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ORIGINAL PAPER

An imaginary refined count for some real rational curves

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Abstract

In 2015, Mikhalkin introduced a refined count for the real rational curves in a toric surface which pass through a set consisting of real points and pairs of complex conjugated points chosen generically on the toric boundary of the surface. He then proved that the result of this refined count depends only on the number of pairs of complex conjugated points on each toric divisor. Using the tropical geometry approach and the correspondence theorem, we address the computation of the refined count when the pairs of complex conjugated points are chosen purely imaginary and belonging to the same component of the toric boundary. Despite the non-genericity, we relate this refined count for purely imaginary values to the refined invariant of Mikhalkin for generic values. That allows us to extend the relation between these classical refined invariants and the tropical refined invariants from Block–Göttsche.

Keywords Tropical geometry · refined invariants · Toric surfaces

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1 Introduction

1.1 Curves in a toric surface and their enumeration

Let N be a lattice of rank 2 and $\Delta = (n_j) \subset N$ be a multiset of m primitive lattice vectors, whose total sum is zero. We consider rational curves of degree Δ inside the toric surface $\mathbb{C}\Delta$ obtained from the fan Σ_Δ in $N_\mathbb{R}$ spanned by Δ. The complex conjugation makes it into a real surface. In this setting, we choose a generic configuration $\mathcal{P}$ of $m-1$ points inside $\mathbb{C}\Delta$, and look for rational curves of degree Δ passing through this configuration. We have a finite number of complex solutions. If we denote by $\mathcal{S}^\mathbb{C}(\mathcal{P})$ the set of solutions, it is a classical result that the cardinal $|\mathcal{S}^\mathbb{C}(\mathcal{P})|$ is independent of the point configuration $\mathcal{P}$. Its value is denoted by N_Δ.
Over the real field, the situation is different. If we choose a conjugation invariant configuration of points $\mathcal{P}$, called a *real configuration*, meaning that it consists of real points and pairs of conjugated points, and denote by $\mathcal{S}^\mathbb{R}(\mathcal{P})$ the set of real rational curves passing through $\mathcal{P}$, the value of the cardinal $|\mathcal{S}^\mathbb{R}(\mathcal{P})|$ depends on the choice of $\mathcal{P}$. However, Welschinger (2005) showed that for del Pezzo surfaces, if the curves are counted with an appropriate sign, the count of real solutions depends only on the number s of pairs of conjugated points in the configuration, yielding an invariant denoted by $W_{\Delta,s}$.
While the values of N_Δ were already known, those of $W_{\Delta,s}$ were computed roughly at the same time their invariance was proven. In Mikhalkin (2005), G. Mikhalkin proved a correspondence theorem along with a lattice path algorithm that provided a way of computing both invariants N_Δ and $W_{\Delta,0}$ (only real points in the configuration) using the tropical geometry approach. Later E. Shustin (Shustin 2006a, b) also used tropical geometry to compute the Welschinger invariants $W_{\Delta,s}$ for any value of s.
To compute the values of N_Δ and $W_{\Delta,0}$, Mikhalkin counts tropical curves solution to the analog tropical enumerative problem with two specific choices of integer multiplicity. Following his computation, Block and Göttsche (2016) proposed a way of combining these integer multiplicities by refining them into a Laurent polynomial multiplicity, which gives back the values of N_Δ and $W_{\Delta,0}$ when evaluated at ± 1 respectively. This new refined multiplicity was proved in Itenberg and Mikhalkin (2013) to

give a tropical invariant. This choice of multiplicity seems to appear in a growing number of situations, while its meaning in classical geometry remains quite mysterious. Conjecturally, this invariant is suspected to coincide with the refinement of Severi degrees by the χ_{-y}-genera proposed by L. Göttsche and V. Shende in Göttsche and Shende (2014). This invariant bears also similarities with some Donaldson-Thomas wall-crossing invariants considered by Kontsevich and Soibelman (2008).

1.2 Refined enumerative geometry

Now, let $\mathcal{P}_0$ be a real configuration of m points taken on the toric boundary of $\mathbb{C}\Delta$, and such that each irreducible component of the toric boundary contains a number of points equal to the number of vectors in Δ directing the corresponding ray in the fan Σ_Δ, e.g. for the projective plane, there are d points per axis. If the toric boundary has components labeled from 1 to p, let s_i be the number of pairs of conjugated points on the i-th component, with $|s| = \sum_1^p s_i$. We assume there is at least one real point in the configuration. The Viète formula ensures that there exists a curve not containing the boundary as a component and passing through the configuration $\mathcal{P}_0$ only if the configuration is subject to the *Menelaus condition*, which we therefore assume: the product of the coordinates given by the monomials $\iota_{n_i}\omega \in M$ of the points is equal to ± 1, with a sign depending on the degree.

Example 1.1 In the projective plane, the coordinates on the coordinate axis are respectively given by z, $\frac{1}{w}$ and $\frac{w}{z}$. For a degree d curve, the product of the coordinates of its $3d$ intersection points with the boundary has value $(-1)^d$.

For a point p_i on the boundary of $\mathbb{C}\Delta$, let $-p_i$ denote the opposite point. Notice that for a pair of complex points $\{p_i, \overline{p_i}\}$, the opposite pair $\{-p_i, -\overline{p_i}\}$ is the same pair precisely when p_i is purely imaginary. Let $\mathcal{S}(\mathcal{P}_0)$ be the set of oriented real rational curves of degree Δ that contain either p_i or $-p_i$ for every $p_i \in \mathcal{P}_0$, meaning that the curve passes through one point of each pair of opposite real points, and through both points of one of the opposite pairs of complex conjugated points. Let $\varphi : \mathbb{C}P^1 \to \mathbb{C}\Delta$ be a parametrized curve, with orientation given by $S \subset \mathbb{C}P^1 \backslash \mathbb{R}P^1$. Recall from Mikhalkin (2017) that there is a sign $\sigma(S, \varphi) = \pm 1$ associated to (S, φ), whose definition is recalled in Sect. 4.2, and a quantum index $k(S, \varphi) \in \frac{1}{2}\mathbb{Z}$, whose definition is recalled in Sect. 3.1. We then set

$$R_\Delta(\mathcal{P}_0) = \frac{1}{4} \sum_{(S,\varphi)\in\mathcal{S}(\mathcal{P}_0)} \sigma(S, \varphi) q^{k(S,\varphi)} \in \mathbb{Z}[q^{\pm 1/2}].$$

Theorem 1.2 (Mikhalkin 2017) *As long as the configuration $\mathcal{P}_0$ is generic, the Laurent polynomial $R_\Delta(\mathcal{P}_0)$ only depends on $s = (s_1, \ldots, s_p)$.*

The above Laurent polynomial only depending on s is denoted by $R_{\Delta,s}$.

Remark 1.3 It is important for the result and for its proof to consider not only curves passing through the points of $\mathcal{P}_0$ but also through the opposite points, otherwise the invariance might fail. Moreover, points in $\mathcal{P}_0$ are not assumed to be purely imaginary, though it may appear to be so in Mikhalkin (2017).

In the case of a totally real configuration of points, *i.e.* $s_i = 0$, using the correspondence theorem, Mikhalkin proved that the invariant $R_{\Delta,(0)}$ coincides, up to a normalization, with the tropical refined invariant $N_\Delta^{\partial,\mathrm{trop}}$. This invariant is obtained as follows. (For a more complete description, see Sect. 4.1.) We consider tropical curves of degree $\Delta \subset N$. For each vector n_j in Δ, one chooses an oriented line directed and oriented by n_j, and such that the configuration of lines is generic. Then we count rational tropical curves of degree Δ whose unbounded ends are contained in the chosen lines, using the Block-Göttsche multiplicities from Block and Göttsche (2016). The result does not depend on the chosen configuration, and is denoted by $N_\Delta^{\partial,\mathrm{trop}}$.

1.3 Results

We now address the enumerative problem exposed in the previous section with two additional assumptions: first, the complex points are chosen on the same boundary component, i.e. all s_i but one are zero, and secondly, the complex points are chosen to be purely imaginary. This is a highly non generic choice for several reasons: because arguments are all chosen equal, and because a purely imaginary pair of conjugated points is equal to its symmetric pair. We denote by $\widehat{\mathcal{P}}_0$ a point configuration that satisfies these above assumptions. By a generic choice of $\widehat{\mathcal{P}}_0$, we mean generic among the configurations that satisfy these assumptions.
Assume that the only nonzero s_i is s_1. The first result of this paper relates the refined count $R_\Delta(\widehat{\mathcal{P}}_0)$ for a generic choice of $\widehat{\mathcal{P}}_0$ *close to the tropical limit*, to the tropical refined invariants $N_{\Delta(s)}^{\partial,\mathrm{trop}}$. Here, $\Delta(s)$ is the family $(\Delta\backslash\{n_1^{2s_1}\})\cup\{(2n_1)^{s_1}\}$: $2s_1$ copies of n_1 are replaced by s_1 copies of $2n_1$. Choosing the constraints *close to the tropical limit* means that the coordinates of the points p_i are chosen the form $\alpha_i t^{v_i}$ for some very large t. One recovers the result from Mikhalkin (2017) by taking $s_1 = 0$.

Theorem 1.4 *For a generic choice of $\widehat{\mathcal{P}}_0$ close to the tropical limit, one has*

$$R_\Delta(\widehat{\mathcal{P}}_0) = \frac{(q^{1/2}-q^{-1/2})^{m-2-s_1}}{(q-q^{-1})^{s_1}} N_{\Delta(s)}^{\partial,trop} = \frac{(q^{1/2}-q^{-1/2})^{m-2-2s_1}}{(q^{1/2}+q^{-1/2})^{s_1}} N_{\Delta(s)}^{\partial,trop}$$

The second result of the paper proves that there exists choices of $\widehat{\mathcal{P}}_0$ which are generic viewed as a choice of $\mathcal{P}_0$. This relates the refined count $R_\Delta(\widehat{\mathcal{P}}_0)$ to the value of the refined invariant $R_{\Delta,(s_1,0,\dots,0)}$, despite the fact that the choice of $\widehat{\mathcal{P}}_0$ can appear to be highly non-generic.

Theorem 1.5 *For a choice of $\widehat{\mathcal{P}}_0$ which is a regular value of the map that sends a curve to the coordinates of its boundary points, one has*

$$R_{\Delta,(s_1,0,\dots,0)} = 2^{s_1} R_\Delta(\widehat{\mathcal{P}}_0).$$

Moreover, a generic choice of $\widehat{\mathcal{P}}_0$ close to the tropical limit is such a choice.

This theorem has two corollaries. First, both theorems lead to a computation of the classical refined invariant $R_{\Delta,(s_1,0,\dots,0)}$ by relating it to the tropical refined invariant $N^{\partial,\text{trop}}_{\Delta(s)}$. This is the content of Corollary 4.4. This extends the relation from Theorem 7 in Mikhalkin (2017), which deals with the case $s=0$. Secondly, we obtain that the refined count $R_\Delta(\widehat{\mathcal{P}}_0)$ does not depend on the choice of $\widehat{\mathcal{P}}_0$ as long as it is generic. The factor 2^{s_1} corresponds to the fact that a pair of purely imaginary conjugated points is equal to its opposite pair.
The proof of Theorem 1.4 relies on a tropical correspondence theorem. We use the version of the correspondence theorem from Shustin (2006b), and more precisely Lemmas 2.8 and 3.2. It is also possible to adapt the correspondence theorem from Mikhalkin (2005) or Tyomkin (2017). The latter is done in Blomme (2020).

Remark 1.6 The case where the complex points are not located on a single boundary component is much more delicate to deal with. Corollary 4.4 can be considered as an easy particular case of the general relation between $R_{\Delta,s}$ and $N^{\partial,\text{trop}}_{\Delta(s)}$. However, the statement is almost identical: to obtain it, the s_1 occuring in exponent should be replaced by $|s|=\sum s_k$. It is proven in Blomme (2020) using an adapted version of the correspondence theorem from Tyomkin (2017). In this general case, the complex points can no longer assumed to be purely imaginary, and this is why Theorems 1.4 and 1.5 are not achievable via the results of Blomme (2020). In particular, Theorem 1.4 is not a subcase of any result in Blomme (2020).

The paper is organized as follows. In the second section we recall the standard definitions related to tropical curves and the tropicalization. In the third section we recall the definition of the quantum index and its computation in few cases. The last section is devoted to the enumerative problems leading to the definition of $R_{\Delta,(s_1,0,\dots,0)}$ and $N^{\partial,\text{trop}}_{\Delta(s)}$ and the proof of Theorems 1.4 and 1.5.

2 Tropical curves and real tropical curves

2.1 Tropical curves

We here briefly recall the basics about abstract tropical curves, which are abstract metric graphs, parametrized tropical curves, which are abstract tropical curves endowed with a map to $N_\mathbb{R}$, and plane tropical curves, which are the images of the latter.

Real abstract tropical curves

Let $\overline{\Gamma}$ be a finite connected graph without bivalent vertices. Let $\overline{\Gamma}^0_\infty$ be the set of 1-valent vertices of $\overline{\Gamma}$, and $\Gamma=\overline{\Gamma}\backslash\overline{\Gamma}^0_\infty$. If m denotes the cardinal of $\overline{\Gamma}^0_\infty$, its elements are labeled with integers from $[\![1;m]\!]$. The non-compact edges resulting from the eviction of 1-valent vertices are called *ends*, also labeled by $[\![1;m]\!]$. The set of bounded edges is denoted by Γ^1_b.

Definition 2.1 Let $l : \gamma \in \Gamma_b^1 \mapsto |\gamma| \in \mathbb{R}_+^* =]0; +\infty[$ be a function, called length function. It endows Γ with the structure of a metric graph by choosing an isometry between a bounded edge γ and $[0; |\gamma|]$, and between an end and $[0; +\infty[$. The obtained metric graph Γ is called an *abstract tropical curve*.

An isomorphism between two abstract tropical curves Γ and Γ' is an isometry $\Gamma \to \Gamma'$. In particular an automorphism of Γ does not necessarily respect the labeling of the ends since it only respects the metric. Therefore, an automorphism of Γ induces a permutation of the set $I = [\![1; m]\!]$ of ends.

Definition 2.2 Let Γ be an abstract tropical curve. A *real structure* on Γ is an involutive isometry $\sigma : \Gamma \to \Gamma$. A *real abstract tropical curve* is an abstract tropical curve enhanced with a real structure.

The real structure also induces an involution on the set of ends $I = [\![1; m]\!]$ of Γ. The fixed ends are called *real ends* and the pairs of exchanged ends are called the *conjugated ends*, or *complex ends*. The fixed locus of σ is denoted by $\mathrm{Fix}(\sigma)$. It is a subgraph of Γ.

Real parametrized tropical curves

Let N be a rank two lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$, and $N_\mathbb{R} = N \otimes \mathbb{R}$. We now define parametrized tropical curves in $N_\mathbb{R}$.

Definition 2.3 A *parametrized tropical curve* in $N_\mathbb{R} \simeq \mathbb{R}^2$ is a pair (Γ, h), where Γ is an abstract tropical curve and $h : \Gamma \to \mathbb{R}^2$ is a map satisfying the following requirements:

- For every edge $E \in \Gamma^1$, the map $h|_E$ is affine. If we choose an orientation of E, the value of the differential of h taken at any interior point of E, evaluated on a tangent vector of unit length, is called the slope of h alongside E. This slope must lie in N.
- We have the so called *balancing condition*: at each vertex $V \in \Gamma^0$, if E is an edge containing V, and u_E is the slope of h along E when the edge E is oriented with the vertex V as its source, then

$$\sum_{E:\partial E \ni V} u_E = 0 \in N.$$

Two parametrized curves $h : \Gamma \to N_\mathbb{R}$ and $h' : \Gamma' \to N_\mathbb{R}$ are isomorphic if there exists an isomorphism of abstract tropical curves $\varphi : \Gamma \to \Gamma'$ such that $h = h' \circ \varphi$.

Definition 2.4 A *real parametrized tropical curve* is a triplet (Γ, σ, h), where (Γ, h) is a parametrized tropical curve, σ is a real structure on Γ, and h is σ-invariant: $h \circ \sigma = h$.

Remark 2.5 In particular, two vertices that are exchanged by σ have the same image under h, and two edges that are exchanged by σ have the same slope and the same image. Such edges are called *double edges*. If they are unbounded, we call them a *double end*. Thus, the image $h(\Gamma) \subset N_{\mathbb{R}}$ may not be sufficient to recover Γ and the real structure, since for instance there is no way of distinguishing a double end from a simple end with twice their slope.

If $e \in \Gamma^1_\infty$ is an end of Γ, let $n_e \in N$ be the slope of h alongside e, oriented out of its unique adjacent vertex, *i.e.* toward infinity. The multiset

$$\Delta = \{n_e \in N | e \in \Gamma^1_\infty\} \subset N,$$

is called the *degree* of the parametrized curve. It is a multiset since an element may appear several times. Using the balancing condition, one can show that $\sum_{n_e \in \Delta} n_e = 0$.

Definition 2.6 A parametrized tropical curve is rational if the graph that parametrizes it is a tree.

Plane tropical curves

Definition 2.7 Plane tropical curves are some weighted graphs inside $N_{\mathbb{R}} \simeq \mathbb{R}^2$ obtained equivalently as follows:

- the corner locus of some tropical polynomial:

$$P(x) = \max_{m \in P_\Delta \cap M} (a_m + \langle m, x \rangle),$$

  where P_Δ is some convex lattice polygon, and $a_m \in \mathbb{R}$ are scalars. Edges are endowed with a weight equal to the lattice length of the edge in the dual subdivision of the polygon P_Δ induced by the tropical polynomial P.
- the image of a parametrized tropical curve (Γ, h). An edge E of $h(\Gamma) \subset N_{\mathbb{R}}$ is endowed with a weight equal to the sum of the lattice lengths of the slope of h on the edges γ mapping to E.
- a subgraph of $N_{\mathbb{R}}$ with weight on the edges, whose edges have integer slope, and satisfying the balancing condition: at each vertex V, if u_E denotes the primitive vector in N directing an edge E adjacent to V, and w_E its weight, one has

$$\sum_{E \ni V} w_E u_E = 0.$$

The polygon P_Δ is called the degree of the plane tropical curve. It is associated to the degree Δ of a parametrized tropical curve in the sense that the vectors in Δ are the outing normal vectors to the sides of P_Δ. For more details on plane tropical curves and tropical polynomials, see Brugallé and Shaw (2014).
A plane tropical curve is *irreducible* if it cannot be written as the union of two tropical curves, and it is rational if it is irreducible and is the image of a parametrized rational tropical curve. The distinction between parametrized and plane tropical curves comes from the fact that there are often many ways to parametrize a plane tropical curve. Yet, there is always a canonical way to parametrize *simple plane rational curves* by an abstract rational tropical curve of the right degree.

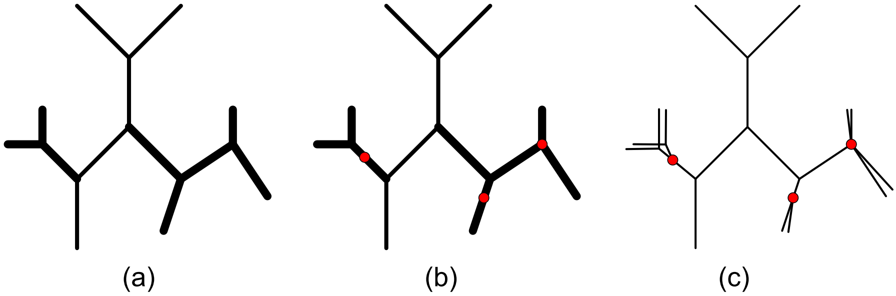


Fig. 1 In **a** a tropical curve with the complex ends and the subgraph Γ_{even} drawn in thick. In **b** some admissible set on the components of Γ_{even}, and in **c** the real tropical curve resulting from the splitting of the curve along the admissible set. The involution exchanges the doubled branches

Proposition 2.8 (Mikhalkin 2005) *Let C be a rational plane tropical curve, and let u_e be a directing primitive lattice vector for each end e, oriented toward infinity. Let w_e be the weight of e. Then C is the image of a unique rational parametrized tropical curve of degree $\Delta = \{w_e u_e\}_e$.*

2.2 Real parametrizations of a plane tropical curve

In this subsection, we extend Proposition 2.8 by describing the possible real rational parametrizations of an irreducible rational plane tropical curve, with ends of weights 1 or 2.

Let C be a rational plane tropical curve with ends of weight 1 or 2. Let $u_1, \ldots, u_r, 2v_1, \ldots, 2v_s$ be the weighted directing vectors of the ends of C, with vectors u_i, v_j being primitive vectors in N. We assume that $r \geq 1$. Let $h : \Gamma \rightarrow N_{\mathbb{R}}$ be the unique rational parametrization of C given by Proposition 2.8, which is of degree $\{u_i, 2v_j\}_{i,j}$. The ends directed by u_i are called *real ends*, and ends directed by $2v_j$ are called *complex ends*. We now describe the parametrizations of C by real parametrized rational curves of degree $\{u_i, v_j^2\}_{i,j}$, which means that now all vectors are primitive, and each end of weight 2 is replaced with two ends of weight 1.

Definition 2.9 The subgraph Γ_{even} of Γ as the subgraph containing all the edges γ satisfying the following: γ splits the curve Γ in two halves, one of them containing only complex ends. Equivalently, Γ_{even} is the complement of the minimal connected subgraph containing all the real ends. See Fig. 1.

Remark 2.10 The subgraph Γ_{even} is the maximal graph on which we can "cut Γ in two" in order to obtain a new graph Γ', used to parametrize C. Notice that on all the edges γ of Γ_{even}, the map h has an even slope: the slope of h belongs to $2N$.

As C admits at least one real end, each connected component $(\Gamma_{\text{even}})_i$ of Γ_{even} contains a unique *stem*. We orient the edges of $(\Gamma_{\text{even}})_i$ away from the stem.

Definition 2.11 A subset of points $\mathcal{R}_i \subset (\Gamma_{\text{even}})_i$ is *admissible* if no point of $\mathcal{R}_i$ is joint to another by an oriented path, and for each end e in $(\Gamma_{\text{even}})_i$, there is at least (and thus exactly one) point of $\mathcal{R}_i$ on the shortest path between the stem and e.

Let $\mathcal{R} = \bigcup_i \mathcal{R}_i$. We then define a real abstract tropical curve $(\Gamma(\mathcal{R}), \sigma)$ with a map $h_{\mathcal{R}} : \Gamma(\mathcal{R}) \to N_{\mathbb{R}}$ that factors through $\Gamma(\mathcal{R}) \to \Gamma \to N_{\mathbb{R}}$ and makes it a real parametrized tropical curve.
Let $\Gamma_{\text{fix}}(\mathcal{R})$ be the closure of the unique connected component of $\Gamma - \mathcal{R}$ not containing any complex end. The abstract tropical curve $\Gamma(\mathcal{R})$ is obtained as the disjoint union of two copies of Γ, glued along $\Gamma_{\text{fix}}(\mathcal{R})$. In other terms, $\Gamma(\mathcal{R}) = \Gamma \coprod_{\Gamma_{\text{fix}}(\mathcal{R})} \Gamma$. It means that we have doubled the components of $\Gamma - \mathcal{R}$ containing the even ends. We denote by $\pi : \Gamma(\mathcal{R}) \to \Gamma$ the map obtained by gluing the identity maps of Γ. The complement of $\Gamma_{\text{fix}}(\mathcal{R})$ in Γ is called the *splitting graph*. It is a subset of Γ_{even}. The splitting graph is maximal if its closure is equal to Γ_{even}. The length function on $\Gamma(\mathcal{R})$ is defined as follows: we consider points of $\mathcal{R}$ as vertices of Γ, then, the length of an edge γ of $\Gamma(\mathcal{R})$ is the length of its image $\pi(\gamma)$ if it is an edge of $\Gamma_{\text{fix}}(\mathcal{R})$ and twice the length of $\pi(\gamma)$ otherwise. The involution σ is the automorphism of $\Gamma(\mathcal{R})$ that exchanges the two preimages whenever there are two. The parametrized map $h_{\mathcal{R}} : \Gamma(\mathcal{R}) \to N_{\mathbb{R}}$ is the composition of π and h.

Remark 2.12 The map π really looks like a tropical cover, as defined in Cavalieri et al. (2010) and Buchholz and Markwig (2015). However, it is not always the case. This is normal since the purpose of the notion of tropical cover is to mimick ramified covers between complex curves. The map π here plays the role of the quotient map by a real involution, which is not a ramified cover.

Let γ be an edge of $\Gamma(\mathcal{R})$, and $n \in N$ be the slope of $\pi(\gamma)$. Then, one can easily check that the choice of length on $\Gamma(\mathcal{R})$ ensures that $h_{\mathcal{R}}$ has slope n if $\gamma \in \text{Fix}(\sigma)$ and $\frac{n}{2}$ otherwise. However, as the edges of Γ_{even} have an even slope, it is still an element of N. One can check that the balancing condition is still satisfied. Therefore, $(\Gamma(\mathcal{R}), h_{\mathcal{R}}, \sigma)$ is a real parametrized tropical curve, of image C, and of degree $\{u_i, v_j^2\}_{i,j}$.

Proposition 2.13 *Let C be an irreducible rational plane tropical curve of degree $P_\Delta \subset M$ having ends of weight 1 or 2. Let $\Delta \subset N$ be the degree associated to P_Δ consisting only of primitive lattice vectors. let $h : \Gamma \to N_{\mathbb{R}}$ be the unique rational parametrization of C given by Proposition 2.8. Using previous notations, every real rational parametrized curve of degree Δ having the image C is one of the curves* $\Gamma(\mathcal{R})$.

Proof The curves $(\Gamma(\mathcal{R}), h)$ provide real rational parametrizations of C. Conversely, if $h : (\Gamma', \sigma) \to N_{\mathbb{R}}$ is a real rational parametrization of C of degree $\{u_i, v_j^2\}$, then we have quotient curve Γ'/σ defined as follows. As a topological space, Γ'/σ is the quotient by σ. The edge lengths are the same for edges in $\text{Fix}(\sigma)$, and the edge length is divided by two for a pair of exchanged edges. Since h is σ-invariant, we have a quotient map $\tilde{h} : \Gamma'/\sigma \to N_{\mathbb{R}}$ and one can check that the above choice of edge length makes it into a parametrized tropical curve. The assumption on the weights of the ends of C ensures that the conjugated ends of Γ' are mapped to the even ends of C. Their weight is doubled when passing to the quotient. Thus, we get a rational parametrization of C of degree $\{u_i, 2v_j\}_{i,j}$. Therefore, it is isomorphic to Γ. Let $\pi : \Gamma' \to \Gamma'/\sigma \simeq \Gamma$ be the quotient map.

The primitivity assumption on the degree ensures that near infinity, the points of the even ends of Γ have two preimages by π. The other ends only have one. Let $\mathcal{R}$ be the topological boundary of $\pi(\mathrm{Fix}(\sigma))$ inside Γ.

- First of all, $\mathrm{Fix}(\sigma)$ is connected: if $p, q \in \mathrm{Fix}(\sigma)$, there is a unique shortest path in Γ between p and q, this path is then σ-invariant, thus in $\mathrm{Fix}(\sigma)$.
- Let Ξ be a connected component of $\Gamma\backslash\pi(\mathrm{Fix}(\sigma))$, the boundary of Ξ contains exactly one point of $\mathcal{R}$: at least one since $\Xi \neq \Gamma$, and at most one, otherwise the path between these points of $\mathcal{R}$ would lie in Ξ, and we have proven that such a path lies in $\pi(\mathrm{Fix}(\sigma))$. By definition, Ξ only contains only complex ends since real ends belong to $\mathrm{Fix}(\sigma)$, and the construction of Γ_{even} ensures that the point of $\mathcal{R}$ on the boundary of Ξ is in Γ_{even}.
- Finally, we have proven that Γ is composed of $\pi(\mathrm{Fix}(\sigma))$, which is connected and has boundary $\mathcal{R}$, and components Ξ that are attached to $\pi(\mathrm{Fix}(\sigma))$ at those vertices. Thus, the configuration $\mathcal{R}$ is admissible: there is at least one point of $\mathcal{R}$ on the shortest path between the stem and a complex end since the stem is in $\mathrm{Fix}(\sigma)$ and the end is not, and there is at most one since $\mathrm{Fix}(\sigma)$ is connected.

Finally, the set $\mathcal{R}$ being admissible, the graph Γ' is recovered as the curve $\Gamma(\mathcal{R})$. □

2.3 Moment of an edge

Recall from the introduction that ω is some fixed generator of $\Lambda^2 M$, *i.e.* a non-degenerated 2-form on N. It extends to a volume form on $N_{\mathbb{R}} \simeq \mathbb{R}^2$. Let $e \in \Gamma^1_\infty$ be an end oriented toward infinity, directed by n_e.

Definition 2.14 The moment of e is the scalar

$$\mu_e = \omega(n_e, p) \in \mathbb{R},$$

where $p \in e$ is any point on the edge e. We similarly define the moment of a bounded edge if we specify its orientation.

The moment of a bounded edge is reversed when its orientation is reversed. Intuitively, the moment of an end is just a way of measuring its position alongside a transversal axis. Thus, fixing the moment of an end amounts to impose on the curve that it goes through some point at infinity. Following this observation, the moment has also a definition in complex toric geometry, where it corresponds to the coordinate of the intersection point of the curve with the toric divisor. Let

$$\varphi : t \in \mathbb{C}P^1 \longmapsto \chi \prod_1^m (t - \alpha_j)^{n_j} \in N \otimes \mathbb{C}^* = N_{\mathbb{C}^*},$$

be a parametrized rational curve. Given a basis (e_1, e_2) of N, letting $n_i = a_i e_1 + b_i e_2$, the notation $z^{n_i} \in N_{\mathbb{C}^*}$ means the point of coordinates (z^{a_i}, z^{b_i}). The product is given by the product coordinates by coordinates. In other words,letting (e_1^*, e_2^*) be the dual basis of M and $a = \chi(e_1^*), b = \chi(e_2^*)$, the parametrized curve is then given as follows:

$$\varphi(t) = \left(a \prod_1^m (t-\alpha_i)^{a_i}, b \prod_1^m (t-\alpha_i)^{b_i} \right) \in (\mathbb{C}^*)^2.$$

This is a curve of degree $\Delta = (n_j) \subset N$. The complex moment at a point $p_j = \varphi(\alpha_j)$ is the evaluation at α_j of the monomial $\iota_{n_j}\omega \in M$ at the corresponding point:

$$\mu_j = \left(\varphi^* \iota_{n_j}\omega\right)(\alpha_j) = \chi(\iota_{n_j}\omega) \prod_i (\alpha_j - \alpha_i)^{\omega(n_j,n_i)}.$$

The complex moment corresponds to the coordinate of p_j on the toric divisor D inside the toric surface $\mathbb{C}\Delta$ defined from Δ. For a primitive vector n in the fan Σ_Δ associated to Δ, the coordinate on the corresponding toric divisor is the monomial $\iota_n\omega \in M$.

An easy computation (generalized to the Weil reciprocity law for non-rational curves) gives us the following relation between the moments:

$$\prod_{i=1}^m \mu_i = \pm 1,$$

with a sign depending on the multiset Δ. We could also prove the relation using Viète formula. In the tropical world we have an analog called the *tropical Menelaus Theorem*, which gives a relation between the moments of the ends of a parametrized tropical curve.

Proposition 2.15 (Tropical Menelaus Theorem (Mikhalkin 2017)). *For a parametrized tropical curve of degree Δ, we have*

$$\sum_{n_e \in \Delta} \mu_e = 0.$$

In the tropical case as well as in the complex case, a configuration of m points on the toric divisors is said to satisfy the *Menelaus condition* if this relation is satisfied. Notice that in tropical case, the moment of a point depends on the vector of M used to compute it: it is of lattice length 1 if the point is real, and of lattice length 2 for a non-real point. In particular, for the complex ends, which are of weight 2, the monomials used to compte the moment in the complex world and in the tropical world do not agree. They differ by a factor 2, since the unique complex end in the tropical world corresponds to a pair of complex conjugated points.

2.4 Moduli space of tropical curves and refined multiplicity of a simple tropical curve

Let (Γ, h) be a parametrized tropical curve such that Γ is trivalent, and has no *flat vertex*. A flat vertex is a vertex whose outgoing edges have their slope contained in a common line. This means that for any two outgoing edges of respective slopes u, v, we have $\omega(u, v) = 0$. In particular, when the curve is trivalent, no edge can have a zero slope since it would imply that its end points are flat vertices. A plane tropical curve is a *simple nodal curve* if the dual subdivision of its Newton polygon consists only of triangles and parallelograms. The unique rational parametrization (given by Proposition 2.8) of a plane rational nodal curve has a trivalent underlying graph, and has no flat vertex.

Definition 2.16 The *combinatorial type* of a tropical curve is the homeomorphism type of its underlying labeled graph Γ, *i.e.* the labeled graph Γ without the metric.

To give a graph a tropical structure, one just needs to specify the lengths of the bounded edges. If the curve is trivalent and has m ends, there are $m-3$ bounded edges, otherwise the number of bounded edges is $m-3-\mathrm{ov}(\Gamma)$, where $\mathrm{ov}(\Gamma)$ is the *overvalence* of the graph. The overvalence is given by $\sum_V(\mathrm{val}(V)-3)$, where V runs over the vertices of Γ, and $\mathrm{val}(V)$ denotes the valence of the vertex. Therefore, the set of curves having the same combinatorial type is parametrized by $\mathbb{R}_{>0}^{m-3-\mathrm{ov}(\Gamma)}$, and the coordinates are the lengths of the bounded edges. If Γ is an abstract tropical curve, we denote by $\mathrm{Comb}(\Gamma)$ the set of curves having the same combinatorial type as Γ.
For a given combinatorial type $\mathrm{Comb}(\Gamma)$, the boundary of $\mathbb{R}_{\geq 0}^{m-3-\mathrm{ov}(\Gamma)}$ corresponds to curves for which the length of an edge is zero, and therefore corresponds to a graph having a different combinatorial type. This graph is obtained by deleting the edge with zero length and merging its end points. We can thus glue together all the cones of the finitely many combinatorial types and obtain the *moduli space* $\mathcal{M}_{0,m}$ *of rational tropical curves with m marked points*. It is a simplicial fan of pure dimension $m-3$, and the top-dimensional cones correspond to trivalent curves. The combinatorial types of codimension 1 are called *walls*.
Given an abstract tropical curve Γ, if we specify the slope of every end, and the position of a vertex, we can define uniquely a parametrized tropical curve $h:\Gamma\to N_\mathbb{R}$. Therefore, if $\Delta\subset N$ denotes the set of slopes of the ends, the *moduli space* $\mathcal{M}_0(\Delta,N_\mathbb{R})$ *of parametrized rational tropical curves of degree* Δ is isomorphic to $\mathcal{M}_{0,m}\times N_\mathbb{R}$ as a fan, where the $N_\mathbb{R}$ factor corresponds to the position of the finite vertex adjacent to the first end.
On this moduli space, we have a well-defined evaluation map that associates to each parametrized curve the family of moments of its ends :

$$\begin{array}{rl}\mathrm{ev}:\mathcal{M}_0(\Delta,N_\mathbb{R}) \longrightarrow & H=\{\sum_1^m \mu_i=0\}\subset\mathbb{R}^m \\ (\Gamma,h)\longmapsto & \mu=(\mu_i)_{1\leq i\leq m}\end{array}.$$

By the tropical Menelaus Theorem, the sum of the moments is 0, so that the map is well-defined. Notice that the evaluation map is linear on every cone of $\mathcal{M}_0(\Delta,N_\mathbb{R})$. Furthermore, both spaces have the same dimension $m-1$. Thus, if Γ is a trivalent curve, the restriction of ev on $\mathrm{Comb}(\Gamma)\times N_\mathbb{R}$ has a determinant well-defined up to sign when H is endowed with some lattice basis, $\mathrm{Comb}(\Gamma)\simeq\mathbb{R}_{\geq 0}^{m-3}$ endowed with its canonical basis, and $N_\mathbb{R}$ is endowed with a basis of N. The absolute value $m_\Gamma^\mathbb{C}$ of the determinant is called the *complex multiplicity* of the curve, well-known to factor into the following product over the vertices of Γ:

$$m_\Gamma^{\mathbb{C}} = \prod_V m_V^{\mathbb{C}},$$

where $m_V^{\mathbb{C}} = |\omega(u, v)|$ if u and v are the slopes of two outgoing edges of V. The balancing condition ensures that $m_V^{\mathbb{C}}$ does not depend on the chosen edges. This multiplicity is the one that appears in the correspondence theorem of Mikhalkin (2005). Notice that the simple parametrized tropical curves are precisely the points of the cones with trivalent graph and non-zero multiplicity. We finally recall the definition of the refined Block-Göttsche multiplicity.

Definition 2.17 The *refined multiplicity* of a simple nodal tropical curve is

$$m_\Gamma^q = \prod_V [m_V^{\mathbb{C}}]_q,$$

where $[a]_q = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}$ is the q-analog of a.

This refined multiplicity is sometimes called the Block-Göttsche multiplicity and intervenes in the definition of the invariant $N_\Delta^{\partial,\text{trop}}$. Notice that the multiplicity is the same for every curve inside a given combinatorial type.

2.5 Tropicalization

Tropicalization of a plane curve We briefly recall the tropicalization process for a family of plane curve. Let C be a plane curve, defined by a polynomial $P_t \in \mathbb{C}\{\{t\}\}[M]$ with coefficients in the field of Puiseux series $\mathbb{C}\{\{t\}\}$, viewed as a family of complex curves which depends on a parameter t. We look for the points of the curve over the Puiseux series, *i.e.* in $N \otimes \mathbb{C}\{\{t\}\}^*$. In a basis of M, the polynomial is given in coordinates by

$$P_t(x, y) = \sum_{(i,j) \in P_\Delta} a_{i,j}(t) x^i y^j.$$

We assume that the coefficients in the corners of the polygon P_Δ are non-zero. Then, we have the associated tropical polynomial

$$\text{Trop}(P_t)(x, y) = \max_{(i,j) \in P_\Delta} \left(\text{val}(a_{i,j}(t)) + ix + jy \right),$$

along with a valuation map, also called *tropicalization map*:

$$\text{Val} : n \otimes z \in N \otimes \mathbb{C}\{\{t\}\}^* \longmapsto n \otimes \text{val}(z) \in \text{Hom}(M, \mathbb{R}) = N_\mathbb{R}.$$

In coordinates, Val is given by the coordinatewise valuation:

$$\text{Val} : (x, y) \in (\mathbb{C}\{\{t\}\}^*)^2 \longmapsto (\text{val}(x), \text{val}(y)) \in \mathbb{R}^2.$$

The Kapranov theorem (Brugallé and Shaw 2014) then ensures that the closure of the image of the vanishing locus of P_t in $(\mathbb{C}\{\{t\}\}^*)^2$ under the valuation map is equal to the tropical curve defined by $\mathrm{Trop}(P_t)$.

Theorem 2.18 (Kapranov) *Let C_{trop} be the tropical curve defined by $Trop(P_t)$. Then, one has*

$$\overline{\mathrm{Val}(C)} = C_{\mathrm{trop}}.$$

Let $\alpha_{i,j} = \mathrm{val}(a_{i,j}(t))$, and $a_{i,j}(t) = t^{\alpha_{i,j}} a^0_{i,j}(t)$. The function $(i, j) \mapsto \alpha_{i,j}$ induces the convex subdivision of P_Δ dual to C_{trop}. Furthermore, to each vertex V, dual to a polygon $P_V \subset P_\Delta$ of the subdivision, is associated a complex curve of equation $\sum_{(i,j)\in P_V} a^0_{i,j}(0)x^i y^j$. These curves are an important part of the correspondence theorem.

Definition 2.19 The following is called a *tropical enhancement*: the data of a real curve C_V of degree P_V for every vertex V of a parametrized tropical curve, such that if V and V' are adjacent, and D, D' are the toric divisor associated to the common side of P_V and $P_{V'}$, canonically identified, the intersection points of C_V and D counted with their order of tangency are the same for $C_{V'}$ and D'.

If we add to a tropical enhancement the data of some curve for each bounded edge of the tropical curve, we get *admissible tropicalization curves*, as mentionned in Shustin (2006b).

Tropicalization of a parametrized curve We finish by reviewing the tropicalization process for a family of parametrized curves. For a more detail exposition, we refer the reader to section 3 of Tyomkin (2012). Let $f_t : \mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}$ be a family of parametrized rational curve. Using the m points $(p_1, \ldots, p_m)$ where f_t is not defined, we get a family of rational curve with m marked points. Let C_0 be the stable limit inside the Deligne-Mumford compactification of the moduli space of marked rational curves $\overline{\mathcal{M}}_{0,m}$. The curve C_0 is a real nodal curve. The tropical curve is defined as the dual graph of the central fiber C_0:

- there is one vertex per irreducible component of C_0,
- two vertices are linked by an edge if the corresponding component share a node, with some edge length fixed by the local model of the family (see (Tyomkin 2012)),
- there is one unbounded end per marked point, adjacent to the vertex corresponding to the component on which the marked point belongs.

The real structure on the family of curves induces a permutation on the set of irreducible components of C_0, which becomes a real structure on the tropical curve Γ. The order of vanishing of the monomials on each component C_V of C_0 gives a map from Γ to $N_{\mathbb{R}}$, leading to a parametrized tropical curve. The order of vanishing being invariant by conjugation, we get a real paramtrized tropical curve.

For each vertex V of Γ, the family of maps f_t induces a map $f_V : C_V \dashrightarrow N_{\mathbb{C}^*}$. These are the parametrized version of the *tropical enhancement* for plane curves, and are also called tropical enhancement.

Remark 2.20 A tropical enhancement corresponds to the data of the central fiber C_0, to which the tropical curve is the dual graph. The purpose of the *correspondence theorem*, is to describe all possible families with the same central fibers, in other words, the possible smoothings, satisfying some constraints.

3 Quantum indices of real curves

We start this section by recalling the theorem about quantum indices by Mikhalkin (2017), restricting ourselves to the case of rational curves. We then compute the quantum indices of some specific curves that appear in the resolution of our enumerative problem.

3.1 The quantum index of a type *I* real curve

Let $\varphi : \mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}$ be a parametrized real rational curve of degree Δ. Let $\alpha \in \mathbb{C}P^1$ be be a point where φ is not defined, and let $n \in N$ be the corresponding weight vector. Recall that the monomial $\iota_n\omega \in M$ is used to compute the complex moment.

Definition 3.1 In the above notations, we say that the rational curve has real or purely imaginary intersection points if $(\varphi^*\iota_n\omega)(\alpha) \in \mathbb{R} \cup i\mathbb{R}$ for every α where φ is not defined.

Remark 3.2 This condition is automatic for $\alpha \in \mathbb{R}P^1$ and thus needs only to be checked for $\alpha \in \mathbb{C}P^1 \backslash \mathbb{R}P^1$.

Recall that we have the logarithmic map

$$\text{Log} : z^n \in N_{\mathbb{C}^*} \longmapsto n \otimes \text{Log}|z| \in N \otimes \mathbb{R}.$$

In a basis of N, it is the logarithm of the absolute value coordinate by coordinate. Similarly we define the argument map

$$\arg : z^n \in N_{\mathbb{C}^*} \longmapsto n \otimes \arg(z) \in N \otimes \mathbb{R}/\pi\mathbb{Z}.$$

Notice that the argument is taken mod π rather than 2π. The parametrized real rational curve $\varphi : \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}^*$ is of type I. Let S be a connected component of $\mathbb{C}P^1 \backslash \mathbb{R}P^1$, inducing a complex orientation of $\mathbb{R}P^1$. By pulling back the volume form ω on $N_{\mathbb{R}}$ to $N_{\mathbb{C}^*}$, we can define the logarithmic area of S:

$$\mathcal{A}_{\text{Log}}(S, \varphi) = \int_S (\text{Log} \circ \varphi)^* \omega.$$

Respectively, the 2-form ω defines a 2-form ω_θ on $N \otimes \mathbb{R}/\pi\mathbb{Z}$. We can pull it back to $N_{\mathbb{C}^*}$ and define the area of the co-amoeba of S:

$$\mathcal{A}_{\arg}(S, \varphi) = \int_S (\arg \circ\varphi)^* \omega.$$

If ω is given in coordinates by $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2$, due to the vanishing of the meromorphic 2-form $\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2}$ restricted to S, one has $\mathcal{A}_{\mathrm{Log}}(S, \varphi) = \mathcal{A}_{\arg}(S, \varphi)$.

We now define the quantum index of a real oriented rational curve. For a complex intersection point p_i with a divisor, let $\varepsilon_i \theta_i \pi = \mathrm{Arg}\left((\varphi^* \iota_{n_i} \omega)(p_i)\right) \in]-\pi; \pi[$, with $\varepsilon_i = \pm 1$, $0 < \theta_i < 1$, and Arg is the classical argument map.

Theorem 3.3 (Mikhalkin 2017) *Let $\varphi : \mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}$ be a real parametrized rational curve. Let S be one of the two connected components of $\mathbb{C}P^1 \backslash \mathbb{R}P^1$. Then, there exists a half-integer $k(S, \varphi)$, called the quantum index of (S, φ), such that*

$$\mathcal{A}_{\mathrm{Log}}(S, \varphi) - \pi^2 \sum_{p_i \in S} \varepsilon_i (2\theta_i - 1) = k(S, \varphi)\pi^2 \in \frac{\pi^2}{2}\mathbb{Z}.$$

In particular, if the curve has real or purely imaginary intersection points with the toric divisors, the sum in the left-hand side vanishes.

The statement, stated as Theorem 1 of Mikhalkin (2017), is only proved for curves with real or purely imaginary intersection points with the toric boundary. To obtain this broader statement, one needs to modify Lemma 28 in Mikhalkin (2017) by moving the geodesics in the torus of arguments so that they pass through a fixed point of $-\mathrm{id}$. Each modifies the area by a term $\pi^2 \varepsilon_i (2\theta_i - 1)$.

3.2 The quantum index near the tropical limit

In Mikhalkin (2017), Mikhalkin proved the following result, that computes the quantum index of curves in a family near the tropical limit, by computing the log-area.

Proposition 3.4 (Mikhalkin 2017) *Let $C^{(t)} = \left(f_t : \mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}\right)$ be a family of type I real parametrized rational curves. Let $h : \Gamma \to N_{\mathbb{R}}$ be the tropical limit of the family. The choice of a component S of $\mathbb{C}P^1 \backslash \mathbb{R}P^1$ induces components S_V of C_V for every vertex $V \in \mathrm{Fix}(\sigma)$, thus inducing complex orientations of the curves C_V. Then, for t large enough,*

$$\mathcal{A}_{\mathrm{Log}}(S, f_t) = \frac{1}{\pi^2} \sum_V \mathcal{A}_{\mathrm{Log}}(S_V, f_V),$$

where the sum is indexed over the vertices of Γ inside $\mathrm{Fix}(\sigma)$.

Remark 3.5 In particular, for one to know the quantum index of curves near the tropical limit, one only needs to know the log-areas of the curves associated to the vertices of the tropical curve, and the chosen components S_W for each of them. This means that the quantum index may be computed in the patchworking construction. Notice that the parametrized curves associated to the vertices may not have real or purely intersection points with the toric boundary. However, this case does not occur in our specific problem where the complex intersection points belong to the same divisor.

The computation near the tropical limit allows us to reduce the calculations needed to compute the log-areas of oriented curves: we only need to compute the log-areas of oriented curves associated to the vertices of a tropical curve. This includes real rational curves with three real intersection points, and real rational curves with two real and two complex conjugated intersection points. Proving that the log-area is well-behaved under the monomial maps, the following lemma reduces the computation even further to two computations, dealt with in the next subsection.

Lemma 3.6 *Let $\varphi : \mathbb{C}C \dashrightarrow N_{\mathbb{C}^*}$ be a type I real curve with a choice of a connected component $S \subset \mathbb{C}C \backslash \mathbb{R}C$, inducing a complex orientation, and let $\alpha : N_{\mathbb{C}^*} \to N' \otimes \mathbb{C}^*$ be a monomial map, associated to a morphism $A^T : M' \to M$. We consider the composition*

$$\psi : \mathbb{C}C \xrightarrow{\varphi} N_{\mathbb{C}^*} \xrightarrow{\alpha} N'_{\mathbb{C}^*}.$$

Let ω and ω' be the volume forms on respectively N and N', dual lattices of M and M', so that we have $A^*\omega' = (\det A)\omega$. Then, we have

$$\int_S (\text{Log} \circ \psi)^* \omega = \det A \int_S (\text{Log} \circ \varphi)^* \omega.$$

Proof We have linear maps $A^T : M' \to M$, $A : N \to N'$. Then,

$$\begin{aligned}
\int_S (\text{Log} \circ \psi)^* \omega' &= \int_S (A \circ \text{Log} \circ \varphi)^* \omega' \\
&= \int_S (\text{Log} \circ \varphi)^* (A^* \omega') \\
&= \det A \int_S (\text{Log} \circ \varphi)^* \omega \text{ since } A^* \omega' = (\det A)\omega.
\end{aligned}$$

□

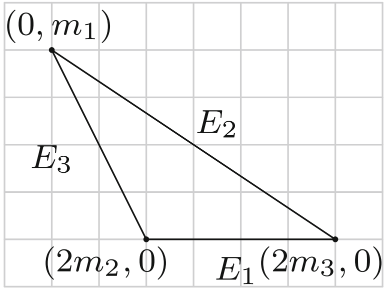
3.3 Local computations

In this section, we compute the log-areas of some auxiliary rational curves. This includes real rational curves with three intersection points with the toric boundary, and real rational curves with two real and two complex intersection points with the toric boundary. We also solve the local enumerative problem of finding oriented real rational curves maximally tangent to the toric boundary passing through some chosen points.

3.3.1 Trivalent real vertex

We first recall the computation of the quantum index of a rational curve with three real punctures. This was dealt with in Mikhalkin (2017).

Fig. 2 The polygon $P_\Delta(m_1, m_2, m_3)$



Lemma 3.7 *Let $\Delta \subset N$ be a family of three non-colinear vectors of total sum 0, and let $P_\Delta \subset M$ be the associated triangle, of lattice area m_Δ. Let $\varphi : \mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}$ be a real parametrized rational curve of degree Δ, thus having a unique real intersection point with maximal tangency with each toric divisor. Then, the quantum index of the curve is $\pm\frac{m_\Delta}{2}$ according to the choice of complex orientation.*

Proof The assumption implies that the curve is the image of a line by a monomial map of determinant m_Δ. Hence, its quantum index is the quantum index of a line, equal to $\pm\frac{1}{2}$, times the determinant of the monomial map which is the lattice area m_Δ of the triangle. □

3.3.2 Quadrivalent complex vertex

We now consider the case of a curve associated with a quadrivalent vertex having two edges exchanged by the involution σ, and two edges fixed. This means that this is a rational curve having two real punctures, and two conjugated ones. In a suitable choice of coordinates, the curve has a degree of the following form. In a basis (e_1, e_2) of N, for $m_1, m_2, m_3 \in \mathbb{N}^*$ with $m_2 < m_3$, let us take

$$\Delta(m_1, m_2, m_3) = \{(m_1, 2m_2); (0, m_3 - m_2)^2; (-m_1, -2m_3)\}.$$

The degree of a planar curve which is parametrized by a curve of degree $\Delta(m_1, m_2, m_3)$ is the lattice polygon in M given by

$$P_\Delta(m_1, m_2, m_3) = \text{Conv}\left((0, m_1), (2m_2, 0), (2m_3, 0)\right).$$

This polygon is drawn on Figure 2. Up to an automorphism of the lattice, every triangle in M having a side of even length is one of the polygons $P_\Delta(m_1, m_2, m_3)$. Let E_i be the side opposite to the i-th vertex in $P_\Delta(m_1, m_2, m_3)$, *i.e.*

$$E_1 = [(2m_2, 0), (2m_3, 0)], \; E_2 = [(2m_3, 0), (0, m_1)], \; E_3 = [(2m_2, 0), (0, m_1)].$$

We denote by $\mathbb{C}E_i$ the associated toric divisor inside the toric surface $\mathbb{C}\Delta(m_1, m_2, m_3)$. Such a curve is the image of a curve of degree $\Delta^{\text{par}} = \{(1, 1), (-1, 1), (0, -1)^2\}$ by a monomial map followed by a translation.

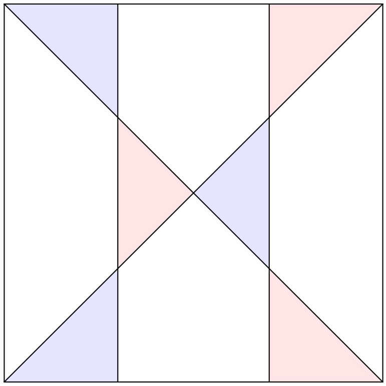


Fig. 3 Co-amoeba of a parabola with order map: -1 for red (triangles with left vertical side), $+1$ for blue (triangles with right vertical side). Central point is 0, the abscissa of the vertical geodesics are $\pm c_k$

Hence, we now consider rational curves of degree Δ^{par}. We assume that the two intersection points with the toric divisor associated to $(0, -1)$ are complex conjugated. Choosing a coordinate on the curve such that these complex points are $\pm i$ and the intersection point with the toric divisor associated to $(1, 1)$ is ∞. There are two such coordinates which differ by their orientation. Thus, the orientation fixes uniquely the coordinate. Up to a multiplicative translation, the oriented curve has a parametrization

$$\varphi : t \in \mathbb{C}P^1 \longmapsto \left(t - c, \frac{t^2+1}{t-c}\right),$$

where $c \in \mathbb{R}$ is the coordinate of the last intersection point with the toric boundary. The reversing of the orientation leads to the change of c by $-c$. Let S be the upper half-plane $\{\mathfrak{Im} t > 0\}$.

Lemma 3.8 *The log-area of the curve* $\varphi : t \in S \longmapsto \left(t - c, \frac{t^2+1}{t-c}\right)$ *is* $2\pi \arctan c$.

Proof The coamoeba along with its order map (i.e. the signed number of antecedents) is depicted on Fig. 3. The order map has value 1 on the blue triangles and -1 on the red ones. The abscissa of the two vertical geodesics are both opposite arguments of the complex intersection points with the boundary. The one with parameter i is

$$\arg(i - c) = \operatorname{arcot}(-c) \in]0; \pi[.$$

This is the co-amoeba of the whole curve. However, as z is a coordinate on the curve, we can restrict to the co-amoeba of the half-curve parametrized by S if we restrict to the triangles in the right half of the square. Therefore, the log-area is equal to

$$\begin{aligned} \mathcal{A}(S, \varphi) &= \operatorname{arcot}(-c)^2 - (\pi - \operatorname{arcot}(-c))^2 \\ &= 2\pi \operatorname{arcot}(-c) - \pi^2 \\ &= 2\pi \arctan c. \end{aligned}$$

□

Remark 3.9 The order map of the co-amoeba, i.e. the function on the argument torus that gives the signed number of antecedents, is found as follows:

- According to Forsgård and Johansson (2015), this function is constant on the complement of the *shell*, which is a union of geodesics in the torus, directed by the vectors of Δ, and whose position is fixed by the arguments of the moments of the points of intersection with the toric boundary.
- The value of the order map changes by one when passing through one of the geodesic of the shell. This defines the order map up to a shift.
- The shift is fixed by the fact that the whole signed area is zero. This leads us to Fig. 3.

4 Tropical enumerative problem and refined curve counting

Let $\Delta \subset N$ be a family of m primitive lattice vectors, with total sum 0, and let Δ be the associated lattice polygon $P_\Delta \subset M_\mathbb{R}$ having m lattice points on its boundary. The toric surface obtained from Δ is still denoted by $\mathbb{C}\Delta$. Let $E_1, \ldots, E_p$ denote the sides of the polygon P_Δ and let $n_1, \ldots, n_p \in N$ be their normal primitive vectors. Let $s \leq \frac{l(E_1)}{2}$ be an integer, $r_1 = l(E_1) - 2s$, and $r_i = l(E_i)$ if $i \geq 2$, so that we have $\sum_1^p r_i + 2s = m$. Let

$$\Delta(s) = (\Delta \backslash \{n_1^{2s}\}) \cup \{(2n_1)^s\} = \{n_1^{r_1}, (2n_1)^s, n_2^{r_2}, \ldots, n_p^{r_p}\}.$$

In other words, for the vector n_1 associated to the side E_1 of P_Δ, $2s$ copies of n_1 inside Δ have been replaced by s copies of $2n_1$. This depends on the choice of the “first" side E_1 which is now fixed, but could be any of them.

4.1 Tropical problem

The tropical curves of degree $\Delta(s)$ have $m - s$ ends, and therefore the moduli space $\mathcal{M}_0(\Delta(s), N_\mathbb{R})$ of parametrized rational tropical curves of degree $\Delta(s)$ in $N_\mathbb{R} \simeq \mathbb{R}^2$ has dimension $m - s - 1$. We have the evaluation map:

$$\text{ev} : \mathcal{M}_0(\Delta(s), N_\mathbb{R}) \longrightarrow H = \left\{ \mu \in \mathbb{R}^{m-s} \text{ s.t. } \sum_1^{m-s} \mu_i = 0 \right\} \subset \mathbb{R}^{m-s},$$

that associates to a parametrized tropical curve the moments of every end. Recall that the sum of the moments is 0 because of the tropical Menelaus Theorem. Let $\mu \in \mathbb{R}^{m-s}$ be a generic family of moments. We look for parametrized rational tropical curves Γ of degree $\Delta(s)$ such that $\text{ev}(\Gamma) = \mu$.
Due to genericity of the choice, as noticed in section 3.1 of Blomme (2019), every parametrized rational tropical curve Γ such that $\text{ev}(\Gamma) = \mu$ is a simple nodal tropical curve and thus has a well-defined refined multiplicity m_Γ^q. We then set

$$N_{\Delta(s)}^{\partial,\mathrm{trop}}(\mu)=\sum_{\mathrm{ev}(\Gamma)=\mu} m_\Gamma^q \in \mathbb{Z}[q^{\pm\frac{1}{2}}].$$

Theorem 4.1 (Blomme 2019) *The value of $N_{\Delta(s)}^{\partial,trop}(\mu)$ is independent of μ provided that it is generic.*

Remark 4.2 The theorem extracted from Blomme (2019) is true for any tropical degree Δ, although we use it here for a degree consisting almost only of primitive vectors.

Remark 4.3 In Göttsche and Schroeter (2019) Göttsche and Schroeter proposed a refined way to count so-called *refined Broccoli curves* having fixed ends, and passing through some fixed configuration of "real and complex" points inside $\mathbb{R}^2$, with some suitable multiplicity.

In the case where there are only marked ends and no marked points, this count coincides with the count of plane tropical curves passing through the configuration with usual Block-Göttsche multiplicities from Definition 2.17 up to a multiplication by a constant term depending on the degree.

4.2 Classical problem

Keeping previous notations, let $\mathcal{P}_0$ (resp.$\widehat{\mathcal{P}_0}$) be a configuration of m points on the toric boundary $\partial\mathbb{C}\Delta$ such that:

- $\mathcal{P}_0$ (resp.$\widehat{\mathcal{P}_0}$) has $s_1=s$ pairs of complex conjugated points (resp. purely imaginary) on $\mathbb{C}E_1$, and r_1 real points on $\mathbb{R}E_1$,
- for each $i\geq 2$, the configuration $\mathcal{P}_0$ (resp.$\widehat{\mathcal{P}_0}$) has r_i real points on $\mathbb{R}E_i$,
- $\mathcal{P}_0$ (resp.$\widehat{\mathcal{P}_0}$) is subject to the Menelaus condition.
- we assume that $\mathcal{P}_0$ (resp.$\widehat{\mathcal{P}_0}$) is generic among such configurations.

For $\mathcal{P}=\mathcal{P}_0$ or $\widehat{\mathcal{P}_0}$, let $\mathcal{S}(\mathcal{P})$ be the set of oriented real rational curves that pass through $\pm p_i$ for every $p_i\in\mathcal{P}$. Such a curve is said to *pass through the configuration $\mathcal{P}$ up to symmetry*. The *up to symmetry* is here to emphasize the fact that we allow our curves to pass through a point or its symmetric. As the curves are oriented, each real curve is counted twice: once with each of its orientations. Notice that if a curve passes through a non-real point, it passes through its conjugate since the curve is real. Moreover, if the complex points are purely imaginary, the conjugate happens to be the opposite. Every oriented curve of $\mathcal{S}(\widehat{\mathcal{P}_0})$ has real or purely imaginary intersection points, and thus a quantum index equal to its log-area up to a factor π^2.

Let $\varphi:\mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}$ be a real parametrized rational curve, oriented by the choice of $S\subset\mathbb{C}P^1\backslash\mathbb{R}P^1$. The logarithmic Gauss map sends a point in $\mathbb{R}P^1$ to the tangent direction to $\mathrm{Log}\,\varphi(\mathbb{R}P^1)\subset N_{\mathbb{R}}$ of its image. We get a map

$$\gamma:\mathbb{R}P^1\to\mathbb{P}^1(N_{\mathbb{R}}).$$

The first space $\mathbb{R}P^1$ is oriented as the boundary of S, while $\mathbb{P}^1(N_{\mathbb{R}})$ is oriented by the choice of ω. The degree of this map is denoted by $\mathrm{Rot}_{\mathrm{Log}}(S,\varphi)\in\mathbb{Z}$. We then set

$$\sigma(S,\varphi)=(-1)^{\frac{m-\mathrm{Rot}_{\mathrm{Log}}(S,\varphi)}{2}}\in\{\pm 1\}.$$

Now let

$$R_\Delta(\mathcal{P}) = \frac{1}{4} \sum_{(S,\varphi)\in\mathcal{S}(\mathcal{P})} \sigma(S,\varphi) q^{k(S,\varphi)} \in \mathbb{Z}[q^{\pm\frac{1}{2}}],$$

be the refined signed count of solutions. The coefficient $\frac{1}{4}$ is here to account for the deck transformations: if $\{f(x, y) = 0\}$ is a curve in $\mathcal{S}(\mathcal{P})$, then $\{f(x, -y) = 0\}$, $\{f(-x, y) = 0\}$, $\{f(-x, -y) = 0\}$ are in $\mathcal{S}(\mathcal{P})$ too. Theorem 1.2 from Mikhalkin (2017) states that the value of $R_\Delta(\mathcal{P}_0)$ does not depend on the generic choice of $\mathcal{P}_0$. As we choose all complex pair of conjugated points on the same toric divisor, the corresponding invariant is in our particular case $R_{\Delta,(s_1,0,\dots,0)}$.

Theorem 1.4 relates the count $R_\Delta(\widehat{\mathcal{P}_0})$ for purely imaginary complex constraints to a tropical count. The choice of $\widehat{\mathcal{P}_0}$ does not a priori allows for a computation of $R_{\Delta,(s_1,0,\dots,0)}$ since it does not correspond to a generic choice of $\mathcal{P}_0$. However, Theorem 1.5 states that this non-generic choice of complex constraints can nevertheless be used to compute the invariant $R_{\Delta,(s_1,0,\dots,0)}$ up to a factor 2^s. Both Theorems now allow one to relate $R_{\Delta,s}$ and $N^{\partial,\mathrm{trop}}_{\Delta(s)}$.

Corollary 4.4 *One has*

$$R_{\Delta,(s_1,0,\dots,0)} = 2^{s_1} \frac{(q^{1/2} - q^{-1/2})^{m-2-s_1}}{(q - q^{-1})^{s_1}} N^{\partial,trop}_{\Delta(s)}.$$

Remark 4.5 The general case for the computation of $R_{\Delta,s}$ is done in Blomme (2020) using a different version of the correspondence theorem.

Moreover, we get the following invariance statement, that does not come from Theorem 1.2 since $\widehat{\mathcal{P}_0}$ is not a generic choice in the sense of Mikhalkin (2017).

Corollary 4.6 *The count $R_\Delta(\widehat{\mathcal{P}}_0)$ does not depend on the choice of $\widehat{\mathcal{P}}_0$ as long as it is generic.*

Proof From Theorems 1.4 and 1.5, we know that the value of $R_\Delta(\widehat{\mathcal{P}}_0)$ is the same for every choice of $\widehat{\mathcal{P}}_0$ which is a regular value of the evaluation map that sends a curve to the coordinates of its intersection points with the toric boundary. The map being algebraic, using Sard's Lemma, the set of regular values contains a Zariski open set inside $(\mathbb{R}^*)^{m-2s-1} \times (\mathbb{C}^*)^s$, and its intersection with $(\mathbb{R}^*)^{m-2s-1} \times (i\mathbb{R}^*)^s$ is non empty. Hence, the intersection is dense inside $(\mathbb{R}^*)^{m-2s-1} \times (i\mathbb{R}^*)^s$. □

5 Proofs of the results

The proof of Theorem 1.4 uses a correspondence theorem. Here, we use the correspondence theorem from Shustin (2006b). More precisely, we use in Shustin (2006b) the Lemmas 2.8 and 3.2. It is also possible to adapt the correspondence theorem from Mikhalkin (2005), or from Tyomkin (2017). The latter is done in Blomme (2020).

The idea of the correspondence theorem is as follows. Let $\widehat{\mathcal{P}}_t$ be a configuration of points depending on a parameter t, chosen as in Sect. 4.2. This means that one is given a collection of series $\pm\zeta_i(t) \in \mathbb{R}((t))^* \cup i\mathbb{R}((t))^*$ corresponding to the coordinates of the points $\pm p_i(t)$ of $\widehat{\mathcal{P}}_t$ on the toric divisors. Let μ be the tropicalization of the point configuration, *i.e.* for a pair of points $\pm p_i(t)$, we have $\mu_i = \text{val}\zeta_i(t)$. The correspondence theorem provides for t large enough a correspondence between the curves of $\mathcal{S}(\widehat{\mathcal{P}}_t)$, which are real parametrized curves of degree Δ, and the parametrized real tropical curves (Γ, h) of degree Δ which have the same image as tropical curves (Γ_0, h_0) of degree $\Delta(s)$ such that $\text{ev}(\Gamma) = \mu$. We here assign a multiplicity to each curve (Γ_0, h_0) of $\text{ev}^{-1}(\mu)$, so that the count of $\text{ev}^{-1}(\mu)$ with these multiplicities gives $R_\Delta(\widehat{\mathcal{P}}_t)$. This multiplicity happens to be proportional to the refined multiplicity of Block-Göttsche, thus leading to the relation stated in Theorem 1.4.
Being given a parametrized tropical curve $h_0 : \Gamma_0 \to N_\mathbb{R}$ of degree $\Delta(s)$ with $\text{ev}(\Gamma_0) = \mu$, the task of computing its multiplicity amounts to two things.

- One needs to find the parametrized real tropical curves (Γ, h) of degree Δ having the same image. This is taken care of by Lemma 5.3.
- Then, find the *tropical enhancements* that allow one to apply the correspondence theorem (see Lemma 3.2 from Shustin 2006b). This means finding a family of curves $(\mathscr{C}_V)_{V\in V(\Gamma)}$ satisfying some compatibility condition, and apply Lemma 2.8 from Shustin 2006b, leading to *admissible tropicalization curves* used in Lemma 3.2 from Shustin 2006b. This technical step of the proof is done in Proposition 5.4, reducing the computation to the elementary case of a curve with with two real punctures and two complex conjugated punctures. The latter is dealt with in Lemma 5.1.

5.1 The case of a curve with four punctures

Before getting to the general results, we study the particular case of the enumerative problem for the degree $\Delta(m_1, m_2, m_3)$. We keep notations from Sect. 3.3.2. We take a real point on $\mathbb{C}E_3$ and two purely imaginary conjugated points on $\mathbb{C}E_1$, and look for real rational curves of degree $\Delta(m_1, m_2, m_3)$, maximally tangent to each toric divisor at the given points. Such a curve has a parametrization of the form

$$\varphi(t) = \left(a(t-c)^{m_1}, b(t-c)^{2m_2}(t^2+1)^{m_3-m_2}\right) \in (\mathbb{C}^*)^2,$$

where c is some real number corresponding to the coordinate of the intersection point with $\mathbb{C}E_3$, and $a, b \in \mathbb{R}^*$. The intersection point with $\mathbb{C}E_2$ corresponds to the coordinate t taking the infinite value. The condition to pass through the specific points are given by the following equations:

$$a(i-c)^{m_1} = i\lambda \in i\mathbb{R}^* \text{ and } \frac{b^{\frac{m_1}{\delta}}}{a^{\frac{2m_2}{d}}}(c^2+1)^{\frac{m_1}{\delta}(m_3-m_2)} = \mu \in \mathbb{R}^*,$$

where $\delta = \gcd(m_1, 2m_2)$ is the integer length of E_3, and $i\lambda$ and μ are the coordinates of the fixed points on their respective divisors.

Lemma 5.1 *The enumerative problem admits the following set of solutions.*

- *If m_1 is odd, there exists precisely $2m_1$ oriented curves solution, two of each quantum index $(m_3 - m_2)(2k + 1 - m_1)$, for $k \in [\![0; m_1 - 1]\!]$. If we also consider curves passing through the symmetric real point, we get $4m_1$ real oriented curves.*
- *If m_1 is even, and $\frac{m_1}{\delta}$ is odd, we get the same result.*
- *If $\frac{m_1}{\delta}$ is even, there are $4m_1$ or 0 oriented curves solution according to the sign of μ.*

Proof We solve the system. The first equation implies that $(i - c)^{m_1} \in i\mathbb{R}$ and thus we can write it $i - c = re^{i\pi\frac{2k+1}{2m_1}}$ with $k \in \mathbb{Z}$ and $r \in \mathbb{R}$. Therefore, we have $i - re^{i\pi\frac{2k+1}{2m_1}} = c \in \mathbb{R}$. Hence,

$$\mathfrak{Im}\left(i - re^{i\pi\frac{2k+1}{2m_1}}\right) = 1 - r\sin\left(\pi\frac{2k+1}{2m_1}\right) = 0 \Rightarrow r = \frac{1}{\sin\left(\pi\frac{2k+1}{2m_1}\right)}$$
$$\Rightarrow c = r\cos\left(\pi\frac{2k+1}{2m_1}\right) = \cot\left(\pi\frac{2k+1}{2m_1}\right) = c_k.$$

We have proven that c can only take a finite number of values $c_k = \cot\left(\pi\frac{2k+1}{2m_1}\right)$, for $k \in [\![0; m_1 - 1]\!]$. For each value of c_k we find a unique a, and then solve for b eventually. Thus, we have proven that up to the action of the real torus $(\mathbb{R}^*)^2$, every real curve having purely imaginary intersection with $\mathbb{C}E_1$, and real intersection with both $\mathbb{C}E_2$ and $\mathbb{C}E_3$ is one of the curves

$$\psi_k : t \longmapsto \left((t - c_k)^{m_1}, (t - c_k)^{2m_2}(t^2 + 1)^{m_3 - m_2}\right).$$

These parametrized curves are the respective images of the curves

$$\varphi_k : t \longmapsto \left(t - c_k, \frac{t^2 + 1}{t - c_k}\right),$$

by the monomial map $\alpha : (z, w) \mapsto (z^{m_1}, z^{m_3+m_2}w^{m_3-m_2})$. Therefore, in order to compute the quantum indices of the oriented curves ψ_k, we just need to compute the Log-areas of the oriented curves φ_k, which were computed in Sect. 3.3.2:

$$k(S, \psi_k) = (m_3 - m_2)(2k + 1 - m_1).$$

□

The logarithmic rotation number of any map $t \mapsto \left(t - c, \frac{t^2+1}{t-c}\right)$ is 0. Applying the monomial map passing from a curve of degree Δ^{par} to a curve of degree $\Delta(m_1, m_2, m_3)$, we get the=at the logarithmic rotation number of the solutions is 0. Thus, the sign with which they are counted is 1. The signed number of solutions is thus

$$4\sum_{k=0}^{m_1-1} q^{(m_3-m_2)(2k+1-m_1)} = 4\frac{q^{m_1(m_3-m_2)} - q^{-m_1(m_3-m_2)}}{q^{m_3-m_2} - q^{-(m_3-m_2)}}.$$

5.2 Auxiliary statements

We now carry the two steps explained at the start of the section. Let $h_0 : \Gamma_0 \to N_\mathbb{R}$ be a parametrized rational tropical curve of degree $\Delta(s)$ such that $\mathrm{ev}(\Gamma_0) = \mu$. Let $C_{\mathrm{trop}} = h_0(\Gamma_0)$ be its image, which is a plane tropical curve. We need to find the real parametrized tropical curves (Γ, h) of degree Δ parametrizing the plane curve C_{trop} and admit a tropical enhancement. The different possible real structures are described in Proposition 2.13.

Lemma 5.2 *The parametrized real tropical curves with a trivalent flat vertex cannot be the tropicalization of a family of parametrized real rational curves passing through $\mathcal{P}_t$.*

Proof Assume that there exists a trivalent flat vertex w, in direction n_1, with two outgoing ends exchanged by the involution. Let $f_t : \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}\{\{t\}\}^*$ be a parametrized real rational curve tropicalizing to (Γ, h). Then, in the real coordinate y such that the two conjugated points have coordinate $\pm i$ and some real point has coordinate ∞, the morphism takes the following form:

$$f(y) = \chi_w t^{h(w)}(y^2+1)^{n_1} \prod_{j\neq 1} \left(\frac{y}{y(p_j)} - 1\right)^{n_j} \in N \otimes \mathbb{C}\{\{t\}\}^*,$$

where p_j are the points where f is not defined, and $\chi_w \in N \otimes \mathbb{R}\{\{t\}\}^*$. The coordinates $y(p_j)$ are Puiseux series with a first order term of negative valuation. In particular, the first order term of $\frac{y}{y(p_j)} - 1$ is -1 if y is taken equal to some Puiseux series of positive valuation.

The moment at $\pm i$ is obtained by evaluating the monomial $\iota_{n_1}\omega \in M$ on the curve at $y = \pm i$. We get two conjugated Puiseux series depending on the parameter t. The first order terms of these Puiseux series are real:

- as noticed, when evaluated at $\pm i$, each term of the product has first order term equal to -1,
- the coefficient $\chi_w(\iota_{n_1}\omega)$ is real,
- the $(y^2+1)^{n_1}$ vanishes after the evaluation of the monomial $\iota_{n_1}\omega$ since the exponent is $\omega(n_1, n_1)$ is 0.

This is absurd since it is supposed to be purely imaginary. Hence, we cannot have any flat vertex. □

Lemma 5.3 *Among the real parametrized tropical curves $h : \Gamma \to N_\mathbb{R}$ of degree Δ with image C_{trop}, at most one may be the tropicalization of a family of parametrized real rational curves passing through $\widehat{\mathcal{P}}_t$. Moreover, this real tropical curve is the tropical curve obtained from Γ_0 with the maximal splitting graph.*

Proof There is an infinite number of parametrized curves with image C_{trop}, obtained by splitting the graph of even edges and described in Proposition 2.13. All the ends of C_{trop} associated to the complex markings are double edges near infinity since they correspond to two distinct marked points. Thus they belong to Γ_{even}. Since all the complex markings are on the same divisor, they have the same direction and they cannot meet at a common vertex. Therefore, the graph Γ_{even} only consists of the complex ends. The only possibility is that the double ends separates itself at a trivalent flat vertex, sent somewhere on the end of C_{trop}. However, this is forbidden by Lemma 5.2. Therefore, all the double ends split and there is a unique possibility. □

We have proven that for C_{trop}, there is a unique real parametrized tropical curve $h : \Gamma \to N_{\mathbb{R}}$ of degree Δ with image C_{trop} that can be the tropicalization of a family of parametrized real rational curves. We now intend to apply the correspondence theorem from Shustin (2006b) to recover the curves solution to the enumerative problem tropicalizing to (Γ, h).

Proposition 5.4 *Let $\widehat{\mathcal{P}}_t$ be a real configuration of points as previously chosen, tropicalizing on a family of moments μ. Let $h_0 : \Gamma_0 \to N_{\mathbb{R}}$ be a parametrized tropical curve of degree $\Delta(s)$ having moments μ, and let $h : \Gamma \to N_{\mathbb{R}}$ be the associated real parametrized tropical curve without flat vertex such that $\text{ev}(\Gamma) = \mu$. Vertices of Γ and Γ_0 are canonically identified. Let $W_1, \ldots, W_s$ be the quadrivalent vertices of Γ, adjacent to the complex ends, let m_{W_i} denote their complex multiplicity as a trivalent vertex of Γ_0.*

Then, there are precisely $2^{m-2s}\prod m_{W_i}$ oriented real curves passing through the configuration $\widehat{\mathcal{P}}_t$ up to symmetry and tropicalizing to (Γ, h). Their contribution to R_Δ is given by

$$m'_\Gamma = 4\prod_1^s \frac{q^{m_{W_i}/2} - q^{-m_{W_i}/2}}{q - q^{-1}} \prod_{V \neq W_i} (q^{m_V/2} - q^{-m_V/2})$$
$$= \frac{4}{(q - q^{-1})^s} \prod_V (q^{m_V/2} - q^{-m_V/2}).$$

Proof We make an induction on the number of vertices. Thus, we initialize with curves Γ having a unique vertex, trivalent or quadrivalent. Following Mikhalkin (2005), to compute the multiplicity, one needs to count (in a suitable way) the local curves over the vertices of Γ, and the number of ways to “glue" them together. In this proof, we do not assume the vectors of Δ to be primitive.

- If there is only one trivalent vertex V in Γ, then we are looking for curves maximally tangent to the toric divisors and passing through two pairs of opposite real points, exactly as in the proof of Theorem 7 in Mikhalkin (2017): there are 4 such curves, which are exchanged by the action of the deck transformation group $\{\pm 1\}^2$ on the associated toric surface. These 4 curves lead to 8 oriented curves. Half of them have logarithmic Gauss degree 1 (and thus $\sigma(\mathbb{R}C) = 1$) and quantum index $\frac{m_V}{2}$, and the other half has degree -1 (and thus $\sigma(\mathbb{R}C) = -1$) and quantum index $-\frac{m_V}{2}$. Therefore the signed contribution is $4(q^{m_V/2} - q^{-m_V/2})$. Notice that we have three pairs of real opposite points. specifying on point in each pair, the number of curves passing through the chosen points may vary. Gathering all the possibilities, the number of solutions is always the same.

– If there is only one quadrivalent vertex W, we look for curves passing through one pair of conjugated imaginary points on one divisor, and a pair of opposite real points. Assume that the degree of the vertex is $\Delta(m_1, m_2, m_3)$. Then, as the unbounded non-real ends are of weight 2, we have $m_3 - m_2 = 1$, and the complex multiplicity is $m_W = 2m_1$. In Lemma 5.1, we have seen that there are always $2m_1$ curves passing through the configuration: either m_1 for each of the real points in the pair, or $2m_1$ and 0. Therefore, there are $4m_1$ oriented curves going through the pair. Moreover, their quantum index is known. The logarithmic rotation number $\mathrm{Rot}_{\mathrm{Log}}$ can be computed thanks to the same monomial map that allowed us to compute their quantum index: if A denotes the matrix of the monomial map, then

$$\mathrm{Rot}_{\mathrm{Log}}(\psi_k) = \det A \times \mathrm{Rot}_{\mathrm{Log}}(\varphi_k).$$

As the logarithmic rotation number of any map $t \mapsto \left(t - c, \frac{t^2+1}{t-c}\right)$ is easily computed to be 0, all the curves have logarithmic rotation number zero and all solutions are counted with the same sign 1. When accounting for both orientations, the desired count is

$$4\sum_{k=0}^{m_1-1} q^{(m_3-m_2)(2k+1-m_1)} = 4\frac{q^{(m_3-m_2)m_1} - q^{-(m_3-m_2)m_1}}{q^{m_3-m_2} - q^{-(m_3-m_2)}} = 4\frac{q^{m_W/2} - q^{-m_W/2}}{q - q^{-1}},$$

since $m_3 - m_2 = 1$, and the complex multiplicity m_W satisfies $m_W = 2m_1$.

Now that the initialization is done, assume that Γ has more than one vertex. Following the steps of Shustin (2006b), we do the following:

- First, find the tropical enhancements, *i.e.* the real curves associated to each vertex of Γ. This is taken care of by Lemma 5.1 for quadrivalent vertices, and by Mikhalkin (2017) for trivalent vertices. This is the first step in finding the *admissible tropicalization curves* for Lemma 3.2 in Shustin (2006b).
- Then, ensure that the curves are compatible in the sense that if two vertices are adjacent, the curves associated to both vertices have the same unique intersection point with the toric divisor associated to the edge between the vertices.
- Use Lemma 2.8 from Shustin (2006b) to finish finding *admissible tropicalization curves*: there is one for each edge of Γ. If the edge is of odd weight, there is a unique tropicalization curve for the edge, and if the edge is of even weight, there are 2: fixing orientations of the curves associated to the endpoints of the edge, there is the orientation preserving way, and the orientation reversing way. This is later referred as “gluing".
- For each tuple of admissible tropicalization curves, Lemma 3.2 from Shustin (2006b) provides a unique curve solution to the enumerative problem. Compute the sign and quantum index for each such solution.

We now carry the explained steps for each case. Let V be a vertex adjacent to two real ends, or one real end and two complex ends. The last edge adjacent to V, which is

bounded, is denoted by γ. Let Γ' be the parametrized tropical curve obtained by deleting this vertex and replacing the edge γ heading to V by an end with same direction. The Menelaus rule allows us to define a pair of real opposite moments associated to the new end. This moment is defined by the condition that the configuration composed by the pairs of points of the edges adjacent to V satisfies the Menelaus condition. We get a new configuration of points $\widehat{\mathcal{P}}'_t$, indexed by the ends of Γ'.
Let $4R$ be the refined count of oriented curves tropicalizing on Γ', passing through the configuration $\widehat{\mathcal{P}}'_t$ up to symmetry. We now have to *glue* together the oriented curves above Γ', and the curves over the vertex V. In each case we inquire for the refined count over the global tropical curve Γ. We make a disjunction over the type of V, which can either be a trivalent one, or a quadrivalent one (*i.e.* one of the vertices W_i), and γ can be an odd or an even edge.

- Assume that V is trivalent, and the bounded edge adjacent to V is odd. Then, there are two real opposite points associated to the unbounded end of Γ' in the configuration, and they are exchanged by the deck transformation group $\{\pm 1\}^2$. Therefore, over Γ', there are $2R$ oriented curves for each point in the pair (both add up to the total $4R$ oriented curves).
  Meanwhile, there are 4 curves above the vertex V, two over each of the real points in the pair. Thus, we get $2 \times 2R$ possible tropical enhancements for each point in the pair. As the edge has odd weight, Lemma 2.8 from Shustin (2006b) produces a unique admissible tropicalization curve for the gluing.

  Moreover, for any possible gluing of an oriented curve and a curve, the orientation of the oriented curve over Γ' extends to the new global curve. The curves above V thus get an orientation. The initialization for the case of a unique trivalent vertex shows that the two oriented curves obtained this way have opposite quantum indices. Moreover, one increases by one the logarithmic rotation number while the other decreases it by one, just as in the proof of Theorem 7 in Mikhalkin (2017). Finally, the total contribution is

$$(q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})2R + (q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})2R = 4(q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})R.$$

- Assume that V is trivalent, and γ is an even edge. We have 4 possibilities exchanged by the deck transformation group, according to the intersection point in the pair, and the position of the curve with respect to the toric divisor. (since the weight is even, the curve stays on the same side) For each of these possibilities, there are R oriented curves over Γ', and just one compatible curve over V. Similarly, each time, Lemma 2.8 from Shustin (2006b) produces two ways of gluing the curves: one that increases the logarithmic rotation number, and one that decreases it. We thus get

$$(q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})R + (q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})R + (q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})R + (q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})R$$
$$= 4(q^{\frac{m_V}{2}} - q^{-\frac{m_V}{2}})R.$$

- Assume that $V = W$ is a quadrivalent vertex, and that γ is an odd edge. Assume that the dual triangle is equivalent to $\Delta(m_1, m_2, m_3)$. Over Γ', there are $2R$ oriented curves for each of the points in the pair, while according to Lemma 5.1, there are m_1 curves over W for each of the intersection point in the pair. Using Lemma 2.8 from Shustin (2006b), in each case the gluing is unique and we get

$$\left(\frac{q^{m_W/2} - q^{-m_W/2}}{q - q^{-1}}\right) 2R + \left(\frac{q^{m_W/2} - q^{-m_W/2}}{q - q^{-1}}\right) 2R = 4\left(\frac{q^{m_W/2} - q^{-m_W/2}}{q - q^{-1}}\right) R.$$

- Finally, if $V = W$ is a quadrivalent vertex and γ is an even edge, there are similarly four cases to consider, each one with R oriented curves over Γ'. This time m_1 is even, and the distribution of the solutions for the curves above the vertex W might be a little trickier. We have seen in Lemma 5.1 that if the boundary points are fixed, there are either $2m_1$ curves above W for one of the points and zero for the other, or m_1 for each of them. In both cases we find the compatible tropical enhancements, and apply Lemma 2.8 from Shustin (2006b) to find all the admissible tropicalization curves. In each case, we still get

$$4\left(\frac{q^{m_W/2} - q^{-m_W/2}}{q - q^{-1}}\right) R.$$

Finally, we recover the formula for m'_Γ. □

Remark 5.5 Concretely, once the curves associated to trivalent and quadrivalent vertices have been found using Lemma 5.1, the proof proceeds exactly as in proof of Theorem 7 in Mikhalkin (2017). The only difference is that we have a different family of curves to choose over the quadrivalent vertices, leading to the vertex multiplicity $\frac{q^{m_W/2}-q^{-m_W/2}}{q-q^{-1}}$ rather than $q^{m_V/2} - q^{-m_V/2}$.

5.3 Proof of Theorems 1.4 and 1.5

For reader's convenience, we restate Theorems 1.4 and 1.5.

Theorem *1.4 For a generic choice of $\widehat{\mathcal{P}_0}$ close to the tropical limit, one has*

$$R_\Delta(\widehat{\mathcal{P}_0}) = \frac{(q^{1/2} - q^{-1/2})^{m-2-s_1}}{(q - q^{-1})^{s_1}} N^{\partial,trop}_{\Delta(s)} = \frac{(q^{1/2} - q^{-1/2})^{m-2-2s_1}}{(q^{1/2} + q^{-1/2})^{s_1}} N^{\partial,trop}_{\Delta(s)}.$$

Proof According to Proposition 5.4, we obtain $R_\Delta(\widehat{\mathcal{P}_t})$ by counting curves with multiplicities $\frac{1}{4}m'_\Gamma$. The multiplicity $\frac{1}{4}m'_\Gamma$ is obtained from m^q_Γ by clearing the denominators of the $m - 2 - s$ vertices and dividing by the terms of the s quadrivalent vertices:

$$\frac{1}{4}m'_\Gamma = \frac{(q^{1/2} - q^{-1/2})^{m-2-s}}{(q - q^{-1})^s} m^q_\Gamma.$$

Therefore, one has

$$R_\Delta(\widehat{\mathcal{P}}_t) = \frac{(q^{1/2}-q^{-1/2})^{m-2-s}}{(q-q^{-1})^s} N_{\Delta(s)}^{\partial,\text{trop}}.$$

The other equality follows from the identity $q-q^{-1} = (q^{1/2}-q^{-1/2})(q^{1/2}+q^{-1/2})$.
□

Theorem 1.5 *For a choice of $\widehat{\mathcal{P}}_0$ which is a regular value of the evaluation map that sends a curve to the coordinates of its boundary points, one has*

$$R_{\Delta,(s_1,0,\dots,0)} = 2^{s_1} R_\Delta(\widehat{\mathcal{P}}_0).$$

Moreover, a generic choice of $\widehat{\mathcal{P}}_0$ close to the tropical limit is such a choice.

Proof For the first statement, using the proof of invariance in section 6.2 of Mikhalkin (2017), any regular value of the following evaluation map may be used to compute $R_{\Delta,(s_1,0,\dots,0)}$:

$$\text{ev} : \mathcal{M}_0(\Delta, N_{\mathbb{C}^*}) \to (\mathbb{R}^*)^{m-2s-1} \times (\mathbb{C}^*)^s,$$

that sends a real oriented parametrized curve of degree Δ to the coordinates of its intersection point with the toric boundary. The factor 2^{s_1} just accounts that for a pair of conjugated purely imaginary points, the opposite pair is the same. Thus, one has to multiply the number of curves by 2^{s_1} to get the invariant $R_{\Delta,(s_1,0,\dots,0)}$ from $R_\Delta(\widehat{\mathcal{P}}_0)$.

For the second statement, we need to show that there exists some purely imaginary regular values. This follows from the tropical computation and the correspondence theorem since the latter relies on the fact that the evaluation map is transverse. □

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