \int_{v\in S^2} |\langle u, v\rangle|\, dv - 2\int_{v\in\Omega^*} |\langle u, v\rangle|\, dv.$$

The first integral on the right-hand side does not depend on u and equals 2π, while in the second one $\langle u, v\rangle$ is positive. Altogether gives

$$\alpha(\Omega) = \pi \int_{u\in\Omega} du - \int_{u\in\Omega, v\in\Omega^*} \langle u, v\rangle\, du\, dv = \pi|\Omega| - \langle c(\Omega), c(\Omega^*)\rangle.$$

In order to express this function in terms of Ω we find a relation between the centroids of Ω and Ω^* using a parametrization of the boundary of Ω. We consider the orientation in Ω given by the unit outward normal to S^2 and let $\gamma(t)$, $0 \le t \le \ell$, be the arc-length parametrization of $\partial\Omega$ with the induced orientation, so that $\vec{T} = \gamma'$ is the unit tangent vector.

Proposition 2.2 *For a spherically convex subset Ω of the sphere S^2 with regular boundary arc-parametrized by $\gamma(t)$ we have the following formulas for the centroids $c(\Omega)$ and $c(\Omega^*)$.*

a)

$$c(\Omega) = \frac{1}{2}\int_0^\ell \gamma(t) \times \gamma'(t)\, dt, \quad c(\Omega^*) = \frac{1}{2}\int_0^\ell k_g(t)\gamma(t)\, dt,$$

where $k_g(t)$ is the geodesic curvature of $\gamma(t)$ and '$\times$' denotes the vector product.

b)

$$c(\Omega) + c(\Omega^*) = \frac{1}{2}\int_0^\ell \gamma'(t) \times \gamma''(t)\, dt = \frac{1}{2}\int_0^\ell k(t)\vec{B}(t)\, dt.$$

where $k(t)$ is the curvature of $\gamma(t)$ and $\vec{B}(t)$ its binormal.

Proof If $u = (x, y, z)$, the first component of $c(\Omega)$ is $\int_\Omega x du$, which is the flow through Ω of the vector field $\vec{X} = (1, 0, 0)$. Since $\vec{X} = \nabla \times \vec{Y}$ with $\vec{Y} = \frac{1}{2}(0, -z, y)$ this component is equal to

$$\frac{1}{2}\int_{\partial\Omega} \langle \vec{T}, \vec{Y}(\gamma(t))\rangle \, dt.$$

Now $\langle \vec{T}, \vec{Y}\rangle$ is the first component of $\gamma(t) \times \gamma'(t)$ and similarly for the other components, so that the first formula in a) is proved. Next, notice that $\gamma^*(t) = \gamma(t) \times \gamma'(t)$ parametrizes the dual curve, the boundary of Ω^*, whence one has as well

$$c(\Omega^*) = \frac{1}{2}\int_0^\ell \gamma^*(t) \times (\gamma^*)'(t)\, dt.$$

Now,

$$\gamma^* \times \gamma^{*\prime} = (\gamma \times \gamma') \times (\gamma \times \gamma')' = (\gamma \times \gamma') \times (\gamma \times \gamma'') = \det(\gamma, \gamma', \gamma'')\gamma = k_g\gamma$$

and a) is proved.

In order to prove b) we simplify the notation writing $\sigma = \gamma \times \gamma' + \gamma^* \times \gamma^{*\prime}$. Denote by $\vec{T}, \vec{N}, \vec{B}$ the Frenet frame of γ. It is easy to see that $\langle \sigma, \vec{T}\rangle = 0$. Also

$$\begin{aligned}\langle \sigma, \vec{N}\rangle &= \frac{1}{k}\langle \gamma \times \gamma', \gamma''\rangle + \frac{1}{k}\langle \det(\gamma, \gamma', \gamma'')\gamma, \gamma''\rangle \\ &= \frac{1}{k}\langle \gamma \times \gamma', \gamma''\rangle - \frac{1}{k}\det(\gamma, \gamma', \gamma'') = 0,\end{aligned}$$

because $\langle \gamma, \gamma'\rangle = 0$ and so $\langle \gamma, \gamma''\rangle = -1$. Now we compute $\langle \sigma, \vec{B}\rangle$,

$$\begin{aligned}\langle \sigma, \vec{B}\rangle &= \langle \sigma, \vec{T} \times \vec{N}\rangle = \frac{1}{k}\langle \sigma, \gamma' \times \gamma''\rangle = \frac{1}{k}\langle \gamma \times \gamma' + \gamma^* \times \gamma^{*\prime}, \gamma' \times \gamma''\rangle \\ &= \frac{1}{k}\langle \gamma \times \gamma', \gamma' \times \gamma''\rangle + \frac{1}{k}\langle \gamma^* \times \gamma^{*\prime}, \gamma' \times \gamma''\rangle \\ &= -\frac{1}{k}\langle \gamma, \gamma''\rangle\langle \gamma', \gamma'\rangle + \frac{1}{k}\langle k_g\gamma, \gamma' \times \gamma''\rangle = \frac{1}{k}(1 + k_g^2).\end{aligned}$$

Since the curve γ is on the unit sphere we have that $k^2 = 1 + k_g^2$; therefore $\langle \sigma, \vec{B}\rangle = k$ and we conclude that

$$\gamma \times \gamma' + \gamma^* \times \gamma^{*\prime} = k\vec{B}.$$

Since $\gamma' \times \gamma'' = \vec{T} \times k\vec{N} = k\vec{B}$ the proposition is proved. □

Using these formulas it is easily checked that if Ω is a spherical cap of angle ω one has

$$\alpha(\Omega) = 2\pi(1-\cos\omega) - \pi\cos^2\omega\sin\omega.$$

2.3 On the set function β

Here we wish to obtain an alternative expression for (1.3) which will lead us to the Crofton–Herglotz formula.

For two different vectors $v_2, v_3 \in S^2$ let $u = v_2 \times v_3/|v_2 \times v_3|$. To specify a basis for $u^\perp$ we write u in spherical coordinates, $u = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, and define

$$e_1 = \frac{\partial}{\partial\varphi} = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi), \qquad e_2 = \frac{1}{\sin\varphi}\frac{\partial}{\partial\theta} = (-\sin\theta, \cos\theta, 0),$$

so that $\{e_1, e_2, u\}$ is a positive orthonormal basis. We write v_2, v_3 in this basis as

$$v_2 = \cos\theta_2 \cdot e_1 + \sin\theta_2 \cdot e_2, \quad v_3 = \cos\theta_3 \cdot e_1 + \sin\theta_3 \cdot e_2.$$

Then v_2, v_3 are parametrized by u, θ_2, θ_3. Keeping in mind that the integral in (1.3) is in fact over the set of all triples of great circles meeting Ω, we have that $\pm v_2$ and $\pm v_3$ count as one and so we consider $0 \leq \theta_2, \theta_3 \leq \pi$ and require that the angles θ_2, θ_3 be within the dihedral angle $\mathcal{D}(\Omega, u)$ determined by Ω and u. Then (u, θ_2, θ_3) is a double parametrization of the set of pairs of great circles meeting Ω. In this parametrization, it is immediate to check that (cf. (Rey Pastor and Santaló 1951, (34.1)))

$$dv_2 dv_3 = |\sin(\theta_3 - \theta_2)| d\theta_2 d\theta_3 du,$$

while

$$|\det(v_1, v_2, v_3)| = |v_2 \times v_3| \cdot |\langle v_1, u\rangle| = |\sin(\theta_3 - \theta_2)| \cdot |\langle v_1, u\rangle|,$$

for all $v_1 \in \tilde{\Omega}$.

Thus

$$\beta(\Omega) = \frac{1}{4}\int_{v_1\in\tilde{\Omega}, u\in S^2, \theta_i\in\mathcal{D}(\Omega,u)} \sin^2(\theta_3 - \theta_2)|\langle v_1, u\rangle| dv_1\, d\theta_2\, d\theta_3\, du.$$

Now the integral with respect to θ_2, θ_3 is easily computed, and denoting as well by $\mathcal{D}(\Omega, u)$ the measure of the dihedral angle we get

$$\beta(\Omega) = \frac{1}{8}\int_{v_1\in\tilde{\Omega},\, u\in S^2} (\mathcal{D}^2(\Omega, u) - \sin^2\mathcal{D}(\Omega, u))|\langle v_1, u\rangle| dv_1\, du.$$

For $\pm u \in \Omega$ one has $\mathcal{D}(\Omega, u) = \pi$ whence the contribution of this part equals $\pi^2\alpha(\Omega)/2$. Thus

$$\beta(\Omega) = \frac{\pi^2}{2}\alpha(\Omega) + \delta(\Omega), \tag{2.8}$$

with

$$\delta(\Omega) = \frac{1}{8}\int_{v\in\tilde{\Omega},\ \pm u\notin\Omega} (\mathcal{D}^2(\Omega, u) - \sin^2\mathcal{D}(\Omega, u))|\langle v, u\rangle|dv\,du.$$

Thus Theorems 1.1 and 1.2 imply

$$M^3 - \frac{1}{4}\pi^3 MF = \int_{P\notin K} \delta(\Omega(P))\,dP.$$

We now insert the definition of $\delta(\Omega(P))$ and use (2.3). If u is the unit direction of the line G, then $\mathcal{D}(\Omega(P), u)$ is the dihedral angle $\mathcal{D}(K, G)$ of K as seen from the line G through P. So the right-hand side above is equal to

$$\begin{aligned}&\frac{1}{2}\int_{E\cap K\neq\emptyset, G\cap K=\emptyset} (\mathcal{D}^2(K, G) - \sin^2\mathcal{D}(K, G))dE\,dG\\ &= \frac{1}{2}\int_{E\cap K\neq\emptyset} dE\left(\int_{G\cap K=\emptyset} (\mathcal{D}^2(K, G) - \sin^2\mathcal{D}(K, G)\right)dG.\end{aligned}$$

Using (2.2) we obtain the classical Crofton–Herglotz formula (see (116) in Blaschke (1955) or (14.33) in Santaló (2004))

$$\int_{G\cap K=\emptyset} (\mathcal{D}^2(K, G) - \sin^2\mathcal{D}(K, G))\,dG = 2M^2 - \frac{\pi^3 F}{2}.$$

Equality (2.8) shows that Theorems 1.1 and Theorem 1.2 are equivalent through the Crofton–Herglotz formula.

3 Some inequalities for convex sets of constant width

In this section we will deal with compact convex sets K of constant width. For each of these sets we have the relation $R + r = a$, where a is the width of K, and r, R are respectively the inradius and the circumradius of K. We denote by S_r and S_R the insphere and the circumsphere of K, respectively. Thus, denoting $c = r/R$, we have

$$r = \frac{ac}{1+c}, \quad R = \frac{a}{1+c}.$$

From Jung's Theorem (Martini et al. 2019, sec. 3.4.2) it follows that $c \geq \sqrt{8/3} - 1 = 0.63...$

Theorem 3.1 *Let K be a compact convex set of constant width a and let $c = r/R$ be the quotient between the inradius and the cirumradius of K. Then*

$$\int L(K \cap E)^2 dE \leq 8\pi^3 a^3 \left(\frac{1}{(1+c)^2} - \frac{1}{12} \right), \tag{3.1}$$

where $L(K \cap E)$ is the perimeter of the planar convex set $K \cap E$. The equality holds for spheres.

Proof First we observe that denoting by $p(u)$, $u \in S^2$, the support function of K and $\eta(u) = p(u) - a/2$ one has

$$\eta(u)^2 = p(u)^2 + a^2/4 - ap(u).$$

Hence

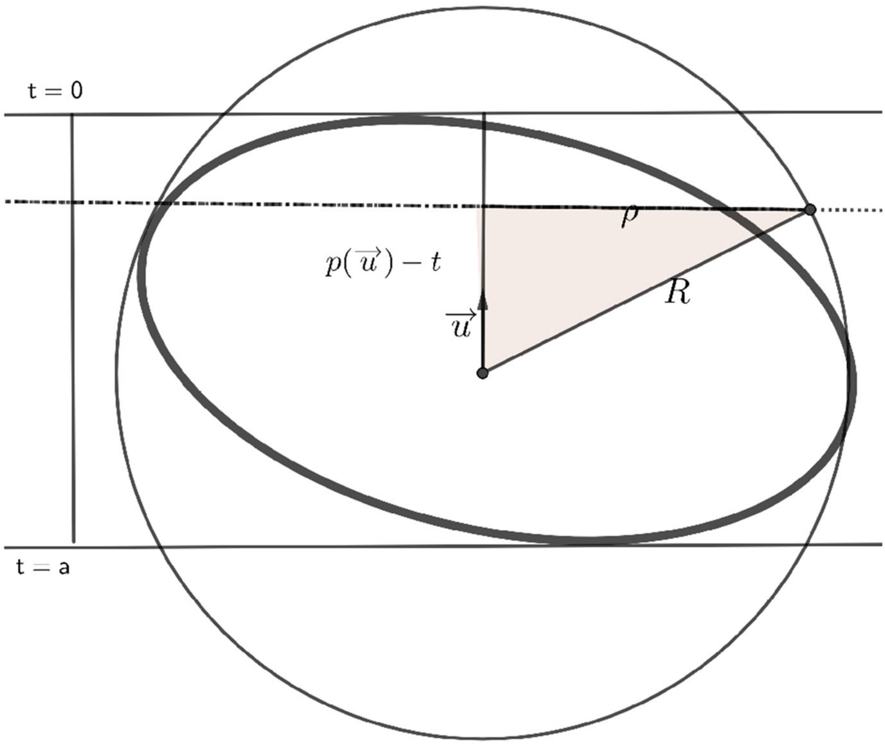
$$\int_{S^2} \eta(u)^2 \, du = \int_{S^2} p(u)^2 \, du + \pi a^2 - 2\pi a^2 \geq 0,$$

and so

$$\int_{S^2} p(u)^2 \, du \geq \pi a^2. \tag{3.2}$$

The set of affine planes in the space is double parametrized by a unit normal vector u and the distance t to the origin. Hence the invariant measure is given by $dE = \frac{1}{2} dt \, du$. Then

$$\begin{aligned} \int L(K \cap E)^2 dE &= \frac{1}{2} \int_{S^2} \int_0^a L(K \cap E)^2 \, dt \, du \leq \frac{1}{2} \int_{S^2} \int_0^a L(S_R \cap E)^2 \, dt \, du \\ &\leq \frac{1}{2} \int_{S^2} \int_0^a \left(2\pi \sqrt{R^2 - (p(u) - t)^2} \right)^2 dt \, du \\ &= \int_{S^2} 2\pi^2 \left(R^2 a - \frac{a^3}{3} + a^2 p(u) - ap(u)^2 \right) du \end{aligned}$$



and by (3.2)

$$\int L(K \cap E)^2 dE \leq 8\pi^3 \left(R^2 a - \frac{a^3}{12} \right) = 8\pi^3 a^3 \left(\frac{1}{(1+c)^2} - \frac{1}{12} \right).$$

□

We note that by Jung's inequality $c \geq \sqrt{8/3} - 1$, the above result implies

$$\int L(K \cap E)^2 dE \leq \frac{7}{3}\pi^3 a^3.$$

Proposition 3.2 *Let K be a compact convex set of constant width a with $c = r/R$ the quotient between the inradius and the cirumradius of K. Then*

$$4\pi^3 a^3 \left(\frac{8}{3} \frac{c^3}{(1+c)^3} - \frac{1}{6} \right) \leq \int_{P \notin K} |\Omega(P)|^2 dP \leq 4\pi^3 a^3 \left(\frac{11 - 3c(3c^2 + c - 3)}{6(1+c)^3} \right),$$

with equalities for spheres, where the lower bound is non negative for $c > 0.657\ldots$.

Proof The right-hand side inequality comes from (3.1) and (1.6) substituting V by V_r, where V_r is the volume of the insphere S_r of K. The left-hand side inequality comes subtracting $4\pi^2 V$ in the easily checked relations

$$8\pi^2 V_r = \int L(S_r \cap E)^2 dE \leq \int L(K \cap E)^2 dE$$

and using the inequality $V \leq V_{a/2}$, where $V_{a/2}$ is the volume of the sphere of radius $a/2$ (see Martini et al. 2019). $\square$

Remark 3.3 We note that in terms of the width only, we have

$$\int_{P \notin K} |\Omega(P)|^2 dP \leq \frac{9}{2}\pi^3 a^3 (\sqrt{6} - 2).$$

Remark 3.4 One can ask if equality

$$\int L(K \cap E)^2 dE = \pi MF - 4\pi^2 V$$

that holds for spheres is also true for compact convex sets of constant width. For this case, with the same kind of arguments used above, we are only able to prove

$$\frac{c^3 - 1}{(1+c)^3} \leq \frac{1}{16\pi^3 a^3}\left(\int L(K \cap E)^2 dE - (\pi MF - 4\pi^2 V)\right)$$
$$\leq \frac{-23c^3 + 3c^2 + 3c + 17}{24(1+c)^3}$$

with equalities for spheres.

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