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Unitals with many involutory translations

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Abstract
If every point of a unital is fixed by a non-trivial translation and at least one translation has order two then the unital is classical (i.e., hermitian).

Keywords Unital · Involution · Translation · Hermitian unital

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1 Introduction

A unital $\mathbb{U} = (U, \mathcal{B})$ of order q is a 2-$(q^3 + 1, q + 1, 1)$-design; i.e., an incidence structure with $|U| = q^3 + 1$ such that every block $B \in \mathcal{B}$ has exactly $q + 1$ points, and any two points in U are joined by a unique block in $\mathcal{B}$. It follows that every point is incident with exactly q^2 blocks. Without loss of generality, we will assume throughout that the blocks are subsets of U. The classical examples of unitals are the hermitian ones: For any prime power q, the points of the hermitian unital $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$ are the absolute points with respect to a suitable polarity of the projective plane $\mathrm{PG}(2, \mathbb{F}_{q^2})$ over the field of order q^2, see (Hughes and Piper 1973, II.8, pp. 57–63) or (Barwick and Ebert 2008, Ch. 2). The blocks are induced by secant lines in $\mathrm{PG}(2, \mathbb{F}_{q^2})$.

A translation (with center c) of a unital $\mathbb{U}$ is an automorphism of $\mathbb{U}$ that fixes the point c and every block through c. We write $\Gamma_{[c]}$ for the group of all translations with

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center c. Much is known for the case where there are at least two points u, v such that $\Gamma_{[u]}$ and $\Gamma_{[v]}$ both have order q, see (Grundhöfer et al. 2021b). If there even exist three non-collinear points with that property then the unital is the classical hermitian one, see (Grundhöfer et al. 2013). We show (Theorem 4.3) that the same conclusion holds if $\Gamma_{[c]}$ has even order for every point c. In fact, it suffices that every point is the center of some non-trivial translation, and that at least one of these translations is an involution.

Polar unitals in Figueroa planes of even order provide examples where the centers of involutory translations form a proper subset not contained in a block (namely, a subunital that is in fact hermitian), see Theorem 5.3. Examples where the set of centers is contained in a block exist in abundance, see Sect. 6.

2 Transitivity

The following elementary observation can be traced back to Gleason; cp. (Dembowski 1968, pp. 190 f), see (Grundhöfer et al. 2013, 4.1) for a (very short) proof.

Lemma 2.1 *Let p be a prime, and let H be a group acting on a finite set X. Assume that for every $x \in X$ there exists in H an element of order p which fixes x but no other element of X. Then H is transitive on X.* □

Now let $\mathbb{U} = (U, \mathcal{B})$ be a unital. A translation of order n is called an n-translation, and $\Gamma_{[c]}^{[n]}$ denotes the set of all n-translations of $\mathbb{U}$ with center c. For each n, let Ω_n be the (possibly empty) set of all centers of n-translations, and $\Gamma^{[n]} := \left\langle \bigcup_{c\in\Omega_n} \Gamma_{[c]}^{[n]} \right\rangle$.

Recall from (Grundhöfer et al. 2013, Theorem 1.3) that a non-trivial translation of a unital fixes no point apart from its center. If Ω_n is not empty, the size of any $\mathrm{Aut}(\mathbb{U})$-orbit in Ω_n is therefore congruent 1 modulo n (because a cyclic group of size n acts semi-regularly on the complement of a point in Ω_n), and every $\mathrm{Aut}(\mathbb{U})$-orbit outside Ω_n has size divisible by n (because the same cyclic group acts semi-regularly on that set).

Lemma 2.2 *Let p be a prime. If $H \le \mathrm{Aut}(\mathbb{U})$ contains a p-translation then H is transitive on the set of all centers of p-translations in H. In particular, we have:*

1. *The group $\Gamma^{[p]}$ generated by all p-translations is transitive on Ω_p.*
2. *For each block B of $\mathbb{U}$ the group generated by all p-translations with center on B is transitive on the set $\Omega_p \cap B$ of centers of p-translations in B. Consequently, the stabilizer of B in $\Gamma^{[p]} \le \mathrm{Aut}(\mathbb{U})$ is transitive on $\Omega_p \cap B$.*
3. *If $n > 1$ is an integer such that Ω_n is not empty, then the group $\Gamma^{[n]}$ is transitive on Ω_n and $\Omega_k = \Omega_n$ holds for each divisor $k > 1$ of n.*

Proof The group H acts on the set of all centers of p-translations in H, and Lemma 2.1 applies. For assertions 1 and 2 we specialize $H = \Gamma^{[p]}$ and $H = \left\langle \bigcup_{x\in B} \Gamma_{[x]}^{[p]} \right\rangle$, respectively. Assertion 3 follows from the fact that Ω_n is contained in the $\mathrm{Aut}(\mathbb{U})$-orbit Ω_k. □

Definition 2.3 Assume that $\mathbb{U}$ admits a translation of order $k > 1$. Let p_k be the smallest prime divisor of k. Then $\Omega_k = \Omega_{p_k}$ holds by Lemma 2.2.3. Put

$K := \{p_k \mid k \in \mathbb{N}, |\Gamma^{[k]}| > 1\}$. Clearly, we obtain a disjoint union $U = \mho \cup \bigcup_{p \in K} \Omega_p$, where $\mho := \{x \in U \mid \Gamma_{[x]} = \{\mathrm{id}\}\}$.

We write $\mathcal{B}_k$ for the set of all blocks of $\mathbb{U}$ that contain at least two points of Ω_k, and consider the incidence structure $\mathbb{U}_k := (\Omega_k, \mathcal{B}_k, \in)$. If Ω_k is not contained in a block then $\mathbb{U}_k$ is a non-trivial linear space.

From Lemma 2.2.1 and Lemma 2.2.2 we know that $\Gamma^{[k]}$ induces a transitive group of automorphisms of $\mathbb{U}_k$, and that for each block $B \in \mathcal{B}_k$ the stabilizer of B in $\Gamma^{[k]}$ acts transitively on $\Omega_k \cap B$.

Recall that a substructure of an incidence structure (with points and blocks) is called *ideally embedded* if the pencils of blocks in the substructure are the same as the pencils in the larger structure.

Lemma 2.4 *Assume that $\mathbb{U}$ is a unital such that every point is the center of some non-trivial translation. For each $p \in K$, the linear space $\mathbb{U}_p$ is then ideally embedded in $\mathbb{U}$. In particular, the set Ω_p is not contained in a block. The action of* $\mathrm{Aut}(\mathbb{U})$ *on Ω_p is faithful.*

If there exists $p \in K$ such that $|\Omega_p \cap B|$ is constant for $B \in \mathcal{B}_p$ then $\Omega_p = U$, and $\Gamma^{[p]}$ is transitive on U. In particular, this happens if $\mathrm{Aut}(\mathbb{U})$ *is two-transitive on Ω_p.*

Proof Consider $p \in K$, and a block $B \in \mathcal{B}$ through a point $x \in \Omega_p$. It suffices to consider the case where there exists a point $y \in B \setminus \Omega_p$. By our assumption, there exists a non-trivial translation τ with center y. Now x^τ is a point different from x, and lies in $\Omega_p \cap B$. So $\mathbb{U}_p$ is ideally embedded in $\mathbb{U}$. Clearly, this implies that Ω_p is not contained in any block.

Assume that $\alpha \in \mathrm{Aut}(\mathbb{U})$ fixes every point in Ω_p. Then α fixes each block that meets Ω_p because $\mathbb{U}_p$ is ideally embedded. Each point x outside Ω_p lies on more than one block meeting Ω_p, so α fixes every point of $\mathbb{U}$, and is trivial.

Now assume that $b := |\Omega_p \cap B|$ is constant for $B \in \mathcal{B}_p$; then $|\Omega_p| = 1 + q^2 b$. If there exists $z \in U \setminus \Omega_p$ then, by our assumption, there is a translation of prime order r and center z. Now r divides both q and $|\Omega_p| = 1 + q^2 b$, and we obtain a contradiction. □

The Figueroa unitals (see Theorem 5.2 below) of even order are examples that show that the condition $\mho = \emptyset$ is necessary in Lemma 2.4: in those unitals the substructure $\mathbb{U}_2$ is not ideally embedded, and there exists an automorphism of order 3 that acts trivially on Ω_2. See also Proposition 4.2.

3 Results from group theory

Lemma 3.1 *Let G be a transitive permutation group on some finite set Ω with more than one element. Suppose that the stabilizer G_x of some $x \in \Omega$ contains an involution τ that is semi-regular on $\Omega \setminus \{x\}$, and that G has a transitive normal subgroup M of odd order. Then the following are equivalent:*

1. *M is abelian.*
2. *M acts regularly (i.e., sharply transitively) on Ω.*

3. *The involution τ acts semi-regularly on M by conjugation (i.e., the centralizer of τ in M is trivial).*

If one (and then any) of these conditions is satisfied then $\langle\tau\rangle M$ is a generalized dihedral group; i.e., conjugation by τ inverts each element of M, and $\langle\tau\rangle M = \tau^M \cup M$.

Proof If M is abelian then the stabilizer M_x also fixes each element of the orbit Ω of x under M. Hence M_x is trivial, and the action of M is regular. If M acts regularly, we identify $m \in M$ with the image x^m. Semi-regularity of τ on Ω then translates into semi-regularity of the automorphism induced by conjugation with τ on M. The map $m \mapsto m^{-1}m^\tau$ is then injective, and bijective since M is finite. Write $m \in M$ as $m = k^{-1}k^\tau$ with $k \in M$, and then calculate $m^{-1} = (k^\tau)^{-1}k = (k^{-1})^\tau k = (k^{-1}k^\tau)^\tau = m^\tau$. Thus the automorphism induced by τ is the anti-automorphism $m \mapsto m^{-1}$, and M is abelian.

If τ induces inversion on M then $m^{-1}\tau m = \tau(\tau m^{-1}\tau)m = \tau m^2$ holds for each $m \in M$. As M has odd order, this means that the coset τM equals the conjugacy class τ^M, and consists of involutions. $\square$

The following result has been proved in (Hering 1972, Theorem 2); cp. (Aschbacher 1973, Theorem 2).

Theorem 3.2 *Assume that the group G acts transitively (but not necessarily faithfully) on some finite set Ω with more than one element. Suppose that the stabilizer G_x of some $x \in \Omega$ has a normal subgroup Q of even order that is semi-regular on $\Omega \setminus \{x\}$. Then the normal closure $S{:=}\langle Q^G\rangle$ either has a transitive normal subgroup of odd order, or acts two-transitively on Ω as one of the groups $\mathrm{SL}_2(\mathbb{F}_{2^e})$, $\mathrm{Sz}(2^{2e-1})$, or $\mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ for suitable $e \geq 2$; the action is the usual two-transitive one.*

The group S itself is then $S = QN$ where N denotes the largest normal subgroup of odd order in S, or S is isomorphic to $\mathrm{SL}_2(\mathbb{F}_{2^e})$, $\mathrm{Sz}(2^{2e-1})$, $\mathrm{SU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ or $\mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$, according to the group induced on Ω. $\square$

By $\mathrm{Sz}(2^{2e-1})$ we denote the Suzuki group — also known as the twisted Chevalley group ${}^2B_2(2^{2e-1})$ — of order $2^{4e-2}(2^{4e-2}+1)(2^{2e-1}-1)$; cp. (Lüneburg 1980, Section 21).

We add some information contained in (Hering 1972) —in particular, see (Hering 1972, Lemma 3)— but not in the statement of the theorem referred to above.

Proposition 3.3 *The kernel of the action of G as considered in Theorem 3.2 is $K = \mathrm{C}_G(S)$.*

If S has a transitive normal subgroup N of odd order then Q acts fixed-point-freely on N/K, so the group N/K induced by N on Ω is a sharply transitive abelian group in that case. The group Q is then (isomorphic to) a Frobenius complement.

If S induces a non-solvable group on Ω then either $K \cap S$ is trivial, or $|K \cap S| = 3$; the latter case can only occur if $S \cong \mathrm{SU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ with odd $e > 1$. $\square$

The Sylow subgroups of a Frobenius complement are either cyclic or generalized quaternion groups; see (Gorenstein 1980, 10.3.1). So Proposition 3.3 yields:

Corollary 3.4 *If $S = \langle Q^G\rangle$ has a transitive normal subgroup N of odd order then the group Q contains exactly one involution. That involution induces inversion on N.*

4 Unitals with involutory translations

By Lemma 2.2, Hering's result (see Theorem 3.2) applies to the permutation group induced by $\Gamma^{[2]}$ on Ω_2 if this set has more than one element.

The following proposition is a small part of the results in (Kantor 1985). For the reader's convenience, we include a proof of the facts that we need here.

Proposition 4.1 *Let $(X, \mathcal{L})$ be a linear space with v points such that each line has k points. Assume that $k > 2$. Consider a group G isomorphic to one of the groups $\mathrm{PSL}_2(\mathbb{F}_q)$, $\mathrm{Sz}(q)$ or $\mathrm{PSU}_3(\mathbb{F}_{q^2}|\mathbb{F}_q)$, for some prime power q. If G acts on X in its usual two-transitive action and by automorphisms of $(X, \mathcal{L})$, then either $v = k$ (and there is just one line in $\mathcal{L}$), or $G \cong \mathrm{PSU}_3(\mathbb{F}_{q^2}|\mathbb{F}_q)$ and $(X, \mathcal{L}) \cong \mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$ is isomorphic to the hermitian unital of order q.*

Proof We assume that there is more than one line, and discuss the three different cases separately.

(PSL) The usual two-transitive action of $G \cong \mathrm{PSL}_2(\mathbb{F}_q)$ is the natural action on the projective line $\mathbb{F}_q \cup \{\infty\}$ via fractional linear transformations. Then $v = q + 1$, and the stabilizer of two points $x, y \in X$ has two orbits of length 1 and at most two other orbits, each of length $\frac{q-1}{\gcd(2,q-1)}$. The line joining x and y contains at least one of those orbits, and $k - 1 \geq \frac{q-1}{\gcd(2,q-1)} + 1 > \frac{v-1}{2}$. So there is no space left for a second line through x, and $v = k$ follows.

The Suzuki groups and the unitary groups need a closer look; we will use the following facts about linear spaces with v points and $k > 2$ points per line:

The number of lines per point in $(X, \mathcal{L})$ is $r = \frac{v-1}{k-1}$. We assume that the linear space has more than one line, so $r > 1$. Fisher's inequality [see (Dembowski 1968, 1.3.8, p. 20)] then says $r \geq k$, and equality holds precisely if the linear space is a projective plane. In the latter case, we have $v = k^2 - k + 1$. If $r > k$ then $v - 1 \geq (k + 1)(k - 1) = k^2 - 1$.

(Sz) The usual two-transitive action of $G \cong \mathrm{Sz}(q)$ is its action on the Suzuki-Tits ovoid; see (Lüneburg 1980, Sect. 21). Then $v = q^2 + 1$ and q is a power of 2. Hence we cannot have $v - 1 = k^2 - k$, and $v - 1 \geq k^2 - 1$ follows.

For any two points $x, y \in X$, the stabilizer $G_{x,y}$ acts on X with two orbits of length 1, every other orbit has length $q-1$; see (Lüneburg 1980, Lemma 21.5). The block joining x and y is thus the union of $\{x, y\}$ with a collection of such orbits of length $q - 1$, and $k \geq (q - 1) + 2 = q + 1$. Now the inequality $q^2 = v - 1 \geq k^2 - 1 \geq (q + 1)^2 - 1$ yields a contradiction.

(PSU) The usual two-transitive action of $G \cong \mathrm{PSU}_3(\mathbb{F}_{q^2}|\mathbb{F}_q)$ is its action on the points of the hermitian unital of order q; see (Higman and McLaughlin 1965), (O'Nan 1972), (Hughes and Piper 1973, Theorem 5.2). Then $v = q^3 + 1$, the fact that q is a prime power yields $v - 1 \neq k^2 - k$, and we have $q^3 = v - 1 \geq k^2 - 1$.

If $q = 2$ then $v = 9$ and $k > 2$ together with $k^2 - 1 \leq v - 1$ yields $k = 3$. Then $(X, \mathcal{L})$ is determined uniquely, we have $(X, \mathcal{L}) \cong \mathcal{H}(\mathbb{F}_4|\mathbb{F}_2) \cong \mathrm{AG}(2, \mathbb{F}_3)$, the affine plane of order 3. We assume $q > 2$ from now on.

The stabilizer $G_{x,y}$ of two points $x, y \in X$ has two orbits of length 1, one orbit of length $q-1$, and every other orbit has length $\frac{q^2-1}{\gcd(3,q+1)}$; see (O'Nan 1972, p. 499). So there are integers $s \in \{0, 1\}$ and $t \geq 0$ such that $k-1 = s(q-1) + t\frac{q^2-1}{\gcd(3,q+1)} + 1$.

Aiming at a contradiction, we assume $t > 0$. Then $k \geq \frac{q^2-1}{\gcd(3,q+1)} + 2 \geq \frac{q^2+5}{3}$. From $q^3 = v - 1 \geq k^2 - 1 \geq \left(\frac{q^2+5}{3}\right)^2 - 1$ we thus obtain $0 \geq q^4 - 9q^3 + 10q^2 + 16 > q^2\big(q(q-9)+10\big)$. So $q < 8$, and $q \in \{3, 4, 5, 7\}$. For $q \in \{3, 4, 7\}$ we have $\gcd(3, q+1) = 1$, and our assumption $t > 0$ yields $q^3 \geq \big((q^2-1)+1\big)^2 = q^4$. This is impossible. For $q = 5$ and $t > 1$, we obtain the contradiction $5^3 \geq \left(2\frac{(5^2-1)}{3}+1\right)^2 = 17^2$. So $t = 1$, and $k - 1 = 4s + 8 + 1$ divides $v - 1 = 5^3$. Both cases for $s \in \{0, 1\}$ lead to a contradiction.

Therefore, we have $t = 0$ and $s = 1$; and the block through x and y is the union of $\{x, y\}$ with the unique orbit of length $q-1$ under $G_{x,y}$. This means that $(X, \mathcal{L})$ is isomorphic to the hermitian unital $\mathcal{H}(\mathbb{F}_{q^2}|\mathbb{F}_q)$. $\square$

Proposition 4.2 *Let $\mathbb{U}$ be a unital of order q, and assume that Ω_2 is not contained in a block. If the group induced by $\Gamma^{[2]}$ on Ω_2 does not have a normal subgroup that acts regularly on Ω_2 then that induced group is isomorphic to* $\mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$*, and $\mathbb{U}_2 = (\Omega_2, \mathcal{B}_2)$ is isomorphic to the hermitian unital of order 2^e, with $e > 1$.*

Proof From Theorem 3.2 we know that the group induced by $\Gamma^{[2]}$ on Ω_2 is isomorphic to one of the groups $\mathrm{SL}_2(\mathbb{F}_{2^e})$, $\mathrm{Sz}(2^{2e-1})$, or $\mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ (for suitable $e \geq 2$), with the usual two-transitive action. From Proposition 4.1 we then know that $\Gamma^{[2]} \cong \mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$, and $\mathbb{U}_2$ is the hermitian unital of order 2^e. $\square$

The situation in Proposition 4.2 actually occurs, for example, in polar unitals of Figueroa planes of even order, see Theorem 5.2 and Theorem 5.3 below. Our Proposition 4.2 is a version for unitals (with translations) of a result (Hering 1976, Theorem 5.2) on projective planes (and elations).

Theorem 4.3 *Let $\mathbb{U} = (U, \mathcal{B})$ be a unital of order q, assume that every point of $\mathbb{U}$ is the center of some non-trivial translation, and that there exists a translation of order* 2*. Then q is a power of* 2*, and the unital $\mathbb{U}$ is the hermitian unital of order q.*

Proof If $q = 2$ then the unital is the hermitian unital of order 2, see (Taylor 1992, 10.16). We assume $q > 2$ in the rest of the proof.

Case A: Assume first that $\Omega_2 = U$, i.e., every point of $\mathbb{U}$ is the center of some involutory translation. If the group $\Gamma^{[2]}$ generated by the involutory translations does not have a regular normal subgroup then Proposition 4.2 says that $\Gamma^{[2]} \cong \mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$, and $\mathbb{U}$ is the hermitian unital of order 2^e.

(Alternatively, this can be derived more directly from Theorem 3.2: the groups $\mathrm{SL}_2(\mathbb{F}_{2^e})$ are excluded since they are triply transitive; the orbits of the two-point-stabilizers in $\mathrm{Sz}(2^{2e-1})$ are too large to yield blocks of the unital. For the groups $\mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ the block through two points consists of those two together with the unique shortest orbit of their stabilizer. Hence $\mathbb{U}$ coincides with the hermitian unital $\mathcal{H}(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ of order 2^e.)

If the group $\Gamma^{[2]}$ has a regular normal subgroup N, then N is abelian (see Proposition 3.3). Consider two blocks B, C through some point. Then the stabilizers N_B and N_C are subgroups of order $q+1$ in N, and $N_B N_C$ is a subgroup of order $(q+1)^2$. So $(q+1)^2$ divides $|N| = q^3+1$, and $q+1$ divides $(q^3+1)/(q+1) = q^2-q+1 = (q+1)(q-1)-(q-2)$. This implies $q-2=0$, contradicting our assumption $q>2$ (in fact, $\Gamma^{[2]}$ on the unital of order 2 does contain a regular normal subgroup, see Remark 4.4).

Case B: Now we assume $\Omega_2 \neq U$, and aim at a contradiction. As every point is the center of a non-trivial translation, there is a prime p such that Ω_p is not empty, and disjoint to Ω_2. We know from Lemma 2.4 that Ω_2 is not contained in a block, and that the group $\Gamma^{[2]}$ generated by all involutory translations acts faithfully on Ω_2.

The action of $\Gamma^{[2]}$ on Ω_2 is as in Theorem 3.2 and Proposition 3.3. If $\Gamma^{[2]}$ is two-transitive on Ω_2 then Lemma 2.4 yields $\Omega_2 = U$, contradicting our present assumptions. If $\Gamma^{[2]}$ is not two-transitive on Ω_2 then $\Gamma^{[2]}$ has an abelian normal subgroup A acting sharply transitively on Ω_2, see Theorem 3.2. From Corollary 3.4 we know that each one of the involutory translations acts by inversion on A, and that $\Gamma_{[x]}$ contains exactly one involution if $x \in \Omega_2$. We denote that involution by j_x. The order of A equals that of Ω_2, so it is odd, and divisible by p.

Consider $a \in A$ and an arbitrary point $x \in \Omega_2$. Then both j_x and $a^{-1} j_x a = j_{x^a}$ fix the block joining x and x^a. So $a^2 = j_x a^{-1} j_x a$ fixes that block, and so does a because squaring is an automorphism of the abelian group $\langle a \rangle$ of odd order.

As $|A| = |\Omega_2|$ is divisible by p, we find an element $a \in A$ of order p. That a fixes a point $c \in \Omega_p$ because $|\Omega_p| \equiv 1 \pmod{p}$. For any $x \in \Omega_2$, we have $a^2 = j_x a^{-1} j_x a$, and $c = c^a$ yields that x^a lies on the block joining x and c because the translations j_x and $a^{-1} j_x a$ fix each block through their respective center. So a fixes every block joining a point of Ω_2 with c. As a fixes no point in Ω_2, the point c is the only point fixed by a.

Let j be any involutory translation. As c is fixed by $a^2 = j a^{-1} j a$, we obtain $c^j = c^{a^{-1} j a} = c^{ja}$, and the point c^j is fixed by a. Since c is the only fixed point of a, we reach the contradiction that the involutory translation j fixes $c \notin \Omega_2$. $\square$

Remark 4.4 Let p be a prime, and let e be a positive integer. Then each non-trivial translation of the hermitian unital $\mathcal{H}(\mathbb{F}_{p^{2e}}|\mathbb{F}_{p^e})$ of order p^e has order p. If $p^e > 2$ then the group $\Gamma^{[p]}$ generated by all translations is simple, coincides with $\mathrm{PSU}_3(\mathbb{F}_{p^{2e}}|\mathbb{F}_{p^e})$, and acts two-transitively on the point set of $\mathcal{H}(\mathbb{F}_{p^{2e}}|\mathbb{F}_{p^e})$; see (Taylor 1992, 10.15, 10.12).

On the hermitian unital of order 2, the group $\Gamma^{[2]}$ behaves in an exceptional way: then the group $\Gamma^{[2]}$ is solvable, and not two-transitive, but it is still the commutator subgroup of $\mathrm{PSU}_3(\mathbb{F}_4|\mathbb{F}_2)$; see (Taylor 1992, 10.15 and the discussion on pp. 123f).

The smallest unital (of order 2) is isomorphic to the affine plane of order 3. What we call a translation of the unital is an affine homology in that plane; this explains the structure of $\Gamma^{[2]} \cong \mathrm{C}_2 \ltimes \mathrm{C}_3^2$, a generalized dihedral group. That group is transitive on the point set, but not transitive on the block set (it preserves each parallel class in the affine plane).

5 Figueroa unitals

Example 5.1 Let r be a prime power. Figueroa (1982) has constructed a projective plane $\mathrm{Fig}(r^3)$ of order r^3 with a pappian subplane $\mathbb{D}$ of order r; see (Hering and Schaeffer 1982) for the case of general r, and (Grundhöfer 1986) for a synthetic construction. The plane $\mathrm{Fig}(r^3)$ is not desarguesian unless $r = 2$. The full group of automorphisms of the subplane $\mathbb{D}$ extends to a group $G \cong \mathrm{P\Gamma L}_3(\mathbb{F}_r)$ of automorphisms of $\mathrm{Fig}(r^3)$. If $r > 2$ then the full automorphism group $\mathrm{Aut}(\mathrm{Fig}(r^3))$ is the direct product of a group of order 3 with said group G; see (Hering and Schaeffer 1982), cp. (Dempwolff 1985b). The cyclic factor is generated by a planar automorphism α (of order 3) that is used in the construction of $\mathrm{Fig}(r^3)$. Dempwolff (Dempwolff 1985a, Theorem B) has noted that every elation of the subplane $\mathbb{D}$ (as an element of the group G) is induced by an elation of $\mathrm{Fig}(r^3)$.

If r is a square, say $r = q^2$, then there is a polarity π of the Figueroa plane of order q^6, and the absolute points of π carry a unital $\mathbb{U}_{\mathrm{Fig}(q^6)}$ of order q^3, see (de Resmini and Hamilton 1998). The unital $\mathbb{U}_{\mathrm{Fig}(q^6)}$ is not hermitian; see (Hui and Wong 2012). In fact, there are O'Nan configurations, see (Tai and Wong 2014). The intersection $\mathbb{H}$ of $\mathbb{U}_{\mathrm{Fig}(q^6)}$ with the subplane $\mathbb{D}$ is isomorphic to the hermitian unital of order q.

The centralizer of the polarity π in $\mathrm{Aut}(\mathrm{Fig}(q^6))$ is the direct product of $\langle\alpha\rangle$ with the centralizer of π in $G \cong \mathrm{P\Gamma L}_3(\mathbb{F}_{q^2})$. In particular, every translation of $\mathbb{H}$ is induced by an elation of the subplane $\mathbb{D}$, and thus by an elation ψ of $\mathrm{Fig}(q^6)$ in the centralizer of π. The restriction of ψ to $\mathbb{U}_{\mathrm{Fig}(q^6)}$ is then a translation of $\mathbb{U}_{\mathrm{Fig}(q^6)}$.

We obtain:

Theorem 5.2 *In the unital $\mathbb{U}_{\mathrm{Fig}(q^6)}$ of order q^3, there is a hermitian subunital $\mathbb{H}$ of order q such that every point of $\mathbb{H}$ is the center of a group of order q consisting of translations. In particular, the point set of $\mathbb{H}$ is contained in Ω_p.*

Let G be the group generated by all elations of the Figueroa plane $\mathrm{Fig}(q^6)$ that leave $\mathbb{U}_{\mathrm{Fig}(q^6)}$ invariant. Then G is isomorphic to the commutator group of $\mathrm{PSU}_3(\mathbb{F}_{q^2}|\mathbb{F}_q)$, and acts two-transitively on the point set of $\mathbb{H}$. □

Theorem 5.3 *Let q be a power of 2. Then every involutory translation of $\mathbb{U}_{\mathrm{Fig}(q^6)}$ has its center in $\mathbb{H}$, and is induced by an elation of the Figueroa plane $\mathrm{Fig}(q^6)$ that leaves $\mathbb{U}_{\mathrm{Fig}(q^6)}$ invariant. In particular, we have $\mathbb{H} = \mathbb{U}_2 = (\Omega_2, \mathcal{B}_2)$, the subunital $\mathbb{H}$ is invariant under $\mathrm{Aut}(\mathbb{U}_{\mathrm{Fig}(q^6)})$, and every non-trivial translation is an involution.*

Proof We write $q = 2^a$ with $a \in \mathbb{N}$; then $\mathbb{U}_{\mathrm{Fig}(q^6)}$ has order 2^{3a} and $\mathbb{H}$ is a unital of order 2^a. The order of any translation of $\mathbb{U}_{\mathrm{Fig}(q^6)}$ divides q, and is thus a power of 2. If the translation is not trivial then Lemma 2.2 yields that its center lies in Ω_2.

From Theorem 5.2 we know that the point set of $\mathbb{H}$ is contained in Ω_2. Assume that Ω_2 is not contained in the subunital $\mathbb{H}$. Then Proposition 4.2 yields that $\Gamma^{[2]}$ induces a group isomorphic to $\mathrm{PSU}_3(\mathbb{F}_{2^{2e}}|\mathbb{F}_{2^e})$ on Ω_2, and $\mathbb{U}_2$ is isomorphic to the hermitian unital of order 2^e. Thus $2^a < 2^e \le 2^{3a}$, and $\mathbb{U}_2 \cong \mathcal{H}(\mathbb{F}_{2^{6a}}|\mathbb{F}_{2^{3a}})$ contains the subunital $\mathbb{H} \cong \mathcal{H}(\mathbb{F}_{2^{2a}}|\mathbb{F}_{2^a})$. According to (Grundhöfer et al. 2021a), the embedding of the unitals is given by an embedding of quadratic field extensions. This leaves only the possibility $e = 3a$, but then $\mathbb{U}_{\mathrm{Fig}(q^6)} = \mathbb{U}_2$ is hermitian, a contradiction. So

$\mathbb{U}_2 = \mathbb{H}$, every translation of $\mathbb{U}_{\mathrm{Fig}(q^6)}$ has its center in $\mathbb{H}$, and $\mathbb{H}$ is invariant under all translations of $\mathbb{U}_{\mathrm{Fig}(q^6)}$.

If τ is any non-trivial translation of $\mathbb{U}_{\mathrm{Fig}(q^6)}$ then the order of τ divides $q^3 = 2^{3a}$. So the center c of τ lies in Ω_2, and is thus a point of $\mathbb{H}$. The elations of $\mathrm{Fig}(q^6)$ with center c that leave $\mathbb{U}_{\mathrm{Fig}(q^6)}$ invariant form a group Θ that acts faithfully on $\mathbb{U}_{\mathrm{Fig}(q^6)}$ and induces the full group of translations of $\mathbb{H}$ with center c.

Let B be a block through c, and consider any point $x \in (\Omega_2 \cap B)\setminus\{c\}$. Since the group of all translations of the hermitian unital $\mathbb{H}$ is transitive on $(\Omega_2 \cap B)\setminus\{c\}$, there exists an involutory elation $\vartheta \in \Theta$ with $x^\tau = x^\vartheta$. Then $\tau\vartheta$ is a translation of $\mathbb{U}_{\mathrm{Fig}(q^6)}$ with center c fixing $x \neq c$, so $\tau\vartheta$ is trivial on $\mathbb{U}_{\mathrm{Fig}(q^6)}$, and $\tau = \vartheta$. □

6 Examples of unitals with few translations

There are unitals with no translations at all, e.g., the Ree unitals, or the presently known unitals of order 6. See (Grundhöfer et al. 2013, 1.8) and (Grundhöfer et al. 2011) for the Ree unitals; the unitals of order 6 are treated in (Krčadinac and Vlahović 2016, 5.1, p. 2888); the information about the automorphisms given there suffices to see that there are no translations. Most of the unitals of order 3 and many unitals of order 4 found by computer do not admit automorphisms of order 3 or 2, respectively, let alone translations; see (Al-Azemi et al. 2014; Krčadinac et al. 2011, Table III, p. 301).

There are unitals of prime power order p^e where Ω_p consists of a single point (and, obviously, Ω_r is empty for every prime $r \neq p$). For instance, consider a Coulter-Matthews plane of order 3^e with even e, defined by a suitable planar monomial; see (Coulter and Matthews 1997). Such a plane admits a unitary polarity; the absolute points carry a unital of order $3^{e/2}$, see (Knarr and Stroppel 2010, 5.2). That unital has a point ∞ with a translation group $\Gamma_{[\infty]}$ of order 3^e, see (Knarr and Stroppel 2010, 6.2). These groups are elementary abelian 3-groups.

If the unital is not classical (this surely happens if a certain Baer subplane is not desarguesian, see (Knarr and Stroppel 2010, 6.8)) then the point ∞ is fixed by every automorphism of the unital, and ∞ is the only center of any translation. For abstract automorphisms (i.e., automorphisms of the unital that are not necessarily induced by collineations of the ambient plane), this follows from a deep result (Grundhöfer et al. 2013); it does not suffice to note that the point ∞ is fixed by every automorphism of the plane.

In planes over finite Dickson semifields, and in planes over twisted fields, one also finds non-classical unitals (polar and otherwise) with exactly one center of translations, see (Hui et al. 2013) and (Grundhöfer et al. 2016, 5.2), respectively.

For each order $q = p^d$ with p prime, Möhler gives a construction of unitals (Möhler 2021c, 4.1) depending on the choice of a suitable family $\mathcal{D}$ of subsets of $\mathrm{SL}_2(\mathbb{F}_q)$ such that either Ω_p is a block and $\Gamma^{[p]} \cong \mathrm{SL}_2(\mathbb{F}_q)$, or the unital is hermitian; see (Möhler 2021a, 3.11). Suitable families $\mathcal{D}$ leading to non-hermitian unitals are known for $q \in \{4, 8\}$, see (Grundhöfer et al. 2016, 2.1, 3.5) and (Möhler 2021c, Sect. 3). In (Möhler 2021b, Corollary 3.8) it is proved: For q even, the set Ω_2 in Grüning's unital ((Grüning 1987), see (Grundhöfer et al. 2016, 5.5) for the description needed here) has size $q+1$, and that unital admits exactly $q+1$ non-trivial translations, each of order 2. In (Möhler 2021b, 4.6, 4.7, 4.8) one finds unitals of order 4 with no translations, unitals of order 4 with $|\Omega_2| = 1$ and $\Gamma^{[2]} \cong \mathrm{C}_2$, and unitals of order 4 with $|\Omega_2| = 1$ and $\Gamma^{[2]} \cong \mathrm{C}_2 \times \mathrm{C}_2$.

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References

Al-Azemi, A., Betten, A., Betten, D. (2014) Unital designs with blocking sets. Discrete Appl. Math. 163(part 2):102–112. https://doi.org/10.1016/j.dam.2013.02.023

Aschbacher, M.: A condition for the existence of a strongly embedded subgroup. Proc. Am. Math. Soc. **38**, 509–511 (1973). https://doi.org/10.2307/2038941

Barwick, S.G., Ebert, G.: Springer Monographs in Mathematics, Unitals in Projective Planes (2008). https://doi.org/10.1007/978-0-387-76366-8

Coulter, R.S., Matthews, R.W.: Planar functions and planes of Lenz–Barlotti class II. Des. Codes Cryptogr. **10**(2), 167–184 (1997). https://doi.org/10.1023/A:1008292303803

de Resmini, M.J., Hamilton, N.: Hyperovals and unitals in Figueroa planes. Eur. J. Combin. **19**(2), 215–220 (1998). https://doi.org/10.1006/eujc.1997.0166

Dembowski, P.: Finite geometries, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol 44. Springer-Verlag, Berlin (1968). https://doi.org/10.1007/978-3-642-62012-6

Dempwolff, U.: PSL$(3, q)$ on projective planes of order q^3. Geom. Dedicata **18**(1), 101–112 (1985a). https://doi.org/10.1007/BF00221208

Dempwolff, U.: (1985b) On the automorphism group of planes of Figueroa type. Rend. Sem. Mat. Univ. Padova 74:59–62. http://www.numdam.org/item?id=RSMUP_1985__74__59_0

Figueroa, R.: A family of not (V, l)-transitive projective planes of order q^3, $q \not\equiv 1$ $(mod\ 3)$ and $q > 2$. Math. Z. **181**(4), 471–479 (1982). https://doi.org/10.1007/BF01182385

Gorenstein, D.: Finite groups, 2nd edn. Chelsea Publishing Co., New York (1980)

Grundhöfer, T.: A synthetic construction of the Figueroa planes. J. Geom. **26**(2), 191–201 (1986). https://doi.org/10.1007/BF01227843

Grüning, K.: A class of unitals of order q which can be embedded in two different planes of order q^2. J. Geom. **29**(1), 61–77 (1987). https://doi.org/10.1007/BF01234988

Grundhöfer, T., Krinn, B., Stroppel, M.J.: Non-existence of isomorphisms between certain unitals. Des. Codes Cryptogr. **60**(2), 197–201 (2011). https://doi.org/10.1007/s10623-010-9428-2

Grundhöfer, T., Stroppel, M.J., Van Maldeghem, H.: Unitals admitting all translations. J. Combin. Des. **21**(10), 419–431 (2013). https://doi.org/10.1002/jcd.21329

Grundhöfer, T., Stroppel, M.J., Van Maldeghem, H.: A non-classical unital of order four with many translations. Discrete Math. **339**(12), 2987–2993 (2016). https://doi.org/10.1016/j.disc.2016.06.008

Grundhöfer, T., Stroppel, MJ., Van Maldeghem, H.: (2021-11-01) Finite subunitals of the hermitian unitals J. Geom., to appear (2021a). arxiv:2201.02701

Grundhöfer, T., Stroppel, MJ., Van Maldeghem, H.: Moufang sets generated by translations in unitals. J. Combin. Des. **30**(2), 91–104 (2021b). https://doi.org/10.1002/jcd.21813 (2021b). arxiv:2008.11445

Hering, C.: On subgroups with trivial normalizer intersection. J. Algebra **20**, 622–629 (1972). https://doi.org/10.1016/0021-8693(72)90075-0

Hering, C.: On projective planes of type VI. In: Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II. Accademia Nazionale dei Lincei, Rome, p 29–53, tenuto a Roma, 3–15 settembre 1973, Atti dei Convegni Lincei, No. 17 (1976)

Hering, C., Schaeffer, HJ.: On the new projective planes of R. Figueroa. In: Combinatorial theory (Schloss Rauischholzhausen, 1982), Lecture Notes in Math., vol 969. Springer, Berlin-New York, p 187–190 (1982). https://doi.org/10.1007/BFb0062982

Higman, D.G., McLaughlin, J.E.: Rank 3 subgroups of finite symplectic and unitary groups. J. Reine. Angew. Math. **218**, 174–189 (1965). https://doi.org/10.1515/crll.1965.218.174

Hughes, D.R., Piper, F.C.: Projective Planes, Graduate Texts in Mathematics, vol. 6. Springer, New York (1973)

Hui, A.M.W., Wong, P.P.W.: Non-classical polar unitals in finite Figueroa planes. J. Geom. **103**(2), 263–273 (2012). https://doi.org/10.1007/s00022-012-0121-7

Hui, A.M.W., Law, H.F., Tai, Y.K., et al.: Non-classical polar unitals in finite Dickson semifield planes. J. Geom. **104**(3), 469–493 (2013). https://doi.org/10.1007/s00022-013-0174-2

Kantor, W.M.: Homogeneous designs and geometric lattices. J. Combin. Theory Ser. A **38**(1), 66–74 (1985). https://doi.org/10.1016/0097-3165(85)90022-6

Knarr, N., Stroppel, M.J.: Polarities and unitals in the Coulter–Matthews planes. Des. Codes Cryptogr. **55**(1), 9–18 (2010). https://doi.org/10.1007/s10623-009-9326-7

Krčadinac, V., Vlahović, R.: New quasi-symmetric designs by the Kramer–Mesner method. Discrete Math. **339**(12), 2884–2890 (2016). https://doi.org/10.1016/j.disc.2016.05.030

Krčadinac, V., Nakić, A., Pavčević, M.O.: The Kramer–Mesner method with tactical decompositions: some new unitals on 65 points. J. Combin. Des. **19**(4), 290–303 (2011). https://doi.org/10.1002/jcd.20277

Lüneburg, H.: Translation Planes. Springer, Berlin (1980)

Möhler, V. Automorphisms of (affine) SL(2,q)-unitals. Innov. Incid. Geom. to appear (2021a) . arxiv:2012.10116

Möhler, V.: Parallelisms and translations of (affine) SL(2, q)-unitals. J. Geom. **112**, 44 (2021b). https://doi.org/10.1007/s00022-021-00611-5

Möhler, V.: Three affine SL(2,8)-unitals. Beitr. Algebra Geom. to appear (2021c). arxiv:2012.10134

O'Nan, M.E.: Automorphisms of unitary block designs. J. Algebra **20**, 495–511 (1972). https://doi.org/10.1016/0021-8693(72)90070-1

Tai, Y.K., Wong, P.P.W.: On the structure of the Figueroa unital and the existence of O'Nan configurations. Discrete Math. **330**, 41–50 (2014). https://doi.org/10.1016/j.disc.2014.04.012

Taylor, D.E.: The geometry of the classical groups, Sigma Series in Pure Mathematics, vol. 9. Heldermann, Berlin (1992)