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Centers of Hecke algebras of complex reflection groups

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Abstract

We provide a dual version of the Geck–Rouquier Theorem (Geck and Rouquier in Finite Reductive Groups (Luminy, 1994), Progr. Math., vol. 141, Birkhäuser Boston, Boston, pp. 251–272, 1997) on the center of an Iwahori–Hecke algebra, which also covers the complex case. For the eight complex reflection groups of rank 2, for which the symmetrising trace conjecture is known to be true, we provide a new faithful matrix model for their Hecke algebra H. These models enable concrete calculations inside H. For each of the eight groups, we compute an explicit integral basis of the center of H.

1 Introduction

Let W be a finite complex reflection group and H the associated generic Hecke algebra, defined over the Laurent polynomial ring $R = \mathbb{Z}[u_1^{\pm}, \ldots, u_k^{\pm}]$, where $\{u_i\}_{1 \leqslant i \leqslant k}$ is a set of parameters whose cardinality depends on W. In 1999, Malle (1999, §5) proved that H is split semisimple when defined over the field $F = \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is a root of u_i of rank N_W, for some specific $N_W \in \mathbb{N}$. By Tits' deformation theorem (Geck and Pfeiffer 2000, Theorem 7.4.6), the specialization $v_j \mapsto 1$ induces a bijection $\mathrm{Irr}(H \otimes_R F) \to \mathrm{Irr}(W)$.

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A natural question is how the irreducible representations behave after specializing the parameters u_i to arbitrary complex numbers. If the specialized Hecke algebra is semisimple, Tits' deformation theorem still applies; the simple representations of the specialized Hecke algebra are parametrized again by $\mathrm{Irr}(W)$. However, if the specialized algebra is not semisimple one needs to find another way to parametrise the irreducible representations. One main obstacle in this direction is the lack of the description of the center $Z(H)$ of the Hecke algebra H (see Geck and Pfeiffer 2000, Lemma 7.5.10).

Apart from the real case (Geck and Rouquier 1997), there is not yet a known precise description of the center of the generic Hecke algebra H in the complex case, except for the groups G_4 and $G(4, 1, 2)$ provided by Francis (2007). In this paper, we introduce a new general method for computing an R-basis of the center of H. Let $\mathcal{B}$ be an R-basis of H. Expressing an arbitrary element $z \in Z(H)$ as a linear combination of $\mathcal{B}$, the conditions $sz = zs$, one for each generator s of H, give an R-linear system, whose solution describes a basis of $Z(H)$ as linear combinations of the elements of $\mathcal{B}$. This elementary approach has the following three difficulties:

- Calculations inside the Hecke algebra are very complicated, even for products of the form sb and bs, for $b \in \mathcal{B}$. In the last four years, there is a progress on this direction (see, for example, Boura et al. 2020a, b; Chavli and Chlouveraki 2022). However, it is still unclear if one can completely automate such calculations, since all the attempts so far use a lot of (long) computations made by hand.
- Solving the aforementioned R-linear system is not always easy.
  It is known that Gaussian elimination can suffer from coefficient explosion over a field of rational functions in several variables, such as the quotient field of the ring R of Laurent polynomials. As a result, the choices of dependent and independent variables can be crucial, since wrong choices can lead to dead ends.
- Even if the R-linear system can be solved (over the fraction field), the solution cannot be expected to lie in R. In fact, for the case of G_{12} a first attempt solving this system provided us with a solution not in R (details can be found in the project's webpage, https://www.eirinichavli.com/center.html).

However, we show that it is possible to overcome all these difficulties, as demonstrated by our results for particular examples. A natural first example is the smallest exceptional group, the group G_4. For this group we use a new approach on constructing an R-basis $\mathcal{B}$ (see Example 2.4), which allows us to work with a faithful matrix representation of H, rather than its usual presentation. Hence, we can

- automate calculations inside the Hecke algebra of G_4 and, in particular, compute the elements sb and bs from above,
- solve the R-linear system $sz = zs$, and
- verify that all solutions we obtain for z are in fact R-linear combinations of $\mathcal{B}$.

Our next goal is to explain the coefficients of these R-linear combinations. This allows us to describe the center $Z(H)$ for other complex reflection groups without relying on the solution of the R-linear system. In order to explain our findings, we first need to revisit the real case.

Let W be a real reflection group and H its associated Iwahori–Hecke algebra, which admits a standard basis $\{T_w : w \in W\}$. Denote by $\mathrm{Cl}(W)$ the set of conjugacy classes

of W, and choose a set of representatives $\{w_C \in C \mid C \in \mathrm{Cl}(W)\}$ such that each element w_C has minimal length in its class C. It can be shown that there exist uniquely determined polynomials $f_{w,C} \in R$, independent of the choice of the minimal length representatives w_C, the so-called *class polynomials* (see Geck and Pfeiffer 2000, §8.2), such that

$$\chi(T_w) = \sum_{C \in \mathrm{Cl}(W)} f_{w,C}\, \chi(T_{w_C})$$

for all $\chi \in \mathrm{Irr}(W)$. In other words, the column $\big(\chi(T_w)\big)_\chi$ of character values of T_w is an R-linear combination of the columns $\big(\chi(T_{w_C})\big)_\chi$ of the basis elements of H corresponding to the conjugacy class representatives of minimal length.

Clearly, for any choice $\{v_C \in C \mid C \in \mathrm{Cl}(W)\}$ of conjugacy class representatives, the square matrix $\big(\chi(T_{v_C})\big)_{\chi,C}$ of character values is invertible as it specializes to the character table of W. Hence, for each $w \in W$, there are uniquely determined coefficients $\zeta_{w,C}$ such that

$$\chi(T_w) = \sum_{C \in \mathrm{Cl}(W)} \zeta_{w,C}\, \chi(T_{v_C}).$$

However, the coefficients $\zeta_{w,C}$ cannot be expected to belong to R. That is why it is crucial to choose minimal length class representatives.

We denote by $\{T_w^\vee : w \in W\}$ the dual basis of $\{T_w : w \in W\}$ with respect to the standard symmetrising form (for the definition of the dual basis see, for example, Geck and Pfeiffer 2000, Definition 7.1.1). The following theorem has been shown by Geck and Rouquier (1997, §5.1), cf. Geck and Pfeiffer (2000, Theorem 8.2.3 and Corollary 8.2.4).

Theorem 1.1 *Let W be a finite real reflection group. The elements*

$$y_C = \sum_{w \in W} f_{w,C}\, T_w^\vee, \quad C \in \mathrm{Cl}(W),$$

form a basis of the center $Z(H)$.

We now examine the complex case. We first assume that the Hecke algebra H admits a symmetrising trace τ. Let $\{b_w : w \in W\}$ be a basis of H as R-module and let $\{b_w^\vee : w \in W\}$ be its dual basis with respect to τ. The lack of length function of a complex reflection group W cannot guarantee us the existence of class polynomials in the sense of the real case. However, it remains true that with respect to any choice of conjugacy class representatives v_C, for each element $w \in W$ there are coefficients $\zeta_{w,C} \in F$, which express the column $(\chi(b_w))_\chi$ of character values as a linear combination of the columns $\big(\chi(b_{v_C})\big)_\chi$. In this paper, we prove the following:

Theorem 1.2 *Let W be a finite complex reflection group. For any choice of conjugacy class representatives v_C, the elements*

$$y_C = \sum_{w \in W} \zeta_{w,C}\, b_w^\vee, \quad C \in \mathrm{Cl}(W),$$

form a basis of the center $Z(H \otimes_R F)$.

This theorem generalizes the Theorem 1.1 of Geck–Rouquier in such a way that it includes the complex case, at the expense of working over F in place of R. At the same time, it gains us some flexibility in terms of choosing the conjugacy class representatives. There is a dual version of the theorem that provides even more flexibility, as follows.

For each class $C \in \mathrm{Cl}(W)$, we choose again a representative $v_C \in C$ and we define coefficients $g_{w,C} \in F$ by the condition

$$\chi(b_w^\vee) = \sum_{C \in \mathrm{Cl}(W)} g_{w,C}\, \chi(b_{v_C}^\vee), \quad \text{for all } \chi \in \mathrm{Irr}(H \otimes_R F).$$

Theorem 1.3 *Let W be a finite complex reflection group. For any choice of conjugacy class representatives* v_C, *the elements*

$$z_C = \sum_{w \in W} g_{w,C}\, b_w, \quad C \in \mathrm{Cl}(W),$$

form a basis of the center $Z(H \otimes_R F)$.

As an illustration, and to state the fact that for the real case Theorems 1.2 and 1.3 give new bases, different from the one of Geck–Rouquier, we apply these theorems to the Coxeter group of type A_2 and express the resulting basis of $Z(H)$ as a linear combination of the standard basis $\{T_w : w \in W\}$ of H.

Example 1.4 Let W be a finite Coxeter group of type A_2 with generators s, t such that $sts = tst$. Then $\{1, s, st\}$ is a set of minimal length class representatives of W, and $\{1, sts, st\}$ is a set of maximal length class representatives. The Iwahori–Hecke algebra H is generated by elements T_s and T_t and defined over the ring $R = \mathbb{Z}[u_1^\pm, u_2^\pm]$. We set $c = u_1 + u_2$ and $d = -u_1 u_2$ (where only d is invertible in R). Then $T_s^2 = c\, T_s + d$ and $T_t^2 = c\, T_t + d$.

We have the following:

- Theorem 1.1, with minimal length class representatives yields the basis

$$T_1, \qquad d^{-1}(T_s + T_t) + d^{-2}\, T_{sts}, \qquad d^{-2}(T_{st} + T_{ts}) + cd^{-3}\, T_{sts}$$

- Theorem 1.2, with maximal length class representatives yields the basis

$$T_1, \qquad d^{-2}(T_s + T_t) + d^{-3}\, T_{sts}, \qquad -cd^{-2}(T_s + T_t) + d^{-2}(T_{st} + T_{ts})$$

- Theorem 1.3, with minimal length class representatives yields the basis

$$T_1, \qquad (T_s + T_t) + d^{-1}\, T_{sts}, \qquad (T_{st} + T_{ts}) + cd^{-1}\, T_{sts}$$

- Theorem 1.3, with maximal length class representatives yields the basis

$$T_1, \qquad d(T_s + T_t) + T_{sts}, \qquad -c(T_s + T_t) + (T_{st} + T_{ts})$$

Note that in all cases the coefficients of the basis elements of $Z(H)$ belong to $R = \mathbb{Z}[u_1^{\pm}, u_2^{\pm}]$, and that in the last case they even lie in the polynomial ring $\mathbb{Z}[u_1, u_2]$. □

It is worth mentioning here that Theorems 1.2 and 1.3 have a general proof, which does not use the case-by-case analysis, based on the classification of complex reflection groups (Shephard and Todd 1954).

As mentioned before, Theorems 1.2 and 1.3 provide bases of the center of H over the splitting field F. Choosing an arbitrary basis of the Hecke algebra and arbitrary class representatives, one cannot expect to obtain a basis of the center of H over R. In fact, for G_{12}, there is a choice of conjugacy class representatives v_C, where not all the coefficients $g_{w,C}$ of Theorem 1.3 belong to R (for details, see the project's web page, https://www.eirinichavli.com/center.html). However, for some choice of a basis $\{b_w : w \in W\}$ and of class representatives v_C, it might turn out that the coefficients $g_{w,C}$ of Theorem 1.3 belong to R, which means that we obtain a basis of the center $Z(H)$.

In fact, in this paper we show that for the groups $G_4, \ldots, G_8, G_{12}, G_{13}, G_{22}$, i.e., for all exceptional groups of rank 2, whose associated Hecke algebra is known to be symmetric, we can make such choices so that we obtain a basis of the center $Z(H)$. We conjecture that this is true for all complex reflection groups that satisfy the symmetrising trace Conjecture 2.3.

The next section of this paper explains in detail how we make these choices. The final section contains our main results. In our calculations we used some programs written in GAP, which one can find in the project's webpage (https://www.eirinichavli.com/center.html).

2 Choosing a basis

2.1 Hecke algebras

A complex reflection group W is a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by *pseudo-reflections* (these are non-trivial elements of W whose fixed points in $\mathbb{C}^n$ form a hyperplane). Real reflection groups, also known as *finite Coxeter groups* are particular cases of complex reflection groups.

We denote by K the *field of definition* of W, that is the field generated by the traces on $\mathbb{C}^n$ of all the elements of W. If $K \subseteq \mathbb{R}$, then W is a finite Coxeter group, and if $K = \mathbb{Q}$, then W is a Weyl group.

A complex reflection group W is *irreducible* if it acts irreducibly on $\mathbb{C}^n$ and, if that is the case, we call n the *rank* of W. Each complex reflection group is a direct product of irreducible ones and, hence, the study of reflection groups reduces to the irreducible case. The classification of irreducible complex reflection groups is due to Shephard and Todd (1954) and it is given by the following theorem.

Theorem 2.1 *Let $W \subset \mathrm{GL}_n(\mathbb{C})$ be an irreducible complex reflection group. Then, up to conjugacy, W belongs to precisely one of the following classes:*

- *The symmetric group S_{n+1}.*
- *The infinite family $G(de, e, n)$, where $d, e, n \in \mathbb{N}^*$, such that $(de, e, n) \neq (1, 1, n)$ and $(de, e, n) \neq (2, 2, 2)$, of all $n \times n$ monomial matrices whose non-zero entries are de-th roots of unity, while the product of all non-zero entries is a d-th root of unity.*
- *The 34 exceptional groups G_4, G_5, ..., G_{37} (ordered with respect to increasing rank).*

Let W be a complex reflection group. We denote by B the *complex braid group* associated to W, as defined in Broué et al. (1998, §2 B). A pseudo-reflection s is called *distinguished* if its only nontrivial eigenvalue on $\mathbb{C}^n$ equals $\exp(-2\pi\sqrt{-1}/e_s)$, where e_s denotes the order of s in W. Let S denote the set of the distinguished pseudo-reflections of W.

For each $s \in S$ we choose a set of e_s indeterminates $u_{s,1}, \ldots, u_{s,e_s}$, such that $u_{s,j} = u_{t,j}$ if s and t are conjugate in W. We denote by R the Laurent polynomial ring $\mathbb{Z}[u_{s,j}, u_{s,j}^{-1}]$. The *generic Hecke algebra* H associated to W with parameters $u_{s,1}, \ldots, u_{s,e_s}$ is the quotient of the group algebra $R[B]$ of B by the ideal generated by the elements of the form

$$(\sigma - u_{s,1})(\sigma - u_{s,2}) \cdots (\sigma - u_{s,e_s}), \tag{2.1}$$

where s runs over the conjugacy classes of S and σ over the set of *braided reflections* associated to the pseudo-reflection s (for the standard notion of a braided reflection associated to s one can refer to Broué et al. (1998, §2 B). It is enough to choose one relation of the form described in (2.1) per conjugacy class, since the corresponding braided reflections are conjugate in B.

We obtain an equivalent definition of H if we expand the relations (2.1). More precisely, H is the quotient of the group algebra $R[B]$ by the elements of the form

$$\sigma^{e_s} - a_{s,e_s-1}\sigma^{e_s-1} - a_{s,e_s-2}\sigma^{e_s-2} - \cdots - a_{s,0}, \tag{2.2}$$

where $a_{s,e_s-k} := (-1)^{k-1} f_k(u_{s,1}, \ldots, u_{s,e_s})$ with f_k denoting the k-th elementary symmetric polynomial, for $k = 1, \ldots, e_s$. Therefore, in the presentation of H, apart from the *braid relations* coming from the presentation of B, we also have the *positive Hecke relations*:

$$\sigma^{e_s} = a_{s,e_s-1}\sigma^{e_s-1} + a_{s,e_s-2}\sigma^{e_s-2} + \cdots + a_{s,0}. \tag{2.3}$$

We notice now that $a_{s,0} = (-1)^{e_s-1} u_{s,1} u_{s,2} \ldots u_{s,e_s} \in R^\times$. Hence, σ is invertible in H with

$$\sigma^{-1} = a_{s,0}^{-1}\sigma^{e_s-1} - a_{s,0}^{-1}a_{s,e_s-1}\sigma^{e_s-2} - a_{s,0}^{-1}a_{s,e_s-2}s^{e_s-3} - \cdots - a_{s,0}^{-1}a_{s,1}. \tag{2.4}$$

We call relations (2.4) the *inverse Hecke relations*.

If W is a real reflection group, H is known as the *Iwahori–Hecke algebra* associated with W (for more details about Iwahori–Hecke algebras one may refer, for example, to Geck and Pfeiffer (2000), §4.4). Iwahori–Hecke algebras admit a *standard basis* $(T_w)_{w \in W}$ indexed by the elements of W (see Bourbaki 2005, IV, §2). Broué, Malle and Rouquier conjectured a similar result for complex reflection groups (Broué et al. 1998, §4).

Conjecture 2.2 (The BMR freeness conjecture) *The algebra H is a free R-module of rank $|W|$.*

This conjecture is now a theorem, thanks to work of several people who used a case-by-case analysis approach, in order to prove the case of all irreducible complex reflection groups. A detailed state of the art of the proof can be found in Boura (2020b, Theorem 3.5).

A symmetrising trace on a free algebra is a trace map τ that induces a non-degenerate bilinear form, meaning that the determinant of the matrix $(\tau(bb'))_{b, b' \in \mathcal{B}}$ is a unit in the ring over which we define the algebra for some (and hence every) basis $\mathcal{B}$ of the algebra. For Iwahori–Hecke algebras there exists a unique symmetrising trace, given by $\tau(T_w) = \delta_{1_w}$ Bourbaki (2005, IV, §2). Broué, Malle and Michel conjectured the existence of a symmetrising trace also for non-real complex reflection groups (Broué et al. 1999, §2.1, Assumption 2(1)).

Conjecture 2.3 (The BMM symmetrising trace conjecture) *There exists a linear map $\tau : H \to R$ such that:*

(1) *τ is a symmetrising trace, that is, we have $\tau(h_1 h_2) = \tau(h_2 h_1)$ for all $h_1, h_2 \in H$, and the bilinear map $H \times H \to R$, $(h_1, h_2) \mapsto \tau(h_1 h_2)$ is non-degenerate.*
(2) *τ becomes the canonical symmetrising trace on $K[W]$ when $u_{s,j}$ specialises to $\exp(2\pi\sqrt{-1}\, j/e_s)$ for every $s \in S$ and $j = 1, \ldots, e_s$.*
(3) *τ satisfies*

$$\tau(T_{b^{-1}})^* = \frac{\tau(T_{b\pi})}{\tau(T_\pi)}, \quad \textit{for all } b \in B(W), \tag{2.5}$$

where $b \mapsto T_b$ denotes the restriction of the natural surjection $R[B] \to H$ to B, $x \mapsto x^*$ is the automorphism of R given by $u_{s,j} \mapsto u_{s,j}^{-1}$ and π the element $z^{|Z(W)|}$, with z being the image of a suitable generator of the center of B inside H.

Since we have the validity of the BMR freeness conjecture, we know by Broué et al. (1999, §2.1) that if there exists such a linear map τ, then it is unique. If this is the case, we call τ the *canonical symmetrising trace on H*.

Malle and Michel (2010, Proposition 2.7) proved that if the Hecke algebra H admits a basis $\mathcal{B} \subset B$ consisting of braid group elements that satisfies certain properties (among them that $1 \in \mathcal{B}$), then Condition 2.5 is equivalent to:

$$\tau(T_{x^{-1}\pi}) = 0, \quad \text{for all } x \in \mathcal{B} \setminus \{1\}. \tag{2.6}$$

Apart from the real case, the BMM symmetrising trace conjecture is known to hold for a few exceptional groups and for the infinite family (detailed references can be found in Chavli and Chlouveraki 2022, Conjecture 3.3).

We now describe the representation theory of Hecke algebras. In Malle (1999, §5) Malle associates to each complex reflection group W a positive integer N_W and he defines for each $s \in S$, a set of e_s indeterminates $v_{s,1}, \ldots, v_{s,e_s}$ by the property

$$v_{s,j}^{N_W} = \exp(-2\pi\sqrt{-1}\, j/e_s) u_{s,j}. \tag{2.7}$$

We denote by F the field $\mathbb{C}(v_{s,j})$ and by extension of scalars we obtain the algebra $FH := H \otimes_R F$, which is split semisimple (Malle 1999, Theorem 5.2). By Tits' deformation theorem (Geck and Pfeiffer 2000, Theorem 7.4.6), the specialization $v_{s,j} \mapsto 1$ induces a bijection $\text{Irr}(FH) \to \text{Irr}(W)$.

Models of irreducible representations of the Hecke algebra FH associated with certain irreducible complex reflection groups have been computed by Malle and Michel (2010), and are readily available in Jean Michel's development version (Michel 2015) of the CHEVIE package (Geck et al. 1996). In this paper, we use these models to evaluate the irreducible characters on some particular elements, when constructing an explicit basis of the center of the Hecke algebra.

2.2 Coset table

Let W be a complex reflection group with associated Hecke algebra H defined over R. The goal of this paper is to provide, at least in some examples, a basis of the center $Z(H)$ of H as R-module. For this purpose, we first find a basis $\mathcal{B}$ of the Hecke algebra and then we describe the basis elements of $Z(H)$ as linear combinations of elements of $\mathcal{B}$.

In this section we explain the method we use in order to find a basis $\mathcal{B}$ for the exceptional groups $G_4, \ldots, G_8, G_{12}, G_{13}, G_{22}$. These groups are exactly the exceptional complex reflection groups of rank 2 for which we know, apart from the validity of the BMR freeness conjecture 2.2, the validity of the BMM symmetrising trace conjecture 2.3 as well. Our method is not an algorithm and we cannot be sure it works in general since, as we will see in a while, one needs to make some crucial choices, which are a product of experimentation and experience. At the end of this section we give in detail the example of G_4, where the reader can see thoroughly our methodology and arguments.

We recall that W is generated by distinguished pseudo-reflections s. We choose a particular generator s_0 of W and we denote by W' the parabolic subgroup of W generated by s_0 and by H' the subalgebra of H generated by σ_0. The action of W on the cosets of W' defines a graph with vertex set $\{W'x \mid x \in W\}$ and edges $W'x \xrightarrow{s} W'xs$, for s running over the generators of H. In the project's webpage (https://www.eirinichavli.com/center.html) the reader can find these graphs for the examples we are dealing with in this paper.

We now choose class representatives x_i as follows: We choose some representatives as the *anchor coset representatives*, and we pick representatives for the remaining cosets along the spanning tree. We always choose $x_1 = 1$.

The coset representatives x_i are in fact explicit words in generators of W. These generators are in one to one correspondence with generators of the Hecke algebra H, by sending s to σ. Hence, we can obtain from the elements x_i corresponding elements inside the Hecke algebra, which we also denote by x_i.

Our goal now is to prove that this chosen set $\{x_i : i = 1, \ldots, |W/W'|\}$ is a basis of H as H'-module. Since H is a free H'-module of dimension $|W/W'|$ (see, for example, Chavli and Chlouveraki 2022), we only have to prove that $\{x_i\}$ is a spanning set for H. By construction, we always have $x_1 = 1$ and, hence, it is enough to prove that for each x_i, the elements $x_i.\sigma$ are linear combinations of the form $\sum_i h_i \cdot x_i$, where $h_i \in H'$.

In order to prove that, we construct a *coset table*, where we list this linear combination not only for the elements $x_i.\sigma$, but also for the elements $x_i.\sigma^{-1}$. The reason we calculate these extra linear combinations is because they are prerequisite to calculating the linear combinations for the elements $x_i.\sigma$.

In order to fill the coset table we use a program created in GAP. This program uses the positive and inverse Hecke relations (2.3) and (2.4) and also some simple hand-calculations. On the project's webpage (https://www.eirinichavli.com/center.html) one can find these programs for the aforementioned exceptional groups of rank 2.

The completed coset table and the fact that H' is a free R-module provides a basis for the Hecke algebra H over R as follows: $\mathcal{B} = \{x_i,\ \sigma_0 x_i, \ldots, \sigma_0^{e_s-1} x_i\ :\ i = 1, \ldots, |W/W'|\}$.

Our goal now is to express every element of the Hecke algebra as R-linear combination of the elements of $\mathcal{B}$. In order to do that, we use again the completed coset table. The expression of $x_i.\sigma$ as linear combination of the form $\sum_i h_i \cdot x_i$, $h_i \in H'$ yields a representation ρ for H as a free module over the subalgebra H'. We use the matrix models of this representation in order to compute the image of each word x in generators σ (and their inverses) inside the Hecke algebra H as the image of the vector

$$1_H = 1_{H'}x_1 + 0x_2 + 0x_3 + \cdots + 0x_{|W/W'|} = (1_{H'}, 0, 0, \ldots, 0)$$

under the product of the matrices corresponding to the letters of the word x. We can see this as follows: Let x be the word $\sigma_{i_1}^{m_1} \cdots \sigma_{i_r}^{m_r}$, where $m_1, \ldots, m_r \in \mathbb{Z}^*$. Then the image of x inside H can be computed as the product $1_H \cdot \rho(\sigma_{i_1})^{m_1} \cdots \rho(\sigma_{i_r}^{m_r})$. This is a vector with coefficients in H' and, hence, it corresponds to an H'-linear combination of x_i's and, hence, to an R-linear combination of elements in $\mathcal{B}$.

In order to make this clearer to the reader, we give the following example of the exceptional group G_4:

Example 2.4 Let $G_4 = \langle s_1, s_2 \mid s_1^3 = s_2^3 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$. The Hecke algebra H of W is defined over $R = \mathbb{Z}[u_1^{\pm}, u_2^{\pm}, u_3^{\pm}]$ and it admits the following presentation:

$$H = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2,\ \sigma_1^3 = a\sigma_1^2 + b\sigma_1 + c,\ \sigma_2^3 = a\sigma_2^2 + b\sigma_2 + c \rangle,$$

with suitable $a, b, c \in R$. For the sake of brevity, we set $\sigma_1' = \sigma_1^{-1}$ and $\sigma_2' = \sigma_2^{-1}$. Let $z := (s_1 s_2)^3 \in Z(W)$ and let $w := s_1 s_2 s_1 = s_2 s_1 s_2$. We have $z = (s_1 s_2)^3 = (s_1 s_2 s_1)(s_2 s_1 s_2) = w^2$. We also denote by W' the parabolic subgroup of W generated by s_1 and by H' the subalgebra of H generated by σ_1.

As we have already mentioned, the action of W on the cosets of W' defines a graph with vertex set $\{W'x \mid x \in W\}$ and edges $W'x \xrightarrow{u} W'xu$, labelled by $u \in \{s_1, s_2\}$. The following diagram shows this action graph, except for a few edges:
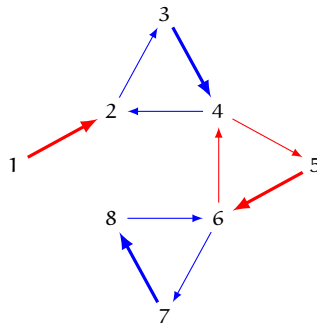
Here, instead of labeling the edges with u, we use the colours red and blue. More precisely, blue edges belong to generator s_1 and red edges belong to generator s_2. Fat edges indicate a spanning tree of the coset graph. The vertices are labeled $1, 2, \ldots, 8$, corresponding to coset representatives $x_1 = 1$, $x_2 = s_2$, $x_3 = s_2 s_1 s_2$, $x_4 = s_2 s_1 s_2 s_1$, $x_5 = zx_1$, $x_6 = zx_2$, $x_7 = zx_3$, $x_8 = zx_4$. We make this choice of representatives as follows: We choose $x_1 = w^0$, $x_3 = w^1$, $x_5 = w^2$ and $x_7 = w^3$ as anchor coset representatives and we pick representatives for the remaining cosets along the spanning tree: $x_{2\ell} = x_{2\ell-1} u$, $\ell = 1, 2, 3, 4$ with generator $u = s_1, s_2$ as in the edge connecting coset $2\ell-1$ to coset 2ℓ in the coset graph.

By sending s_1 and s_2 to σ_1 and σ_2, respectively we can obtain from the elements x_i, z and w corresponding elements inside the Hecke algebra, which we also denote by x_i, z and w. The elements z and w are also central elements in H (see Bessis 2015, Theorem 12.8 and Broué et al. 1998, Theorem 2.2.4).

As we have explained earlier, in order to prove that the set $\{x_i, , i = 1, \ldots, 8\}$ is a basis of H as H'-module we construct the following coset table.

x_i, $i = 1, \ldots, 8$	$x_i . \sigma_1$	$x_i . \sigma_2$	$x_i . \sigma_1'$	$x_i . \sigma_2'$
$x_1 = z^0$	$\sigma_1 \cdot x_1$	x_2	$\sigma_1' \cdot x_1$	
$x_2 = z^0 \, \sigma_2$	$\sigma_1' \cdot x_3$		$x : \sigma_1'$	x_1
$x_3 = z^0 \, \sigma_2 \sigma_1 \sigma_2$	x_4	$\sigma_1 \cdot x_3$	$\sigma_1 \cdot x_2$	$\sigma_1' \cdot x_3$
$x_4 = z^0 \, \sigma_2 \sigma_1 \sigma_2 \sigma_1$	$x_4 : \sigma_1$	$\sigma_1' \cdot x_5$	x_3	$x_4 : \sigma_2'$
$x_5 = z^1$	$\sigma_1 \cdot x_5$	x_6	$\sigma_1' \cdot x_5$	$\sigma_1 \cdot x_4$
$x_6 = z^1 \, \sigma_2$	$\sigma_1' \cdot x_7$	$x_6 : \sigma_2$	$x_6 : \sigma_1'$	x_5
$x_7 = z^1 \, \sigma_2 \sigma_1 \sigma_2$	x_8	$\sigma_1 \cdot x_7$	$\sigma_1 \cdot x_6$	$\sigma_1' \cdot x_7$
$x_8 = z^1 \, \sigma_2 \sigma_1 \sigma_2 \sigma_1$	$x_8 : \sigma_1$		x_7	

In this table, some notation is used as short-hand for more complex expressions. More precisely:

- $x_i : u$ indicates that $x_i.u$ can be computed from other entries in the table by using the relation $u = a + bu' + cu'u'$. This relation is obtained from the positive Hecke relation (2.3) if we multiply both sides with $u'u'$. We see how we use such a relation in the example of $x_4.\sigma_1$. We have: $x_4.\sigma_1 = ax_4 + bx_4.\sigma_1' + cx_4.\sigma_1'\sigma_1'$. We now notice that $x_4.\sigma_1' = x_3$. Therefore, $x_4.\sigma_1 = ax_4 + bx_3 + cx_3.\sigma_1'$. Since $x_3.\sigma_1' = \sigma_1.x_2$ we have that $x_4.\sigma_1 = ax_4 + bx_3 + c\sigma_1.x_2$.
- Similarly, $x : u'$ indicates that $x.u'$ can be computed by using the inverse Hecke relation (2.4) $u' = c^{-1}u^2 - ac^{-1}u - bc^{-1}$.

It remains now to compute the entries $x_1.\sigma_2'$, $x_2.\sigma_2$, $x_8.\sigma_2$ and $x_8.\sigma_2'$. We compute these four excluded cases as follows, using braid relations and existing entries from the coset table.

$$x_2.\sigma_2 = x_2.(\sigma_2\sigma_1\sigma_2)\sigma_2'\sigma_1' = x_2.(\sigma_1\sigma_2\sigma_1)\sigma_2'\sigma_1' = x_4.\sigma_2'\sigma_1'$$
$$x_8.\sigma_2' = x_8.(\sigma_2'\sigma_1'\sigma_2')\sigma_2\sigma_1 = x_8.(\sigma_1'\sigma_2'\sigma_1')\sigma_2\sigma_1 = x_6.\sigma_2\sigma_1$$
$$x_1.\sigma_2' = x_1 : \sigma_2'$$
$$x_8.\sigma_2 = x_8 : \sigma_2$$

The completed coset table and the fact that H' is a free R-module with basis $\{1, \sigma_1, \sigma_1^2\}$ proves the following:

Proposition 2.5 *With the above notation*

(i) H *is a free* H'*-module with basis* $\{x_i : i = 1, \ldots, 8\}$.
(ii) H *is a free* R*-module with basis* $\{b_j : j = 1, \ldots, 24\} = \{x_i,\ \sigma_1 x_i,\ \sigma_1^2 x_i\ :\ i = 1, \ldots, 8\}$.

We will now express every element of the Hecke algebra as R-linear combination of the elements of $\{b_j : j = 1, \ldots, 24\}$, using the following representation for H as a free module over the subalgebra H', which comes from the completed coset table:

$$\rho(\sigma_1) := \left[\begin{array}{cccc|cccc}
\sigma_1 & . & . & . & . & . & . & . \\
. & . & \sigma_1' & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & c\sigma_1 & b & a & . & . & . & . \\
\hline
. & . & . & . & \sigma_1 & . & . & . \\
. & . & . & . & . & . & \sigma_1' & . \\
. & . & . & . & . & . & . & 1 \\
. & . & . & . & . & c\sigma_1 & b & a
\end{array}\right]$$

$\rho(\sigma_2)$

$$:= \left[\begin{array}{cccc|cccc}
. & 1 & . & . & . & . & . & . \\
. & . & -bc^{-1} & . & -ac^{-1}\sigma_1'' & -bc^{-2}\sigma_1' & -ac^{-2}\sigma_1'' & c^{-2}\sigma_1'' \\
. & . & \sigma_1 & . & . & . & . & . \\
\hline
. & . & . & . & \sigma_1' & . & . & . \\
. & . & . & c\sigma_1 & b & a & . & . \\
. & . & . & . & . & 1 & . & . \\
c^3\sigma_1^2 & bc^2\sigma_1^2 & bc^2 + b^2c\sigma_1 & bc\sigma_1^2 & -a^2c + b^2\sigma_1 & ac & -a^2 + a\sigma_1 & a
\end{array}\right]$$

where $\sigma_1'' = \sigma_1^{-2}$. We recall that $\sigma' = \sigma^{-1} = -bc^{-1} - ac^{-1}\sigma - c^{-1}\sigma^2$ and, hence, $\sigma'' = -bc^{-1}\sigma^{-1} - ac^{-1} - c^{-1}\sigma_1 = -bc^{-1}(-bc^{-1} - ac^{-1}\sigma - c^{-1}\sigma^2) - ac' - c'\sigma_1$. Therefore, the entries of the matrices $\rho(\sigma_1)$ and $\rho(\sigma_2)$ involve only positive powers of σ_1.

In the following example, one can see how we can use ρ in order to express every element of the Hecke algebra as an R-linear combination of elements in the basis $\{b_j : j = 1, \ldots, 24\} = \{x_i, \ \sigma_1 x_i, \ \sigma_1^2 x_i : i = 1, \ldots, 8\}$.

$$\begin{aligned}
\sigma_1^2 \sigma_2^2 &= (1_{H'}, 0, 0, 0, 0, 0, 0, 0) \cdot \rho(\sigma_1) \cdot \rho(\sigma_1) \cdot \rho(\sigma_2) \cdot \rho(\sigma_2) \\
&= (0, \ 0, \ -bc^{-1}\sigma_1^2, \ 0, \ -ac^{-1}, \ -bc^{-2}\sigma_1, \ -ac^{-2}, \ c^{-2}) \\
&= 0x_1 + 0x_2 - bc^{-1}\sigma_1^2 x_3 + 0x_4 - ac^{-1}x_5 - bc^{-2}\sigma_1 x_6 - ac^{-2}x_7 + c^{-2}x_8 \\
&= -bc^{-1}b_9 - ac^{-1}b_{13} - bc^{-2}b_{17} - ac^{-2}b_{19} + c^{-2}b_{22}.
\end{aligned}$$

$\square$

3 Center

Let W be a complex reflection group and let H be the associated Hecke algebra, defined over the Laurent polynomial ring $R = \mathbb{Z}[u_{s,j}, \ u_{s,j}^{-1}]$. We recall that F denotes the splitting field $\mathbb{C}(v_{s,j})$, with the indeterminates $v_{s,j}$ as defined in (2.7). We also recall that FH denotes the algebra $H \otimes_R F$.

As we have mentioned in Sect. 2, we have the validity of the BMR freeness conjecture 2.2. As a result, we can fix a basis $\mathcal{B} = \{b_w : w \in W\}$ of H, indexed by the elements of W. We now assume also the validity of the BMM symmetrising trace conjecture 2.3, meaning that H admits a unique symmetrising trace τ. We denote by $\mathcal{B}^\vee = \{b_w^\vee : w \in W\}$ the dual basis of $\mathcal{B}$ with respect to τ, uniquely determined by the condition $\tau(b_{w_1} b_{w_2}^\vee) = \delta_{w_1, w_2}$ for all $w_1, w_2 \in W$.

We denote now by $\mathrm{Cl}(W)$ the set of conjugacy classes of elements of W. For each class $C \in \mathrm{Cl}(W)$, we choose a representative $w_C \in C$. The square matrix $\left(\chi(b_w)\right)_{\chi, C}$ of character values is invertible as it specializes to the character table of W. Hence, for each $w \in W$, there are uniquely determined coefficients $f_{w,C} \in F$ by the condition

$$\chi(b_w) = \sum_{C \in \mathrm{Cl}(W)} f_{w,C} \, \chi(b_{w_C})$$

for all $\chi \in \mathrm{Irr}(FH)$. These coefficients depend on the choice of the elements w_C and of the basis $\mathcal{B}$.

Remark 3.1 Let W be a real reflection group. Then, we choose $\mathcal{B}$ to be the standard basis $\{T_w : w \in W\}$ and w_C an element of minimal length in C. We know (Geck and Pfeiffer 2000, §8.2.2 and §8.2.3) that the coefficients $f_{w,C}$ are independent of the actual choice of the elements w_C and they belong to R. In this case, $f_{w,C}$ are known as *class polynomials*.

The following theorem has been shown by Geck and Rouquier in the real case (see Geck and Rouquier 1997, §5.1 or Geck and Pfeiffer 2000, Theorem 8.2.3 and Corollary 8.2.4). However, the proof in general for every complex reflection group is slightly different from the original, as the prior existence of class polynomials in the complex case cannot be assumed.

Theorem 3.2 *Let W be a complex reflection group. The elements*

$$y_C = \sum_{w \in W} f_{w,C}\, b_w^{\vee}, \quad C \in \mathrm{Cl}(W),$$

form a basis of the center $Z(FH)$.

Proof For each class $C \in \mathrm{Cl}(W)$, we define the function $f_C \colon H \to F$, $b_w \mapsto f_{w,C}$. We first prove that f_C is a trace function. For each $b_w \in \mathcal{B}$ and for each $\chi \in \mathrm{Irr}(FH)$ we have:

$$\chi(b_w) = \sum_{C \in \mathrm{Cl}(W)} f_{w,C}\, \chi(b_{w_C}) = \sum_{C \in \mathrm{Cl}(W)} f_C(b_w)\, \chi(b_{w_C})$$
$$= \sum_{C \in \mathrm{Cl}(W)} \chi(b_{w_C})\, f_C(b_w).$$

Therefore,

$$\chi = \sum_{C \in \mathrm{Cl}(W)} \chi(b_{w_C}) f_C, \quad \text{for all } \chi \in \mathrm{Irr}(FH). \tag{3.1}$$

The matrix $(\chi(b_{w_C}))_{\chi,C}$ specializes to the character table of W, which has a nonzero determinant and, hence, it is invertible. We denote its inverse by $(i(C, \chi))_{\chi,C}$. Therefore, it follows from (3.1) that

$$f_C = \sum_{\chi \in \mathrm{Irr}(FH)} i(C, \chi)\, \chi,$$

which proves that f_C is a trace function.

We now prove that the set $\{f_C\}_C$ is a basis of the space of trace functions on FH. Since the algebra FH is split semisimple, we know (Geck and Pfeiffer 2000, Exercise 7.4(b)) that the set $\mathrm{Irr}(FH)$ is a basis for the space of trace functions on FH. Therefore, from Eq. (3.1) we conclude that $\{f_C\}_C$ is a spanning set of this space. We now prove that the set $\{f_C\}_C$ is a linearly independent set of trace functions. For this purpose, it suffices to prove that $f_C(b_{w_{C'}}) = \delta_{C,C'}$. By definition, $f_C(b_{w_{C'}}) = f_{w,C'}$ and $f_{w,C'}$ is the coefficient of the column of the character value $\chi(b_{w_{C'}})$, when the column $\chi(b_w)$ is expressed as a linear combination of the character values $\chi(b_{w_C})$. We have

$$\chi(b_{w_C}) = 1 \cdot \chi(b_{w_C}) + 0,$$

therefore $f_{w,C'} = 0$, for all $C' \neq C$ and $f_{w,C} = 1$. We conclude that the set $\{f_C\}_C$ is a basis of the space of trace functions on FH.

We now prove that the set $\{y_C\}_C$ is a basis of the center $Z(FH)$. Since we assume that the algebra FH admits a symmetrising trace, we can apply (Geck and Pfeiffer 2000, Lemma 7.1.7), which states that the set $\{f_C^*\}_C$ is a basis of $Z(FH)$, where f_C^* denotes the dual of $f_C \in \mathrm{Hom}_F(FH, F)$. We have:

$$f_C^* = \sum_w f_C(b_w) b_w^{\vee} = \sum_w f_{w,C} b_w^{\vee} = y_C$$

Therefore, the set $\{y_C\}_C$ is a basis of the center $Z(FH)$. $\qquad\square$

We now prove a *dual* version of Theorem 3.2. For each class $C \in \mathrm{Cl}(W)$, we choose again a representative $w_C \in C$ and we define coefficients $g_{w,C} \in F$ by the condition

$$\chi(b_w^{\vee}) = \sum_{C \in \mathrm{Cl}(W)} g_{w,C}\, \chi(b_{w_C}^{\vee}), \quad \text{for all } \chi \in \mathrm{Irr}(FH).$$

Theorem 3.3 *The elements*

$$z_C = \sum_{w \in W} g_{w,C}\, b_w, \quad C \in \mathrm{Cl}(W),$$

form a basis of the center $Z(FH)$.

Proof For each class $C \in \mathrm{Cl}(W)$, we define the function $g_C \colon H \to F$, $b_w^{\vee} \mapsto g_{w,C}$. As in the proof of Theorem 3.2, we prove that $\{g_C\}_C$ is a basis of the space of trace functions on FH. For each $b_w \in \mathcal{B}$ and for each $\chi \in \mathrm{Irr}(FH)$ we have:

$$\chi(b_w^{\vee}) = \sum_{C \in \mathrm{Cl}(W)} g_{w,C}\, \chi(b_{w_C}^{\vee}) = \sum_C g_C(b_w^{\vee}) \chi(b_{w_C}^{\vee}) = \sum_C \chi(b_{w_C}^{\vee})\, g_C(b_w^{\vee})$$

Therefore,

$$\chi = \sum_{C \in \mathrm{Cl}(W)} \chi(b_{w_C}^{\vee}) g_C, \quad \text{for all } \chi \in \mathrm{Irr}(FH). \tag{3.2}$$

The matrix $(\chi(b_{w_C}^{\vee}))_{\chi,C}$ is invertible, since it specializes to the character table of W, which has a nonzero determinant. We denote its inverse by $(j(C,\chi))_{\chi,C}$ and Eq. (3.2) becomes:

$$g_C = \sum_{\chi \in \mathrm{Irr}(FH)} j(C,\chi)\,\chi$$

and, hence, g_C is a trace function. Using the same arguments again as in the proof of Theorem 3.2, we have $g_C(b_{C'}^\vee) = \delta_{C,C'}$ and, hence, together with Eq. (3.2) we conclude that the set $\{g_C\}_C$ is a basis of the space of trace functions on FH.

We now prove that the set $\{z_C\}_C$ is a basis of the center $Z(FH)$. Since we assume that the algebra FH admits a symmetrising trace, we can apply (Geck and Pfeiffer 2000, Lemma 7.1.7), which states that the set $\{g_C^*\}_C$ is a basis of $Z(FH)$, where g_C^* denotes the dual of $g_C \in \mathrm{Hom}_F(FH, F)$. We have:

$$g_C^* = \sum_w g_C(b_w^\vee)(b_w^\vee)^\vee = \sum_w g_{w,C} b_w = z_C$$

Therefore, the set $\{z_C\}_C$ is a basis of the center $Z(FH)$.   □

Note that the elements z_C depend on the choice of the basis $\mathcal{B}$ and of the class representatives w_C, $C \in Cl(W)$.

We focus now on the exceptional groups we have described in Sect. 2, namely the groups G_n, where $n \in \{4, \ldots, 8, 12, 13, 22\}$. In Sect. 2 we have made a particular choice of a basis $\mathcal{B}$ and we managed with the help of the coset table to express each element of the Hecke algebra as linear combination of elements in $\mathcal{B}$. In particular, we can express any product bb', with $b, b' \in \mathcal{B}$ as a linear combination of the elements in $\mathcal{B}$. We use this linear combination in order to give another proof the BMM symmetrising trace conjecture 2.3 (this conjecture is known to hold for these complex reflection groups, as we have mentioned in Sect. 2.1).

We define a linear map $\tau \colon H \to R$ by setting $\tau(\sum_{b \in \mathcal{B}} \alpha_b b) = \alpha_1$. We can now calculate the Gram matrix $A = (\tau(bb'))_{b,b' \in \mathcal{B}}$ and prove the following:

Proposition 3.4 *Let W be one of the groups G_n, where $n \in \{4, \ldots, 8, 12, 13, 22\}$. With the above choice of $\mathcal{B}$, we have:*

(i) *The matrix A is symmetric and its determinant is a unit in R.*
(ii) *Condition (2.6) is satisfied.*

We note here that in the project's webpage (https://www.eirinichavli.com/center.html) we give explicitly for each group the determinant of the Gram matrix A.

Our goal is to find the coefficients $g_{w,C}$. By their definition, we need to find the dual basis $\mathcal{B}^\vee$. More precisely, we need to express each $b_i^\vee \in \mathcal{B}^\vee$ as R-linear combination in elements of $\mathcal{B}$. Let $b_i^\vee = a_1^i b_1 + a_2^i b_2 + \cdots + a_{|W|}^i b_{|W|}$ be this linear combination. Then, we have the following:

$$\begin{pmatrix} \tau(b_i^\vee b_1) \\ \tau(b_i^\vee b_2) \\ \vdots \\ \tau(b_i^\vee b_{|W|}) \end{pmatrix} = A \begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_{|W|}^i \end{pmatrix}$$

By Proposition 3.4(i) A is invertible in R. Moreover, we know that the dual basis is determined by the condition $\tau(b_k^\vee b_j) = \delta_{kj}$. Therefore:

$$\begin{pmatrix} a_1^i \\ a_2^i \\ \vdots \\ a_{|W|}^i \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \; i - \text{place} \; = \text{i-th column of } A^{-1}$$

Having now expressed each $b_i^\vee \in \mathcal{B}^\vee$ as R-linear combination in elements of $\mathcal{B}$, it remains to make a choice of class representatives w_C, which will allow us to calculate the coefficients $g_{w,C}$. We give for each group this particular choice of representatives in the webpage of this project (https://www.eirinichavli.com/center.html). The following result is the main theorem of this paper.

Theorem 3.5 *Let W be the exceptional group G_n, where $n \in \{4, \ldots, 8, 12, 13, 22\}$. There exists a choice of a basis $\{b_w : w \in W\}$ of the Hecke algebra H of W and a choice of conjugacy class representatives $\{w_C : C \in \mathrm{Cl}(W)\}$, such that the coefficients $g_{w,C}$ belong to R and, hence, the set $\{z_C : C \in \mathrm{Cl}(W)\}$ is a basis of $Z(H)$.*

For a better understanding of the above theorem, we revisit the example of G_4.

Example 3.6 Let W be the complex reflection group G_4. In the Example 2.4 we saw that the associated Hecke algebra H admits the basis

$$\{b_{3k+m} : k = 0, \ldots, 7, \; m = 1, 2, 3\},$$

defined as $b_{3k+m} = \sigma_1^{m-1} x_{k+1}$, where $x_1 = 1$, $x_2 = \sigma_2$, $x_3 = \sigma_2\sigma_1\sigma_2$, $x_4 = \sigma_2\sigma_1\sigma_2\sigma_1$, $x_5 = zx_1$, $x_6 = zx_2$, $x_7 = zx_3$, $x_8 = zx_4$. We now make the following choice for the class representatives w_C:

$$w_{C_1} = b_1, \; w_{C_2} = b_{10}, \; w_{C_3} = b_{13}, \; w_{C_4} = b_{15}, \; w_{C_5} = b_{22},$$
$$w_{C_6} = b_{23}, \; w_{C_7} = b_{24}.$$

The evaluation of the irreducible characters of H on the dual basis $\{b_i^\vee\}$ yields the elements $g_{i,C}$ and hence an explicit basis of $Z(FH)$:

$$z_1 = ac^3(b_3 + b_5) + (abc^2 + c^3)(b_6 + b_7) + (bc^2 + ab^2c - a^2c^2)b_8$$
$$+ ac^2(b_9 + b_{11}) + (abc + c^2)b_{12} + (2bc + ab^2 - a^2c)b_{14}$$
$$+ c(b_{18} + b_{19}) + b\, b_{20} + b_{24}$$
$$z_2 = c^3 b_3 + bc^2(b_6 + b_7) + b^2c\, b_8 + bc\, b_{12} + (-ac + b^2)b_{14}$$
$$+ c\, b_{17} - a\, b_{20} + b_{21} + b_{23}$$
$$z_3 = c(b_{14} + b_{16}) - a\, b_{19} + b_{20} + b_{22}$$

$$z_4 = c^2 b_5 - ac\, b_8 + c(b_9 + b_{11}) - a\, b_{14} + b_{15}$$
$$z_5 = b_{13}$$
$$z_6 = c(b_2 + b_4) - a\, b_7 + b_8 + b_{10}$$
$$z_7 = b_1$$

As we can see, the elements $g_{i,C}$ are actually elements in R and, hence, the set $\{z_1, \dots, z_7\}$ is indeed a basis of $Z(H)$. □

Based on these examples and the fact that complex reflection groups generalize the properties of real reflection groups, we believe that one can find in this way a basis $\{z_C : C \in Cl(W)\}$ of $Z(H)$ for each complex reflection group W. Therefore, we state the following conjecture:

Conjecture 3.7 *Let W be a complex reflection group. There exists a choice of a basis $\{b_w : w \in W\}$ of the Hecke algebra H of W, and a choice of conjugacy class representatives $\{w_C : C \in Cl(W)\}$ such that the construction of Theorem 3.3 yields polynomial coefficients $g_{w,C} \in R$, and hence a basis $\{z_C : C \in Cl(W)\}$ of $Z(H)$.*

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The project's webpage. https://www.eirinichavli.com/center.html

Note added in proof We mention here that, after this work was completed, Hu and Shi (2022) proved the aforementioned conjecture for the groups of type $G(d, 1, n)$.