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ORIGINAL PAPER

On homogeneous η-Einstein almost cosymplectic manifolds

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Abstract
We prove that every compact, homogeneous η-Einstein almost cosymplectic manifold is a cosymplectic manifold.

Keywords Homogeneous almost contact metric manifold · Almost cosymplectic manifold · Goldberg conjecture

Mathematics Subject Classification 53C25 · 53C15 · 53C30

1 Introduction

In Cappelletti Montano and Pastore (2010), Cappelletti Montano and Pastore proved that every compact *Einstein almost cosymplectic* manifold $(M, \varphi, \xi, \eta, g)$ with *Killing* Reeb vector field ξ is necessarily a *cosymplectic manifold*. This result can be considered as an odd-dimensional counterpart of the famous Goldberg conjecture concerning compact Einsten almost Kähler manifolds. This conjecture claims that such a manifold must be actually a Kähler manifold. We refer the reader to Sect. 2 for basic definitions concerning almost contact metric and almost cosymplectic manifolds. Under the same assumption on ξ, they showed that the same conclusion holds if the Einstein condition is replaced by the η-*Einstein* condition, namely:

$$Ric = ag + b\,\eta \otimes \eta, \quad a, b \in \mathbb{R},$$

provided the constant a is positive, see Cappelletti Montano et al. (2010, Theorem 4.2). Their proof is based on a structure result for the leaves of the foliation orthogonal to the Reeb vector field, due to de Leon and Marrero (1997) and Sekigawa's result

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confirming the validity of the Golberg conjecture for compact almost Kähler manifolds having non-negative scalar curvature (Sekigawa 1987).

The purpose of this note is to show that these results can be improved in the homogeneous setting; indeed, we shall prove the following:

Theorem 1.1 *Let $(M, \varphi, \xi, \eta, g)$ be a compact, homogeneous η-Einstein almost cosymplectic manifold. Then M is a cosymplectic manifold.*

As usual, homogeneity means that there exists a Lie group G acting transitively on the manifold M, preserving the underlying geometric structure (φ, ξ, η, g).

2 Preliminaries

We recall that an an almost contact metric structure (φ, ξ, η, g) on an odd-dimensional manifold consists of a $(1, 1)$ tensor field φ, a 1-form η, a vector field ξ and a Riemannian metric g satisfying the following conditions:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where X, Y are arbitrary smooth vector fields. The vector field ξ is called the *Reeb vector field* of the almost contact metric structure. This notion corresponds to that of almost Hermitian structure in complex geometry. We shall assume that all manifolds under considerations are C^∞ and connected.

When the tensor field φ is *parallel* with respect to the Levi-Civita connection, one gets a structure which is analogous to a Kähler structure. In this case $(M, \varphi, \xi, \eta, g)$ is called a *cosymplectic manifold*. We remark that another relevant and widely studied class of almost contact metric structures which is deeply related to the Kähler class is that of Sasakian manifolds. These two kind of geometric structures are radically different since for symplectic manifolds the 1-form η is necessarily *closed*, while for Sasakian manifolds it is a contact form. We refer the reader to Blair's monograph (Blair 2010) and to the survey (Cappelletti-Montano et al. 2013) for motivation and background about these concepts and a detailed account concerning relevant results in the literature.

Like in complex geometry, there is a meaningful weaker notion of *almost cosymplectic structure*, corresponding to that of *almost Kähler* structure: this is an almost contact metric structure characterized by the following requirements:

$$d\eta = 0, \quad d\Phi = 0,$$

where Φ is the fundamental 2-form of the underlying almost contact metric structure, defined as usual according to:

$$\Phi(X, Y) = g(X, \varphi Y).$$

Given an almost cosymplectic manifold $(M, \varphi, \xi, \eta, g)$, we shall denote by D the kernel of η, which is a integrable distribution in the sense of Frobenius. The following basic identity is known (cf. Olszak 1981):

$$Ric(\xi, \xi) = -||\nabla\xi||^2. \tag{2.1}$$

For every smooth vector field $Z \in \mathfrak{X}(M)$, we shall denote by A_Z the $(1, 1)$-tensor field defined by $A_Z(Y) = -\nabla_Y Z$. We recall that if Z is a Killing vector field, then for every $Y \in \mathfrak{X}(M)$ it holds:

$$Ric(Z, Y) = -div(A_Z Y) - tr(A_Z \circ A_Y), \tag{2.2}$$

see for instance (Kobayashi and Nomizu 1963, Ch. VI, Prop. 5.1).

A vector field Z will be called an *infinitesimal automorphism* of an almost contact metric manifold M provided its flow preserves the underlying geometric structure. In particular, such a Z is a Killing vector field. We recall the following useful fact:

Lemma 2.1 *Let $Z \in \mathfrak{X}(M)$ be an infinitesimal automorphism of an almost cosymplectic manifold. Then $\eta(Z)$ is a constant function.*

Proof Indeed, by Cartan's formula, since η is closed, one has:

$$0 = \mathcal{L}_Z\eta = (d \circ i_Z + i_Z \circ d)\eta = d\eta(Z).$$

3 Proof of the result

Let $(M, \varphi, \xi, \eta, g)$ be a compact, homogeneous η-Einstein almost cosymplectic manifold. Denote by $\mathfrak{g}$ the Lie algebra of a Lie group G acting smoothly and transitively on M as a group of automorphisms. By assumption, the Ricci tensor of g is of the form

$$Ric = ag + b\eta \otimes \eta, \quad a, b \in \mathbb{R}.$$

First, we show that $a + b = 0$. Fix $p \in M$ and set $v := \xi_p$; then, by homogeneity, v can be extended to an infinitesimal automorphism Z of M. Indeed, the mapping

$$A \in \mathfrak{g} \mapsto A^*_p \in T_pM$$

is surjective; here A^* denotes the fundamental vector field generated by A by means of the action, cf. e.g. Kobayashi et al. (1963, Ch. I). So it suffices to take $Z = A^*$, where $A \in \mathfrak{g}$ is chosen so that $A^*_p = v$.

Now, according to Lemma 2.1, we have $\eta(Z) = 1$, so that:

$$Ric(Z, \xi) = ag(Z, \xi) + b\eta(Z)\eta(\xi) = a + b.$$

Therefore, since Z is a Killing vector field, using (2.2), we get:

$$\int_M (a + b)dV = -\int_M tr(A_\xi \circ A_Z)dV = 0,$$

where the last equality is justified by the fact that A_ξ is a symmetric operator (being $d\eta = 0$), while A_Z is a skew-symmetric operator.

We have thus obtained that $a + b = 0$, which means that $Ric(\xi, \xi) = 0$, whence ξ is parallel according to (2.1). Moreover,

$$Ric = a(g - \eta \otimes \eta).$$

If $a = 0$, then M is Ricci-flat, so that the quoted Cappelletti-Montano and Pastore's result about the Einstein case applies, yielding that M is a cosymplectic manifold. So we can assume $a \neq 0$. Actually, we can show by an argument similar to the one used above that $a > 0$; indeed, choose now $v \in D_p$, $v \neq 0$, and consider again an infinitesimal automorphism Z such that $Z_p = v$. Then Lemma 2.1 guarantees that Z is actually a section of D. Using again (2.2) yields:

$$\int_M Ric(Z, Z)dV = -\int_M tr(A_Z)^2 dV \geq 0,$$

whence

$$a \int_M g(Z, Z)dV \geq 0,$$

so that $a > 0$. Hence, we are in a position to apply the result by Cappelletti Montano and Pastore concerning the η-Einstein case, yielding again that the structure of M is cosymplectic.

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