



PBW filtration and monomial bases for Demazure modules in types A and C

George Balla¹ · Ghislain Fourier¹ · Kunda Kambaso¹

Received: 21 June 2022 / Accepted: 31 August 2022 / Published online: 17 September 2022
© The Author(s) 2022

Abstract

We characterise the symplectic Weyl group elements such that the FFLV basis is compatible with the PBW filtration on symplectic Demazure modules, extending type A results by the second author. Surprisingly, the number of such elements depends not on the type A or C of the Lie algebra but on the rank only.

Keywords PBW filtration · FFLV basis · Demazure modules

Mathematics Subject Classification 05E10 · 14M15 · 17B10

1 Introduction

Let \mathfrak{g} be a complex simple, finite-dimensional Lie algebra. For a dominant integral weight λ , let $V(\lambda)$ denote the simple, finite-dimensional \mathfrak{g} -module of highest weight λ and let $v_\lambda \in V(\lambda)$ be a fixed highest weight vector. Let $\text{gr } V(\lambda)$ denote the associated graded module with respect to the PBW filtration on $V(\lambda)$. These degenerate modules have been studied extensively in the last decade, mainly also due to the induced degenerate flag variety, defined by Feigin (2012) as a highest weight orbit of an action of a degenerate group on $\text{gr } V(\lambda)$. One of the achievements of this framework so far is a “new” monomial basis of $\text{gr } V(\lambda)$ and hence of $V(\lambda)$ for \mathfrak{sl}_n , \mathfrak{sp}_{2n} (Feigin et al. 2011a, b). These monomials are naturally assigned with lattice points in a normal polytope $P(\lambda) \subset \mathbb{R}_{\geq 0}^N$, N the number of positive roots, known as the FFLV polytope.

✉ George Balla
balla@art.rwth-aachen.de

Ghislain Fourier
fourier@art.rwth-aachen.de

Kunda Kambaso
kambaso@art.rwth-aachen.de

¹ Chair of Algebra and Representation Theory, RWTH Aachen University, Pontdriesch 10-16, 52062 Aachen, Germany

We seek to understand if this basis is compatible with Demazure modules. The second author has provided in Fourier (2016) the compatibility for Demazure modules in type A corresponding to a class of Weyl group elements that avoid the patterns 4231 and 2413. These are called triangular Weyl group elements. Formulated differently, a particular face of the FFLV polytope indexes a basis of the graded Demazure modules for triangular Weyl group elements. The PBW-degenerated Schubert varieties in that context were also studied. The main goal of the current paper is to extend these results to a more general setting that also includes Demazure modules and Schubert varieties in type C .

Let R^- denote the set of negative roots of \mathfrak{g} . We call a subset $A \subset R^-$, *FFLV-admissible* if it is closed under taking sums inside R^- and stable for a *join* operation (see Definition 3). This join operation allows for a straightening law on monomials. For example, the sets corresponding to the triangular subsets studied in Fourier (2016) are admissible but admissibility is not restricted to triangular subsets, as for example, in type A every single element set is admissible. We consider a submodule of $V(\lambda)$ corresponding to an FFLV-admissible set A , which we denote by $V_A(\lambda)$, generated through v_λ by the subalgebra of \mathfrak{g} defined by A . It turns out that the induced PBW filtration is compatible with the one on $V(\lambda)$ (Proposition 14). Let $\text{gr } V_A(\lambda)$ denote the associated graded submodule. Let W denote the Weyl group of \mathfrak{g} . In the special case $A = A_w$ for a Weyl group element $w \in W$, one has the isomorphism of our submodules for FFLV-admissible sets to Demazure modules, i.e.,

$$V_A(\lambda) \cong V_w(\lambda),$$

via twisting by w (see Sect. 5.1).

Our first interest is in constructing a monomial basis for $V_A(\lambda)$ and $\text{gr } V_A(\lambda)$ and hence for the corresponding Demazure modules $V_w(\lambda)$ and $\text{gr } V_w(\lambda)$. For this, we will consider the following faces of the polytope $P(\lambda)$: for $A \subset R^-$, let $P_A(\lambda)$ denote the face of $P(\lambda)$ obtained by setting the coordinates $s_\beta = 0$ for all $\beta \notin A$. Let $S_A(\lambda)$ denote the set of lattice points in $P_A(\lambda)$. We prove:

Theorem 1 *Let \mathfrak{g} be $\mathfrak{sl}_{n+1}(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$, $A \subset R^-$ an FFLV-admissible set, and let w be a Weyl group element such that A_w is FFLV-admissible, then:*

- (a) *for all λ, μ , dominant integral weights, $S_A(\lambda) + S_A(\mu) = S_A(\lambda + \mu)$.*
- (b) *the set $S_A(\lambda)$ parametrizes a monomial basis of the PBW-graded submodule $\text{gr } V_A(\lambda) \subset \text{gr } V(\lambda)$ and hence of $V_A(\lambda)$.*
- (c) *the set $S_A(\lambda)$ parametrizes a monomial basis of the PBW-graded Demazure module $\text{gr } V_w(\lambda)$ and hence of $V_w(\lambda)$.*

The proof we are providing does not depend on A being induced from a Weyl group element. Hence the theorem is true more generally for FFLV-admissible sets when \mathfrak{g} is $\mathfrak{sl}_{n+1}(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$. Our approach in proving part (b) of this theorem relies on having defining relations for the modules $\text{gr } V(\lambda)$ as constructed in Feigin et al. (2011a, b) (see Theorem 19). A direct consequence is the following and explains our interest in considering FFLV-admissible subsets: if $A \subset R^-$ is not FFLV-admissible and λ is regular, then $S_A(\lambda)$ doesn't parametrize (in a "natural" way) a monomial basis of

gr $V_A(\lambda)$ (Lemma 20). We should note, that a Kogan face construction as in Kogan (2000), Kirichenko et al. (2012) and Biswal and Fourier (2015) is not applicable, as in contrast to that case, the coordinates are assigned positive roots and not simple roots (with respect to a reduced expression of the longest Weyl group element).

We would also like to note that the polytopes constructed in Kambaso (2022) parametrizing monomial bases for submodules studied there, are in general not faces of the FFLV polytope $P(\lambda)$ in types A and C. On the other hand, monomial bases similar to those discussed here have been constructed in Makhlin (2019) for orthogonal modules in type B. However, our approach is not applicable in that situation, since defining relations for type B are still missing.

It turns out that the submodules $V_A(\lambda)$ are favourable in the sense of Feigin et al. (2017), so we obtain flat PBW-degenerated symplectic Schubert varieties which degenerate further into toric varieties (Proposition 22). Moreover, we consequently describe a monomial basis for the homogeneous coordinate rings of the PBW-degenerate Schubert varieties labelled by PBW-semistandard tableaux (Proposition 24).

Our paper is organised as follows: in Sect. 2, we recall basic notation on Lie algebras while in Sect. 3, we introduce FFLV-admissible subsets and show that they are compatible with Dynkin automorphisms. In Sect. 4, we describe the correspondence between FFLV-admissible subsets and triangular Weyl group elements and in Sect. 5 we describe the monomial basis for our submodules and give the geometric interpretation in terms of PBW-degenerated Schubert varieties.

2 Preliminaries: notation on Lie algebras

We follow Humphreys (2012). Let \mathfrak{g} be the simple, complex Lie algebra of classical type A_n, B_n, C_n or D_n . We fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Having fixed this decomposition, we denote by $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots, R^+ and R^- , the sets of positive roots and negative roots respectively.

- For $\mathfrak{g} = \mathfrak{sl}_{n+1}$ (of type A_n) and $1 \leq i \leq j \leq n$, we abbreviate $\alpha_{i,j} := \alpha_i + \dots + \alpha_j$. Then $R_{A_n}^+ = \{\alpha_{i,j} \mid 1 \leq i \leq j \leq n\}$.
- For $\mathfrak{g} = \mathfrak{so}_{2n+1}$ (of type B_n), $\alpha_{i,j}$ as above and $\alpha_{i,\bar{j}} := \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_n$, we have $R_{B_n}^+ = \{\alpha_{i,j}, \alpha_{k,\bar{l}} \mid 1 \leq i \leq j \leq n, 1 \leq k < l \leq n\}$.
- For $\mathfrak{g} = \mathfrak{sp}_{2n}$ (of type C_n), $\alpha_{i,j}$ as above and $\alpha_{i,\bar{j}} := \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{n-1} + \alpha_n$ (with $\alpha_{i,n} = \alpha_{i,\bar{n}}$), one has $R_{C_n}^+ = \{\alpha_{i,j}, \alpha_{i,\bar{j}} \mid 1 \leq i \leq j \leq n\}$.
- Finally, for $\mathfrak{g} = \mathfrak{so}_{2n}$ (of type D_n), $\alpha_{i,j}$ as before, $\alpha_{i,\bar{j}} := \alpha_i + \dots + \alpha_{n-2} + \alpha_j + \dots + \alpha_n$, the set of positive roots is given by $R_{D_n}^+ = \{\alpha_{i,j}, \alpha_{k,\bar{l}} \mid 1 \leq i \leq j \leq n-1, 1 \leq k < l \leq n\}$.

For an element $\beta = \sum a_i \alpha_i \in \sum \mathbb{Z} \alpha_i$ we denote $\text{supp } \beta = \{i \mid a_i \neq 0\}$ and further say α and β are linked, if $\text{supp } \alpha + \beta$ defines a connected subdiagram of the Dynkin diagram.

For $\beta \in R^+$, let e_β and f_β denote fixed basis elements in the root spaces of \mathfrak{g} of weights β and $-\beta$ respectively. We will use short notation $e_{i,j}$ resp. $f_{i,\bar{j}}$ for e_β and f_β where $\beta = \alpha_{i,j}$ or $\beta = \alpha_{i,\bar{j}}$. For $\beta \in R^+$, h_β will denote the corresponding co-root.

Let the set of (dominant) integral weights of \mathfrak{g} be denoted by P (resp. P^+). The fundamental weights are denoted $\{\omega_1, \dots, \omega_n\}$, they generate the monoid P^+ . For $\lambda \in P^+$, we denote by $V(\lambda)$ the simple, finite-dimensional highest weight module of highest weight λ , with a highest weight vector v_λ .

For a Lie algebra \mathfrak{a} , let $U(\mathfrak{a})$ denote the universal enveloping algebra. Let (z_1, \dots, z_l) be an ordered basis of \mathfrak{a} , then the PBW filtration on $U(\mathfrak{a})$ is defined as

$$U(\mathfrak{a})_s := \langle z_{i_1} \cdots z_{i_r} \mid r \leq s, z_{i_j} \in \mathfrak{a} \rangle_{\mathbb{C}}.$$

The famous PBW theorem states that the associated graded algebra is the symmetric algebra $S(\mathfrak{a})$. On a cyclic $U(\mathfrak{a})$ -module M with generator $m \in M$, one has the induced filtration $M_s := U(\mathfrak{a})_s \cdot m$ and the associated graded module $\text{gr } M$ is a cyclic module for $S(\mathfrak{a})$.

We denote by W the Weyl group of \mathfrak{g} , it is generated by the simple reflections s_i corresponding to the simple roots α_i . The Weyl group acts on the weights in P and on the weights of $V(\lambda)$, leaving the dimension of the weight spaces invariant. For $w \in W$, the extremal weight space of weight $w(\lambda)$ in $V(\lambda)$ is therefore one-dimensional and we denote by $v_{w(\lambda)}$ a basis element in this weight space.

Let $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$ be a Borel subalgebra of \mathfrak{g} . The Demazure module corresponding to $w \in W$ is defined to be the \mathfrak{b} -module $V_w(\lambda) := U(\mathfrak{b}) \cdot v_{w(\lambda)}$ generated by the element $v_{w(\lambda)}$. Notice that when w is the longest element in W , the vector space $V_w(\lambda)$ coincides with $V(\lambda)$.

Now let \mathfrak{g} be of type X and τ an automorphism of the Dynkin diagram, \mathfrak{g}^τ be the fixed point algebra of type X^τ . Let R_X^- and $R_{X^\tau}^-$ denote negative roots of \mathfrak{g} and \mathfrak{g}^τ respectively. In particular, we will consider the following automorphisms and fixed point algebras. The automorphism of type:

- (a) A_{2n-1} , and corresponding fixed point algebra which is of type C_n .
- (b) D_{n+1} , and corresponding fixed point algebra being of type B_n .

We introduce two partial orders on R^- . The first one is classical:

$$\beta < \gamma \Leftrightarrow \gamma - \beta \in \sum \mathbb{Z}_{\geq 0} \alpha_i.$$

Let δ be a sum of two negative roots, then

$$\mathcal{P}(\delta) := \begin{cases} (\delta, 0) & \text{if } -\delta \text{ is the highest root of a Dynkin subdiagram,} \\ \{(\beta_1, \beta_2) \in (R^- \cup \{0\})^2 \mid \beta_1 + \beta_2 = \delta\} & \text{else.} \end{cases}$$

We define a partial order \leq on $\mathcal{P}(\delta)$ by

$$(\beta_1, \beta_2) \leq (\gamma_1, \gamma_2) \Leftrightarrow \gamma_1 \leq \beta_1, \beta_2 \leq \gamma_2.$$

For clarity, we include here a short example of the above partial order on $\mathcal{P}(\delta)$. In type A_3 , consider the pair of negative roots $(\alpha_{1,2}, \alpha_{2,3})$, then one has $(\alpha_{1,3}, \alpha_{2,2}) \in$

$\mathcal{P}(\alpha_{1,2} + \alpha_{2,3})$, and $(\alpha_{1,2}, \alpha_{2,3}) < (\alpha_{1,3}, \alpha_{2,2})$. Consider type C_3 and the pair $(\alpha_{1,\bar{2}}, \alpha_{1,3})$, then $(\alpha_{1,\bar{2}}, \alpha_{1,3}) < (\alpha_{1,\bar{1}}, \alpha_{2,3}) \in \mathcal{P}(\alpha_{1,\bar{2}} + \alpha_{1,3})$.

Definition 1 Let $(\beta_1, \beta_2) \in R^- \times R^-$, then we define the **join** of β_1 and β_2 :

$$\beta_1 \oplus \beta_2 := \{(\gamma_1, \gamma_2) \mid (\gamma_1, \gamma_2) \in \mathcal{P}(\beta_1 + \beta_2) \mid (\beta_1, \beta_2) < (\gamma_1, \gamma_2)\}.$$

By abuse of notation, we denote $\beta_1 \oplus \beta_2$ also the set of roots in $\beta_1 \oplus \beta_2$.

Example 1 To explain the join of two roots, we work it out for types A and C:

1. Suppose $\beta_1 = \alpha_{i_1, i_2}, \beta_2 = \alpha_{j_1, j_2}$ with $i_1 \leq j_1, i_2, j_2 \in I$, then:

$$\beta_1 \oplus \beta_2 = \begin{cases} \emptyset & \text{if } i_1 = j_1 \text{ or } i_2 < j_1 - 1 \text{ or } j_2 \leq i_2, \\ \{(\alpha_{i_1, j_2}, \alpha_{j_1, i_2})\} & \text{if } j_1 - 1 \leq i_2 < j_2. \end{cases}$$

2. Suppose $\beta_1 = \alpha_{i_1, \bar{i}_2}, \beta_2 = \alpha_{j_1, \bar{j}_2}$ with $i_1 \leq j_1, i_2, j_2 \in I$, then:

$$\beta_1 \oplus \beta_2 = \begin{cases} \emptyset & \text{if } i_2 < j_1, \\ \{(\alpha_{i_1, \bar{j}_1}, \alpha_{j_2, \bar{i}_2}), (\alpha_{i_1, \bar{j}_2}, \alpha_{j_1, \bar{i}_2})\} & \text{if } j_1 \leq j_2 < i_2, \\ \{(\alpha_{i_1, \bar{j}_1}, \alpha_{i_2, \bar{j}_2})\} & \text{if } j_1 \leq i_2 \leq j_2. \end{cases}$$

3. For $\beta_1 = \alpha_{i_1, i_2}, \beta_2 = \alpha_{j_1, \bar{j}_2}$ with $i_1 \leq j_1, i_2, j_2 \in I$, one has:

$$\beta_1 \oplus \beta_2 = \begin{cases} \emptyset & \text{if } i_1 \leq i_2 < j_1 - 1 \leq j_2, \\ \{(\alpha_{i_1, \bar{j}_1}, \alpha_{j_2, i_2}), (\alpha_{i_1, \bar{j}_2}, \alpha_{j_1, i_2})\} & \text{if } j_1 \leq j_2 \leq i_2, \\ \{(\alpha_{i_1, \bar{j}_2}, \alpha_{j_1, i_2}), (\alpha_{i_1, \bar{j}_1}, 0)\} & \text{if } j_1 \leq i_2 = j_2 - 1, \\ \{(\alpha_{i_1, \bar{j}_2}, \alpha_{j_1, i_2})\} & \text{if } j_1 \leq i_2 < j_2 - 1, \\ \{(\alpha_{i_1, \bar{j}_2}, 0)\} & \text{if } i_1 \leq i_2 = j_1 - 1 \leq j_2. \end{cases}$$

4. Lastly, for $\beta_1 = \alpha_{i_1, i_2}, \beta_2 = \alpha_{j_1, \bar{j}_2}$ with $j_1 < i_1, i_2, j_2 \in I$, we have:

$$\beta_1 \oplus \beta_2 = \begin{cases} \emptyset & \text{if } i_1 \leq i_2 < j_2 - 1, \\ \{(\alpha_{j_1, \bar{i}_1}, \alpha_{j_2, i_2})\} & \text{if } i_1 \leq j_2 \leq i_2 \text{ or } j_1 \leq j_2 < i_1 \leq i_2, \\ \{(\alpha_{j_1, \bar{i}_1}, 0)\} & \text{if } i_1 \leq i_2 = j_2 - 1. \end{cases}$$

The second order on R^- is defined as follows:

Let $\alpha = \sum a_i \alpha_i, \beta = \sum b_i \alpha_i \in R^-$. Let $s = \min\{i \mid a_i \neq 0\}$. We define $\alpha > \beta$ if

$$\beta = \alpha + \alpha_s \quad \text{and} \quad a_s = 1$$

or

$$\beta = \alpha - \alpha_j \quad \text{for some } j \geq s.$$

The transitive closure of these cover relations defines an order $<$ on R^- .

Example 2 Two examples to explain the order:

1. In type A: $\alpha_{i,j} > \alpha_{k,\ell} \Leftrightarrow i \leq k, j \leq \ell$.
2. In type C: $\alpha_{i,\bar{j}} > \alpha_{k,\bar{\ell}} \Leftrightarrow i \leq k, j \geq \ell$.

We note here, that if $\beta_1 > \beta_2$, then γ_1 and γ_2 are non-comparable with respect to $>$ for all $(\gamma_1, \gamma_2) \in \beta_1 \oplus \beta_2$. The motivation for the second order is due to the definition of a Dyck path for Lie algebras of type A or C from Feigin et al. (2011a, b):

Definition 2 A Dyck path p is a sequence of positive roots $(\beta_1, \dots, \beta_s)$ such that

- β_1 is a simple root α_i , we denote $s(p) = i$.
- β_s is either a simple root α_i or additionally in type C equals $\alpha_{i,\bar{i}}$, we denote $e(p) = i$ or $e(p) = n$ (in the latter case).
- for all $1 \leq i < s$: $\beta_i > \beta_{i+1}$.

3 FFLV-admissible sets and fixed point subalgebras

We will introduce the main combinatorial object of our study. We will explain in Sect. 5, how this definition is motivated by the study of certain monomial bases of simple, finite-dimensional modules.

Definition 3 A subset $A \subseteq R^-$ is called **FFLV-admissible** (short **admissible**) if

1. $\mathfrak{n}_A = \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$ is a Lie subalgebra.
2. For all pairs α, β such that $\alpha \geq \beta \in A$ one has $\alpha \oplus \beta \in A$.

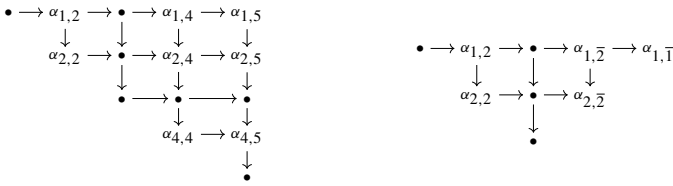
We note, that if A is admissible, then

$$\mathfrak{g}_A := \mathfrak{h} \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$$

is a Lie subalgebra in \mathfrak{g} .

Example 3 We provide a short list of examples here, and refer to more example classes to Sect. 4:

1. There are two trivial admissible subsets, \emptyset and R^- .
2. The next obvious examples to be considered are subsets with one root $A = \{\alpha\}$. Then A is admissible if and only if α is the highest root of the subdiagram:
 - (a) In type A, every subset $\{\alpha\}$ is admissible.
 - (b) In type C, $\{\alpha\}$ is admissible if and only if $\alpha = \alpha_{i,j}$ with $j < n$ or $\alpha = \alpha_{i,\bar{i}}$.
3. In type C, the minimal admissible set containing $\alpha_{i,\bar{j}}$ with $i \neq j$ can be read off from Example 1, it is $\{\alpha_{i,\bar{j}}, \alpha_{i,\bar{i}}, \alpha_{j,\bar{j}}\}$.
4. Slightly larger, non-trivial examples are the subsets given by the roots appearing in the following grid for type A_5 and type C_3 respectively.



Proposition 2 Let $A \subset R^-$ be admissible and $S \subset A$. Let $\text{supp } S = \bigcup_{\alpha \in S} \text{supp } \alpha$ and $S_i \subset S$ be a connected component. Then θ_{S_i} , the highest root of the Dynkin subdiagram of S_i belongs to A .

Proof The statement follows for roots of the form $\alpha_{i,j}$ with $i, j \in I$ straight from the definition. Now, let $\beta \in A$ and suppose $\beta \oplus \beta \neq \emptyset$. Let $S = \text{supp } \beta$, then $\theta_S \in \beta \oplus \beta$. □

3.1 Fixed point subalgebras and admissible sets

We discuss here how admissible sets and fixed point sets of Dynkin automorphisms are compatible. Recall the Lie algebra \mathfrak{g} of type X and the corresponding fixed point algebra \mathfrak{g}^τ of type X^τ for a fixed Dynkin automorphism τ . We have two notions of admissibility for \mathfrak{g}^τ : the first one is given by Definition 3. For the second one, we denote for a τ -invariant $A \subset R_X^-$ the set of orbits by $A^\tau \subset R_{X^\tau}^-$. Recall that every subset of $R_{X^\tau}^-$ is of the form A^τ for a unique $A \subset R_X^-$.

Definition 4 We call $A^\tau \subset R_{X^\tau}^-$ **induced admissible** if $A \subset R_X^-$ is admissible.

The following proposition gives the equivalence of the two notions:

Proposition 3 Let $A^\tau \subset R_{X^\tau}^-$, then A^τ is admissible if and only if A^τ is induced admissible.

Proof Let $A = \tau(A)$ be admissible and $\beta_1^\tau, \beta_2^\tau \in A^\tau$ with $\beta_1^\tau \geq \beta_2^\tau$. We choose minimal representatives β_1, β_2 with respect to $>$. Suppose $(\gamma_1^\tau, \gamma_2^\tau) \in \beta_1^\tau \oplus \beta_2^\tau$, e.g. $\gamma_1^\tau + \gamma_2^\tau = \beta_1^\tau + \beta_2^\tau$ and $\gamma_1^\tau \leq \beta_1^\tau, \beta_2^\tau \leq \gamma_2^\tau$. We choose minimal representatives γ_1, γ_2 , then

$$\gamma_1 + \gamma_2 = \beta_1 + \beta_2 \quad \text{and} \quad \gamma_1 \leq \beta_1, \beta_2 \leq \gamma_2$$

and hence $(\gamma_1, \gamma_2) \in \beta_1 \oplus \beta_2$. A is admissible and τ invariant, so $(\gamma_1^\tau, \gamma_2^\tau) \in \beta_1^\tau \oplus \beta_2^\tau$.

Suppose $A = \tau(A)$ and A^τ is admissible. Similarly as before, one shows for $\beta_1, \beta_2 \in A$ that $\beta_1 \oplus \beta_2 \subset A$ if both are the minimal or both are the maximal elements in their orbit. We are left with β_1 minimal and β_2 maximal, $\beta_1 > \beta_2$ and suppose $(\gamma_1, \gamma_2) \in \beta_1 \oplus \beta_2$. Then $\gamma_1^\tau + \gamma_2^\tau = \beta_1^\tau + \beta_2^\tau$ and $\gamma_1^\tau \leq \beta_1^\tau, \beta_2^\tau \leq \gamma_2^\tau$ as those properties are true for γ_1, γ_2 . Since A^τ is admissible, $\{\gamma_1^\tau, \gamma_2^\tau\} \in A^\tau$ and so $\{\gamma_1, \gamma_2\} \in A$. □

3.2 Saturated sets

Definition 5 For a subset $S \subset R^-$, we define the **join closure** \overline{S}^\oplus of S to be the minimal subset in R^- , containing S and for any two roots $\alpha, \beta \in \overline{S}^\oplus$, $\alpha \oplus \beta \in S$.

This is certainly well-defined and a non-trivial subset, for example in type A , if $S = \{\alpha\}$, then $S = \overline{S}^\oplus$ while the closure of the set of all (negative) simple roots is R^- . Admissible subsets are by definition closed under join.

We consider the join closure of particular subsets:

Proposition 4 Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\mathbf{p} = (\alpha_{i_1, j_1}, \dots, \alpha_{i_s, j_s})$ a Dyck path, such that consecutive roots are linked. Then

$$\overline{\mathbf{p}}^\oplus = \{\alpha_{i_k, j_\ell} \mid 1 \leq k, \ell \leq n\}.$$

Proof Let $1 \leq k < n$, then

$$\alpha_{i_k, j_k} \oplus \alpha_{i_{k+1}, j_{k+1}} = \{\alpha_{i_k, j_{k+1}}, \alpha_{j_k, i_{k+1}}\}.$$

Iterating this procedure gives the proposed set. We are left to show, that this is closed under join: consider $\alpha_{i_k, j_\ell}, \alpha_{i_r, j_t}$ with $i_k \leq i_r$ and $i_k, i_r \in \{i_1, \dots, i_s\}$, $j_\ell, j_t \in \{j_1, \dots, j_s\}$. Then by Example 1 we see that

$$\alpha_{i_k, j_\ell} \oplus \alpha_{i_r, j_t} = \emptyset$$

unless $i_r - 1 \leq j_\ell < j_t$, while in this case

$$\alpha_{i_k, j_\ell} \oplus \alpha_{i_r, j_t} = \{\alpha_{i_k, j_t}, \alpha_{j_\ell, i_r}\}$$

which is in the proposed set. □

Proposition 5 Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and \mathbf{p} be a Dyck path, such that consecutive roots are linked, then

$$\overline{\mathbf{p}}^\oplus = (\overline{T_{\mathbf{p}}})^\tau \text{ where } T_{\mathbf{p}} = \bigcup_{\alpha^\tau \in \mathbf{p}} \{\gamma \in \alpha^\tau\} \subset R_{\mathbb{A}_{2n-1}}^-.$$

Proof Let $\alpha^\tau, \beta^\tau \in R^-$, then we can read off from Example 1:

$$\alpha^\tau \oplus \beta^\tau = \{\gamma_1 \oplus \gamma_2 \mid \gamma_1 \in \alpha^\tau, \gamma_2 \in \beta^\tau\}^\tau.$$

Applying this to \mathbf{p} yields

$$(\overline{T_{\mathbf{p}}})^\tau \subseteq \overline{\mathbf{p}}^\oplus$$

since by construction $\tau(\overline{T_{\mathbf{p}}}) = \overline{T_{\mathbf{p}}}$.

We are left to show that the left hand side is closed under join. So let $\alpha^\tau, \beta^\tau \in (\overline{T_{\mathbf{p}}^\oplus})^\tau$, then the first equation in the proof shows that $\alpha^\tau \oplus \beta^\tau \subset (\overline{T_{\mathbf{p}}^\oplus})^\tau$, because $\overline{T_{\mathbf{p}}^\oplus}$ is closed under join. \square

The following will be a crucial observation in the proof of the main Theorem 19.

Proposition 6 *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}, \mathfrak{sp}_{2n}$ and let \mathbf{p} be a Dyck path, such that consecutive roots are linked. Let $\alpha, \beta \in \overline{\mathbf{p}}^\oplus$ and suppose $\gamma = \alpha - \beta \in R^+$. Then*

$$(\overline{\mathbf{p}}^\oplus + \gamma) \cap R^- \subset \overline{\mathbf{p}}^\oplus.$$

Proof For $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\mathbf{p} = (\alpha_{i_1, j_1}, \dots, \alpha_{i_s, j_s}), \gamma$ could be only one of the following

$$\{\alpha_{i_k, i_{\ell-1}}, \alpha_{j_{k-1}, j_\ell} \mid k < \ell\}.$$

Observing, that $\overline{\mathbf{p}}^\oplus$ is the set of all roots starting in i_k and ending in j_ℓ completes the proof in this case.

Turning to $\mathfrak{g} = \mathfrak{sp}_{2n}$, we again observe that $\overline{\mathbf{p}}^\oplus$ is admissible and hence induced admissible. Any difference of two roots in $\overline{\mathbf{p}}^\oplus$ is in fact the orbit of a difference between two roots in \mathfrak{sl}_{2n} . We can therefore reduce the stability of $\overline{\mathbf{p}}^\oplus$ to the \mathfrak{sl}_{n+1} -case. \square

4 Triangular Weyl group elements

In this section, we consider particular subsets $A \subset R^+$. Let $w \in W$, then

$$A_w := w^{-1}(R^+) \cap R^-.$$

We are aiming to classify the $w \in W$, such that A_w is admissible.

Definition 6 Let $\mathbf{i} := i_1 i_2 i_3 i_4$ with $i_j \in \{1, \dots, 4\}$ pairwise distinct. We call $\tau \in S_N$ **\mathbf{i} -avoiding** (*pattern avoiding*) if there is no quadruple $j_1 < j_2 < j_3 < j_4$ such that $(\tau(j_1), \tau(j_2), \tau(j_3), \tau(j_4))$ is ordered as \mathbf{i} .

4.1 Type A

The case of admissible sets originating from Weyl group elements in type A has been treated in Fourier (2016). We recall here the original definition from Fourier (2016) and Chirivi et al. (2021):

Definition 7 Let $w \in S_{n+1}$, then w is called **triangular** if w does not contain patterns of the form 4231 and 2413.

We should note here some interesting facts on triangular elements in type A:

1. The number of triangular elements is given by the sequence A032351 in OEIS, OEIS (2018a). The first few elements suggest, that the number might be close to $n!$ but this is not true for large n .

2. There are interesting elements in S_n introduced in Dehy and Yu (2001), which are also triangular elements as discussed in Fourier (2016).
3. The “smallest” non-triangular element is $s_1s_2s_3$ in S_4 .
4. There is a subset of triangular elements called rectangular elements introduced in Chirivi et al. (2021). These avoid the patterns 2431 and 4213 in addition. They are counted by the integer sequence A006012 in OEIS, OEIS (2018b).

The following can be found in (Fourier 2016, Proposition 1) and it explains the restriction of our focus to triangular Weyl group elements.

Proposition 7 *Let $w \in S_{n+1}$, then $w^{-1}(R^+) \cap R^-$ is admissible if and only if w is triangular.*

4.2 Type C

We turn to the symplectic case and its Weyl group W_{C_n} . We consider the symmetric group S_{2n} with the involutive element

$$w_0 : \{1, \dots, 2n\} \longrightarrow \{1, \dots, 2n\}, i \mapsto 2n + 1 - i.$$

With the help of w_0 we define an involution on S_{2n} by

$$\sigma : S_{2n} \longrightarrow S_{2n}, \tau \mapsto w_0 \circ \tau \circ w_0.$$

The symplectic Weyl group W_{C_n} is generated by s_1, \dots, s_n subject to the relations

$$s_i^2 = 1, (s_i s_j)^2 = 1 \text{ for } |i - j| > 1, (s_i s_{i+1})^3 = 1 \text{ for } i < n - 1, (s_{n-1} s_n)^4 = 1.$$

This symplectic Weyl group is isomorphic to the subgroup of σ -invariant elements in S_{2n} , that is $\tau = w_0 \circ \tau \circ w_0$, via the following identification ι

$$s_i \mapsto t_i t_{2n-i} \text{ for } i \leq n - 1; s_n \mapsto t_n.$$

Again we are interested in the $w \in W_{C_n}$ such that A_w is admissible. Our goal is

Lemma 8 *There is a bijection from the set $\{w \in W_{C_n} \mid A_w \text{ is admissible}\}$ to the set $\{w \in W_{A_n} \mid A_w \text{ is admissible}\}$.*

Before giving the proof, we use the lemma to define

Definition 8 An element $w \in W_{C_n}$ is called **symplectic triangular** if $\iota(w)$ is triangular for type A.

Due to the facts on triangular elements, we deduce

Corollary 9 *The number of symplectic triangular elements is given by the sequence A032351 in OEIS (2018a).*

One could ask whether there is also bijection of the admissible sets of R_{C_n} to the admissible sets of R_{A_n} . It turns out, that there are 8 admissible subsets in type C_2 and only 7 admissible subsets in type A_2 ; in type A_2 all Weyl group elements are triangular while in type C_2 , the list of symplectic triangular Weyl group elements is

$$\text{id}, s_1, s_2, s_1s_2, s_2s_1s_2, s_1s_2s_1s_2.$$

In type A_2 , there is one admissible set which is not obtained from a Weyl group element, namely $\{\alpha_1 + \alpha_2\}$, while in type C_2 , there are two $\{\alpha_{1,\bar{1}}\}$ and $\{\alpha_2, \alpha_{1,\bar{1}}\}$.

We turn to the proof of Lemma 8, which has three steps:

Proposition 10 *Fix a pattern $\mathbf{i} \subset \{1, \dots, 4\}$. There is a mapping between the elements from S_{2n} that avoid \mathbf{i} , and the elements from S_{n+1} that avoid \mathbf{i} .*

Proof Let $\tau = (j_1, \dots, j_{2n}) \in S_{2n}$ and define $\tilde{\tau}$ to be the sequence where all $j > n + 1$ are replaced by $n + 1$. Let $\underline{\tau}$ denote the sequence where all $n + 1$ in $\tilde{\tau}$ are deleted except the leftmost. Then $\underline{\tau} \in S_{n+1}$. It follows immediately, that if τ avoids the pattern \mathbf{i} , then $\underline{\tau}$ avoids \mathbf{i} as well. □

In the next step, we prove the surjectivity:

Proposition 11 *Consider a pattern $\mathbf{i}^* := i_1i_2i_3i_4$ with $i_j \in \{1, \dots, 4\}$ pairwise distinct such that $i_1 > i_2, i_4$ and $i_3 > i_2, i_4$. Then there is a surjection between the elements in S_{2n} that are σ -invariant and avoid the pattern \mathbf{i}^* and the elements in S_{n+1} that avoid the pattern \mathbf{i}^* .*

Proof Suppose $\tau \in S_{2n}$ is invariant under σ and avoids \mathbf{i}^* . Let $\underline{\tau}$ be the pattern constructed from τ as in the proof of Proposition 10. Let $1 \leq j_1 < j_2 < j_3 < j_4 \leq n + 1$ be such that $\underline{\tau}(j_1, j_2, j_3, j_4)$ is the pattern \mathbf{i}^* . Then either $j_4 < n + 1$, in which case this τ would have the same pattern, which is a contradiction or $j_4 = n + 1$ and we set $n + 1 \leq s \leq 2n$ the leftmost element in τ greater or equals $n + 1$. Then $\tau(j_1, j_2, j_3, s)$ is the pattern \mathbf{i}^* . We conclude that $\underline{\tau}$ avoids the pattern \mathbf{i}^* . □

For the bijection of Lemma 8, it remains to prove the injectivity:

Proof Let $w \in S_{n+1}$ be an element that avoids the patterns 3142 and 4231. We will construct the preimage of this element in the following way. Consider the position of $n + 1$ in the presentation of w . Let k denote the number of elements left of $n + 1$ in this presentation. Let j be the element at position $i < n + 1$. Construct an element w' as follows. We leave every element j at position i and every element of the form $2n + 1 - j$ (left of $n + 1$), we put at position $2n + 1 - i$. Now we are left with the element $n + 1$ and the elements to its right. These are mapped in the following way. The ordering of the elements $1, \dots, n$ determines the ordering of $n + 1, \dots, 2n$, because of the involution. Now replace all $n + 1$ by the elements $n + 1, \dots, 2n$ which are not placed before, in that order. These are $n - k$ elements. So the elements to the right of $n + 1$ are just shifted by $n - k$ to the right. □

From Proposition 3 and Lemma 8, we can deduce the following analogue to Proposition 7:

Corollary 12 *Let $w \in W_{C_n}$, then $w^{-1}(R^+) \cap R^-$ is admissible if and only if w is symplectic triangular.*

5 Application

We discuss in this section the application we have in mind for defining admissible sets. For this we will recall Demazure modules, the PBW filtration and the FFLV monomial bases for simple, finite-dimensional modules.

5.1 Submodules for admissible sets

We fix \mathfrak{g} and an admissible set $A \subset R^-$, then $\mathfrak{h} \oplus \mathfrak{n}_A^-$ is a subalgebra and hence we can consider $U(\mathfrak{h} \oplus \mathfrak{n}_A^-)$ as a subalgebra in $U(\mathfrak{h} \oplus \mathfrak{n}^-)$. We further fix a dominant, integral weight λ and a highest weight vector $v_\lambda \in V(\lambda)$. We define

$$V_A(\lambda) := U(\mathfrak{n}_A^-) \cdot v_\lambda.$$

By construction, this is a $\mathfrak{h} \oplus \mathfrak{n}_A^-$ -module but in fact, it can be extended to a module for

$$\mathfrak{h} \oplus \mathfrak{n}_A^+ \oplus \mathfrak{n}_A^-$$

where \mathfrak{n}_A^+ is the maximal subalgebra in \mathfrak{n}^+ such that the above is invariant under \mathfrak{n}_A^+ . In the special case $A = A_w$, one has

$$V_A(\lambda) \cong V_w(\lambda)$$

via the twist $w : U(\mathfrak{h} \oplus \mathfrak{n}_A^+ \oplus \mathfrak{n}_A^-) \longrightarrow U(\mathfrak{b})$. We want to analyse the modules $V_A(\lambda)$ and especially asking for a monomial basis, e.g. a set of monomials in $U(\mathfrak{n}_A^-)$. Note, that in the special case of Demazure modules there are monomial bases known, while we explain in the next section that we are interested in a monomial basis defined by a homogeneous lexicographic order and such bases for Demazure modules are not known in general.

5.2 PBW filtration and sub objects

For a Lie algebra \mathfrak{a} , we explain briefly that the PBW filtration on $U(\mathfrak{a})$ is compatible with the filtration on the universal enveloping algebra of a subalgebra. In the case of submodules, one has to ask for mild assumptions which are satisfied in all relevant cases.

Proposition 13 *Let $\mathfrak{b} \subset \mathfrak{a}$ be a subalgebra, then $U(\mathfrak{b})_s = U(\mathfrak{a})_s \cap U(\mathfrak{b})$, i.e., the PBW filtration is compatible with subalgebras.*

Proof It is clear, that $U(\mathfrak{b})_s \subset U(\mathfrak{a})_s \cap U(\mathfrak{b})$. We fix an ordered basis (x_1, \dots, x_s) of \mathfrak{b} and extend with (y_1, \dots, y_t) to a basis of \mathfrak{a} . We denote the induced basis of $U(\mathfrak{b})$ by $\{x_I \mid I \in \mathbb{Z}_{>0}^s\}$. $U(\mathfrak{a})$ is a free $U(\mathfrak{b})$ -module with basis given by ordered monomials in (y_1, \dots, y_t) and hence a basis of $U(\mathfrak{a})$ is given by $\{x_I y_J \mid I \in \mathbb{Z}_{>0}^s, J \in \mathbb{Z}_{>0}^t\}$.

Suppose now $p \in U(\mathfrak{b})_r \setminus U(\mathfrak{b})_{r-1}$, then $p = \sum_I c_I x_I$ with $|I| \leq r$ and at least one I with $|I| = r$. Suppose $p \in U(\mathfrak{a})_{r-1}$, then $p = \sum_{I,J} c_{I,J} x_I y_J$ with $|I| + |J| \leq r - 1$ and by construction there is $c_{I,J} \neq 0$ for some J with $|J| \neq 0$. But the freeness of $U(\mathfrak{a})$ and the equality $\sum_I c_I x_I = \sum_{I,J} c_{I,J} x_I y_J$ provide a contradiction. This implies that $U(\mathfrak{b})_s = U(\mathfrak{a})_s \cap U(\mathfrak{b})$. \square

We turn to PBW filtration of sub-modules, say $\mathfrak{b} \subset \mathfrak{a}$ a subalgebra and V a cyclic \mathfrak{a} -module with generator v . Here one can apply the PBW filtration to

$$V_{\mathfrak{b}} := U(\mathfrak{b}) \cdot v \subset V.$$

In contrast to the case of subalgebras, this filtration is in general not compatible with the filtration on V . Let $\mathfrak{a} = \langle x, y, z \rangle_{\mathbb{C}}$ be the abelian three dimensional Lie algebra and $\mathfrak{b} = \langle x, y \rangle_{\mathbb{C}}$. We consider the $U(\mathfrak{a})$ -module defined by the ideal $(x^2, y^2, z^2, xz, yz, xy - z)$. Then the degree vector of $\text{gr } V$ is $(1, 3, 0, \dots)$. While the submodule through 1 generated by \mathfrak{b} has degree vector $(1, 2, 1, 0 \dots)$. We see, that if the vanishing ideal is not homogeneous, the PBW filtration is not compatible. One needs a few more properties to have the PBW filtration on modules compatible with submodules.

Let for the moment \mathfrak{g} be a simple, finite-dimensional complex Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and let $\mathfrak{b} \subset \mathfrak{n}^-$ be a subalgebra such that $\mathfrak{b} = \bigoplus_{\alpha \in R^-} \mathfrak{b} \cap \mathfrak{g}_{\alpha}$. Let λ be a dominant integral weight for \mathfrak{g} and $V(\lambda)$ the irreducible highest weight module with highest weight λ , v_{λ} a highest weight generator.

Proposition 14 *With this setup, one has $\text{gr } V_{\mathfrak{b}}(\lambda) \subset \text{gr } V(\lambda)$, the PBW filtration on $V(\lambda)$ is compatible with the filtration on the subalgebra.*

Proof We follow here the proof in Chirivi et al. (2021). The PBW filtration on $V(\lambda)$ is induced by the filtration on each weight space $V_{\mu}(\lambda)$ and for the associated graded module, it is enough to compare the degree of each monomial in root vectors. Now let $0 \neq f_{\beta_1} \cdots f_{\beta_s} \cdot v_{\lambda} \in V_{\alpha}(\lambda)$ with $f_{\beta_i} \in \mathfrak{b}$. Suppose there exists $f_{\gamma_1} \cdots f_{\gamma_r} \cdot v_{\lambda} = f_{\beta_1} \cdots f_{\beta_s} \cdot v_{\lambda}$ for some $r < s$. From Ohsugi and Hibi (2002), we deduce that there are two kinds of exchange relations on roots. The first one is homogeneous and for our purpose, we can ignore it here. The second is not homogeneous and reads as $\alpha + \beta = \gamma$ for some roots. Suppose now there are i, j, k with $\beta_i + \beta_j = \gamma_k$. But this implies that $f_{\gamma_k} \in \mathfrak{b}$, since \mathfrak{b} is a subalgebra. Hence we deduce that $f_{\gamma_1} \cdots f_{\gamma_r} \in U(\mathfrak{b})$ and hence the PBW filtrations are compatible. \square

5.3 The FFLV bases

In Feigin et al. (2011a, b), monomial bases of simple, finite-dimensional modules for complex Lie algebras of type A and C were introduced:

Let $\lambda = \sum m_i \omega_i \in P^+$, then for type A (resp. type C), the following polytope has been defined in Feigin et al. (2011a) (resp. Feigin et al. 2011b):

$$P(\lambda) = \left\{ (x_{\alpha}) \in \mathbb{R}_{\geq 0}^{\#R^+} \mid \forall \text{ Dyck paths } \mathbf{p}, \sum_{\alpha \in \mathbf{p}} x_{\alpha} \leq m_{s(\mathbf{p})} + \cdots + m_{e(\mathbf{p})} \right\}$$

We denote $S(\lambda) = P(\lambda) \cap \mathbb{Z}^{\#R^+}$, the lattice points. In the mentioned paper it is proved that

$$\left\{ \prod_{\alpha} f_{\alpha}^{s_{\alpha}} \cdot v_{\lambda} \mid \mathbf{s} \in S(\lambda) \right\}$$

is (for any chosen order in the monomials) a basis of $V(\lambda)$. This basis has an interesting property as we will explain here:

Theorem 15 (Feigin et al. (2011a, b)) *The following is a basis of $gr V(\lambda)$:*

$$\left\{ \prod_{\alpha} f_{\alpha}^{s_{\alpha}} \cdot v_{\lambda} \mid \mathbf{s} \in S(\lambda) \right\}.$$

There are several monomial bases known for $V(\lambda)$, such as the basis constructed by Gelfand and Tsetlin (1950) and its generalization (Molev (2006)), Lusztig bases (Lusztig (1990)), string bases (Littelmann (1998); Berenstein and Zelevinsky (1992)) but none of these bases is compatible with the PBW filtration.

An important property of the assigned polytopes is the following

Theorem 16 (Feigin et al. (2011a, b)) *Let $\lambda, \mu \in P^+$, then $P(\lambda)$ is a lattice polytope and moreover:*

$$P(\lambda) + P(\mu) = P(\lambda + \mu) \text{ and } S(\lambda) + S(\mu) = S(\lambda + \mu),$$

where $+$ denotes the Minkowski sum.

5.4 Monomial bases for submodules

We turn to admissible sets $A \subset R^-$ and the natural projection of the polytopes $P(\lambda)$. We define

$$\pi_A : \mathbb{R}^{\#R^+} \longrightarrow \mathbb{R}^{\#A}, \quad e_{\alpha} \mapsto \begin{cases} e_{\alpha} & \text{if } \alpha \in -A, \\ 0 & \text{else,} \end{cases}$$

and

$$P_A(\lambda) := \pi_A(P(\lambda)) \text{ which implies } S_A(\lambda) = \pi_A(S(\lambda)).$$

As we are considering faces of the polytope $P(\lambda)$, we have by Theorem 16:

Corollary 17 *For all $\lambda, \mu \in P^+$ and every admissible subset A one has*

$$P_A(\lambda) + P_A(\mu) = P_A(\lambda + \mu) \text{ and } S_A(\lambda) + S_A(\mu) = S_A(\lambda + \mu).$$

We list a few useful relations:

Proposition 18 *Let $1 \leq i \leq n$ and let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ or \mathfrak{sp}_{2n} . Let $1 < k \leq i \leq j < \ell \leq n$ (resp. $\ell < n$ for type C), then one has the following relation*

$$(f_{1,j} f_{k,\ell} + c f_{1,\ell} f_{k,j}) \cdot v_{\omega_i} = 0 \in V(\omega_i).$$

In type C, there are more relations that will be needed in the following:

1. for $1 < k \leq \ell, j \leq n$, one has

$$(f_{1,j} f_{k,\bar{\ell}} + c_1 f_{1,\bar{\ell}} f_{k,j} + c_2 f_{1,\bar{k}} f_{\ell,j}) \cdot v_{\omega_i} = 0 \in V(\omega_i)$$

for some $c_r \in \mathbb{C} \setminus \{0\}$, where $f_{\ell,j} = 1$ for $\ell = j + 1$ and 0 for $\ell > j + 1$.

2. for $1 < k \leq \ell < j \leq n$, one has

$$(f_{1,\bar{j}} f_{k,\bar{\ell}} + c_1 f_{1,\bar{\ell}} f_{k,\bar{j}} + c_2 f_{1,\bar{k}} f_{\ell,\bar{j}}) \cdot v_{\omega_i} = 0 \in V(\omega_i)$$

for some $c_r \in \mathbb{C} \setminus \{0\}$.

Proof For $\mathfrak{g} = \mathfrak{sl}_{n+1}$ one has $f_{1,\ell}^2 \cdot v_{\omega_i} = 0$, resp. for $\mathfrak{g} = \mathfrak{sp}_{2n}$ one has $f_{1,\ell-1} \cdot v_{\omega_i} = 0$ and $f_{1,\bar{1}}^2 \cdot v_{\omega_i} = 0 \in V(\omega_i)$. Certainly, for any monomial $\mathbf{m} \in U(\mathfrak{g})$:

$$\mathbf{m} \cdot f_{1,\bar{1}}^2 \cdot v_{\omega_i} = 0.$$

- For the first relation, one sets $\mathbf{m} = e_{j+1,\ell} e_{1,k-1}$.
- For the second relation, one sets $\mathbf{m} = e_{1,k-1} e_{j+1,\bar{j+1}} e_{1,\ell-1} e_{1,j}$.
- For the third relation, one sets $\mathbf{m} = e_{1,k-1} e_{1,j-1} e_{1,\ell-1}$.

The computation is straightforward and can be found in Feigin et al. (2011a, b). □

We give an equivalent statement of the proposition. For $\alpha \succeq \beta \in R^-$ one has

$$f_\alpha f_\beta \cdot v_{\omega_i} + \sum c f_{\gamma_1} f_{\gamma_2} \cdot v_{\omega_i} = 0.$$

where the sum is over $(\gamma_1, \gamma_2) \in \mathcal{P}(\alpha + \beta)$ with $(\alpha, \beta) \leq (\gamma_1, \gamma_2)$.

The main motivation for the paper is the following:

Theorem 19 *Let \mathfrak{g} be of type A or C, $\lambda \in P^+, A \subset R^-$ admissible, then*

$$\left\{ \prod_{\alpha \in A} f_\alpha^{s_\alpha} \cdot v_\lambda \mid \mathbf{s} \in S_A(\lambda) \right\}$$

is a basis of gr $V_A(\lambda)$ and hence (for any chosen ordering in the monomials) a basis for $V_A(\lambda)$.

A remark before the proof: a proof for the \mathfrak{sl}_{n+1} -case has been provided in Fourier (2016) and then Proposition 3 allows to reduce the study for \mathfrak{sp}_{2n} to \mathfrak{sl}_{n+1} .

Proof Let $\lambda = \sum m_i \omega_i \in P^+$. Since $S_A(\lambda) \subset S(\lambda)$,

$$\left\{ \prod_{\alpha \in A} f_\alpha^{s_\alpha} \cdot v_\lambda \mid \mathbf{s} \in S_A(\lambda) \right\}$$

is linearly independent, due to Theorem 15. It remains to prove, that this set spans $V_A(\lambda)$. Clearly,

$$\left\{ \prod_{\alpha \in A} f_\alpha^{s_\alpha} \cdot v_\lambda \mid \mathbf{s} \in \mathbb{Z}_{\geq 0}^{\#A} \right\}$$

is a spanning set and we have to prove a straightening law to reduce to $S_A(\lambda)$. Let $\mathbf{s} \in \mathbb{Z}_{\geq 0}^{\#A}$ and suppose there is a Dyck path $\mathbf{p} \subset A$, such that consecutive roots are linked and

$$\sum_{\alpha \in \mathbf{p}} s_\alpha > m_{s(\mathbf{p})} + \dots + m_{e(\mathbf{p})}.$$

We will show that $f^{\mathbf{s}} \cdot v_\lambda$ is in the span of

$$\{f^{\mathbf{s}} \cdot v_\lambda \mid \mathbf{s} \in S_A(\lambda)\}.$$

First, we consider \mathfrak{sl}_{n+1} , then $\mathbf{p} = (\alpha_{i_1, j_1}, \dots, \alpha_{i_s, j_s})$ with $i_1 \leq \dots \leq i_s, j_1 \leq \dots \leq j_s$. We have seen in Proposition 4, that the full subgrid $\bar{\mathbf{p}}^\oplus$ of roots α_{i_k, j_l} is contained in A .

In Feigin et al. (2011a), a homogeneous order $<$ on $\mathbb{Z}^{\#R^-}$ is defined, we consider its restriction to $\mathbb{Z}^{\#A_{\mathbf{p}}}$. It is shown using operators from $\{e_\beta \mid \beta \in R^+\}$ applied on $f^{\sum_{\alpha \in \mathbf{p}} s_\alpha} \cdot v_\lambda (= 0 \in V(\lambda))$ that

$$f^{\mathbf{s}} \cdot v_\lambda = \sum_{\mathbf{t} < \mathbf{s}} c_{\mathbf{t}} f^{\mathbf{t}} \cdot v_\lambda$$

for some $c_{\mathbf{t}}$. A further inspection of the proof in Feigin et al. (2011a) shows, that the needed operators e_β are exactly given by $\beta = \gamma_1 - \gamma_2$ with $\gamma_i \in \bar{\mathbf{p}}^\oplus$. We conclude from Proposition 6, that each \mathbf{t} on the right hand side is supported on $\bar{\mathbf{p}}^\oplus \subset A$. Concluding, for each violated Dyck path we have a straightening law with respect to $<$. Iterating this procedure is a finite process, showing that

$$\{f^{\mathbf{s}} \cdot v_\lambda \mid \mathbf{s} \in S_A(\lambda)\}$$

is a spanning set for $V_A(\lambda)$.

We turn to $\mathfrak{g} = \mathfrak{sp}_{2n}$. It turns out, that the same argument as for \mathfrak{sl}_{n+1} is valid here, thanks to Proposition 3. The homogeneous order for \mathfrak{sl}_{n+1} is naturally extended to \mathfrak{sp}_{2n} in Feigin et al. (2011b).

Similarly as above, we just have to show that for each Dyck path \mathbf{p} , there is a set of differential operators in $U(\mathfrak{n}^+)$ that are acting on a given subset $A_{\mathbf{p}}^\tau$ and to generate a straightening law. Feigin et al. (2011b) provides the straightening law and we have

to check, whether this is supported on A only. But this follows as for \mathfrak{sl}_{n+1} from Proposition 5 and, with a close inspection of the proof in Feigin et al. (2011b), from Proposition 6. \square

With the following lemma, we justify the discussion of admissible sets. Reasonably, we restrict ourselves to $A \subset R^-$ such that \mathfrak{n}_A^- is a Lie subalgebra.

Lemma 20 *Suppose $A \subset R^-$ is not admissible, then there exists $i \in I$ such that the lattice points in the face $P_A(\omega_i)$ do not parametrize a basis of $V_A(\omega_i)$.*

Proof Suppose A is not admissible, then there exists $\beta_1 \succ \beta_2$ such that $(\gamma_1, \gamma_2) \in \beta_1 \oplus \beta_2$ but $\{\gamma_1, \gamma_2\} \notin A$. Suppose $\text{supp } \beta_1 \cap \text{supp } \beta_2 = \emptyset$ but $\beta_1 \oplus \beta_2 \neq \emptyset$, then $\beta_1 \oplus \beta_2 = \{\beta_1 + \beta_2\}$, but this is in A since \mathfrak{n}_A^- is a Lie subalgebra. In both cases, there exists i with

$$f_{\beta_1} f_{\beta_2} \cdot v_{\omega_i} \neq 0 \text{ and } f_{\gamma_1} f_{\gamma_2} \cdot v_{\omega_i} \neq 0.$$

γ_1, γ_2 are non-comparable with respect to \succ , and hence, due to the definition of Dyck paths, $e_{\gamma_1} + e_{\gamma_2} \in S(\omega_i) \setminus S_A(\omega_i)$. Proposition 18 on the other hand implies, that $f_{\beta_1} f_{\beta_2} \cdot v_{\omega_i}$ is a nontrivial linear combination involving $f_{\gamma_1} f_{\gamma_2} \cdot v_{\omega_i}$. Since $S(\omega_i)$ parametrizes a linearly independent subset

$$f_{\beta_1} f_{\beta_2} \cdot v_{\omega_i} \notin \text{span} \left\{ \prod_{\alpha \in A} f_{\alpha}^{s_{\alpha}} \cdot v_{\omega_i} \mid \mathbf{s} \in S_A(\omega_i) \right\}.$$

\square

5.5 Geometric interpretation

In this section, we are giving a geometric interpretation of the results on monomial bases. The main motivation is due to the admissible sets of the form A_w for some $w \in W$. In this context, the module $V_A(\lambda)$ is isomorphic to the Demazure module $V_w(\lambda)$ and one obtains an action of $\mathfrak{n}^+ \oplus \mathfrak{h}$ on $\text{gr } V_w(\lambda)$ (resp. $w^{-1}(\mathfrak{n}^+ \oplus \mathfrak{h})$ on $\text{gr } V_A(\lambda)$).

5.5.1 Favourable modules and flat degenerations

We begin by briefly explaining the notion of a favourable module following Feigin et al. (2017). Let M be a cyclic finite-dimensional complex vector space acted upon by a complex algebraic unipotent group \mathbb{U} satisfying certain conditions. First, this module is cyclic for $U(\mathfrak{n})$, the universal enveloping algebra of \mathfrak{n} , the nilpotent Lie algebra corresponding to \mathbb{U} . For a fixed basis of \mathfrak{n} and a fixed homogeneous ordering \leq on monomials in this basis, we obtain a filtration of M and the associated graded space has one-dimensional leafs only. The monomials of the basis of $\text{gr } M$ are called essential (Fang et al. (2017)) and we denote $\text{es } M \subset \mathbb{Z}_{\geq 0}^N$ the set of the exponents of these monomials. M is called favourable if

- there exists a convex polytope $P(M) \subset \mathbb{R}_{\geq 0}^N$ such that its set of lattice points, $S(M)$, coincides with the set of essential multi-exponents $\text{es } (M)$.

- for all $n \geq 1$, consider $U(\mathfrak{n}) \cdot (m^{\otimes n}) \subset M \otimes \cdots \otimes M$ and demand $\dim U(\mathfrak{n}) \cdot (m^{\otimes n}) = |nS(M)|$.

In the following, we show that the submodule $V_A(\lambda) \subset V(\lambda)$ is favourable. For a fixed admissible subset A , let G_A denote the corresponding connected, simply connected Lie group of the Lie algebra \mathfrak{g}_A . Notice that G_A is a unipotent subgroup of G , the connected, simply connected Lie group of the Lie algebra \mathfrak{g} . We have:

Lemma 21 *For A an admissible set, $V_A(\lambda)$ is a favourable G_A -module for $\lambda \in P^+$ regular.*

Proof We follow the proof of (Fourier 2016, Lemma 5). It suffices to consider the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let the roots in R^+ be ordered as follows:

$$\alpha_{1,\bar{1}} \succ \alpha_{1,\bar{2}} \succ \alpha_{2,\bar{2}} \succ \cdots \succ \alpha_{1,\bar{n}} \succ \cdots \succ \alpha_{n,\bar{n}} \succ \alpha_{1,n-1} \succ \cdots \succ \alpha_{n-1,n-1} \succ \cdots \succ \alpha_{1,1}.$$

We consider the restriction of this order on A . By (Feigin et al. 2017, Theorem 11.8), choosing the above order and taking the induced homogeneous reverse lexicographic order on monomials in $S(\mathfrak{n}^-)$, implies that the \mathfrak{sp}_{2n} -module $V(\lambda)$ is favourable. In particular, we have $S(\lambda) = es(V(\lambda))$. Let $\mathfrak{s} \in \mathbb{Z}_{\geq 0}^N$, with $s_\alpha = 0$ for all $\alpha \notin A$. Suppose \mathfrak{s} is essential for $V(\lambda)$. By definition, and through the trivial embedding $\mathbb{Z}_{\geq 0}^{\#A} \hookrightarrow \mathbb{Z}_{\geq 0}^N$, it follows that \mathfrak{s} is essential for $V_A(\lambda)$. It therefore follows that $S_A(\lambda) \subset es(V_A(\lambda))$. The opposite inclusion $es(V_A(\lambda)) \subset S_A(\lambda)$ holds by reason of dimension, which gives the equality.

Furthermore, we have an injective homomorphism of modules $V_A(\lambda + \mu) \hookrightarrow V_A(\lambda) \otimes V_A(\mu)$, obtained by restricting the embedding $V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes V(\mu)$. From this, and by Corollary 17, we have that for all $\lambda, \mu \in P^+$:

$$\dim U(\mathfrak{n}_A^-) \cdot v_\lambda \otimes v_\mu = |S_A(\lambda + \mu)| = |S_A(\lambda) + S_A(\mu)|.$$

The proof is completed by considering the case $\lambda = \mu$, and extending by induction to the general case. □

We denote the projective varieties

$$X_A(\lambda) := \overline{G_A \cdot [v_\lambda]} \hookrightarrow \mathbb{P}(V(\lambda)).$$

and

$$X_A^a(\lambda) := \overline{G_A^a \cdot [v_\lambda]} \hookrightarrow \mathbb{P}(\text{gr } V_A(\lambda)).$$

where G_A^a denotes $\#A$ copies of the additive group \mathbb{G}_a , which is a Lie group associated with the Lie algebra of the vector space \mathfrak{n}_A^- with a trivial bracket. Note that the latter and hence G_A^a act on $\text{gr } V_A(\lambda)$.

Definition 9 For A admissible and $\lambda \in P^+$ regular, we call $X_A^a(\lambda)$ the PBW degenerate variety corresponding to $X_A(\lambda)$.

While the definition is reasonable for all types, the following proposition depends on the existence of a suitable basis.

Proposition 22 *Let \mathfrak{g} be of type \mathbb{A}, \mathbb{C} . Let $\lambda \in P^+$ be regular, and A be an admissible set, then the variety $X_A(\lambda)$ degenerates flatly into the PBW degenerate variety $X_A^a(\lambda)$ and further into a toric variety.*

Proof It follows from (Feigin et al. 2017, Theorem 8.1) since the module $V_A(\lambda)$ is favourable (according to Lemma 21). \square

Remark 1 In view Proposition 22, whenever A is an admissible set given as A_w for some Weyl group element w and $\lambda \in P^+$ regular, we obtain flat degenerations of the Schubert variety $X_w(\lambda)$ into a PBW degenerate Schubert variety $X_w^a(\lambda)$ and further into a toric variety. More particularly, we obtain flat PBW and toric degenerations of symplectic Schubert varieties, adding to the similar scenario for Schubert varieties of type \mathbb{A} (Fourier (2016)).

5.5.2 PBW-semistandard tableaux bases for coordinate rings

Feigin introduced in Feigin (2011) PBW-semistandard tableaux to describe a monomial basis of the homogeneous coordinate ring of the PBW-degenerate flag variety in type \mathbb{A} , extended by the first author in Balla (2022) to type \mathbb{C} . Within our setup, it is natural to describe the subset of those tableaux parametrizing a basis of the homogeneous coordinate ring of the PBW degenerate Schubert variety $X_w^a(\lambda)$ whenever w corresponds to an admissible subset A .

For a Young diagram Y_λ corresponding to a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$, let μ_c denote the length of the c -th column.

A type \mathbb{A} PBW-semistandard tableau is a filling of the Young diagram Y_λ of shape $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$, with numbers $T_{r,c} \in \{1, \dots, n + 1\}$ satisfying the properties:

- if $T_{r,c} \leq \mu_c$, then $T_{r,c} = r$,
- if $r_1 < r_2$ and $T_{r_1,c} \neq r_1$, then $T_{r_1,c} > T_{r_2,c}$,
- for any $c > 1$ and any r , there exists $r' \geq r$ such that $T_{r',c-1} \geq T_{r,c}$.

A type \mathbb{C} PBW-semistandard tableau is a filling of the Young diagram Y_λ of shape $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$, with numbers $T_{r,c} \in \{1, \dots, n, \bar{n}, \dots, \bar{1}\}$ satisfying all the three properties above and the following extra property:

- if $T_{r,c} = r$, and $\exists r'$ such that $T_{r',c} = \bar{r}$, then $r' < r$.

Example 4 Consider \mathfrak{g} of type \mathbb{A}_3 . Then the full set of type \mathbb{A} PBW-semistandard tableaux of shape $\lambda = \omega_1 + \omega_2$ is the following set of tableaux with entries in $\{1, 2, 3, 4\}$:

$$\begin{matrix} \boxed{1} \boxed{1} & \boxed{1} \boxed{2} & \boxed{1} \boxed{1} & \boxed{1} \boxed{2} & \boxed{1} \boxed{3} & \boxed{3} \boxed{1} & \boxed{3} \boxed{2} & \boxed{3} \boxed{3} & \boxed{4} \boxed{1} & \boxed{4} \boxed{2} \\ \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} \\ \hline \boxed{4} \boxed{3} & \boxed{4} \boxed{4} & \boxed{4} \boxed{1} & \boxed{4} \boxed{2} & \boxed{4} \boxed{3} & \boxed{4} \boxed{4} & \boxed{1} \boxed{1} & \boxed{1} \boxed{2} & \boxed{1} \boxed{3} & \boxed{1} \boxed{4} \\ \boxed{2} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \end{matrix}$$

By renaming the entries to read $\{1, 2, \bar{2}, \bar{1}\}$, it can be seen that the last four tableaux are not symplectic. So without them, we recover the respective tableaux for \mathfrak{g} of type \mathbb{C}_2 .

Proposition 23 (Feigin 2012; Balla 2022) *Let \mathfrak{g} be of type \mathbb{A}, \mathbb{C} . For $\lambda = \sum_{k=1}^n m_k \omega_k$ a dominant integral weight, the set $\{\prod_{\alpha} f_{\alpha}^{s_{\alpha}} \cdot v_{\lambda} \mid \mathbf{s} \in S(\lambda)\}$ is in a weight preserving one-to-one correspondence with the set of PBW-semistandard tableaux of shape λ .*

The idea for the proof of the above proposition is the following. If a root vector $f_{\alpha_{i,j}}$ appears in $\prod_{\alpha} f_{\alpha}^{s_{\alpha}} \cdot v_{\lambda}$, with $\mathbf{s} \in S(\lambda)$, then there is a box in the i -th row of the corresponding tableau containing the entry $j + 1$. Additionally for type \mathbb{C} , if there is a root vector of the

form $f_{\alpha_{i,\bar{j}}}$ in $\prod_{\alpha} f_{\alpha}^{s_{\alpha}} \cdot v_{\lambda}$, with $\mathbf{s} \in S(\lambda)$, there should be an entry \bar{j} in some box in the i -th row. Details on how these boxes are arranged to form the PBW-semistandard tableaux can be found in Balla (2022).

Now we turn to the case of an admissible set A . We define

Definition 10 An A -permissible type \mathbb{A} PBW-semistandard tableau of shape λ is a type \mathbb{A} PBW-semistandard tableau of shape λ , that satisfies the following extra condition:

(a) for all $\alpha_{i,j} \notin A$, if $j \geq \mu_c$ and $r = i$ then $T_{r,c} \neq j + 1$.

An A -permissible type \mathbb{C} PBW-semistandard tableau of shape λ is a type \mathbb{C} PBW-semistandard tableau of shape λ , that satisfies (a) above and the following extra condition:

(b) for all $\alpha_{i,\bar{j}} \notin A$, $T_{r,c} \neq \bar{j}$ for all c and r such that $r = i$.

Example 5 Consider an admissible set of type \mathbb{A}_3 ; $A = \{\alpha_{1,2}, \alpha_2, \alpha_{1,3}, \alpha_{2,3}\}$. Then the following is the full set of A -permissible type \mathbb{A} PBW-semistandard tableaux of shape $\lambda = (2, 1)$:

$$\begin{array}{cccccccc}
 \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \end{array} & \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 & \end{array} & \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 2 & \end{array} & \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 2 & \end{array} & \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & \end{array} \\
 \begin{array}{|c|c|} \hline 4 & 4 \\ \hline 2 & \end{array} & \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 3 & \end{array} & \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 3 & \end{array} & \begin{array}{|c|c|} \hline 4 & 4 \\ \hline 3 & \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 4 & \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & \end{array} & \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 4 & \end{array}
 \end{array}$$

The following is an analogue of Proposition 23 for admissible sets.

Proposition 24 For an admissible set A , and for $\lambda = \sum_{k=1}^n m_k \omega_k$ a dominant integral weight, the set $\{\prod_{\alpha} f_{\alpha}^{s_{\alpha}} \cdot v_{\lambda} \mid \mathbf{s} \in S_A(\lambda)\}$ is in a weight preserving one-to-one correspondence with the set of A -permissible PBW-semistandard tableaux of shape λ .

Proof We claim that the bijection of Proposition 23 still holds when restricted to the case of an admissible set A . Indeed, suppose $\alpha_{i,j} \notin A$ with $j \geq \mu_c$ and $r = i$. Then clearly $T_{r,c} \neq j + 1$ in the corresponding tableau. And likewise for any $\alpha_{i,\bar{j}} \notin A$, it follows that $T_{r,c} \neq \bar{j}$ for all columns c and rows r such that $r = i$. This together with Theorem 19 imply the result.

Acknowledgements The first and third authors are supported by the Deutscher Akademischer Austauschdienst (DAAD, German Academic Exchange Service) scholarship program: Research Grants-Doctoral Programs in Germany [program-ID 57440921]. The work of the second author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through [project-ID 286237555-TRR 195].

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Balla, G.: Symplectic PBW degenerate flag varieties; PBW tableaux and defining equations. *Transform. Groups* (2022). <https://doi.org/10.1007/s00031-022-09725-9>
- Berenstein, A., Zelevinsky, A.: *String Bases for Quantum Groups of Type A_r*. Kyoto University. Research Institute for Mathematical Sciences [RIMS] (1992)
- Biswal, R., Fourier, G.: Minuscule Schubert varieties: poset polytopes, PBW-degenerated Demazure modules, and Kogan faces. *Algebra Represent. Theory* **18**(6), 1481–1503 (2015). <https://doi.org/10.1007/s10468-015-9548-5>
- Chirivì, R., Fang, X., Fourier, G.: Degenerate Schubert varieties in type A. *Transform. Groups* **26**, 1189–1215 (2021)
- Dehy, R., Yu, R.W.T.: Degeneration of Schubert varieties of SL_n/B to toric varieties. *Ann. Inst. Fourier (Grenoble)* **51**(6), 1525–1538 (2001)
- Fang, X., Fourier, G., Littelmann, P.: Essential bases and toric degenerations arising from birational sequences. *Adv. Math.* **312**, 107–149 (2017). <https://doi.org/10.1016/j.aim.2017.03.014>
- Feigin, E.: Degenerate flag varieties and the median Genocchi numbers. *Math. Res. Lett.* **18**(6), 1163–1178 (2011)
- Feigin, E.: \mathbb{C}_q^n degeneration of flag varieties. *Sel. Math.* **18**(3), 513–537 (2012)
- Feigin, E., Fourier, G., Littelmann, P.: PBW filtration and bases for irreducible modules in type A_n . *Transform. Groups* **16**(1), 71–89 (2011a)
- Feigin, E., Fourier, G., Littelmann, P.: PBW filtration and bases for symplectic lie algebras. *Int. Math. Res. Not.* **2011**(24), 5760–5784 (2011b)
- Feigin, E., Fourier, G., Littelmann, P.: Favourable modules: filtrations, polytopes, newton-okounkov bodies and flat degenerations. *Transform. Groups* **22**(2), 321–352 (2017)
- Fourier, G.: PBW-degenerated demazure modules and schubert varieties for triangular elements. *J. Comb. Theory Ser. A* **139**, 132–152 (2016)
- Gelfand, I.M., Tsetlin, M.L.: Finite-dimensional representations of the group of unimodular matrices. In *Dokl. Akad. Nauk SSSR* **71**, 825 (1950)
- Humphreys, J.E.: *Introduction to Lie Algebras and Representation Theory*, vol. 9. Springer Science & Business Media, Berlin (2012)
- Kambaso, K.: Homogeneous bases for Demazure modules. *Commun. Algebra* **50**(7), 2934–2953 (2022)
- Kirichenko, V.A., Smirnov, E.Y., Timorin, V.A.: Schubert calculus and Gelfand–Tsetlin polytopes. *Uspekhi Mat. Nauk* **67**(4(406)), 89–128 (2012). <https://doi.org/10.1070/RM2012v067n04ABEH004804>
- Kogan, M.: *Schubert geometry of flag varieties and Gelfand-Cetlin theory*. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)—Massachusetts Institute of Technology (2000)
- Littelmann, P.: Cones, crystals, and patterns. *Transform. Groups* **3**(2), 145–179 (1998)
- Lusztig, G.: Canonical bases arising from quantized enveloping algebras. *J. Am. Math. Soc.* **3**(2), 447–498 (1990)
- Makhlín, I.: FFLV-type monomial bases for type B. *Algebraic Comb.* **2**(2), 305–322 (2019)
- Molev, A.I.: Gelfand–Tsetlin bases for classical lie algebras. *Handb. Algebra* **4**, 109–170 (2006)
- OEIS Foundation Inc.: The on-line encyclopedia of integer sequences (2018a). <https://oeis.org/A032351>
- OEIS Foundation Inc.: The on-line encyclopedia of integer sequences (2018b). <https://oeis.org/A006012>
- Ohsugi, H., Hibi, T.: Quadratic initial ideals of root systems. *Proc. Am. Math. Soc.* **130**(7), 1913–1922 (2002)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.