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Semiaffine stable planes

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Abstract
A locally compact stable plane of positive topological dimension will be called semi-affine if for every line L and every point p not in L there is at most one line passing through p and disjoint from L. We show that then the plane is either an affine or projective plane or a punctured projective plane (i.e., a projective plane with one point deleted). We also compare this with the situation in general linear spaces (without topology), where P. Dembowski showed that the analogue of our main result is true for finite spaces but fails in general.

Keywords Stable plane · Semiaffine · Affine line · Projective line

Mathematics Subject Classification 51H10 · 51M30

1 Introduction

We recall the *basic notions*, compare Löwen (1976), Löwen (1981), Grundhöfer and Löwen (1995) and Salzmann (1967). A *stable plane* is a pair $(M, \mathcal{L})$ consisting of a set M of *points* and a set $\mathcal{L}$ of subsets $L \subseteq M$, called *lines*, such that any two points $p, q \in M$ are joined by a unique line $L = p \vee q \in \mathcal{L}$. Moreover, it is required that both M and $\mathcal{L}$ carry locally compact topologies of positive topological dimension such that the operation $\vee\colon M \times M \to \mathcal{L}$ of *joining* is continuous and the opposite operation $\wedge$ of *intersection* sending a pair of intersecting lines to their point of intersection is continuous and its domain of definition is open. The last mentioned property is called *stability of intersection*. It distinguishes stable planes from spatial geometries,

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where pairs of intersecting lines may be approximated by pairs of skew lines. To avoid trivialities, it is also assumed that there is a *quadrangle*, i.e., four points no three of which are on one line.

If two lines always intersect, then we have a *topological projective plane*. This happens if and only if M is compact. See Salzmann et al. (1995) for many examples. An abundance of non-projective examples is obtained by taking the trace geometry induced on any proper open subset of the point set in a topological projective plane. Apart from these *(projectively) embeddable* examples, there are non-embeddable ones. They are generally harder to construct, and this explains why only two-dimensional examples are known, compare 2.2 below.

The *pencil* of all lines passing through a point p is denoted $\mathcal{L}_p$; it is always compact and connected, see Löwen (1976, 1.17, 1.14). As a consequence, a non-compact line L is disjoint from at least one line through any point $p \notin L$. This is seen by looking at the map $x \to x \vee p$ from L to $\mathcal{L}_p$.

In Löwen (1981), we studied special cases of Euclid's *parallel axiom*, which stipulates that given a line L and a point $p \notin L$, there exists exactly one line $K \in \mathcal{L}_p$ that is disjoint from L. This line is then said to be *parallel* to L. Also every line is considered as being parallel to itself. If the parallel axiom holds universally, then the plane is called an *affine plane.* If the parallel axiom is satisfied for a fixed line A and for each point not on A, then we say that the line A is *affine.*

An affine line all of whose parallels are also affine will be called a *biaffine* line. Not all affine lines are biaffine. Indeed, if we take a projective plane $(P, \mathcal{L})$ and delete from P a closed subset of some line W, then what we obtain is called an *almost projective plane*. If we have deleted a proper subset $X \subseteq W$ containing more than one point, then any line $A \neq W$ originally containing a point of X will be affine, and the remainder $W \backslash X$ is parallel to A, but is not affine. So A is affine, but not biaffine. Two intersecting affine lines may have a common parallel, as shown by lines A_1 and A_2 of the kind just considered, the common parallel being $W \backslash X$. However, we observe:

Lemma 1.1 *On the set $\mathcal{A}$ of all affine lines of a stable plane, the parallelity relation is an equivalence.*

Proof We have to show that two parallels of an affine line do not intersect. Indeed, this is part of the definition of an affine line. □

Lemma 1.2 *There is a continuous map $P \colon \mathcal{A} \times M \to \mathcal{L}$ that sends a pair (A, p) to the unique parallel of A passing through p.*

Proof The spaces M and $\mathcal{L}$ are second countable by Löwen (1976, 1.9), hence it suffices to show sequential continuity. So let $(A_n, p_n) \to (A, p)$ in $\mathcal{A} \times M$. If $P(A_n, p_n)$ converges to a line L, then stability of intersection implies that either $p \in A$ and $L = A$ or $p \notin A$ and L is a line passing through p which is disjoint from A. So in either case, $L = P(A, p)$. Now by Löwen (1976, 1.17), the set of all lines containing one of the points p_n or p is compact, hence every subsequence of $P(A_n, p_n)$ has a subsequence converging to $P(A, p)$. This implies convergence of the entire sequence. □

A line is L called *projective* if it meets every other line. A line is projective if and only if it is compact, because we have a continuous and open embedding $p \mapsto p \vee q$ of L into the compact and connected pencil $\mathcal{L}_q$ of any point $q \notin L$.

Finally, we need the notion of a *pointwise coaffine line*, introduced in Löwen (1981). These can be characterized as non-projective lines intersecting only all projective lines. We refer to Löwen (1981) for an explanation of the term 'pointwise coaffine', which describes how these lines appear in the so-called opposite plane, a kind of weak dual. In a *punctured projective plane*, i.e., a projective plane with just one point deleted, all lines are either projective or pointwise coaffine, and both types occur. In fact, this property characterizes punctured projective planes as is easily seen. We give a proof, because this fact does not seem to be recorded in the literature:

Proposition 1.3 *If every line of a stable plane is either projective or pointwise coaffine then the plane is either a projective or a punctured projective plane. The converse is also true.*

Proof We may assume that non-projective (hence non-compact) lines exist. Every non-compact line has at least one parallel through any given point outside. That line is not projective and must be pointwise coaffine, hence it is the unique non-compact line passing through that point. Thus we have a set of pairwise disjoint biaffine lines covering the point set. Adding a point at infinity, we obtain a topological projective plane, compare Löwen (1981, Theorem 2.2). □

In the terminology of Löwen (1981), punctured projective planes are also called *coaffine planes*. Here is another related fact, taken from Löwen (1981, 1.3).

Proposition 1.4 *If some pencil $\mathcal{L}_p$ of a stable plane consists entirely of projective lines, then the plane is projective.*

Proof Every line L not containing p is homeomorphic to the compact pencil $\mathcal{L}_p$ via join and intersection, hence L is projective. □

Our focus here will be on *semiaffine* stable planes, that is, planes where for every non-incident point-line pair (p, L) there is at most one line containing p and disjoint from L. We have the following

Proposition 1.5 *In a semiaffine stable plane, every line is either affine or projective.*

Proof If all lines passing through $p \notin L$ intersect L, then L is homeomorphic to the compact pencil $\mathcal{L}_p$, hence L is projective. If this never happens for a given line L, then L is affine. □

2 Semiaffine planes

Lemma 2.1 *If a projectively embeddable stable plane $(M, \mathcal{L})$ contains two intersecting affine lines, one of which is biaffine, then $(M, \mathcal{L})$ is an affine plane.*

Proof Suppose that $(M, \mathcal{L})$ is embedded in the topological projective plane $(P, \mathcal{K})$. Recall from the introduction that this means that M is an open subset of P, and that the lines $L \in \mathcal{L}$ are the nonempty intersections $L = K \cap M$, $K \in \mathcal{K}$. Note that then L contains more than one point.

For $L \in \mathcal{L}$, let $\bar{L} \in \mathcal{K}$ be the unique line containing L. If $\bar{L}\backslash L$ contains distinct points a, b, then any point $p \in M\backslash L$ lies on two distinct lines $p \vee a$ and $p \vee b$ disjoint from L. Hence L is affine if and only if $\bar{L}\backslash L$ consists of a single point ∞_L. The parallels of L in the stable plane $(M, \mathcal{L})$ are then precisely the lines $K \in \mathcal{L}$ such that $\bar{K}$ contains ∞_L. If L is biaffine, it follows that $\bar{K}\backslash K = \{\infty_L\}$ holds for these lines. Let $\mathcal{A}_L$ be the set of all parallels K of L. Passing to the corresponding lines $\bar{K}$, we obtain a subset $\overline{\mathcal{A}_L} \subseteq \mathcal{K}_{\infty_L}$, and then we see that M is the union of $\overline{\mathcal{A}_L}$ minus the point ∞_L.

Now let $B \notin \mathcal{A}_L$ be another affine line. Then $\infty_B \neq \infty_L$, and the line $W \in \mathcal{K}$ joining these two points does not belong to $\overline{\mathcal{A}_L}$, because it contains two points outside M. On the other hand, all other lines C in the pencil of ∞_L must belong to $\overline{\mathcal{A}_L}$ because their intersection q with $\bar{B}$ belongs to $B \subseteq M$, whence $q \vee \infty_L$ induces a parallel of L. It follows that $\overline{\mathcal{A}_L} = \mathcal{K}_{\infty_L}\backslash\{W\}$, and hence that $M = P\backslash W$ is an affine plane. In fact, $(P, \mathcal{K})$ is the projective completion of this affine plane. □

In Löwen (1981, 5.4), peculiar examples of stable planes are constructed. They depend on a real function α with suitable properties and are called E_α. They have point set $M = \mathbb{R}^2$, and their lines are all homeomorphic to $\mathbb{R}$. There is one special line C in E_α [the y-axis, called Y_0 in Löwen (1981)], and the remaining lines are divided into those not meeting C and those meeting C, with very different behaviour. Precisely the lines of the latter type are affine. They are in fact biaffine, because their parallels are obtained by translation in y-direction. The plane is not affine, because the lines not meeting C are not affine. Using Lemma 2.1, we infer the following

Corollary 2.2 *The two-dimensional stable planes E_α referred to above are examples of non-embeddable planes.* □

The planes E_α can be extended by adding a point at infinity to every line meeting C, see Löwen (1981, 5.4). The added points together form an additional line D. One obtains examples $\overline{E_\alpha}$ of stable planes containing two pointwise coaffine lines C and D without being coaffine (i.e., punctured projective) planes. This property shows directly that the extended planes are non-embeddable. We could have told this before, since an open embedding of the extended plane would induce an open embedding of the original one.

Lemma 2.3 *Suppose that A and B are two intersecting biaffine lines in a stable plane. Then every parallel of A intersects every parallel of B.*

Proof Let A' and B' be parallels of A and B, respectively. Then A' and B' are affine by assumption. If they are disjoint, i.e., parallel, then this violates Lemma 1.1. □

This is false if A and B are merely affine. Indeed, if then A' intersects B', deleting the intersection point from the given plane yields a counterexample.

Proposition 2.4 *If A and B are two intersecting biaffine lines in a stable plane $(M, \mathcal{L})$, then the map $A \times B \to M$ that sends (a, b) to the point $P(a, B) \wedge P(b, A)$ is a homeomorphism.*

Proof By Lemma 2.3 we have an inverse map sending $p \in M$ to $(P(B, p) \wedge A, P(A, p) \wedge B)$; it is continuous by Lemma 1.2. □

Corollary 2.5 *If a stable plane contains two intersecting biaffine lines A, B, then it does not contain a projective line.*

Proof If L is a projective line, then by Lemma 2.3 there is a continuous map $L \to B$ sending $p \in L$ to $P(A, p) \wedge B$. This map is surjective because L meets every line. However, L is compact and B is not. □

In none of the two preceding assertions, the assumption can be weakened from 'biaffine' to 'affine'; every almost projective plane that is neither projective nor punctured projective nor affine yields counterexamples. The following lemma is obvious from the definitions.

Lemma 2.6 *If all lines of a stable plane are either affine or projective, then every affine line is biaffine.* □

Theorem 2.7 *The semiaffine locally compact positive-dimensional stable planes are precisely the following:*

1. *affine planes*
2. *projective planes*
3. *punctured projective planes.*

Proof Clearly, the planes listed are all semiaffine. For the converse assertion, recall that all lines are either affine or projective by Proposition 1.5. We only need to consider the mixed case, where both affine and projective lines exist. We claim that the plane is punctured projective. By Lemma 2.6, all affine lines are biaffine, and by Lemma 2.5, no two of them intersect. We may then embed our plane in a compact projective plane by adding a point ∞ to the point set and replacing every affine line A by $A \cup \{\infty\}$. See Löwen (1981, Theorem 2.2) for details on the continuity properties of the extended plane.

Instead of this completion argument, we may use the notion of a coaffine point introduced in Löwen (1981), as follows: By the previous remarks, every point is incident with precisely one affine line, and all other lines passing through that point are projective. This means that every point is coaffine. Then the plane is pointwise coaffine and hence coaffine, i.e., punctured projective. □

We take this opportunity to give a complete statement of Proposition 1.8 of Löwen (1981), which was mutilated by the publisher after proofreading.

Proposition 2.8 (Löwen 1981) *Let $(M, \mathcal{L})$ be a stable plane whose lines are manifolds.*

(a) *If $C \in \mathcal{L}$ is pointwise coaffine, then the complement $(M \backslash C, \mathcal{L} \backslash \{C\})$ has a point set homeomorphic to Euclidean space $\mathbb{R}^{2^{n+1}}$, where $0 \leq n \leq 3$, and all lines are homeomorphic to $\mathbb{R}^{2^n}$.*

(b) *If there is a second pointwise coaffine line D, then the opposite plane $(M, \mathcal{L})^*$ has the same topological properties.*

The proof given in Löwen (1981) is correct. The opposite plane has point set $\mathcal{K}$, the set of all compact lines of $(M, \mathcal{L})$, and its lines are the partial pencils $\mathcal{K}_p$.

3 Appendix: Linear spaces

All our assertions make sense in the more general situation of *linear spaces*, where there is no topology, and we shall examine which of them remain true. Some strong tools, in particular those related to compactness, are no longer available, and examples are hard to construct. This is why at least one question remains open. Yet also some strong results are known.

We define a linear space to be a pair $(M, \mathcal{L})$ consisting of a set M and a collection $\mathcal{L}$ of subsets of cardinality at least 3, called lines, such that any two points are joined by a unique line, and such that every point is on at least 3 lines. Linear spaces include all affine spaces with lines of size at least 3 and all projective spaces. For these as well as many others, dimension is meaningful and is either infinite, or takes arbitrary integer values $d \geq 2$. However, if we assume the existence of affine or projective lines (defined in the same manner as before), then this indicates planar behaviour. Kreuzer (1993) introduces a notion of semiaffine linear spaces that makes sense in higher definitions, but we shall not adopt his definition.

A few results from Sect. 2 remain true without the topological assumptions. Notably, this holds for Lemma 1.1 (parallelity is an equivalence among affine lines) and its consequence Lemma 2.3 (parallels of intersecting biaffine lines always intersect). The proofs are valid without change. All remaining results that we obtained for stable planes rely on topological arguments for their proofs, and we have stressed the places where this occurs in the previous sections.

One assertion that definitely fails is the analogue of Proposition 1.5, which asserts that a semiaffine linear space contains only affine and projective lines. An easy counterexample is the projectively embedded plane obtained from any affine plane by adding a single point x at infinity in the projective closure. Lines passing through x are projective, but those not containing x are not affine, because they do not possess parallels containing x. As a consequence, the analogue of Theorem 2.7 fails. In addition to this standard counterexample, Dembowski (1962) constructed a plethora of other counterexamples by a processs of free extension. Also Proposition 1.4 fails without topology: in the above example, the pencil of x consists of projective lines. We do not know the answer to the following.

Problem Does Theorem 2.7 hold for linear spaces with the stronger assumption that all lines are either affine or projective?

Things are somewhat different in finite linear spaces. There, existence of a projective or affine line implies that all pencils of points outside this line have equal cardinalities, and this may sometimes replace our compactness arguments. In fact, we have the following.

Theorem 3.1 *Let $(M, \mathcal{L})$ be a finite linear space as defined above. If all lines are either affine or projective, then $(M, \mathcal{L})$ is an affine plane or a projective plane or a punctured projective plane.*

Proof Only the mixed case (with lines of both kinds) needs to be discussed. No line is affine and projective at the same time, and every affine line is biaffine. Let q be the number of points on an affine line A. For every point x outside A, we have the unique parallel to A containing x. The other lines in the pencil $\mathcal{L}_x$ meet A, hence the pencil has $q + 1$ elements. Repeating this argument with another line parallel to A one sees that every pencil has cardinality $q + 1$. It follows that a line is affine or projective according as its cardinality is q or $q + 1$, respectively.

Now we can use the arguments from Lemmas 2.3 and 2.5 (with homeomorphisms replaced by bijections) to conclude that no projective lines exist if there are two intersecting affine lines. The only remaining possibility is that every point is on a unique affine line, and then by adding a common point at infinity to these lines we obtain a projective plane. □

Theorem 3.1 is a special case of a result by Dembowski (1962), which is much harder to prove. It asserts that the only finite semiaffine linear spaces are the finite affine, projective or punctured projective planes and the finite affine planes extended by one point at infinity. In other words, the counterexample to Theorem 2.7 exhibited above is the only one in the finite case. A generalization allowing lines with only two points is given by Totten and de Witte (1974).

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