On the existence of four or more curved foldings with common creases and crease patterns

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Abstract

Consider an oriented curve Γ in a domain D in the plane $\mathbf{R}^2$. Thinking of D as a piece of paper, one can make a curved folding in the Euclidean space $\mathbf{R}^3$. This can be expressed as the image of an “origami map” $\Phi : D \rightarrow \mathbf{R}^3$ such that Γ is the singular set of Φ , the word “origami” coming from the Japanese term for paper folding. We call the singular set image $C := \Phi(\Gamma)$ the *crease* of Φ and the singular set Γ the *crease pattern* of Φ . We are interested in the number of origami maps whose creases and crease patterns are C and Γ , respectively. Two such possibilities have been known. In the authors’ previous work, two other new possibilities and an explicit example with four such non-congruent distinct curved foldings were established. In this paper, we determine the possible values for the number N of congruence classes of curved foldings with the same crease and crease pattern. As a consequence, if C is a non-closed simple arc, then $N = 4$ if and only if both Γ and C do not admit any symmetries. On the other hand, when C is a closed curve, there are infinitely many distinct possibilities for curved foldings with the same crease and crease pattern, in general.

Keywords Curved folding · Origami · Flat surfaces · Developable surfaces

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Fig. 1 A given crease pattern Γ (left) and its realization (right) as a curved folding along the crease C

1 Introduction

The geometry of curved foldings is, nowadays, an important subject not only from the viewpoint of mathematics but also from the viewpoint of engineering. Works on this topic have been published by numerous authors; for example, (Demaine et al. 2015, 2018; Duncan and Duncan 1982; Fuchs and Tabachnikov 1999; Huffman 1976; Kilian et al. 2008) and (Fuchs and Tabachnikov 2007, Lecture 15) are fundamental references.

Drawing a curve Γ in $\mathbf{R}^2$, we think of a tubular neighborhood of Γ as a piece of paper. Then we can fold this along Γ , and obtain a curved folding in the Euclidean space $\mathbf{R}^3$, which is a developable surface (as a subset of $\mathbf{R}^3$) whose singular set image is a space curve $C (\subset \mathbf{R}^3)$. In this paper, we fix an orientation of C and denote by $-C$ the image of the same curve with the opposite orientation. We denote by $|C|$ without considering its orientation.

In this paper, we focus on curved foldings which are produced from a single curve Γ in $\mathbf{R}^2$ satisfying the following properties:

- (i) The length of Γ is equal to that of C (Fuchs and Tabachnikov 1999, Page 29 (5)). Moreover, the two curves have a bijective correspondence by an arc-length parametrization.
- (ii) The curvature functions of C and Γ have no zeros (Fuchs and Tabachnikov 1999, Page 29 (3)).
- (iii) The curves C and Γ have no self-intersections.
- (iv) The absolute value of the curvature function of Γ is less than the curvature function of C at each point of Γ (Fuchs and Tabachnikov 1999, Page 28 (1)).

If a curved folding satisfies (i)–(iii) and the following condition (stronger than (iv)), then it is said to be *admissible* (cf. (2.4) and also (2.10)).

- (iv') The maximum of the absolute value of the curvature function of Γ is less than the minimum of the curvature function of C (Fuchs and Tabachnikov 1999, Page 28 (1)).

For a pair $(\Gamma, |C|)$ giving a curved folding P , there is another possibility for corresponding curved foldings (see Fuchs and Tabachnikov 1999). Moreover, the authors' previous work (Honda et al. 2020c), two additional possibilities were found when P is admissible, and an explicit example of four non-congruent curved foldings with the

same crease and crease pattern was given. Here, two subsets A, B in $\mathbf{R}^3$ are said to be *congruent* if there exists an isometry T in $\mathbf{R}^3$ such that $B = T(A)$. The purpose of this paper is to further develop the discussions in Honda et al. (2020c). In fact, we are interested in the number of congruence classes of curved foldings with a given pair of crease $|C|$ and crease pattern Γ . Since this number is closely related to the symmetries of $|C|(\subset \mathbf{R}^3)$ and $\Gamma(\subset \mathbf{R}^2)$, we give the following:

Definition 1.1 A subset A of the Euclidean space $\mathbf{R}^k$ ($k = 2, 3$) is said to have a *symmetry* if $T(A) = A$ holds for an isometry T of $\mathbf{R}^k$ which is not the identity map, and T is called a *symmetry* of A . Moreover, if there is a point $\mathbf{x} \in A$ such that $T(\mathbf{x}) \neq \mathbf{x}$, then A is said to have a *non-trivial symmetry* T . On the other hand, a symmetry T of A is called *positive* (resp. *negative*) if T is an orientation preserving (resp. reversing) isometry of $\mathbf{R}^k$.

We denote by $\mathcal{P}(\Gamma, |C|)$ (resp. $\mathcal{P}_*(\Gamma, |C|)$) the set of curved foldings (resp. the set of admissible curved foldings) whose creases and crease patterns are C and Γ satisfying (i)–(iv) (resp. (i)–(iii) and (iv')), see (3.5) for the precise definition. We prove the following, which is a refinement of (Honda et al. 2020c, Theorems A and B):

Theorem A *Let C be the image of an embedded curve which is defined on a bounded closed interval. Then the number $n_{C,\Gamma}$ of the elements in $\mathcal{P}_*(\Gamma, |C|)$ as subsets in $\mathbf{R}^3$ is four if Γ has no symmetries. Otherwise, $n_{C,\Gamma}$ is equal to two.*

Theorem B *Let C be the image of an embedded curve which is defined on a bounded closed interval. Then the number $N_{C,\Gamma}$ of the congruence classes of curved foldings in $\mathcal{P}_*(\Gamma, |C|)$ is less than or equal to $n_{C,\Gamma}$ and satisfies the following:*

- (1) *if C has no symmetries and Γ also has no symmetries, then $N_{C,\Gamma} = 4$,*
- (2) *if not the case in (1), then $N_{C,\Gamma} \leq 2$ holds, and*
- (3) *$N_{C,\Gamma} = 1$ if and only if*
 - (a) *C lies in a plane and has a non-trivial symmetry,*
 - (b) *C lies in a plane and Γ has a symmetry, or*
 - (c) *C does not lie in any plane and has a positive symmetry, and Γ also has a symmetry.*

In Honda et al. (2020c), an example of $\mathcal{P}_*(\Gamma, |C|)$ consisting of four non-congruent subsets in $\mathbf{R}^3$ was given by computing the mean curvature functions along C . However, this approach seems insufficient to prove Theorem B (see Proposition 5.12 in Sect. 5). So, in this paper, we will prepare several new techniques for its proof (see Sects. 3, 4 and 5).

We next consider the case that

- C is a knot (i.e. a simple closed curve) of length $l(> 0)$ in $\mathbf{R}^3$ giving a crease of a curved folding $P \in \mathcal{P}_*(\Gamma, |C|)$, and
- Γ is a curve of length l embedded in $\mathbf{R}^2$ as a crease pattern.

Even when C is closed, Γ may not be a closed curve in general. In fact, if we consider the curved folding along the unit circle whose first angular function (see Sect. 2 for

Fig. 2 A crease pattern (left) corresponding to a curved folding along a circle (right)

the definition) is $\pi/4$ as in Fig. 2, then its crease pattern is the sector of the circle of radius $\sqrt{2}$ whose length is 2π .

We return to the general setting. Let $\gamma(s)$ be an arc-length parametrization of Γ . If C is a closed curve of length l , then the curvature function $\mu(s)$ of $\gamma(s)$ can be extended as an l -periodic function $\tilde{\mu}(s)$ ($s \in \mathbf{R}$). A curved folding $P \in \mathcal{P}_*(C, \Gamma)$ consists of a union of two developable strips along C whose geodesic curvature functions coincide with $\mu(s)$.

Theorem C Let $\mathbf{c} : \mathbf{R} \rightarrow \mathbf{R}^3$ and $\tilde{\gamma} : \mathbf{R} \rightarrow \mathbf{R}^2$ be regular curves parametrized by arc-length such that

- (1) $\mathbf{c}(s) = \mathbf{c}(s + l)$,
- (2) the curvature function $\tilde{\mu} : \mathbf{R} \rightarrow \mathbf{R}$ of $\tilde{\gamma}$ induces a function $\mu : \mathbf{R}/l\mathbf{Z} \rightarrow \mathbf{R}$ defined on the one dimensional torus $\mathbf{R}/l\mathbf{Z}$ satisfying

$$0 < \max_{w \in \mathbf{R}/l\mathbf{Z}} \mu(w) < \kappa(s) \quad (s \in [0, l)),$$

where $\kappa(s)$ is the curvature function of $\mathbf{c}(s)$.

We set $C := \mathbf{c}([0, l])$ and $\Gamma := \tilde{\gamma}([0, l])$. Then there exist four continuous families $\{P_{\mathbf{x}}^i\}_{\mathbf{x} \in C}$ ($i = 1, 2, 3, 4$) of curved foldings in $\mathcal{P}_*(\Gamma, |C|)$ satisfying the following properties:

- (a) The set $\mathcal{P}_*(\Gamma, |C|)$ coincides with the set $\{P_{\mathbf{x}}^i; i \in \{1, 2, 3, 4\}, \mathbf{x} \in C\}$.
- (b) Suppose that C is not a circle and Γ is not a subset of a circle. Then, for each $P_{\mathbf{x}}^i$ ($i \in \{1, 2, 3, 4\}, \mathbf{x} \in C$), its congruence class

$$\Lambda_{\mathbf{x}}^i := \{Q \in \mathcal{P}_*(\Gamma, |C|); Q \text{ is congruent to } P_{\mathbf{x}}^i\}$$

is finite. In particular, quotient of $\mathcal{P}_*(\Gamma, |C|)$ by congruence relation $\mathcal{P}_*(\Gamma, |C|)$ contains uncountably many curved foldings which are not congruent to each other.

- (c) Suppose that C and μ have no symmetries (cf. Definition 3.3). Then for each $i \in \{1, 2, 3, 4\}$ and $\mathbf{x} \in C$, the set $\Lambda_{\mathbf{x}}^i$ consists of a single element, that is, any two curved foldings in $\mathcal{P}_*(\Gamma, |C|)$ are mutually non-congruent.

In Theorem C, we do not need to assume that the crease pattern Γ is a closed curve in $\mathbf{R}^2$. However, the most interesting case is that C and Γ are both closed: At the end of Sect. 6, we give a concrete example of a $\mathcal{P}_*(\Gamma, |C|)$ containing uncountably many congruence classes.

Theorems A, B and C can be considered as the analogues for cuspidal edges along a space curve given in (Honda et al. 2020a, Theorems III and IV) and (Honda et al. 2020b, Theorem 1.8), respectively. However, even if Γ and C admit real analytic parametrizations, Theorems A, B and C do not directly follow from the corresponding assertions for cuspidal edges.

2 Preliminaries

We let C be the image of an embedded space curve whose length is l . When C is a non-closed curve, it has a parametrization $\mathbf{c} : \mathbb{I}_l \rightarrow \mathbf{R}^3$ with arc-length parameter, where $\mathbb{I}_l := [-l/2, l/2]$. On the other hand, if C is closed (i.e. a knot), then it can be parametrized by a curve $\mathbf{c} : \mathbb{T}_l^1 \rightarrow \mathbf{R}^3$ with arc-length parameter, where $\mathbb{T}_l^1 := \mathbf{R}/l\mathbf{Z}$. Since we treat the bounded closed interval $\mathbb{I}_l$ and the one dimensional torus $\mathbb{T}_l^1$ at the same time, we set

$$J := \mathbb{I}_l \text{ or } \mathbb{T}_l^1.$$

As explained in the introduction, C has the orientation induced by the parametrization $\mathbf{c}$. We let $-C$ be the curve C whose orientation is reversed, and $|C|$ denotes the curve C ignoring its orientation. We consider special strips along C , which are “developable strips” along C . Let J_0 be a set which is homeomorphic to J .

Definition 2.1 A *developable strip* along C is a C^∞ -embedding $f : U \rightarrow \mathbf{R}^3$ defined on a tubular neighborhood U of $J_0 \times \{0\}$ in $J_0 \times \mathbf{R}$ such that

- $J_0 \ni u \mapsto f(u, 0) \in \mathbf{R}^3$ parametrizes C ,
- there exists a unit vector field $\xi_f(u)$ of f along C (called a *ruling vector field*) such that f can be expressed as

$$f(u, v) = f(u, 0) + v\xi_f(u) \quad ((u, v) \in U), \text{ and}$$

- the Gaussian curvature of f vanishes on U identically.

The developable strip f represents a map germ along C . We identify this induced map germ with f itself if it creates no confusion. Hereafter, we assume that the curvature function of C never vanishes, and we denote by $\mathbf{e}(u)$, $\mathbf{n}(u)$ and $\mathbf{b}(u)$ the unit tangent, unit principal normal and unit bi-normal vector fields associated with the parametrization

$$\mathbf{c}_f : J_0 \ni u \mapsto f(u, 0) \in \mathbf{R}^3 \tag{2.1}$$

of C , respectively. With these notations, we can express ξ_f as

$$\xi_f(u) = \cos \beta_f(u) \mathbf{e}(u) + \sin \beta_f(u) \left(\cos \alpha_f(u) \mathbf{n}(u) + \sin \alpha_f(u) \mathbf{b}(u) \right). \quad (2.2)$$

This $\alpha_f : J_0 \rightarrow \mathbf{R}$ is called the *first angular function*, and $\beta_f : J_0 \rightarrow \mathbf{R}$ is called the *second angular function* of f .

In this paper, we consider the developable strips satisfying

$$0 < |\cos \alpha_f(u)| < 1 \quad (u \in J_0). \quad (2.3)$$

We denote by $\mathcal{D}(C)$ the set of developable strip germs along C satisfying (2.3). Moreover, $f \in \mathcal{D}(C)$ is said to be *admissible* if it satisfies the following stronger condition

$$0 < |\kappa_f(u) \cos \alpha_f(u)| < \min_{w \in J_0} \kappa_f(w) \quad (u \in J_0), \quad (2.4)$$

which corresponds to the condition (iv') in the introduction, where κ_f is the curvature function of $\mathbf{c}_f$ (cf. (2.1)). Let $\mathcal{D}_*(C)$ be the set of admissible developable strip germs along C . By definition,

$$\mathcal{D}_*(C) \subset \mathcal{D}(C)$$

holds. We set

$$\mathcal{D}(|C|) := \mathcal{D}(C) \cup \mathcal{D}(-C), \quad \mathcal{D}_*(|C|) := \mathcal{D}_*(C) \cup \mathcal{D}_*(-C).$$

By (2.3), we can choose the first angular function α_f so that

$$0 < |\alpha_f(u)| < \frac{\pi}{2} \quad (u \in J_0). \quad (2.5)$$

The Gaussian curvature of f vanishes identically if and only if

$$\det(f_u(u, 0), \xi_f(u), \xi'_f(u)) = 0 \quad \left(\xi'_f(u) := \frac{d\xi_f(u)}{du} \right),$$

which is equivalent to the formula

$$\cot \beta_f(u) = \frac{\alpha'_f(u) + |\mathbf{c}'_f(u)| \tau_f(u)}{|\mathbf{c}'_f(u)| \kappa_f(u) \sin \alpha_f(u)}, \quad (2.6)$$

where $\tau_f(u)$ is the torsion function of $\mathbf{c}_f(u)$. In particular, we may assume that

$$0 < \beta_f(u) < \pi \quad (u \in J_0). \quad (2.7)$$

Throughout this paper, we assume (2.5) and (2.7) for $f \in \mathcal{D}(C)$. In particular, $\xi_f(u)$ satisfies

$$\xi_f(u) \cdot \mathbf{n}(u) > 0 \quad (u \in J_0). \quad (2.8)$$

Such a ξ_f is called the *normalized ruling vector field* of f . Then

$$\mathbf{N}_f(u) := \cos \alpha_f(u) \mathbf{n}(u) + \sin \alpha_f(u) \mathbf{b}(u)$$

gives the unit co-normal vector field of f along C satisfying $\mathbf{N}_f \cdot \xi_f > 0$. We set

$$\mu_f(u) := \frac{\mathbf{c}_f''(u) \cdot \mathbf{N}_f(u)}{|\mathbf{c}_f'(u)|^2} = \kappa_f(u) \cos \alpha_f(u), \quad (2.9)$$

which is a positive-valued function giving the geodesic curvature of C as a curve on the surface f . We call μ_f the *geodesic curvature function* of f along C . Since $\mu_f(u) > 0$, (2.5) and (2.4) reduce to the conditions

$$(0 <) \mu_f(u) < \kappa_f(u) \quad (u \in J_0), \quad (2.10)$$

and

$$(0 <) \mu_f(u) < \min_{w \in J_0} \kappa_f(w) \quad (u \in J_0), \quad (2.11)$$

respectively. Let l be the total arc-length of C . Then the parameter u of $\mathbf{c}_f$ can be expressed as $u = u(s)$ ($s \in J$) so that $\mathbf{c}(s) := \mathbf{c}_f(u(s))$ ($s \in J$) gives the arc-length parametrization of C . In this situation, the function defined by

$$\hat{\mu}_f(s) := \mu_f(t(s)) \quad (s \in J)$$

is called the *normalized geodesic curvature function* of f . We now fix a point

$$\mathbf{x}_0 \in C, \quad (2.12)$$

which is the midpoint of C if $J = \mathbb{I}_l$ and is an arbitrarily chosen point if $J = \mathbb{T}_l^1$.

Definition 2.2 Let $f \in \mathcal{D}(C)$. We call $f(s, v)$ a *normal form* of a developable strip if it is defined on a tubular neighborhood of $J \times \{0\}$ in $\mathbf{R}^2$ and

$$J \ni s \mapsto f(s, 0) \in \mathbf{R}^3$$

gives an arc-length parametrization of C . Since we will use s to denote the arc-length parameter of C , we use the parametrization $f(s, v)$ when f is a normal form. (We shall denote such developable strips using capital letters to emphasize that they are written in normal forms.)

Suppose that $F(s, v)$ is a normal form. As seen in Honda et al. (2020c), the restriction $H(s)$ of the mean curvature function of $F(s, v)$ to the curve $\mathbf{c}(s)$ ($s \in J$) satisfies

$$|H(s)| = \frac{\kappa(s)^2 \sin^2 \alpha_F(s) + (\alpha'_F(s) + \tau(s))^2}{2\kappa(s) |\sin \alpha_F(s)|}, \quad (2.13)$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion function of $\mathbf{c}(s)$. By (2.5), the right-hand side of (2.13) never vanishes. In particular, F has no umbilics and the ruling direction $\xi_F(s) := F_v(s, 0)$ along C points in the (uniquely determined) asymptotic direction of F . Hence, the germ of the normal form of F is determined by the base point $\mathbf{x}_0 \in C$ and the first angular function α_F .

From now on, we again assume that u is a general parameter, that is, it may not be an arc-length parameter of C , in general. Let J_0 be a set which is homeomorphic to J . We let

$$C_{\pi/2}^\infty(J_0) \quad (2.14)$$

be the set of C^∞ -functions defined on J_0 whose images lie in the set $(-\pi/2, \pi/2) \setminus \{0\}$. Then the first angular function $\alpha_f(u)$ of a developable strip¹ $f(u, v)$ belongs to this class $C_{\pi/2}^\infty(J_0)$.

Remark 2.3 When $J_0 = [b, c]$ ($b < c$), we set $\mathbf{c}^\sharp(u) := \mathbf{c}_f(b + c - u)$, which has the same image as $\mathbf{c}_f(u)$ (cf. (2.1)) but has the opposite orientation. Then

$$\mathbf{e}^\sharp(u) := -\mathbf{e}(b + c - u), \quad \mathbf{n}^\sharp(u) := \mathbf{n}(b + c - u), \quad \mathbf{b}^\sharp(u) := -\mathbf{b}(b + c - u) \quad (2.15)$$

are the unit velocity vector, the unit principal normal vector and the unit bi-normal vector of the curve $\mathbf{c}^\sharp(u)$, respectively. If we denote by $\tau_f(u)$ the torsion function of $\mathbf{c}_f(u)$, then

$$\kappa^\sharp(u) := \kappa_f(b + c - u), \quad \tau^\sharp(u) := \tau_f(b + c - u) \quad (2.16)$$

coincide with the curvature and torsion functions of $\mathbf{c}^\sharp(u)$, respectively. For each $f \in \mathcal{D}(C)$, we set

$$f^\sharp(u, v) := f(b + c - u, v), \quad (2.17)$$

and call this the *reverse* of f . By definition, $f^\sharp$ has the same image as f , and the involution $\mathcal{D}(|C|) \ni f \mapsto f^\sharp \in \mathcal{D}(|C|)$ is canonically induced. By (2.15), $\xi_f^\sharp(u) := \xi_f(b + c - u)$ gives the normalized ruling vector field of $f^\sharp$ if this is so of $\xi_f(u)$ for f . Then the first and second angular functions $\alpha^\sharp$, $\beta^\sharp$ of $f^\sharp$ satisfy

¹ Since we have used the capital letters F , G to express the normal forms of developable strips, we use lower-case letters f , g to express general developable strips.

$$\alpha^\sharp(u) = -\alpha_f(b + c - u), \quad \beta^\sharp(u) = \pi - \beta_f(b + c - u), \quad (2.18)$$

respectively.

Definition 2.4 Let J_i ($i = 1, 2$) be two sets which are homeomorphic to J , and let $f_i : J_i \times (-\varepsilon_i, \varepsilon_i) \rightarrow \mathbf{R}^3$ ($i = 1, 2$) be two developable strips along C , where $\varepsilon_i > 0$. Then f_2 is said to be *image equivalent* (resp. *right equivalent*) to f_1 if there exists a positive number $\delta (< \min(\varepsilon_1, \varepsilon_2))$ such that $f_1(J_1 \times (-\delta, \delta))$ coincides with $f_2(J_2 \times (-\delta, \delta))$ (resp. f_2 coincides with $f_1 \circ \varphi$ for a diffeomorphism $\varphi : J_1 \times (-\delta, \delta) \rightarrow J_2 \times (-\delta, \delta)$).

Recall that $s \mapsto \mathbf{c}(s)$ ($s \in J$) gives an arc-length parametrization of C such that $\mathbf{c}(0) = \mathbf{x}_0$. The following assertion holds.

Proposition 2.5 Let $F, G \in \mathcal{D}(|C|)$ be normal forms (cf. Definition 2.2) satisfying $F(0, 0) = G(0, 0) = \mathbf{x}_0$. Then the following two assertions are equivalent:

- (1) $F = G$ or $F = G^\sharp$,
- (2) F is image equivalent to G .

In particular, for each $\alpha \in C_{\pi/2}^\infty(J)$, there exists a unique normal form

$$F^\alpha \in \mathcal{D}(C) \quad (2.19)$$

satisfying $F^\alpha(0, 0) = \mathbf{x}_0$ (cf. (2.12)) whose first angular function is α .

Proof Replacing G by $G^\sharp$, we may assume that $F, G \in \mathcal{D}(C)$. (In fact, $G^\sharp$ is a normal form if so is G .) It is obvious that (1) implies (2). So it is sufficient to show the converse. Since each normal form of a developable strip is determined by its base point, the arc-length parametrization of C and the first angular function, (2) implies (1). $\square$

We also prove the following assertion:

Proposition 2.6 For each $f \in \mathcal{D}(C)$, there exists a unique normal form $F \in \mathcal{D}(C)$ such that

- (1) $F(0, 0) = \mathbf{x}_0$ (if $J = \mathbb{I}$, this holds automatically), and
- (2) F is right equivalent (cf. Definition 2.4) to f ,

where $\mathbf{x}_0 \in C$ is the base point given in (2.12). Moreover, $\hat{\mu}_f = \hat{\mu}_F = \mu_F$ hold.

We call this F the *normal form associated with f of base point $\mathbf{x}_0$* .

Proof Applying (Umebara and Yamada 2017, Lemma B.5.3), we can take a curvature line coordinate system (s, v) of f such that $s \mapsto f(s, 0)$ parametrizes C and $df(\partial/\partial v)$ points in the ruling direction. In this situation, we may assume that $f(0, 0) = \mathbf{x}_0$ and s is the arc-length parameter of C . We then adjust v so that $|f_v(s, 0)| = 1$ for $s \in J$. Since the image of each v -curve gives a straight line, this parametrization $f(s, v)$ gives the normal form associated to f . $\square$

We next prepare the following two lemmas, which will be applied in the later discussions.

Lemma 2.7 *Let $F_i \in \mathcal{D}(C)$ ($i = 1, 2$) be normal forms satisfying $F(0, 0) = G(0, 0) = \mathbf{x}_0$, and let $\alpha_i \in C_{\pi/2}^\infty(J)$ be its first angular functions. If $\alpha_2 - \alpha_1$ does not have any zeros on J , then the ruling direction of F_1 (cf. (2.19)) is linearly independent of that of F_2 at each point of C .*

Proof We let $\xi_i(s)$ ($i = 1, 2$) be the normalized unit ruling vector fields associated with $F^{\alpha_i}(s, v)$. Since $\alpha_2(s) \neq \alpha_1(s)$, two vectors

$$\cos \alpha_1(s) \mathbf{n}(s) + \sin \alpha_1(s) \mathbf{b}(s), \quad \cos \alpha_2(s) \mathbf{n}(s) + \sin \alpha_2(s) \mathbf{b}(s)$$

in $\mathbf{R}^3$ are linearly independent for each $s \in J$. So we obtain the assertion. $\square$

Proposition 2.8 *Let $f \in \mathcal{D}(C)$. If T is a symmetry of C (cf. Definition 1.1), then $T \circ f$ belongs to $\mathcal{D}(|C|)$. Moreover, if f is a normal form, then so is $T \circ f$, and $dT(\xi_f)$ gives the normalized ruling vector field of $T \circ f$. Furthermore, the normalized geodesic curvature function of $T \circ f$ coincides with that of f .*

Proof We denote by F the normal form associated with f of base point $\mathbf{x}_0$ (cf. Proposition 2.6). It is sufficient to show the assertion holds for F . Since the property that the geodesic curvature has no zeros is preserved by isometries of $\mathbf{R}^3$, the first assertion is obtained. Since $s \mapsto T \circ F(s, v)$ gives an arc-length parametrization of $|C|$, $T \circ F(s, v)$ is a normal form. Since the principal normal vector field is common in C and $-C$ (cf. (A.1)), it can be easily seen that $dT(\mathbf{n}(s))$ gives the principal normal vector field of $T \circ \mathbf{c}(s)$, and (2.8) yields that

$$dT(\xi_F(s)) \cdot dT(\mathbf{n}(s)) = \xi_F(s) \cdot \mathbf{n}(s) > 0 \quad (2.20)$$

along C , which implies the second assertion. Since geodesic curvatures on surfaces are geometric invariants up to $\pm$ -ambiguities, the geodesic curvature of $T \circ F$ coincides with $\sigma \mu_F$, where $\sigma \in \{1, -1\}$. Moreover, (2.20) implies that $\sigma = 1$, proving the last assertion. $\square$

We set

$$\Omega_\varepsilon^+ := J \times (0, \varepsilon), \quad \Omega_\varepsilon^- = J \times (-\varepsilon, 0), \quad \Omega_\varepsilon := J \times (-\varepsilon, \varepsilon). \quad (2.21)$$

Proposition 2.9 *For $f, g \in \mathcal{D}(|C|)$, the following five conditions are equivalent:*

- (1) f is right equivalent to g as map germs,
- (2) $F = G$ or $F = G^\sharp$, where F and G are the normal forms associated with f and g satisfying $F(0, 0) = G(0, 0) = \mathbf{x}_0$, respectively,
- (3) $F(\Omega_\varepsilon^+) = G(\Omega_\varepsilon^+)$ for sufficiently small $\varepsilon(> 0)$,
- (4) $F(\Omega_\varepsilon) = G(\Omega_\varepsilon)$ for sufficiently small $\varepsilon(> 0)$,
- (5) f is image equivalent (cf. Definition 2.4) to g as map germs.

Proof By Proposition 2.6, (1) implies (2), because $F(0, 0) = G(0, 0) = \mathbf{x}_0$. Obviously (2) implies (3). Moreover, (3) implies (4), because $F(s, v)$ and $G(s, v)$ are ruled strips and are real analytic with respect to the parameter v . On the other hand, it is obvious that (5) is equivalent to (4). If (4) holds, then Proposition 2.5 yields that $F = G$ or $F = G^\sharp$. In particular, (1) is obtained. $\square$

We next prove the following assertion, which is a refinement of (Honda et al. 2020c, Lemma 1.2).

Proposition 2.10 *Let $F_i \in \mathcal{D}(C)$ ($i = 1, 2$) be developable strips written in normal forms satisfying $F(0, 0) = G(0, 0) = \mathbf{x}_0$. If their normalized ruling vector fields are linearly independent at each point of C , then $F_1(\Omega_\varepsilon) \cap F_2(\Omega_\varepsilon)$ coincides with C for sufficiently small $\varepsilon(> 0)$.*

Proof Without loss of generality, we may assume that F_1 and F_2 are defined on a tubular neighborhood of $J \times \{0\}$ in $J \times \mathbf{R}$ and $F_1(s, 0) = F_2(s, 0)$ holds for $s \in J$. Suppose that the assertion fails. Then there exist

- two sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ on J , and
- two sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ on $(-1/n, 1/n)$

such that

$$F_1(s_n, u_n) = F_2(t_n, v_n), \quad (s_n, u_n) \neq (t_n, v_n). \quad (2.22)$$

Here, $s_n \neq t_n$ holds. (In fact, if not, then $F_1(s_n, u_n) = F_2(t_n, v_n)$ implies $u_n \xi_1(s_n) = v_n \xi_2(s_n)$, where ξ_i ($i = 1, 2$) is the normalized ruling vector field of F_i . However, since $\{\xi_1(s_n), \xi_2(s_n)\}$ is linearly independent, the fact $u_n \xi_1(s_n) = v_n \xi_2(s_n)$ implies $u_n = v_n = 0$, which contradicts the fact $(s_n, u_n) \neq (t_n, v_n)$.)

Since J is compact, we may assume that the limits $\lim_{n \rightarrow \infty} s_n = s_\infty$ and $\lim_{n \rightarrow \infty} t_n = t_\infty$ exist and $s_\infty, t_\infty \in J$. Then by (2.22), we have $F_1(s_\infty, 0) = F_2(t_\infty, 0)$. Since C has no self-intersections, we have $s_\infty = t_\infty$. By (2.22), we can write

$$\frac{\mathbf{c}(s_n) - \mathbf{c}(t_n)}{s_n - t_n} = p_n \xi_1(s_n) - q_n \xi_2(t_n) \quad \left(p_n := \frac{-u_n}{s_n - t_n}, \quad q_n := \frac{-v_n}{s_n - t_n} \right). \quad (2.23)$$

If $n \rightarrow \infty$, then the left-hand side of (2.23) converges to the vector $\mathbf{c}'(s_\infty) (= \mathbf{e}(s_\infty))$. Thus, we can conclude that the limits $\lim_{n \rightarrow \infty} p_n = p_\infty$ and $\lim_{n \rightarrow \infty} q_n = q_\infty$ exist such that

$$\mathbf{e}(s_\infty) = p_\infty \xi_1(s_\infty) - q_\infty \xi_2(s_\infty). \quad (2.24)$$

In particular, we have

$$(p_\infty, q_\infty) \neq (0, 0). \quad (2.25)$$

We set

$$\mathbf{e}_\infty := \mathbf{e}(s_\infty), \quad \mathbf{n}_\infty := \mathbf{n}(s_\infty), \quad \mathbf{b}_\infty := \mathbf{b}(s_\infty)$$

and

$$A_1 := \alpha_1(s_\infty), \quad A_2 := \alpha_2(s_\infty), \quad B_1 := \beta_1(s_\infty), \quad B_2 := \beta_2(s_\infty),$$

where α_i and β_i ($i = 1, 2$) are the first and second angular functions of F_i , respectively. Multiplying $\mathbf{n}_\infty$ and $\mathbf{b}_\infty$ to (2.24) via inner products, we have

$$p_\infty \cos A_1 - q_\infty \cos A_2 = 0, \quad p_\infty \sin A_1 - q_\infty \sin A_2 = 0.$$

By (2.25), we have

$$0 = \cos A_1 \sin A_2 - \cos A_2 \sin A_1 = \sin(A_1 - A_2). \quad (2.26)$$

On the other hand, since $F_1, F_2 \in \mathcal{D}(C)$ (cf. (2.3)) and $\xi_1(s_\infty)$ is linearly independent of $\xi_2(s_\infty)$, we have

$$0 < |A_1 - A_2| < |A_1| + |A_2| < \pi,$$

which contradicts (2.26). We remark that a similar argument for developable surfaces is used in (Murata and Umehara 2009, Section 5). $\square$

Using the same technique, we can prove the following assertion:

Proposition 2.11 *Let $F, G \in \mathcal{D}(|C|)$ be developable strips written in normal forms. Then $F(\Omega_\varepsilon^+)$ does not meet $G(\Omega_\varepsilon^-)$ for sufficiently small $\varepsilon(> 0)$.*

Proof Replacing G by $G^\sharp$, we may assume that $F, G \in \mathcal{D}(C)$. Moreover, replacing $G(s, v)$ by $G(s + b, v)$ for a suitable $b \in [0, l)$ if necessary, we may assume that F, G are defined on a tubular neighborhood of $J \times \{0\}$ in $J \times \mathbf{R}$ and $F(s, 0) = G(s, 0)$ holds for $s \in J$. We denote by ξ_F and ξ_G the normalized vector fields of F and G , respectively. Then we may assume that F and G are defined on a tubular neighborhood of $J \times \{0\}$ in $J \times \mathbf{R}$. Suppose that the assertion fails. Then there exist

- two sequences $\{s_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ on J , and
- two sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ on $(0, 1/n)$ and $(-1/n, 0)$, respectively,

such that $F(s_n, u_n) = G(t_n, v_n)$ and $(s_n, u_n) \neq (t_n, v_n)$. In this situation, we can show that $s_n \neq t_n$. (In fact, if $s_n = t_n$, then $F(s_n, u_n) = G(t_n, v_n)$ implies $u_n \xi_F(s_n) = v_n \xi_G(s_n)$ and so we have

$$u_n \xi_F(s_n) \cdot \mathbf{n}(s_n) = v_n \xi_G(s_n) \cdot \mathbf{n}(s_n).$$

Here, $\xi_F(s_n) \cdot \mathbf{n}(s_n)$ and $\xi_G(s_n) \cdot \mathbf{n}(s_n)$ are positive (cf. (2.8)). So this contradicts the facts $u_n \in (0, 1/n)$ and $v_n \in (-1/n, 0)$.)

Since J is compact, we may assume that the limits $\lim_{n \rightarrow \infty} s_n = s_\infty$ and $\lim_{n \rightarrow \infty} t_n = t_\infty$ exist and $s_\infty, t_\infty \in J$. Since C has no self-intersections, we have $s_\infty = t_\infty$. By (2.22), we can write

$$\frac{\mathbf{c}(s_n) - \mathbf{c}(t_n)}{s_n - t_n} = p_n \xi_F(s_n) - q_n \xi_G(t_n) \quad \left(p_n := \frac{-u_n}{s_n - t_n}, \quad q_n := \frac{-v_n}{s_n - t_n} \right). \quad (2.27)$$

If $n \rightarrow \infty$, then the left-hand side of (2.23) converges to the vector $\mathbf{c}'(s_\infty) (= \mathbf{e}(s_\infty))$. Thus, we can conclude that the limits $\lim_{n \rightarrow \infty} p_n = p_\infty$ and $\lim_{n \rightarrow \infty} q_n = q_\infty$ exist such that

$$\mathbf{e}(s_\infty) = p_\infty \xi_F(s_\infty) - q_\infty \xi_G(s_\infty). \quad (2.28)$$

Since the left hand side does not vanish, we have

$$(p_\infty, q_\infty) \neq (0, 0). \quad (2.29)$$

Taking the inner product of $\mathbf{n}(s_\infty)$ to (2.28), we have

$$0 = p_\infty \xi_F(s_\infty) \cdot \mathbf{n}(s_\infty) - q_\infty \xi_G(s_\infty) \cdot \mathbf{n}(s_\infty). \quad (2.30)$$

Since $u_n > 0$ and $v_n < 0$, we have $p_\infty q_\infty \leq 0$. Since $\xi_F(s_\infty) \cdot \mathbf{n}(s_\infty)$ and $\xi_G(s_\infty) \cdot \mathbf{n}(s_\infty)$ are positive, (2.29) implies that the right hand side of (2.30) does not vanish, a contradiction. $\square$

3 The dual developable strips and curved foldings

In this section, we explain curved foldings using pairs of developable strips:

Definition 3.1 Let J_i ($i = 1, 2$) be bounded closed intervals of $\mathbf{R}$. Two functions $\mu_i : J_i \rightarrow \mathbf{R}$ ($i = 1, 2$) are said to be *equi-affine equivalent* if there exists a diffeomorphism $\varphi : J_1 \rightarrow J_2$ of the form

$$\varphi(u) = \sigma u + d \quad (\sigma \in \{1, -1\}, \quad d \in \mathbf{R})$$

such that $\mu_2 \circ \varphi = \mu_1$. (By definition, if μ_2 is equi-affine equivalent to μ_1 , then the length of the interval J_2 must be equal to that of J_1 .)

In the case of $J_1 = J_2 = [b, c]$ ($b < c$) and $(\mu :=) \mu_1 = \mu_2$, the map $\varphi : J_1 \rightarrow J_1$ is called a *symmetry* of μ if φ is not the identity. (There is at most one possibility for such a φ , which must have the expression $\varphi(u) = b + c - u$. So, if such a φ exists, μ satisfies $\mu(u) = \mu(b + c - u)$ on $[b, c]$.)

Lemma 3.2 For each $i \in \{1, 2\}$, let $\gamma_i : \mathbb{I}_l \rightarrow \mathbf{R}^2$ ($l > 0$) be a regular curve without self-intersections parametrized by arc-length. Suppose that the curvature functions of γ_1 and γ_2 are positive-valued. Then the following two assertions are equivalent:

- (1) the curvature function of γ_2 is equi-affine equivalent to that of γ_1 ,
 (2) there exists an isometry T of $\mathbf{R}^2$ such that $T(\gamma_1(\mathbb{I}_l))$ coincides with $\gamma_2(\mathbb{I}_l)$.

Proof We suppose (1). We set $\gamma_1^\sharp(s) := S \circ \gamma_1(-s)$ ($s \in \mathbb{I}_l$), where S is the reflection with respect to a straight line in $\mathbf{R}^2$. We denote by μ_i ($i = 1, 2$) the curvature function of γ_i . If μ_2 is equi-affine equivalent to μ_1 , then μ_2 coincides with the curvature function of γ_1 or $\gamma_1^\sharp$. By the fundamental theorem of curves in the Euclidean plane (cf. (Umehara and Yamada 2017, Chapter 2)), there exists an isometry T in $\mathbf{R}^2$ such that $T \circ \gamma_1 = \gamma_2$ or $T \circ \gamma_1^\sharp = \gamma_2$, which implies (2).

Conversely, we suppose (2). Since γ_1 and γ_2 have no self-intersections, such an isometry T is uniquely determined. Since γ_1 and γ_2 are parametrized by arc-length, either $T \circ \gamma_1(s) = \gamma_2(s)$ or $T \circ \gamma_1(s) = \gamma_2(-s)$ holds on $\mathbb{I}_l$. Since the curvature functions of γ_1 and γ_2 are positive-valued,

$$T \circ \gamma_1(s) = \gamma_2(s) \quad (\text{resp. } T \circ \gamma_1(s) = \gamma_2(-s))$$

holds on $\mathbb{I}_l$ if T is an orientation preserving (resp. reversing) isometry of $\mathbf{R}^3$, which implies $\mu_1(s) = \mu_2(s)$ (resp. $\mu_1(s) = \mu_2(-s)$) for $s \in \mathbb{I}_l$. So (1) holds. $\square$

Let $\mu : \mathbb{T}_l^1 \rightarrow \mathbf{R}$ be a C^∞ -function on a one dimensional torus $\mathbb{T}_l^1 := \mathbf{R}/l\mathbf{Z}$ ($l > 0$). Then an a -periodic function $\tilde{\mu} : \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$\tilde{\mu} := \mu \circ \pi \tag{3.1}$$

is called the *lift* of the function μ , where $\pi : \mathbf{R} \rightarrow \mathbb{T}_l^1$ is the canonical projection.

Definition 3.3 We set $J_i := \mathbf{R}/l_i\mathbf{Z}$ ($l_i > 0$, $i = 1, 2$). Two functions $\mu_i : J_i \rightarrow \mathbf{R}$ ($i = 1, 2$) are said to be *equi-affine equivalent* if there exists a diffeomorphism $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ of the form

$$\varphi(u) = \sigma u + d \quad (\sigma \in \{1, -1\}, d \in \mathbf{R})$$

such that $\tilde{\mu}_2 \circ \varphi = \tilde{\mu}_1$, where $\tilde{\mu}_i$ ($i = 1, 2$) are the lifts of the functions μ_i . When $(\mu :=) \mu_1 = \mu_2$ and $J_1 = J_2$, the function μ has a *symmetry* φ , if φ is non-trivial, that is, either $\sigma = -1$ or $d \notin l\mathbf{Z}$ holds, where $l := l_1 (= l_2)$. (If μ is a non-constant continuous function, then φ can be a candidate for symmetries of μ only when d/l is a rational number.)

The following is an analogue of Lemma 3.2 for $\mathbb{T}_l^1$.

Lemma 3.4 Let J_i ($i = 1, 2$) be two bounded closed intervals, and let $\gamma_i : J_i \rightarrow \mathbf{R}^2$ be plane curves of length l parametrized by arc-length. Suppose that each curvature function of γ_i ($i = 1, 2$) has a C^∞ -extension $\tilde{\mu}_i : \mathbf{R} \rightarrow \mathbf{R}$ which is the lift of an l -periodic function $\mu_i : \mathbb{T}_l^1 \rightarrow \mathbf{R}$. Then the following two assertions are equivalent:

- (1) the function μ_2 is equi-affine equivalent to μ_1 ,
 (2) there exist a plane curve $\tilde{\gamma} : \mathbf{R} \rightarrow \mathbf{R}^2$ and an orientation preserving isometry T of $\mathbf{R}^3$ such that $\gamma_1(J_1)$ and $T \circ \gamma_2(J_2)$ are subarcs of $\tilde{\gamma}(\mathbf{R})$.

Proof We suppose (1). Since each curvature function of γ_i ($i = 1, 2$) can be extended as an l -periodic C^∞ -function $\tilde{\mu}_i$ on $\mathbf{R}$, the curve γ_i is extended as a regular curve $\tilde{\gamma}_i : \mathbf{R} \rightarrow \mathbf{R}^2$ whose curvature function is $\tilde{\mu}_i$. If (1) holds, then there exist $\sigma \in \{1, -1\}$ and $d \in [0, l)$ such that

$$\tilde{\mu}_1(s) = \tilde{\mu}_2(\sigma s + d) \quad (s \in \mathbf{R}).$$

By the fundamental theorem of plane curves, $\tilde{\gamma}_2(s)$ coincides with $T \circ \tilde{\gamma}_1(\sigma s + d)$, where T is an orientation preserving isometry of $\mathbf{R}^2$. By setting $\tilde{\gamma} := \tilde{\gamma}_1$, (2) is obtained. On the other hand, the converse assertion can be proved easily. $\square$

We now define the “geodesic equivalence relation” on $\mathcal{D}(|C|)$ as follows:

Definition 3.5 Let C be a non-closed space curve (i.e. $C := \mathbf{c}(\mathbb{I}_l)$) or a closed curve (i.e. $C := \mathbf{c}(\mathbb{T}_l^1)$) of total length l embedded in $\mathbf{R}^3$. Two developable strip germs $f, g \in \mathcal{D}(|C|)$ are said to be *geodesically equivalent* if the normalized geodesic curvature function $\hat{\mu}_f : J \rightarrow \mathbf{R}$ is equi-affine equivalent to $\hat{\mu}_g : J \rightarrow \mathbf{R}$, where $J = \mathbb{I}_l$ or $J = \mathbb{T}_l^1$.

The following assertion holds:

Proposition 3.6 Let $f, g \in \mathcal{D}(|C|)$. If f and g are right equivalent, then they are geodesically equivalent.

Proof By replacing g by $g^\sharp$, we may assume that $f, g \in \mathcal{D}(C)$. We denote by F and G the normal forms associated with f and g satisfying $F(0, 0) = G(0, 0) = \mathbf{x}_0$, respectively. Then by Corollary 1.10, F coincides with G , which implies $\mu_f = \mu_F = \mu_G = \mu_g$. $\square$

Later, we will see that the geodesic equivalence relation is useful for constructing curved foldings with a given crease and crease pattern. Based on this, we give the following:

Definition 3.7 For $f \in \mathcal{D}(|C|)$, a developable strip germ $g \in \mathcal{D}(|C|)$ is called an *isomer* of f if

- (1) g is geodesically equivalent to f , but
- (2) g is not right equivalent to f .

Remark 3.8 The above definition of isomers is an analogue for that for cuspidal edges (cf. Honda et al. 2020a). In the case of cuspidal edges, (1) was replaced by the condition that the first fundamental forms of two surfaces are isometric. (It should be remarked that all developable surfaces are mutually locally isometric.)

We now give a tool to construct isomers of a given developable strip. Let $\tilde{C}$ be an embedded curve in $\mathbf{R}^3$ which is homeomorphic to C and has the same total length as C .

Definition 3.9 Let $\tilde{\mathbf{c}}(s)$ ($s \in J$) be the arc-length parametrization of $\tilde{C}$, and let $\tilde{\kappa}(s)$ be its curvature function. Then $\tilde{\mathbf{c}}$ is said to be *compatible* to $f \in \mathcal{D}(C)$ if it satisfies

$$|\hat{\mu}_f(s)| < \tilde{\kappa}(s) \quad (s \in J), \quad (3.2)$$

where $\hat{\mu}_f$ is the normalized geodesic curvature function of f .

Let J_0 be a set which is homeomorphic to J . The following proposition plays an essential role in considering the relationship of developable strips and curved foldings.

Proposition 3.10 *Let $f : U \rightarrow \mathbf{R}^3$ be a developable strip belonging to $\mathcal{D}(C)$ satisfying $J_0 \times \{0\} \subset U$ and $f(J_0 \times \{0\}) = C$. Let $u_0 \in J_0$ be the point such that $f(u_0, 0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is the base point (cf. (2.12)) of C . Suppose that $\tilde{\mathbf{c}}(s)$ ($s \in J$) is the arc-length parametrization of $\tilde{C}$ which is compatible to f . Then there exist a tubular neighborhood $V(\subset U)$ of $J_0 \times \{0\}$ in $J_0 \times \mathbf{R}$ and developable strips g_+, g_- belonging to $\mathcal{D}(\tilde{C})$ such that*

- (1) $g_+(u_0, 0) = g_-(u_0, 0) = \tilde{\mathbf{c}}(0)$ (this condition is automatically satisfied if J_0 is a closed bounded interval),
- (2) $J_0 \ni u \mapsto g_+(u, 0) = g_-(u, 0) \in \mathbf{R}^3$ gives a parametrization of $\tilde{C} := \tilde{\mathbf{c}}(J)$,
- (3) the first angular functions $\alpha_{\pm}(u)$ of $g_{\pm}(u, v)$ satisfy $\alpha_+ = -\alpha_-$,
- (4) $\mu_f = \mu_{g_{\pm}}$ on J_0 (cf. (2.9)),
- (5) $\alpha_f(u)$ has the same sign as $\alpha_{g_+}(u)$ for each $u \in J_0$, and
- (6) g_+ and g_- are normal forms if the same is true of f .

Proof Since C has total arc-length l , we can take the arc-length parametrization $u = u(s)$ ($s \in J$) so that $u_0 = u(0)$ and $\mathbf{c}(s) := \mathbf{c}_f(u(s))$ ($s \in J$) parametrizes C . Since $\tilde{\mathbf{c}}$ is compatible to f , $\tilde{\mathbf{c}}(u) := \tilde{\mathbf{c}}(s(u))$ ($u \in J_0$) gives a parametrization of $\tilde{C}$ defined on J_0 such that

$$\mu_f(u) < \tilde{\kappa}(u) \quad (u \in J_0), \quad (3.3)$$

where $\tilde{\kappa}(u)$ is the curvature function of $\tilde{\mathbf{c}}(u)$. Then there exists a unique function $\tilde{\alpha} : J_0 \rightarrow (-\pi/2, \pi/2)$ such that

$$\tilde{\kappa}(u) \cos \tilde{\alpha}(u) = \mu_f(u), \quad \tilde{\alpha}(u) \alpha_f(u) \geq 0$$

for each $u \in J_0$. By the compatibility of $\tilde{\mathbf{c}}$, the function $\sin \tilde{\alpha}$ never vanishes. Thus, we can define the second angular function $\tilde{\beta} : J_0 \rightarrow (0, \pi)$ so that

$$\cot \tilde{\beta}_{\pm}(u) := \frac{\tilde{\alpha}'(u) \pm |\tilde{\mathbf{c}}'(u)| \tilde{\tau}(u)}{|\tilde{\mathbf{c}}'(u)| \tilde{\kappa}(u) \sin \tilde{\alpha}(u)},$$

where $\tilde{\kappa}(u)$ and $\tilde{\tau}(u)$ are the curvature and torsion functions of $\tilde{\mathbf{c}}(u)$ ($u \in J_0$), respectively. We set

$$g_{\pm}(u, v) := \tilde{\mathbf{c}}(u) + \tilde{\xi}_{\pm}(u),$$

$$\tilde{\xi}_{\pm}(u) := \cos \tilde{\beta}_{\pm}(u) \tilde{\mathbf{e}}(u) + \sin \tilde{\beta}_{\pm}(u) \left(\cos \tilde{\alpha}(u) \tilde{\mathbf{n}}(u) \pm \sin \tilde{\alpha}(u) \tilde{\mathbf{b}}(u) \right),$$

where $\tilde{\mathbf{e}}$, $\tilde{\mathbf{n}}$ and $\tilde{\mathbf{b}}$ are the unit tangent vector field, the unit principal normal vector field and the unit bi-normal vector field of $\tilde{C}$, respectively. Since $u_0 = u(0)$, we obtain (1). It can be easily checked that $g_{\pm}$ satisfy (2)–(5). Finally, if f is written in a normal form, then $u = s$ holds for $s \in J$, and $\tilde{C}(u) = \tilde{C}(s)$ is parametrized by arc-length. In particular, $g_{\pm}(u, v) = g_{\pm}(s, v)$ give normal forms. So (6) is obtained. $\square$

Corollary 3.11 *Let $f : U \rightarrow \mathbf{R}^3$ be a developable strip belonging to $\mathcal{D}(C)$, where U is a tubular neighborhood of $J_0 \times \{0\}$ in $J_0 \times \mathbf{R}$. Then there exist a tubular neighborhood $V(\subset U)$ of $J_0 \times \{0\}$ in $J_0 \times \mathbf{R}$ and a developable strip $g : V \rightarrow \mathbf{R}^3$ such that*

- (1) $f(u, 0) = g(u, 0)$ for each $u \in J_0$,
- (2) $\alpha_g(u) = -\alpha_f(u)$ for each $u \in J_0$,
- (3) g is uniquely determined from f as a strip germ along C , and
- (4) g is a normal form if so is f .

We call this g the *dual* of f and denote it by $\check{f}$. Then an involution $\mathcal{D}(C) \ni f \mapsto \check{f} \in \mathcal{D}(C)$ is induced. Moreover, we have

$$\alpha_{\check{f}}(u) = -\alpha_f(u) \quad (u \in J_0). \quad (3.4)$$

Proof of Corollary 3.11 Setting $\tilde{C}(u) := f(u, 0)$, we can apply Proposition 3.10 because $f \in \mathcal{D}(C)$. Then the absolute value of the first angular function of g_{-} coincides with that of $f \in \mathcal{D}(C)$, but the sign is opposite. Thus, g_{-} gives the desired developable strip. $\square$

By (2) of Corollary 3.11, $\check{f}$ has the same geodesic curvature function as f , and we obtain the following:

Proposition 3.12 *For each $f \in \mathcal{D}(C)$, the dual $\check{f}$ is an isomer of f .*

Moreover, we have the following:

Proposition 3.13 *Let $F, G \in \mathcal{D}(C)$ be normal forms satisfying $F(0, 0) = G(0, 0)$. If $\mu_F = \mu_G$ and $F(0, 0) = G(0, 0)$, then either $G = F$ or $G = \check{F}$ holds.*

Proof Since $\mu_F = \mu_G$, we have $\cos \alpha_F = \cos \alpha_G$, which implies $\alpha_F = \pm \alpha_G$. Since the normal form of a developable strip is determined by its first angular function and its base point, we have $G = F$ or $G = \check{F}$. $\square$

Corollary 3.14 *Let $F \in \mathcal{D}(C)$ be a normal form. If T is a symmetry of C , then $T \circ \check{F}$ is also a normal form giving the dual of $T \circ F$.*

Proof Since F is a normal form, so is $\check{F}$. By Proposition 2.8, $T \circ \check{F}$ also gives a normal form. So we have

$$\mu_{T \circ F} = \mu_F = \mu_{\check{F}} = \mu_{T \circ \check{F}}$$

Fig. 3 The images of F and its dual given in Example 3.15

and $T \circ F(0, 0) = T \circ \check{F}(0, 0)$. Then Proposition 3.13 implies $T \circ \check{F}$ coincides with $T \circ F$ or its dual. Since $T \circ \check{F}(\Omega_\varepsilon)$ meets $T \circ F(\Omega_\varepsilon)$ only along C (cf. (3.6)), we obtain the conclusion. $\square$

Example 3.15 We fix a positive number $l > 0$. Then

$$\mathbf{c}_1(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right) \quad \left(|s| \leq \frac{l}{2} \right)$$

gives a helix with arc-length parameter satisfying $\kappa = \tau = 1/2$. We set $C_1 := \mathbf{c}_1([-l/2, l/2])$. We fix a constant $\alpha \in (-\pi/2, 0) \cup (0, \pi/2)$ and let $F \in \mathcal{D}(C_1)$ be a normal form whose first angular function is identically equal to α . Then $T \circ \check{F} = F$ holds, where T is the 180° -rotation with respect to the normal line of C_1 at the origin. Figure 3 shows the images of F and $\check{F}$.

We consider the case that $J = \mathbb{I}_l$. Regarding Lemma 3.2, we give the following:

Definition 3.16 The image Γ of a regular curve $\gamma : \mathbb{I}_l \rightarrow \mathbf{R}^2$ parametrized by arc-length is called a *generator* of a strip $f \in \mathcal{D}(C)$ ($C := \mathbf{c}(\mathbb{I}_l)$) if the curvature function of $\gamma(s)$ is equi-affine equivalent to the normalized geodesic curvature function $\hat{\mu}_f(s)$ ($s \in \mathbb{I}_l$) of f .

We next consider the case that $J = \mathbb{T}_l^1$. Regarding Lemma 3.4, we give the following:

Definition 3.17 The image Γ of a regular curve $\gamma : \mathbb{T}_l^1 \rightarrow \mathbf{R}^2$ parametrized by arc-length is called a *generator* of the strip $f \in \mathcal{D}(C)$ ($C := \mathbf{c}(\mathbb{T}_l^1)$) if the curvature function of $\gamma(s)$ can be smoothly extended as an l -periodic function which is the lift of a function which is equi-affine equivalent to the normalized curvature function $\hat{\mu}_f(s)$ ($s \in \mathbb{T}_l^1$) of f .

A generator Γ of f has an ambiguity of isometric motions in the plane $\mathbf{R}^2$. Since f is a developable surface, it can be developed to a plane, and the curve C is deformed to a plane curve which is congruent to Γ . If Γ has no self-intersections, then f can be obtained as a deformation of an immersed developable strip from a tubular neighborhood of Γ to the image of f (cf. Umehara and Yamada 2017, Theorem B.6.4). According to (Fuchs and Tabachnikov 1999), we define “origami maps” as follows:

Definition 3.18 The *origami map* Φ_f induced by $f \in \mathcal{D}(C)$ is defined by

$$\Phi_f(u, v) := \begin{cases} f(u, v) & (v \geq 0), \\ \check{f}(u, v) & (v < 0), \end{cases}$$

Fig. 4 Angular functions β_L and β_R on the crease pattern of Φ_f

where $\check{f}$ is the dual of f . In this setting, C is called the *crease* of Φ_f , and a generator Γ of f is called a *crease pattern* of the origami map Φ_f . (Since the normalized curvature function μ is common in f and $\check{f}$, Γ gives a generator of $\check{f}$. When J_0 is a bounded closed interval, the congruence class of the crease pattern of Φ_f is uniquely determined.)

We set

$$\mathcal{O}(C) := \{\Phi_f; f \in \mathcal{D}(C)\}, \quad \mathcal{O}_*(C) := \{\Phi_f; f \in \mathcal{D}_*(C)\}$$

and

$$\mathcal{O}(|C|) := \mathcal{O}(C) \cup \mathcal{O}(-C), \quad \mathcal{O}_*(|C|) := \mathcal{O}_*(C) \cup \mathcal{O}_*(-C),$$

which are the sets of origami maps and the sets of *admissible origami maps* along C and $|C|$, respectively. Figure 1 indicates the second angular functions β and $\check{\beta}$ of f and $\check{f}$, respectively. Here,

$$\check{\Phi}_f := \Phi_{\check{f}}$$

is called the *adjacent origami map* with respect to Φ_f . Obviously, the union of the images of Φ_f and $\check{\Phi}_f$ coincides with the union of the images of f and $\check{f}$.

The following fact is known:

Fact 3.19 (Fuchs and Tabachnikov 1999, see also Honda et al. 2020c) *A curved folding P along a curve $|C|$ satisfying (i),(ii),(iii) and (iv) (resp. (iv')) in the introduction is realized as the image of Φ_F for a certain normal form $F \in \mathcal{D}(C)$ (resp. $F \in \mathcal{D}_*(C)$). Moreover, Γ corresponds to the crease pattern of Φ_F .*

Remark 3.20 More precisely, the condition (ii) in the introduction implies $\cos \alpha_F > 0$ and (iv) (resp. (iv')) in the introduction corresponds to the condition $\cos \alpha_F \neq 1$ (resp. $\max_{u \in J_0} |\cos \alpha_F(u)| < 1$). Moreover, μ_F coincides with the curvature function

with respect to the arc-length parametrization $\gamma : J \rightarrow \mathbf{R}^2$ of the generator Γ . We set $\beta_L := \beta$ and $\beta_R := \pi - \check{\beta}$, where β and $\check{\beta}$ are the second angular functions of F and $\check{F}$, respectively. Then β_L (resp. β_R) gives the left-ward (resp. right-ward) angular

function of the ruling direction from the tangential direction $\gamma'(s)$ in the plane $\mathbf{R}^2$, see Fig. 4.

Here, we set

$$\begin{aligned}\mathcal{O}(\Gamma, |C|) &:= \{\Phi_f; \text{ the generator of } f \in \mathcal{D}(|C|) \text{ is } \Gamma\}, \\ \mathcal{O}_*(\Gamma, |C|) &:= \{\Phi_f; \text{ the generator of } f \in \mathcal{D}_*(|C|) \text{ is } \Gamma\}\end{aligned}$$

and

$$\mathcal{P}(\Gamma, C) := \{\text{Im}(\Phi_f); f \in \mathcal{O}(\Gamma, C)\}, \quad \mathcal{P}_*(\Gamma, C) := \{\text{Im}(\Phi_f); f \in \mathcal{O}_*(\Gamma, C)\}, \quad (3.5)$$

where $\text{Im}(\Phi_f)$ is the image of the strip germ Φ_f along C . Then $\mathcal{P}(\Gamma, C)$ can be considered as the set of curved foldings whose crease and crease pattern are C and Γ , respectively. The set $\mathcal{P}_*(\Gamma, C)$ consists of admissible curved foldings along C defined as in the introduction. The following assertion is obvious by the definition of the map $\Phi : \mathcal{D}(|C|) \rightarrow \mathcal{O}(|C|)$ and Lemma 3.24.

The following assertion implies that we can fold a pair $\Phi_F(\Omega_\varepsilon)$, $\check{\Phi}_F(\Omega_\varepsilon)$ of curved foldings at the same time whenever $\varepsilon(> 0)$ is sufficiently small:

Proposition 3.21 *For each normal form $F \in \mathcal{D}(|C|)$,*

$$F(\Omega_\varepsilon) \cap \check{F}(\Omega_\varepsilon) = C \quad (3.6)$$

holds. Moreover, the induced origami map Φ_F has no self-intersections.

Proof Since the ruling vector of F is linearly independent of $\check{F}$ at each point of C , the first assertion follows from Proposition 2.10. So we prove the second assertion. If not, (3.6) implies that $F(\Omega_\varepsilon^+)$ must meet $F(\Omega_\varepsilon^-)$, which is impossible. $\square$

Proposition 3.22 *Let $F, G \in \mathcal{D}(|C|)$ be normal forms. If $\Phi_F(\Omega_\varepsilon)$ coincides with $\Phi_G(\Omega_\varepsilon)$ for sufficiently small ε , then F is right equivalent to G .*

Proof We denote by F, G the normal forms associated with f, g satisfying $F(0, 0) = G(0, 0)$, respectively. It is sufficient to show the assertions hold for F and G . We suppose $\Phi_F(\Omega_\varepsilon) = \Phi_G(\Omega_\varepsilon)$. By replacing F (resp. G) by $F^\sharp$ (resp. $G^\sharp$), Proposition 2.8 yields that $F, G \in \mathcal{D}(C)$ without loss of generality. Then we have $F(\Omega_\varepsilon^+) \cup \check{F}(\Omega_\varepsilon^-) = G(\Omega_\varepsilon^+) \cup \check{G}(\Omega_\varepsilon^-)$. By Proposition 2.11, $F(\Omega_\varepsilon^+) \cap \check{G}(\Omega_\varepsilon^-)$ is the empty set, and so we have $F(\Omega_\varepsilon^+) = G(\Omega_\varepsilon^+)$. By Proposition 2.9, we can conclude that $F = G$. $\square$

We now prove the following:

Theorem 3.23 *The map $\Phi : \mathcal{D}(|C|) \ni f \mapsto \Phi_f \in \mathcal{O}(|C|)$ has the following properties:*

- (1) $f, g \in \mathcal{D}(|C|)$ are right equivalent if and only if $\Phi_F(\Omega_\varepsilon)$ coincides with $\Phi_G(\Omega_\varepsilon)$, where F, G are normal forms associated with f and g satisfying $F(0, 0) = G(0, 0) = \mathbf{x}_0$, respectively.
- (2) If Φ_f and Φ_g ($f, g \in \mathcal{D}(|C|)$) have the same crease pattern, then f and g are geodesically equivalent.
- (3) For each $f \in \mathcal{D}(|C|)$, the crease pattern of $\check{\Phi}(f) (= \Phi(\check{f}))$ coincides with that of $\Phi(f)$.
- (4) Let T be a symmetry of C , then $T \circ \Phi_F = \Phi_{T \circ F}$ holds for each normal form $F \in \mathcal{D}(|C|)$.
- (5) Let $f, g \in \mathcal{D}(|C|)$. If $\Phi_F(\Omega_\varepsilon)$ is congruent to $\Phi_G(\Omega_\varepsilon)$, then there exists a symmetry T of C such that g is right equivalent to $T \circ f$.

Before proving this assertion, we prepare the following lemma:

Lemma 3.24 *Let C be a non-closed space curve (i.e. J_0 is a bounded closed interval), and let Γ be a simple arc in $\mathbf{R}^2$ which is a generator of a developable strip $F \in \mathcal{D}(C)$ written in a normal form. Then the following two assertions are equivalent:*

- Γ has a symmetry (cf. Definition 1.1),
- the geodesic curvature function μ_F of F has a symmetry (cf. Definition 3.3).

Proof Since $\hat{\mu}_F$ coincides with the curvature function of Γ , the conclusion follows from Lemma 3.2. $\square$

Proof of Theorem 3.23 We denote by F, G the normal forms associated with f, g satisfying $F(0, 0) = G(0, 0)$, respectively. It is sufficient to show the assertions hold for F and G . Then F is defined on a tubular neighborhood of $J \times \{0\}$ in $J \times \mathbf{R}$. If F, G are right equivalent, then it is obvious that the images of Φ_F, Φ_G coincide. The converse of this assertion follows from Proposition 3.22.

We now prove (2). In the case that J is a bounded closed interval, (2) follows from Lemma 3.24. We then consider the case that J is a one dimensional torus $\mathbb{T}_l$ ($l > 0$). If Φ_F and Φ_G have the same crease pattern Γ , then the curvature function of Γ can be extended as a smooth l -periodic function on $\mathbf{R}$, and coincides with the lift of common normalized geodesic curvature function of F and G . So F and G are geodesically equivalent.

On the other hand, (3) is obvious from the definition of $\check{\Phi}(F)$.

We next prove (4). Let T be a symmetry of C , then, by Corollary 3.14, we have

$$T \circ \Phi_F(\Omega_\varepsilon) = T \circ F(\Omega_\varepsilon^+) \cup T \circ \check{F}(\Omega_\varepsilon^-) = \Phi_{T \circ F}(\Omega_\varepsilon).$$

So $T \circ \Phi_F$ is right equivalent to $\Phi_{T \circ F}$ by (1). Since F is a normal form, we have $T \circ \Phi_F = \Phi_{T \circ F}$.

Finally, we prove (5). Suppose that $\Phi_F(\Omega_\varepsilon)$ is congruent to $\Phi_G(\Omega_\varepsilon)$. Then there exists an isometry T of $\mathbf{R}^3$ such that $T \circ \Phi_G(\Omega_\varepsilon)$ coincides with $\Phi_F(\Omega_\varepsilon)$. Since $T \circ \Phi_G(\Omega_\varepsilon) = \Phi_{T \circ G}(\Omega_\varepsilon)$, Proposition 3.22 implies that $T \circ G$ is right equivalent to F . So we obtain (5). $\square$

4 The inverses and inverse duals

In this section, we set $J = \mathbb{I}_l (= [-l/2, l/2])$ ($l > 0$), that is, C is a non-closed space curve. Let $I := [b, c]$ ($b < c$) be a closed bounded interval on $\mathbf{R}$.

Proposition 4.1 *Let $f : U \rightarrow \mathbf{R}^3$ be a developable strip belonging to $\mathcal{D}_*(C)$, where U is a tubular neighborhood of $I \times \{0\}$ ($\subset I \times \mathbf{R}$). Then there exist a tubular neighborhood $V(\subset U)$ of $I \times \{0\}$ in $I \times \mathbf{R}$ and two maps $f_*, \check{f}_* : V \rightarrow \mathbf{R}^3$ such that*

- (1) f_* and $\check{f}_*$ belong to $\mathcal{D}_*(-C)$,
- (2) $f_*(u, 0) = \check{f}_*(u, 0) = f(-u, 0)$ for each $u \in I$,
- (3) the first angular function α_{f_*} takes the same sign as α_f and satisfies

$$\kappa_f(b + c - u) \cos \alpha_{f_*}(u) = \kappa_f(u) \cos \alpha_f(u), \quad (4.1)$$

where $\mathbf{c}_f(u) := f(u, 0)$ ($u \in I$) and $\kappa_f(u)$ is its curvature function,

- (4) $\alpha_{\check{f}_*}(u) = -\alpha_{f_*}(u)$,
- (5) f_* and $\check{f}_*$ are normal forms if f is.

Moreover, such two maps f_* and $\check{f}_*$ are uniquely determined from f as map germs along C .

We call f_* the *inverse* of f , and $\check{f}_*$ the *inverse dual* of f (cf. Honda et al. 2020c). By definition,

$$\mu_{f_*}(u) = \mu_{\check{f}_*}(u) = \mu_f(u), \quad (4.2)$$

and so each generator of f gives a generator of f_* (and of $\check{f}_*$).

Proof Since f is admissible (i.e. $f \in \mathcal{D}_*(C)$), we have

$$0 < \kappa_f(u) \cos \alpha_f(u) < \min_{u \in I} \kappa_f(u) \quad (u \in I).$$

We note that $\mathbf{c}^\sharp(u) := \mathbf{c}_f(b + c - u)$ ($u \in I$) gives the parametrization of $\tilde{C} := -C$. Then $\kappa_f(b + c - u)$ is the curvature function of $\mathbf{c}^\sharp(u)$. Since $\mu_f := \kappa_f \cos \alpha_f$ is the curvature function of the generator of f , we have

$$0 < \mu_f(u) < \min_{u \in I} \kappa_f(u) = \min_{u \in I} \kappa_f(b + c - u) \quad (u \in I). \quad (4.3)$$

So $\mathbf{c}^\sharp$ is compatible to f . Thus, there exists a developable strip $g_+ \in \mathcal{D}_*(-C)$ (resp. $g_- \in \mathcal{D}_*(-C)$) satisfying (1)–(6) of Proposition 3.10. In particular, the first angular function of g_+ (resp. g_-) is positive (resp. negative). Moreover, by (4.3), g_+ and g_- are belonging to $\mathcal{D}_*(-C)$. Then it can be easily checked that $f_* := g_+$ and $\check{f}_* := g_-$ satisfy (1)–(5) of Proposition 4.1. In fact,

$$\mu_{f_*}(u) = \kappa_f(b + c - u) \cos \alpha_{f_*}(u)$$

coincides with $\mu_f(u) (= \kappa_f(u) \cos \alpha_f(u))$ by (4) of Proposition 3.10. $\square$

Remark 4.2 Fix $f \in \mathcal{D}_*(C)$. By (2.18), the first angular function of the inverse dual f_* takes the opposite sign as that of $f^\sharp$.

We have the following:

Proposition 4.3 *If $g \in \mathcal{D}_*(|C|)$ is geodesically equivalent to $f \in \mathcal{D}_*(C)$, then g is right equivalent to one of $\{f, \check{f}, f_*, \check{f}_*\}$.*

Proof Replacing g by $g^\sharp$, we may assume that $f, g \in \mathcal{D}_*(|C|)$. We denote by F, G the normal forms associated with f, g satisfying $F(0, 0) = G(0, 0)$, respectively. Then the assumption that g is geodesically equivalent to f implies $\mu_F = \mu_G$. By Proposition 3.13 we have $G = F$ or $G = \check{F}$. On the other hand, if $G \in \mathcal{D}_*(-C)$, then, replacing C by $-C$ and applying Proposition 3.13, we can conclude $G = F_*$ or $G = \check{F}_*$. $\square$

Moreover, we can prove the following:

Proposition 4.4 *For each normal form $F \in \mathcal{D}_*(C)$, the ruling direction of the inverse F_* is linearly independent of that of F at each point of C . In particular,*

$$F(\Omega_\varepsilon) \cap F_*(\Omega_\varepsilon) = C \quad (4.4)$$

holds for each sufficiently small $\varepsilon(> 0)$.

Proof Without loss of generality, we may replace f by its normal form F . Since the first angular function of $(F_*)^\sharp$ has the opposite sign of that of F (cf. (2.18)), Lemma 2.7 yields that the ruling direction of F_* is linearly independent of that of F along C . The last assertion is a consequence of Proposition 2.10. $\square$

Using this proposition, we can prove the following:

Proposition 4.5 *For $f \in \mathcal{D}_*(C)$, the following three assertions are equivalent:*

- (1) f is right equivalent to $\check{f}_*$,
- (2) $\check{f}$ is right equivalent to f_* ,
- (3) the normalized geodesic curvature $\hat{\mu}_f$ of f has a symmetry (cf. Definition 3.1).

Proof The equivalency of (1) and (2) is obvious. So it is sufficient to show that (1) is equivalent to (3). We may assume that f is a normal form and denote it by F . Then $F(s, v)$ is defined on a tubular neighborhood of $\mathbb{I}_l \times \{0\}$ in $\mathbb{R}^2$, where l is the total arc-length of C . Suppose that μ_F has a symmetry. Since $\mu_F(s) = \mu_F(-s)$ for $s \in \mathbb{I}_l$, $F^\sharp$ is geodesically equivalent to F . Since $F^\sharp \in \mathcal{D}_*(-C)$, Proposition 3.13 yields that $F^\sharp$ coincides with F_* or $\check{F}_*$. However, $F^\sharp$ never coincides with F_* by Proposition 4.4. So we have $F^\sharp = \check{F}_*$.

Conversely, we suppose (1). Then $F^\sharp = \check{F}_*$ holds. By (4.4), we have

$$\mu_F(-s) = \mu_{F^\sharp}(s) = \mu_{\check{F}_*}(s) = \mu_F(s),$$

which implies that μ_F has a symmetry. $\square$

Fig. 5 Images of Φ_1 and Φ_2 (left) and images of $\{\Phi_j\}_{j=1,\dots,4}$ (right)

In the above discussions, the following assertion was also obtained.

Corollary 4.6 *Let $F \in \mathcal{D}_*(C)$ be a normal form. If the geodesic curvature μ_F has a symmetry, then $\check{F}_* = F^\sharp$ holds.*

Theorem 4.7 *Let $f \in \mathcal{D}_*(C)$ and n_f the number of right equivalence classes of f , $\check{f}$, f_* and $\check{f}_*$. If the normalized geodesic curvature of f has no symmetries, then $n_f = 4$, otherwise $n_f = 2$.*

Proof We may assume that f is a normal form and denote it by F . Suppose that μ_F has a symmetry. By Proposition 4.5, we have $F = \check{F}_*$ and $\check{F} = F_*$, and $n_F = 2$. On the other hand, suppose that $n_F < 4$. If necessary, replacing F by one of $\{\check{F}, F_*, \check{F}_*\}$, we may assume that F coincides with one of $F, F_*, \check{F}_*$. By (3.6) and (4.4), F must coincide with $\check{F}_*$. By Proposition 4.5, μ_F has a symmetry. $\square$

Remark 4.8 In Example 3.15, the curvature function of C and the angular function of F are constant, $\check{F}_* = F^\sharp$ and $\check{F} = F^\sharp$ hold. So in this case, $n_F = 2$ holds.

Proof of Theorem A We fix a curved folding $P \in \mathcal{P}_*(\Gamma, C)$ arbitrarily. Then there exists a normal form $F \in \mathcal{D}_*(C)$ such that $P = \Phi_F$, and so $n_{C,\Gamma} = n_F$. Moreover,

$$\Phi_F(\Omega_\varepsilon), \Phi_{\check{F}}(\Omega_\varepsilon), \Phi_{F_*}(\Omega_\varepsilon), \Phi_{\check{F}_*}(\Omega_\varepsilon)$$

produce all candidates of curved foldings. We let Γ be a generator of F . Since Γ has no self-intersections, the symmetries of Γ correspond to the symmetries of the geodesic curvature function μ_F (cf. Lemma 3.24). By (1) of Theorem 3.23, the number $n_{C,\Gamma}$ of elements in $\mathcal{P}_*(\Gamma, C)$ coincides with the number of distinct subsets in $\{F(\Omega_\varepsilon), \check{F}(\Omega_\varepsilon), F_*(\Omega_\varepsilon), \check{F}_*(\Omega_\varepsilon)\}$. So we obtain Theorem A by Theorem 4.7. $\square$

Example 4.9 We consider a one-quarter of the unit circle given by

$$\mathbf{c}(s) := (\cos s, \sin s, 0) \quad \left(|s| \leq \frac{\pi}{4}\right)$$

and

$$\alpha(s) := \frac{\pi}{4} - \frac{s}{2}, \quad T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $T(C) = C$, and the developable strip $F := F^\alpha$ along $C := \mathbf{c}([-\pi/4, \pi/4])$ with the first angular function α induces four associated origami maps

$$\Phi_1 := \Phi_F, \quad \Phi_2 := \check{\Phi}_F, \quad \Phi_3 := T \circ \Phi_F, \quad \Phi_4 := T \circ \check{\Phi}_F.$$

Since the normalized geodesic curvature $\mu_F = \cos \alpha$ does not have symmetries, the four curved foldings are distinct. In fact, Fig. 5 left (resp. right) indicates the images of Φ_1 and Φ_2 (resp. $\Phi_1, \dots, \Phi_4$), by which we can observe the four curved foldings $\Phi_1(\Omega_\varepsilon), \dots, \Phi_4(\Omega_\varepsilon)$ are distinct as subsets of $\mathbb{R}^3$ although they are congruent to each other.

5 The congruence classes of isomers of developable strips

We first consider the case that C admits a symmetry:

Lemma 5.1 *Suppose that C lies in a plane Π , and let T_0 be the reflection with respect to Π . Then $T_0 \circ f = \check{f}$ holds for each $f \in \mathcal{D}(C)$.*

Proof We may assume that C lies in the xy -plane in $\mathbb{R}^3$. The reflection T_0 maps $(x, y, z) \in \mathbb{R}^3$ to $(x, y, -z) \in \mathbb{R}^3$. Since $\mathbf{b} = (0, 0, 1)$ and the second angular function of $\check{f}$ coincides with that of f , the assertion follows by a direct calculation. $\square$

Let $\mathbf{c}(s)$ ($s \in \mathbb{I}_l$) be the arc-length parametrization of C , where $\mathbb{I}_l := [-l/2, l/2]$. The following assertion plays an important role in the latter discussions:

Theorem 5.2 *Let $F \in \mathcal{D}_*(C)$ be a normal form, and let T be a non-trivial symmetry of C . Then, the following two assertions hold:*

- (1) *Suppose that T is a positive symmetry. Then $T \circ F(s, v) = F_*(s, v)$. Moreover, in this setting, if μ_F has a symmetry (cf. Definition 1.1), then $T \circ F(-s, v) = \check{F}(s, v)$.*
- (2) *Suppose that T is a negative symmetry. Then $T \circ F(s, v) = \check{F}_*(s, v)$ holds. Moreover, if μ_F has a symmetry, then $T \circ F(-s, v) = F(s, v)$.*

We remark that the developable strip F^α given in Example 3.15 satisfies (1).

Proof By a suitable motion in $\mathbb{R}^3$, we may assume that $F(0, 0) = \mathbf{0}$ and T is an orthogonal matrix. Then its determinant $\sigma := \det(T)$ is equal to 1 (resp. -1) if T is a positive (resp. negative) symmetry of C .

Since C admits a non-trivial symmetry, we have $\kappa(-s) = \kappa(s)$, where $\kappa(s)$ is a curvature function of C with respect to the arc-length parametrization $\mathbf{c}$ of C on $\mathbb{I}_l$. By the definition of $\alpha_*(s)$, we have

$$\kappa(-s) \cos \alpha_*(s) = \mu(s) = \kappa(s) \cos \alpha_F(s) = \kappa(-s) \cos \alpha_F(s),$$

that is, $\cos \alpha_* = \cos \alpha_F$. By (4) of Proposition 4.1, we can conclude $\alpha_* = \alpha_F$. Then we have

$$T\xi_F = \cos \beta_F T\mathbf{e} + \sin \beta_F (\cos \alpha_* T\mathbf{n} + \sin \alpha_* T\mathbf{b}),$$

where β_F is the second angular function of F and $\mathbf{e}$, $\mathbf{n}$, $\mathbf{b}$ are the unit tangent vector field, the unit principal normal vector field and the unit bi-normal vector field of C , respectively.

By (2.15) and (A.1), we have

$$T\mathbf{n}(s) = \mathbf{n}(-s) = \mathbf{n}^\sharp(s), \quad T\mathbf{b}(s) = -\sigma \mathbf{b}(-s) = \sigma \mathbf{b}^\sharp(s)$$

and

$$\begin{aligned} T\xi_F(s) &= \cos(\pi - \beta_F(s))\mathbf{e}^\sharp(s) \\ &\quad + \sin(\pi - \beta_F(s))(\cos(\sigma\alpha_*(s))\mathbf{n}^\sharp(s) + \sin(\sigma\alpha_*(s))\mathbf{b}^\sharp(s)). \end{aligned}$$

By Proposition 2.8, $T\xi_F(s)$ gives the normalized ruling vector field of $T \circ F$. Thus $T \circ F(s, v)$ belongs to $\mathcal{D}_*(-C)$, and its first and second angular functions are given by $-\sigma\alpha_*(s)$ and $\pi - \beta_F(s)$, respectively. Since the geodesic curvature function of $T \circ F(s, v)$ coincides with $\hat{\mu}(s)$ (cf. Proposition 2.8), $T \circ F$ must coincide with F_* (resp. $\check{F}_*$) if T is a positive (resp. negative) symmetry of C .

We next suppose that μ_F has a symmetry. By Proposition 4.1, $F^\sharp = \check{F}_*$ holds. If $\sigma = 1$, then

$$T \circ F(s, v) = F_*(s, v) = \check{F}^\sharp(s, v) = \check{F}(-s, v),$$

which implies the second assertion of (1). Similarly, considering the case of $\sigma = -1$, we also obtain the second assertion of (2). $\square$

Corollary 5.3 *Let $F \in \mathcal{D}_*(C)$ be a normal form. If C has a non-trivial symmetry T , then $\{F(\Omega_\varepsilon), \check{F}(\Omega_\varepsilon)\}$ coincide with $\{T \circ F_*(\Omega_\varepsilon), T \circ \check{F}_*(\Omega_\varepsilon)\}$ for sufficiently small $\varepsilon(> 0)$.*

Proof By Theorem 5.2, if T is positive (resp. negative), then $T \circ F(s, v) = F_*(s, v)$ (resp. $T \circ F(s, v) = \check{F}_*(s, v)$) and $T \circ \check{F}(s, v) = \check{F}_*(s, v)$ (resp. $T \circ \check{F}(s, v) = F_*(s, v)$). $\square$

The following assertion is a refinement of (Honda et al. 2020c, Lemma 3.2):

Proposition 5.4 *Let $F \in \mathcal{D}_*(C)$ be a normal form. Then $F(\Omega_\varepsilon)$ is congruent to $\check{F}(\Omega_\varepsilon)$ for sufficiently small $\varepsilon > 0$ if and only if*

- (1) C lies in a plane, or
- (2) C has a positive symmetry and μ_F also has a symmetry.

Proof By Lemma 5.1 and (2) of Theorem 5.2, (1) or (2) implies that $F(\Omega_\varepsilon)$ is congruent to $\check{F}(\Omega_\varepsilon)$. To show the converse assertion, we suppose that $F(\Omega_\varepsilon)$ is congruent to $\check{F}(\Omega_\varepsilon)$. Then, there exists an isometry T on $\mathbf{R}^3$ such that

$$T \circ \check{F}(\Omega_\varepsilon) = F(\Omega_\varepsilon). \quad (5.1)$$

By Lemma 5.1, we may assume that C does not lie in any plane. It is sufficient to show (2). Since T must be non-trivial, that is, it reverses the orientation of C (cf. Proposition A.1). If T is a negative symmetry of C , then by Theorem 5.2, we have

$$\check{F}(\Omega_\varepsilon) = T \circ F(\Omega_\varepsilon) = T \circ F^\sharp(\Omega_\varepsilon) = F_*(\Omega_\varepsilon).$$

By Proposition 2.9, we have $\check{F} = F_*$, and by Proposition 4.5, Γ has a symmetry. Then, by Lemma 3.24 and Theorem 5.2, we have

$$\check{F}(\Omega_\varepsilon) = T \circ F(\Omega_\varepsilon) = F(\Omega_\varepsilon),$$

contradicting (3.6). So T is a positive symmetry of C and $T \circ F = F_*$ by Theorem 5.2. By (5.1), we have

$$F_*(\Omega_\varepsilon) = T \circ F(\Omega_\varepsilon) = \check{F}(\Omega_\varepsilon),$$

which implies $F_* = \check{F}$. By Proposition 4.5, μ_F has a symmetry. So we obtain (2). $\square$

Proposition 5.5 *Let $F \in \mathcal{D}_*(C)$ be a normal form. Then, for sufficiently small $\varepsilon(>0)$, $F(\Omega_\varepsilon)$ is congruent to $\check{F}_*(\Omega_\varepsilon)$ if and only if*

- (1) C has a negative symmetry, or
- (2) μ_F has a symmetry.

Proof By (2) of Theorem 5.2 and Corollary 4.6, (1) or (2) implies that $F(\Omega_\varepsilon)$ is congruent to $\check{F}_*(\Omega_\varepsilon)$. So it is sufficient to show the converse. We suppose that $F(\Omega_\varepsilon)$ is congruent to $\check{F}_*(\Omega_\varepsilon)$. We also suppose that μ_F has no symmetries. By Proposition 4.5, $F(\Omega_\varepsilon) \neq \check{F}_*(\Omega_\varepsilon)$ holds. So C must have a symmetry T such that $T \circ F(\Omega_\varepsilon) = \check{F}_*(\Omega_\varepsilon)$. If T is not non-trivial, then C lies in a plane Π and T is the reflection with respect to Π . By Lemma 5.1, we have

$$\check{F}(\Omega_\varepsilon) = T \circ F(\Omega_\varepsilon) = \check{F}_*(\Omega_\varepsilon),$$

which implies $F(\Omega_\varepsilon) = F_*(\Omega_\varepsilon)$. However, this is impossible by Proposition 4.4. By Proposition A.1 in the appendix, we may assume T is a non-trivial symmetry of C , that is, T is either a positive or negative symmetry. If T is a positive symmetry of C , Theorem 5.2 yields $T \circ F_* = F$ and

$$F(\Omega_\varepsilon) = T \circ F_*(\Omega_\varepsilon) = \check{F}(\Omega_\varepsilon),$$

contradicting (3.6). So T must be a negative symmetry of C . $\square$

Similarly, the following assertion holds:

Proposition 5.6 *Let $F \in \mathcal{D}_*(C)$ be a normal form. Suppose that C does not lie in any plane in $\mathbf{R}^3$. Then for sufficiently small $\varepsilon(> 0)$, $F(\Omega_\varepsilon)$ is congruent to $F_*(\Omega_\varepsilon)$ if and only if C has a positive symmetry.*

Proof If C has a positive symmetry, then (1) of Theorem 5.2 implies that $F(\Omega_\varepsilon)$ is congruent to $F_*(\Omega_\varepsilon)$. So it is sufficient to prove the converse. Suppose that there exists an isometry T satisfying $T \circ F(\Omega_\varepsilon) = F_*(\Omega_\varepsilon)$. Since $F(\Omega_\varepsilon) \neq F_*(\Omega_\varepsilon)$ (cf. Proposition 4.4), T is a non-trivial symmetry of C . If T is a negative symmetry, Theorem 5.2 yields that $T \circ F_* = \check{F}$. Then we have

$$F(\Omega_\varepsilon) = T \circ F_*(\Omega_\varepsilon) = \check{F}(\Omega_\varepsilon),$$

which contradicts (3.6). So T must be a positive symmetry of C . $\square$

We prove the following assertion:

Theorem 5.7 *Let $F \in \mathcal{D}_*(C)$ be a normal form. If the normalized geodesic curvature of F has no self-intersections, then the number N_F of congruence classes of*

$$F(\Omega_\varepsilon), \quad \check{F}(\Omega_\varepsilon), \quad F_*(\Omega_\varepsilon), \quad \check{F}_*(\Omega_\varepsilon)$$

satisfies $N_F \leq n_F$ (the definition of n_F is given in Theorem 4.7) and also the following properties:

- (1) *If C has no symmetries and μ_F has no symmetries, then $N_F = 4$.*
- (2) *If not the case in (1), then $N_F \leq 2$ holds.*
- (3) *Moreover $N_F = 1$ holds if and only if*
 - (a) *C lies in a plane and has a non-trivial symmetry,*
 - (b) *C lies in a plane and μ_F has a symmetry, or*
 - (c) *C has a positive symmetry, and μ_F has a symmetry.*

Proof Since two non-congruent subsets of $\mathbf{R}^3$ are distinct, the relation $N_F \leq n_F$ is obvious. We suppose (1). Since C has no symmetry, C does not lie in any plane. If $N_F < 4$, then replacing F by $\check{F}$, F_* or $\check{F}_*$, we may assume that $F(\Omega_\varepsilon)$ is congruent to $G(\Omega_\varepsilon)$ for sufficiently small $\varepsilon(> 0)$, where G is $\check{F}$, F_* or $\check{F}_*$. If $G = \check{F}$, then Proposition 5.4 implies that C has a positive symmetry, a contradiction. If $G = F_*$, then by Proposition 5.6, C has a positive symmetry, a contradiction. If $G = \check{F}_*$ then by Proposition 5.5, C has a positive symmetry or Γ has a symmetry, which is also a contradiction. So we obtain (1).

We now prove (2). If μ_F has a symmetry, then $N_F \leq 2$ follows from Theorem 4.7. So we may assume that C has a symmetry T . If T is trivial, then C lies in a plane and T is the reflection with respect to the plane (cf. Lemma 5.1). Then, we have

$$T \circ F(\Omega_\varepsilon) = \check{F}(\Omega_\varepsilon), \quad T \circ F_*(\Omega_\varepsilon) = \check{F}_*(\Omega_\varepsilon)$$

for sufficiently small $\varepsilon > 0$, and so $N_F \leq 2$ is obtained. We next consider the case that T is non-trivial. Then Corollary 5.3 implies that $N_F \leq 2$.

Finally, we prove (3). We consider the case (a) or (b). Then C lies in a plane, and $F(\Omega_\varepsilon)$ (resp. $F_*(\Omega_\varepsilon)$) is congruent to $\check{F}(\Omega_\varepsilon)$ (resp. $\check{F}_*(\Omega_\varepsilon)$) by Lemma 5.1. If (a) happens, then C admits a positive symmetry by Proposition A.2. So (1) of Theorem 5.2 implies that $F(\Omega_\varepsilon)$ is congruent to $F_*(\Omega_\varepsilon)$, and $N_F = 1$ is obtained. If (b) happens, then $F(\Omega_\varepsilon) = \check{F}_*(\Omega_\varepsilon)$ by Proposition 4.5. So we have $N_F = 1$.

We next consider the case (c). By (1) of Theorem 5.2, $F(\Omega_\varepsilon)$ is congruent to $F_*(\Omega_\varepsilon)$. Moreover, since μ_F has a symmetry, $F(\Omega_\varepsilon)$ (resp. $\check{F}(\Omega_\varepsilon)$) is congruent to $\check{F}_*(\Omega_\varepsilon)$ (resp. $F_*(\Omega_\varepsilon)$), so we have $N_F = 1$.

Conversely, we suppose $N_F = 1$. Then $F(\Omega_\varepsilon)$ must be congruent to $\check{F}_*(\Omega_\varepsilon)$, and so, Proposition 5.5 implies that

- (i) C has a negative symmetry, or
- (ii) Γ has a symmetry.

If C lies in a plane, then (i) corresponds to (a) and (ii) corresponds to (b). So we may assume that C does not lie in any plane. Since $N_F = 1$, $F(\Omega_\varepsilon)$ also congruent to $\check{F}(\Omega_\varepsilon)$. Since C is non-planar, (2) of Proposition 5.4 holds, which implies (c). $\square$

We now prove Theorem B in the introduction:

Proof of Theorem B We fix a curved folding $P \in \mathcal{P}_*(\Gamma, C)$ arbitrarily. Then there exists a normal form $F \in \mathcal{D}_*(C)$ such that $P = \Phi_F$ and so $N_{C,\Gamma} = N_F$. Then Γ gives a generator of F . By Lemma 3.24, the condition that Γ has a symmetry can be replaced by the condition that $\hat{\mu}_F$ has a symmetry. By Theorem 3.23, Theorem B is obtained as a corollary of Theorem 5.7. $\square$

As an application, we can prove the following:

Corollary 5.8 *Suppose that C does not lie in any plane. Let $f \in \mathcal{D}_*(C)$. If the derivatives of the curvature function κ_f of C and the geodesic curvature function μ_f of f both do not vanish at the midpoint of C , then $N_f = 4$ and the number of the congruence classes of curved foldings in $\mathcal{P}_*(\Gamma, |C|)$ is also four, where Γ is a generator of f .*

Proof We let l be the common total length of C and Γ . We denote by κ (resp. μ) the curvature function of C (resp. Γ) with respect to the arc-length parameter on $\mathbb{I}_l$. Since κ (resp. μ) is positive-valued, the existence of a symmetry of C (resp. Γ) implies

$$\kappa(-s) = \kappa(s), \quad (\text{resp. } \mu(-s) = \mu(s)) \quad (s \in \mathbb{I}_l).$$

Differentiating it at $s = 0$, we have

$$\kappa'(0) = 0, \quad (\text{resp. } \mu'(0) = 0). \quad (5.2)$$

We may assume that f is defined on a tubular neighborhood of $I \times \{0\}$ in $I \times \mathbf{R}$. In the parametrization $u \mapsto f(u, v)$ of C , we suppose that the point $s = 0$ corresponds to the point $u = u_0 \in I$. Then (5.2) is equivalent to the condition that $d\kappa_f(u)/du$ (resp. $d\mu_f(u)/du$) does not vanish at $u = u_0$. So, we obtain the conclusion. $\square$

Remark 5.9 For a given curve in $\mathbf{R}^2$ or $\mathbf{R}^3$, it is difficult to judge that it has a symmetry or not, in general. So we gave Corollary 5.8, which can be considered as a practical criterion on whether a given developable strip (or a curved folding) induces mutually non-congruent isomers or not, without use of the arc-length parametrization of C . In fact, let $f \in \mathcal{D}_*(C)$ be a developable strip as in the proof of Corollary 5.8. As a preliminary step, one should show that the value $u_0 \in I$ giving the midpoint $\mathbf{c}_f(u_0)$ of $C := \mathbf{c}_f(I)$ lies in a certain subinterval I_1 in I . (This subinterval is best to be taken as small as possible.) After that if one can show that $d\kappa_f(u)/du$ and $d\mu_f(u)/du$ do not vanish at the same time on I_1 , then Corollary 5.8 implies that $N_f = 4$.

Example 5.10 (Honda et al. 2020c) Consider the space curve

$$\mathbf{c}(s) = \left(\arctan s, \frac{\log(1+s^2)}{\sqrt{2}}, s - \arctan s \right) \quad (s \in I),$$

where $I := [1/10, 9/10]$. We set $\alpha(s) := \pi(s+10)/24$, and consider the developable strip $F(s, t)$ whose first angular function is $\alpha(s)$. It can be easily checked that F belongs to $\mathcal{D}_*(C)$. Then the images of F , $\check{F}$, F_* and $\check{F}_*$ belongs to distinct congruence classes as shown in Honda et al. (2020c). This fact can be easily checked by applying Corollary 5.8. In fact, the curvature and torsion functions of $\mathbf{c}$ are given by $\kappa = \tau = \sqrt{2}/(1+s^2)$. Since τ is non-constant, the image of $\mathbf{c}$ does not lie in any plane. Moreover, since $\kappa' > 0$ everywhere and $\mu := \kappa \cos \alpha$ satisfies

$$\mu'(1/2) = -\frac{96 \sin(\pi/16) + 5\pi \cos(\pi/16)}{75\sqrt{2}} < 0.$$

Thus, we have reproved the fact $N_F = 4$. The figures of the images of F , $\check{F}$, F_* and $\check{F}_*$ are given in (Honda et al. 2020c, Fig. 3).

Although, in the above example, the space curve $\mathbf{c}$ has an arc-length parametrization, Corollary 5.8 can be applied without assuming C is parametrized by arc-length:

Example 5.11 We set $d \in (0, \pi/4)$ and $I := [3\pi/8, 5\pi/8]$. Consider the embedded space curve defined by

$$\mathbf{c}(t) := (\cos t, \sin(t+d), t) \quad (t \in I).$$

We let f be the developable strip whose first angular function is $\alpha := \pi/3$ along $C := \mathbf{c}(I)$. Since $d \neq 0$, the torsion function of $\mathbf{c}$ is not identically equal to zero, that is, C does not lie in any plane. Since the curvature function κ of $\mathbf{c}$ is given by

$$\kappa(t) = \left(\frac{3 + \cos 2d + \cos 2t - \cos(2t+2d)}{2(1 + \sin^2 t + \cos^2(t+d))} \right)^{1/2},$$

it has the following expansion

$$\kappa(t) = \frac{1}{2} + \frac{\sin 2t}{2}d + o(d) \quad (5.3)$$

with respect to the parameter d , where $o(d)$ is a term of order higher than d . Since the maximum of $\kappa/2 (= \kappa \cos \alpha)$ on I is less than the minimum of $\kappa(t)$ on I for sufficiently small d , we may assume that f belongs to $\mathcal{D}_*(C)$ (in fact, this holds if $d \leq \pi/5$). By (5.3), we have

$$\kappa'(t) = (\cos 2t)d + o(d).$$

Since $\cos 2t$ is negative on I , the derivative κ' is negative on I for sufficiently small d . Since $\mu(t) := \kappa(t)/2$ is the geodesic curvature function of C along f , the derivatives κ' and μ' do not vanish at the midpoint of C . So, by Corollary 5.8, the images of f , $\check{f}$, f_* and $\check{f}_*$ are mutually non-congruent for sufficiently small $d(> 0)$.

In the authors' previous work, non-congruence of the images of F , $\check{F}$, F_* and $\check{F}_*$ in Example 5.10 was shown by computing mean curvature functions of their associated developable strips. However, this argument does not work in general, since the mean curvature functions of f , $\check{f}$, $(\check{f}_*)^\sharp$ and $(f_*)^\sharp$ for $f \in \mathcal{D}_*(C)$ may not take distinct values along C even when their images are non-congruent each other, as follows:

Proposition 5.12 *Let $\kappa(s)$ be a C^∞ -function defined on an interval I containing $s = 0$, and let $\alpha : I \rightarrow (0, \pi/2)$ be a C^∞ -function satisfying*

$$\kappa'(0) < 0, \quad \alpha'(0) \geq 0. \quad (5.4)$$

Then there exists an embedded space curve $\mathbf{c}(s)$ defined on an interval $I_1 \subset I$ containing $s = 0$ satisfying the following properties:

- (1) *$\mathbf{c}(s)$ gives an arc-length parametrization of $C := \mathbf{c}(I_1)$ whose curvature function coincides with κ on I_1 .*
- (2) *Let $F(s, v)$ be the developable strip along C whose first angular function is $\alpha(s)$. Then F belongs to $\mathcal{D}_*(C)$ for sufficiently small choice of I_1 , and the images of F , $\check{F}$, F_* , $\check{F}_*$ are non-congruent each other; however*
- (3) *the restriction of the mean curvature function of $F(s, v)$ on $I_1 \times \{0\}$ is equal to that of $\check{F}_*(-s, v)$.*

The curvature functions of the space curves and the first angular functions of the developable strips given in Example 5.11 satisfy (5.4).

Proof We let $l > 0$ be a number so that $\mathbb{I}_l = [-l/2, l/2]$ is contained in I . Consider a C^∞ -function $\tau(t)$ defined on $\mathbb{I}_l$. Then there exists a unique regular space curve $\mathbf{c} : \mathbb{I}_l \rightarrow \mathbf{R}^3$ whose curvature and torsion functions are $\kappa(s)$ and $\tau(s)$ ($s \in \mathbb{I}_l$), respectively. Let $F(s, v)$ be the normal form of a developable strip whose angular function is $\alpha(s)$. If we choose sufficiently small l , then F belongs to $\mathcal{D}_*(C)$ because $0 < \alpha(0) < \pi/2$. By (5.4), $\mu := \kappa \cos \alpha$ satisfies

$$\mu'(0) = \kappa'(0) \cos \alpha(0) - \kappa(0) \alpha'(0) \sin \alpha(0) < 0.$$

This, with the fact $\kappa'(0) < 0$, F satisfies the assumption of Corollary 5.8, and we obtain (1) and (2).

We next adjust the torsion function τ to obtain (3): The first angular function $A(s)$ of $F_1(s, v) := \check{F}_*(-s, v)$ is given by (cf. (2.16) and (4.1))

$$A(s) = \arccos\left(\frac{\kappa(-s) \cos \alpha(-s)}{\kappa(s)}\right). \quad (5.5)$$

We let $H(s, v)$ and $H_1(s, v)$ be the mean curvature functions of $F(s, v)$ and $F_1(s, v)$, respectively. By (2.13), the condition $H(s, 0) = H_1(s, 0)$ is equivalent to the identity

$$\frac{\kappa^2 + (\alpha' + \tau)^2}{\sin \alpha} = \frac{\kappa^2 + (A' + \tau)^2}{\sin A}, \quad (5.6)$$

which can be considered as the quadratic equation of τ . In fact, (5.6) can be written as

$$B_0(s)\tau^2(s) + 2B_1(s)\tau(s) + B_2(s) = 0,$$

where

$$\begin{aligned} B_0 &:= \csc \alpha - \csc A, & B_1 &:= \alpha' \csc \alpha - A' \csc A, \\ B_2 &:= \left((\alpha')^2 + \kappa^2 \sin^2 \alpha\right) \csc \alpha - \left((A')^2 + \kappa^2 \sin^2 A\right) \csc A. \end{aligned}$$

By (5.5), we have

$$A(0) = \alpha(0), \quad A'(0) = \frac{2\kappa'(0) \cot \alpha(0)}{\kappa(0)} - \alpha'(0),$$

which imply (cf. (5.4))

$$B_0(0) = 0, \quad B_1(0) = \frac{-2 \csc \alpha(0)}{\kappa(0)} \left(\kappa'(0) \cot \alpha(0) - \kappa(0) \alpha'(0) \right) > 0.$$

If we set

$$\tau = \frac{-B_1 + \sqrt{B_1^2 - B_0 B_2}}{B_0} = \frac{-B_2}{B_1 + \sqrt{B_1^2 - B_0 B_2}}, \quad (5.7)$$

then $\lim_{s \rightarrow 0} \tau(s)$ is equal to $-B_2(0)/(2B_1(0))$. In particular, τ gives a C^∞ -function on $\mathbb{I}_l$ for sufficiently small l . So (3) is obtained. $\square$

Fig. 6 The shapes of Γ_1 (left) and Γ_2 (right)

6 The case that C is closed

In this section, we consider the case $J = \mathbb{T}_l^1$, that is, C is a closed embedded curve in $\mathbb{R}^3$. We fix an immersed curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ with arc-length parameter whose curvature function $\mu(s)$ of $\gamma(s)$ is an l -periodic smooth function $\tilde{\mu} : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\max_{s \in [0, l]} |\mu(s)| < \min_{w \in \mathbb{T}_l^1} \kappa(w).$$

We let $\Gamma \subset \mathbb{R}^2$ be the image $\gamma([b_0, b_0 + l])$ of the plane curve γ for some $b_0 \in [0, l]$ such that $\gamma([b_0, b_0 + l])$ has no self-intersections. Here, such a Γ is not uniquely determined, since the possibilities for Γ have the ambiguity of the choice of the value b_0 (Γ and C have the same length).

If $\mathcal{P}_*(\Gamma, C)$ is defined as in the introduction, then, for each $P \in \mathcal{P}_*(\Gamma, C)$, we can write $P = \Phi(F)$ for a developable strip $F \in \mathcal{D}_*(C)$ written in a normal form such that $\mu(s)$ coincides with the lift $\tilde{\mu}_F$ of the normalized curvature function μ_F of F (see the beginning of Sect. 2).

Remark 6.1 The plane curve $\Gamma := \gamma([b_0, b_0 + l])$ is prepared as the crease pattern of curved foldings belonging to $\mathcal{P}_*(\Gamma, C)$. So Γ should not have any self-intersections. However, this property depends on the choice of $b_0 \in [0, l]$. In fact, we consider the trochoid γ defined by

$$\gamma(t) := (2t/3 - \sin t, 1 - \cos t) \quad (t \in \mathbb{R}). \quad (6.1)$$

Then its curvature function is 2π -periodic, and two subarcs $\Gamma_1 = \gamma([0, 2\pi])$ and $\Gamma_2 = \gamma([\pi, 3\pi])$ can be considered as fundamental pieces of the trochoid (cf. Fig. 6). The arc Γ_2 has a self-intersection but Γ_1 does not.

We fix a normal form $F \in \mathcal{D}_*(C)$ such that Γ is a generator of F . Without loss of generality, we may assume that F is defined on a tubular neighborhood of $\mathbb{T}_l^1 \times \{0\}$ in $\mathbb{T}_l^1 \times \mathbb{R}$. We assume that the geodesic curvature function μ_F of F coincides with μ . For each fixed $b \in [0, l]$ and $\sigma \in \{1, -1\}$, we set

$$\tilde{\mathbf{c}}(s) := \mathbf{c}(\sigma s + b),$$

which parametrizes the curve $C := \mathbf{c}(\mathbb{T}_l^1)$. Since $F \in \mathcal{D}_*(C)$, we can apply Proposition 3.10 for the curve $\tilde{C} := \tilde{\mathbf{c}}(\mathbb{T}_l^1)$. Since $\sigma = \pm 1$, we obtain normal forms

$F_b^i \in \mathcal{D}_*(C)$ ($i = 1, 2, 3, 4$) whose first angular functions $\alpha_b^i \in C_{\pi/2}^\infty(\mathbb{T}_l^1)$ satisfy $\alpha_b^1 = -\alpha_b^2 > 0$, $\alpha_b^3 = -\alpha_b^4 > 0$ and

$$\begin{aligned} (\mu_{F_b^1} =) \kappa(s+b) \cos \alpha_b^1(s) &= \mu_F(s), & (\mu_{F_b^2} =) \kappa(s+b) \cos \alpha_b^2(s) &= \mu_F(s), \\ (\mu_{F_b^3} =) \kappa(-s+b) \cos \alpha_b^3(s) &= \mu_F(s), & (\mu_{F_b^4} =) \kappa(-s+b) \cos \alpha_b^4(s) &= \mu_F(s), \end{aligned} \quad (6.2)$$

where $\kappa(s)$ is the curvature function of $\mathbf{c}(s)$ ($s \in \mathbb{T}_l$). By definition, these four developable strips are all geodesically equivalent to F , and satisfy

$$F_b^1(s, 0) = F_b^2(s, 0) = \mathbf{c}(s+b), \quad F_b^3(s, 0) = F_b^4(s, 0) = \mathbf{c}(-s+b)$$

and $F_0^1 = F$. The following assertion holds:

Proposition 6.2 *Let $G \in \mathcal{D}_*(|C|)$ be a normal form. If G is geodesically equivalent to F , then there exist $b \in [0, l)$ and $j \in \{1, 2, 3, 4\}$ such that $G = F_b^j$.*

Proof Since G is geodesically equivalent to F , $\mu_G(\sigma s + b) = \mu_F(s)$ holds for some $\sigma \in \{1, -1\}$ and $b \in \mathbb{T}_l^1$. Replacing F by $F^\sharp$ if necessary, we may assume that $\sigma = 1$. We set $G_1(s, v) := G(s+b, v)$. Then we have $\mu_{G_1}(s, 0) = \mu_F(s, 0)$ and $F, G \in \mathcal{D}_*(C)$. By Proposition 3.13, G_1 must coincide with F or $\tilde{F}$, which proves the assertion because of the definition of the families $\{F_b^j\}_{b \in \mathbb{T}_l^1}$ ($j = 1, 2, 3, 4$). $\square$

Corollary 6.3 *Let $G \in \mathcal{D}_*(|C|)$ be a normal form. If $G(\Omega_\varepsilon)$ is congruent to $F(\Omega_\varepsilon)$ for sufficiently small $\varepsilon(> 0)$, then there exist an isometry T of $\mathbf{R}^3$ and $b \in [0, l)$ and $\sigma \in \{1, -1\}$ such that $G = T \circ F_b^j$.*

Proof Since $G(\Omega_\varepsilon)$ is congruent to $F(\Omega_\varepsilon)$, there exists an isometry T of $\mathbf{R}^3$ such that $G(\Omega_\varepsilon) = T \circ F(\Omega_\varepsilon)$. By Propositions 2.5 and 2.8, $T \circ F$ is right equivalent to G . In particular, there exists a diffeomorphism φ such that $T \circ F \circ \varphi = G$. Since $s \mapsto T \circ F \circ \varphi(s, 0)$ and $s \mapsto G(s, 0)$ are both arc-length parametrizations, there exist $b \in [0, a)$ and $\sigma \in \{1, -1\}$ such that $\varphi(s, 0) = \sigma s + b$. In particular, $T \circ F$ is geodesically equivalent to G , and so we obtain the conclusion applying Proposition 6.2. $\square$

We prove the following assertion:

Theorem 6.4 *Let $F \in \mathcal{D}_*(C)$ and $\{F_b^i\}_{b \in \mathbb{T}_l^1}$ ($i = 1, 2, 3, 4$) be defined as above. Then the following assertions hold:*

- If $g \in \mathcal{D}_*(|C|)$ is geodesically equivalent to F , then there exist $b \in [-l/2, l/2)$ and $j \in \{1, 2, 3, 4\}$ such that g is right equivalent to F_b^j .
- Suppose that C is not a circle and μ_F is not a constant function. Then, for each $\{i, b\}_{(i,b) \in \{1,2,3,4\} \times \mathbb{T}_l^1}$, the set

$$\Lambda_b^i := \left\{ (j, c) \in \{1, 2, 3, 4\} \times \mathbb{T}_l^1 ; \right.$$

$F_c^j(\Omega_\varepsilon)$ is congruent to $F_b^i(\Omega_\varepsilon)$ for sufficiently small $\varepsilon(>0)$

is finite.

(c) Suppose that C and μ_F have no symmetries, then for each $i \in \{1, 2, 3, 4\}$ and $b \in [-l/2, l/2)$, the set Λ_b^i consists of a single element.

Proof Although the strategy of the proof is essentially the same as the corresponding proof for cuspidal edges given in Honda et al. (2020b), we need several modifications, since the real analyticity of C is assumed in Honda et al. (2020b) and the corresponding proof for cuspidal edges does not apply the properties of their normal forms.

The first assertion (a) follows from Proposition 6.2. So we prove (b). Without loss of generality, we may set $F = F_b^i$. Suppose that the congruence classes of Λ_b^i are not finite. By Proposition 6.2, there exist a sequence $F_n := F_{b_n}^{j_n}$ ($n \geq 1$) in $\mathcal{D}_*(|C|)$ such that

- $\{(j_n, b_n)\}_{n=1}^\infty$ consisting of mutually distinct elements in $\{1, 2, 3, 4\} \times \mathbb{T}_l^1$, and
- $F_n(\Omega_\varepsilon)$ ($n \geq 1$) are all congruent to $F(\Omega_\varepsilon)$.

Taking a subsequence of $\{(j_n, b_n)\}_{n=1}^\infty$ if necessary, we may assume that

(A1) $j := j_n$ does not depend on n , and

(A2) $\{b_n\}_{n=1}^\infty$ consists of mutually distinct numbers.

By Corollary 6.3, for each n , there exist an isometry T_n of $\mathbf{R}^3$, $d_n \in [0, l)$ and $\sigma_n \in \{1, -1\}$ such that $F_n(s, v) = T_n \circ F(\sigma_n s + d_n, v)$. Then we have the identity

$$\mu(s) = \mu(\sigma_n s + d_n) \quad (s \in \mathbb{T}_l^1). \quad (6.3)$$

Since the number of possibilities for the signs σ_n is at most two, we may set $\sigma := \sigma_n$. Since μ is not constant, we can take $s_0 \in [-l/2, l/2)$ such that $\mu(s_0)$ is a regular value of the function μ on $\mathbb{T}_l^1$. Substituting $s = s_0$, (6.3) yields $\mu(s_0) = \mu(\sigma s_0 + d_n)$. Since $\mathbb{T}_l^1$ is compact and $\mu(u_0)$ is a regular value of μ , the possibilities for $\{d_n\}_{n=1}^\infty$ are finite, and we may assume $d := d_n$ does not depend on n . So we have

$$F(\sigma' s + b_n, v) = F_n(s, v) = T_n \circ F(\sigma s + d, v)$$

for $n \geq 1$, where $\sigma' = -1$ if $j \in \{1, 2\}$ and $\sigma' = +1$ if $j \in \{3, 4\}$. Substituting $v = 0$, we have $F_n(\sigma' s + b_n, 0) = T_n \circ F(\sigma s + d, 0)$. Since C is not a circle, the number of symmetries of C is finite (see Proposition A.3 in the appendix). So, the number of possibilities for b_n and T_n are also finite, contradicting (A2), and we obtain (b).

Finally, we prove (c). If the assertion fails, then we can choose $(j, c) \in \Lambda$. By Corollary 6.3, there exist T of $\mathbf{R}^3$, $b \in [0, l)$ and $\sigma \in \{1, -1\}$ such that

$$F_c^j(s, v) = T \circ F(\sigma s + b, v). \quad (6.4)$$

Substituting $v = 0$, it can be easily checked that $T(C) = C$. Since C admits no symmetries, T must be the identity map. So (6.4) reduces to $F_c^j(s, v) = F(\sigma s + b, v)$.

Since μ_F coincides with $\mu_{F_c^j}$ (cf. (6.2)), we have $\mu_F(s) = \mu_F(\sigma s + b)$ ($s \in \mathbb{T}_l^1$). Since μ_F has no symmetries, we have $(\sigma, b) = (1, 0)$, that is, $F_c^j = F$ holds, proving the assertion (c). $\square$

Proof of Theorem C Like as in the case of Theorem B, Theorem C is obtained by applying Theorems 6.4 and 3.23. Condition (3) is not used to prove Theorem C, which is required for the definition of the set $\mathcal{P}_*(\Gamma, |C|)$. $\square$

As we have pointed out, the crease pattern Γ of a curved folding along a closed curve C does not necessary to be a closed curve. However, from the viewpoint of curved foldings, the most interesting case is that Γ is a closed curve without self-intersections as well as the crease C :

Example 6.5 Consider a curve

$$\mathbf{c}_m(t) := \left((2 + \cos mt) \cos t, (2 + \cos mt) \sin t, \sin mt \right)$$

lying on a rotationally symmetric torus in $\mathbf{R}^3$. We denote by L_m the length of $\mathbf{c}_m$. Then for each $m \geq 2$, the inequality $\min_{t \in [0, 2\pi]} \kappa_m(t) > 2\pi/L_m$, holds, where $\kappa_m(t)$ is the curvature of $\mathbf{c}_m(t)$. So if Γ is an ellipse of length L_m which is sufficiently close to a circle, then $\mathcal{P}_*(\Gamma, C_m)$ is non-empty. For example, consider an ellipse $\gamma(t) := (\cos t, a \sin t)$ for $a := 6/5$, whose curvature function is given by $\mu(t) = a(\sin^2 t + a^2 \cos^2 t)^{-3/2}$. We set $m = 3$ and

$$\tilde{\gamma}(t) := k\gamma(t), \quad k := L_3/l_\gamma \approx 3.31,$$

where l_γ is the length of γ . Then the two curves $\tilde{\gamma}$ and $\mathbf{c}_3$ have a common length. We reparametrize $\tilde{\gamma}$ by the arc-length parameter s ($0 \leq s \leq L_3$) so that $\tilde{\gamma}(0) = (k, 0)$. Let $F \in \mathcal{D}_*(C_3)$ be the developable strip along C_3 such that $\mu_F(t) = \hat{\mu}(t)$, where $\hat{\mu}(s)$ is the curvature function of $\hat{\gamma}(s) := \tilde{\gamma}(t(s))$. Figure 7 indicates the crease patterns of the curved foldings induced by F (left), F_b^1 (center) and F_{2b}^1 (right) for $b = 1/8$ via the map Φ , respectively (cf. (6.2) and Definition 3.18).

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Fig. 7 The ruling directions on the ellipse of three non-congruent curved folding along C_3

Appendix A. Symmetries of curves

In this appendix, we prove several assertions on symmetries of curves in $\mathbf{R}^2$ or $\mathbf{R}^3$: Let $\mathbf{c} : \mathbb{I}_l \rightarrow \mathbf{R}^3$ be an embedded arc parametrized by arc-length, and denote by C its image, where $\mathbb{I}_l = [-l/2, l/2]$. Let T be a symmetry (cf. Definition 1.1) of C which reverses the orientation of C . Here, we ignore the orientation of C . The following assertion holds obviously.

Proposition A.1 *Suppose the space curve C admits a symmetry T . Then T is an involution, and has the following properties:*

- (1) C lies in a plane, and T is the reflection with respect to the plane, or
- (2) T is a positive or negative non-trivial symmetry of C .

If T is a non-trivial symmetry of C , then $T \circ \mathbf{c}(-s) = \mathbf{c}(s)$ holds. By differentiating, we have

$$\begin{aligned} -T\mathbf{e}(-s) &= \mathbf{e}(s), \quad \kappa(-s) = \kappa(s), \quad T\mathbf{n}(-s) = \mathbf{n}(s), \\ -T\mathbf{b}(-s) &= \sigma\mathbf{b}(s), \quad \tau(-s) = \sigma\tau(s), \end{aligned} \quad (\text{A.1})$$

where $\sigma \in \{+, -\}$ is the sign of $\det(T)$. Using this, we show the following assertion:

Proposition A.2 *The space curve C admits a positive symmetry and a negative symmetry (cf. Definition 1.1) at the same time if and only if C lies in a plane and C has a non-trivial symmetry.*

Proof Suppose that C admits a positive symmetry T_1 and a negative symmetry T_2 at the same time. By (A.1), the torsion function $\tau(s)$ of $\mathbf{c}(s)$ satisfies $\pm\tau(-s) = \tau(s)$. So τ must vanish identically, that is, C lies in a plane. Since the trivial symmetry C is uniquely determined and is the reflection with respect to Π , the symmetry T_1 of C must be non-trivial, and the “only if” part is proved.

To prove the converse, we suppose that C lies in a plane and C has a non-trivial symmetry. Then each point of C is fixed by the reflection T_0 with respect to the plane. We let T_1 be a non-trivial symmetry of C . If T_1 is a positive (resp. negative) symmetry of C , then $T_2 := T_0 \circ T_1$ is a negative (resp. positive) symmetry of C . So we obtain the conclusion. $\square$

Proposition A.3 *Let $\mathbf{c} : \mathbb{T}_l^1 \rightarrow \mathbb{R}^3$ ($l > 0$) be an embedded C^∞ -regular curve. Then $C := \mathbf{c}(\mathbb{T}_l^1)$ has at most finitely many symmetries, unless C is a circle.*

Proof We may assume that $\mathbf{c}(s)$ has an arc-length parametrization and denote by $\kappa(s)$ its curvature function. Since $\mathbf{c}$ is not a circle, κ is not a constant function. By Sard's theorem, the set of critical values of κ is of measure zero. We choose a regular value r of the curvature function κ such that $\kappa^{-1}(r)$ consists of finitely many points $s_1, \dots, s_n \in \mathbb{T}_l^1$. Then we set $\mathbf{x}_i = \mathbf{c}(s_i)$ ($i = 1, \dots, n$). Suppose that there are infinitely many distinct non-trivial symmetries $\{T_j\}_{j=0}^\infty$ of C . Then $T_j(\mathbf{x}_1) \in \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. So, there exists $j_0 \in \{1, \dots, n\}$ such that $T_j(\mathbf{x}_1) = \mathbf{x}_{j_0} = T_1(\mathbf{x}_1)$ for infinitely many $j \geq 2$. Since there exist at most finitely many non-trivial symmetries of C fixing $\mathbf{x}_1$, $\{T_j\}_{j=0}^\infty$ must consist of at most finitely many isometries of $\mathbb{R}^3$ a contradiction. $\square$

Note Added in Proof: In the last section of the following authors' recent paper A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada, *A generalization of Zakalyukin's lemma, and symmetries of surface singularities*, to appear in J. Singul. (arXiv:2104.03505),

the existence of a canonical map from the set of real analytic cuspidal edges along a curve C into the set of curved folding along C is shown. This map is not injective, but it maps a pair of cuspidal edges with the same first fundamental form to a pair of curved folding with the same crease and crease pattern. The existence of the map does not produce any new proofs of our main theorems, but is useful to understand how two objects are related to each other.

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