Infinite groups with many complemented subgroups

M. Ferrara¹ · M. Trombetti²

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Abstract

This paper has two souls. On one side, it is a survey on (infinite) groups in which certain systems of subgroups are complemented (like for instance the abelian subgroups). On another side, it provides generalizations and new, easier proofs of some (un)known results in this area.

Keywords Subgroup lattice · Complemented subgroup · K-group · C-group

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✉ M. Trombetti
marco.trombetti@unina.it

M. Ferrara
maria.ferrara1@unicampania.it

¹ Dipartimento di Matematica e Fisica, Università degli Studi della Campania “Luigi Vanvitelli”,
viale Lincoln 5, Caserta, Italy

² Dipartimento di Matematica e Applicazioni “Renato Caccioppoli”, Università di Napoli Federico II
Complesso Universitario Monte S. Angelo, Via Cintia, Naples, Italy

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1 Introduction

A subgroup H of a group G is *complemented* in G if it admits a *complement*, that is, if there is a subgroup K of G such that $G = \langle H, K \rangle$ and $H \cap K = 1$; should it also happen that $HK = KH$, we say that H is *permutably complemented* in G and K is a *permutable complement* to H in G . A group G is called a *K-group* if all its subgroups are complemented and it is called a *C-group* if all its subgroups are permutably complemented. The structure of arbitrary C-groups is well known and can for instance be found in Theorem 3.2.5 of Schmidt (1994). In particular, abelian C-groups turn out to be direct products of *elementary abelian groups* (i.e., abelian groups in which every non-trivial element has order p , where p is a fixed prime number); notice that in this paper we will often write that an abelian group is a C-group instead of writing that it is the direct product of elementary abelian groups. Although Schmidt (1994) provides a great deal of information on groups in which all subgroups are complemented in some sense, it gives only a very brief account on groups in which certain systems of subgroups are (permutably) complemented. The aim of this survey is essentially to fill this gap, at least for infinite groups.

There are many papers concerning with groups in which particular systems of subgroups are (permutably) complemented (Černikov 1954, Černikov 1967; Emaldi 1971; de Giovanni and Franciosi 1984; Gorchakov 1960, 1962) but some of the oldest ones are in Russian or Italian, and are difficult to retrieve. This fact caused some of the quoted paper to be neglected or misunderstood in subsequent works. For example, the two papers (Gorchakov 1960, 1962) deal with groups whose subgroups of prime order are permutably complemented and show that this condition is equivalent to the requirement that the finite subgroups are permutably complemented, a fact unnoticed even in the masterpiece (Schmidt 1994) (see page 125); moreover, these papers give much more than a characterization in terms of cartesian products (as can be seen in our Sect. 5, which is essentially based on these results). As a matter of fact, Gorchakov (1960, 1962) encompass a great deal of the results of the much later papers Camp-Mora and Monetta (2020) and de Giovanni and Franciosi (1984); actually in Gorchakov (1962) there is an example (see our Example 5.27) contradicting Theorem A of Camp-Mora and Monetta (2020) (see Sect. 7 for more details concerning also their Theorem B). These are just some of the many instances for which we felt such a survey should have been made.

Here, we want to describe in a homogeneous way all the results of those quoted papers, providing new, easier proofs and often even generalizing their results. It should be also pointed out that our statements are sometimes very different from the original ones: this is because we choose clarity over terminology (which is for instance very heavy in Gorchakov (1960, 1962), and even though the original statements may perhaps seem to be more precise in their descriptions, they are basically equivalent to ours

and every little bit of further information can be easily obtained with an immediate application of the structure of C-groups.

Each section deals with specific systems of subgroups and we refer to the introduction of each section for more details.

1.1 Notation

The notation is mostly standard and can be found in Robinson (1972, 1995) and Schmidt (1994). However, for the sake of the survey it is perhaps convenient to have a place in which most of the notation is explained.

Symbol	Description
(m, n)	The greatest common divisor of the non-negative integers m and n
$\mathbb{N}$	$\{1, 2, 3, \dots, 100, 101, \dots\}$
$\mathbb{N}_0$	$\{0, 1, 2, 3, \dots, 100, 101, \dots\}$
$\mathbb{P}$	The set of all prime numbers
$\mathbb{Q}$	The additive group of all rational numbers
Q_8	The quaternion group of order 8
$\mathbb{Z}$	The additive group of all integers
$\mathbb{Z}_n$	The additive group of all integers modulo n
$\text{Sym}(n)$	The symmetric group on n objects
p'	$(p \text{ is a prime number and } \pi \text{ is a set of primes})$
π'	$\mathbb{P} \setminus \{p\}$
$\mathbb{Q}_\pi$	$\mathbb{P} \setminus \pi$
$\mathbb{Q}_p$	The subgroup of $\mathbb{Q}$ generated by all rational numbers whose denominators are powers of primes in π
$H \simeq K$	$\mathbb{Q}_{\{p\}}$
$H \times K$	$(H \text{ and } K \text{ are arbitrary groups})$
$H \ltimes K$	H is isomorphic to K
$\text{o}(x)$	The direct product of H and K
$\pi(G)$	The semidirect product of H acting on K
$H \simeq_G K$	$(G \text{ is a group, } x \in G \text{ and } H, K \leq G)$
H^G	The order of the element x
K_G	The set of all prime numbers p for which G has an element of order p
	H is G -isomorphic to K
	The normal closure of H in G
	The normal core of K in G

Finally, notice that the multiplicative notation will be employed for all “abstract” abelian groups; so we will for instance speak of “direct factor” instead of “direct summand”.

2 Additional results on C-groups

The structure of C-groups, i.e. groups with only permutably complemented subgroups, is well described in Schmidt (1994). Although we do not really need the main result of this section (Theorem 2.1) in the rest of the paper, it seems to come very handy to put things in perspective in Sect. 5.

Recall that a group is said *locally normal* if every finitely generated subgroup is contained in a finite normal subgroup; in particular, every locally normal group is locally finite. The following result is essentially due to Gorchakov and should be compared to Theorem 3.2.4 of Schmidt (1994).

Theorem 2.1 *Let G be a locally normal group. Then G is residually a C-group if and only if it is a C-group.*

Proof Of course, we can assume G is residually a C-group, so it can be embedded into a cartesian product of C-groups. Since G is periodic, it follows from Theorem 3.2.5 of Schmidt (1994) that G' is a direct product of elementary abelian groups.

Let A be any finite G -invariant subgroup of G' . By hypothesis, there is a normal subgroup N of G such that $N \cap A = 1$ and G/N is a C-group. It easily follows that $A \simeq_G AN/N$ is a direct product of G -invariant subgroups of prime order (see for instance Schmidt 1994, Lemmas 3.1.7 and 3.2.3). Since G is locally normal, we have that G' is covered by finite G -invariant subgroups and so

$$G' = \text{Dr}_{i \in I} K_i$$

is a direct product of G -invariant subgroups K_i of prime order.

Notice now that for any prime number p the Sylow p -subgroups of G are elementary abelian since by hypothesis they are locally embeddable in C-groups and so they are locally elementary abelian (see Schmidt 1994, Lemma 3.1.1); since G is locally finite, we also have that every section of G has elementary abelian Sylow p -subgroups.

Let $i' \in I$ and put

$$K = \text{Dr}_{i \in I \setminus \{i'\}} K_i, \quad \overline{G} = G/K,$$

so $\overline{G}' = K_{i'}K/K \simeq K_{i'}$. Since $\overline{C} = C_{\overline{G}}(\overline{G}')$ is nilpotent, it is the direct product of elementary abelian groups. It is now easy to see that $\overline{C} = \overline{L}_{i'} \times \overline{G}'$ for some $\overline{G}$ -invariant subgroup $\overline{L}_{i'}$ and that $\overline{G}/\overline{L}_{i'}$ is a C-group; in fact, it is a (non-abelian) subgroup of the holomorph of $K_{i'}$ with elementary abelian Sylow primary subgroups.

Write

$$G/G' = \text{Dr}_{j \in J} H_j/G',$$

where each H_j/G' is cyclic of prime order. Let $j' \in J$ and put $M_{j'} = \langle H_j : j \in J \setminus \{j'\} \rangle$.

It is clear that the intersection of all L_i 's and all M_j 's is 1, so that G naturally embeds in the cartesian product of the finite C-groups G/L_i 's and G/M_j 's; we need

to check that such a natural embedding is actually into the direct product of these groups. Let $g \in G$. If $g \in G'$, then $gL_i \neq L_i$ for finitely many $i \in I$, and certainly $g \in M_j$ for all j . Suppose $g \in G \setminus G'$. Then $gM_j \neq M_j$ for finitely many $j \in J$. If $gL_i \notin G'L_i/L_i$ for infinitely many $i \in I$, then $\{g^x : x \in G'\}$ is infinite, thus contradicting the fact that G is locally normal. If $L_i \neq gL_i \in G'L_i/L_i$ for infinitely many $i \in I$, then $\{g^x : x \in G \setminus G'\}$ is infinite (notice that $xL_i \notin G'L_i/L_i$ for finitely many $i \in I$, whenever $x \in G \setminus G'$), again contradicting the local normality of G . Putting all together we see that G embeds into the direct product of the finite C-groups G/L_i 's and G/M_j 's. Now, Theorem 3.2.5 of Schmidt (1994) shows that G is a C-group. $\square$

Notice that Example 5.3 shows that the hypothesis of the above theorem cannot be weakened replacing local normality by periodicity. We also remark here that the same example yields that being a C-group is not a local property.

We end this section with an obvious consequence of point (d) of Theorem 3.2.5 of Schmidt (1994).

Lemma 2.2 *Let G be a C-group. Then $G' \cap Z(G) = 1$.*

Corollary 2.3 *Let G be a C-group with a central series. Then G is abelian.*

It is a well known result of Mal'cev that any locally nilpotent group admits a central series (see for instance Robinson 1972, Corollary to Theorem 8.24). Thus the above corollary shows in particular that a locally nilpotent C-group is abelian. Observe that the latter results will be generalized in Sect. 4.

3 Pure subgroups

This section deals with abelian groups whose pure subgroups are complemented. The results presented here are mostly due to Černikov. In fact, here we prove all results obtained in his long paper (Černikov 1954), often with simpler proofs, and we even improve some of those results either by obtaining complete characterizations or by going a little deeper into the analysis.

A subgroup B of an abelian group A is said *pure* in A if $A^n \cap B = A^n$ for all positive integers n , or, equivalently, if every $b \in B$ having a n -th root in A admits a n -th root in B . The main properties of these subgroups can be found for instance in Robinson (1995), here we shortly recall some of them: every direct factor is pure; if A is torsion-free, the subgroup B is pure in A if and only if A/B is torsion-free; the intersection of two pure subgroups of a torsion-free abelian group A is still pure, so it makes sense to consider the pure subgroup generated by a subset of A (a fact that is definitely not true for periodic groups).

It is well known that any bounded pure subgroup of an abelian group is a direct factor and that this is not anymore true (even for periodic abelian groups) when we drop off the “boundedness” of the pure subgroup. For torsion-free abelian groups one can actually provide conditions under which there always exist pure subgroups which are not direct factors. First, look at the following example. Let A be the subgroup

of $\mathbb{Q}_p \times \mathbb{Q}_p$ generated by the set

$$\left\{ \left(\frac{1}{p^n}, \frac{1 + p + \cdots + p^{n-1}}{p^n} \right) : n = 0, 1, 2, \dots, \right\}.$$

It is easy to see that $B = \langle (0, 1) \rangle$ is pure in A and that there are no subgroups K such that $A = B \times K$. Now, let's turn this into a general statement.

Lemma 3.1 *Let G be a torsion-free abelian group of rank 2. If $G = X_1 \times X_2$, where X_1, X_2 are non-divisible and non-isomorphic subgroups, then G contains a pure subgroup which is not a direct factor.*

Proof Of course, since G has rank 2 and $X_1, X_2 \neq 1$, it follows that both X_1 and X_2 are locally cyclic. Let $1 \neq x_1 \in X_1, 1 \neq x_2 \in X_2$ and let X be the pure subgroup generated by $x_1 x_2$. Suppose by contradiction that there exists a subgroup Y such that $G = X \times Y$.

The first case we consider is that in which there are two distinct prime numbers p and q such that X_1 is p -divisible but not q -divisible and X_2 is q -divisible but not p -divisible. It follows that G does not contain any element with both infinite p and q -heights. On the other hand, it may be easily seen that $Y \simeq G/X$ contains such an element and this is a contradiction. Therefore there is some prime number q such that both X_1 and X_2 are not q -divisible.

Assume that there is a prime p such that X_1 is p -divisible but X_2 is not; in particular, $p \neq q$. The factor group G/X is p -divisible and so $Y \leq X_1$. Moreover, x_1 can be chosen with q -height 0 while x_2 with q -height 1. This choice yields that $XX_1/X_1 < X_2X_1/X_1$ and so that XY is strictly contained in G , a contradiction. Therefore X_1 and X_2 are p -divisible for the same primes p .

Now, assume there is an infinite subset S of $\mathbb{P}$ such that the p -height $h_1(p)$ of x_1 is distinct from the p -height $h_2(p)$ of x_2 for all $p \in S$. It follows that there is no element of G having p -height $\max(h_1(p), h_2(p))$ for all $p \in S$. On the other hand, $Y \simeq G/X$ can be easily seen to contain such elements, giving again a contradiction. Thus x_1 and x_2 have distinct finite p -heights for finitely many primes p .

Combining everything we see that X_1 is isomorphic to X_2 , the final contradiction. $\square$

On the other hand, pure subgroups of a divisible group are themselves divisible, so they are direct factors, and next lemma, which is essentially due to Baer (1937), shows that this situation occurs also in some non-trivial cases.

Lemma 3.2 *Let G be a torsion-free abelian group of rank $n \geq 1$. If $G = X_1 \times \cdots \times X_n$ for certain non-trivial, isomorphic subgroups X_i ($1 \leq i \leq n$) then every pure subgroup of G is a direct factor.*

Proof If $n = 1$ the results is immediate; suppose $n > 1$ and let X be a pure subgroup of G . Let $1 \neq x \in X$ and let Y be the pure subgroup generated by x , which is of course of rank 1 and contained in X . If we can prove that $G = Y \times Z$ for some subgroup $Z \simeq Z_1 \times \cdots \times Z_{n-1}$ with $Z_i \simeq X_1$ for all $i \in \{1, \dots, n-1\}$, then we may use induction to show that the pure subgroup $X \cap Z$ is a direct factor of Z and so that $X = Y \times (X \cap Z)$ is a direct factor of G .

Assume hence X of rank 1 and let's prove by induction on n the above statement. The case $n = 1$ can be assumed to be solved by the choice $Z = 1$, so let $n > 1$. Write $x = x_1 x_2 \dots x_n$ where $x_i \in X_i$ for all $i \in \{1, \dots, n\}$ (of course, we may assume $x_1, \dots, x_n \neq 1$). The pure subgroup Y_1 generated by $x_2 \dots x_n$ is (by induction) a direct factor of $X_2 \times \dots \times X_n$, so

$$X_2 \times \dots \times X_n = Y_1 \times W,$$

where the subgroup W is either trivial or isomorphic to direct product of finitely many copies of X_1 . Thus x is contained in $X_1 \times Y_1$ and so also $X \leq X_1 \times Y_1$. Since Y_1 is clearly isomorphic to X_1 , this shows that we may simply assume $n = 2$.

Now, let $\varphi : X_1 \rightarrow X_2$ be an isomorphism, $1 \neq x'_1 \in X_1$ and put $x'_2 = \varphi(x'_1) \in X_2$. Clearly, $x' = (x'_1)^q (x'_2)^r \in X$ for some non-zero integers q and r which may be assumed prime each other; in particular, X is the pure subgroup generated by x' and there are non-zero integers l and m such that $ql - mr = 1$. Let U be the pure subgroup generated by $y' = (x'_1)^m (x'_2)^l$. It is easy to see that $U \cap X = 1$ and that $x'_1, x'_2 \in UX$.

Of course, $X \simeq X_1 \simeq U$, so the pure subgroups U_1 and U_2 generated by x'_1 and x'_2 , respectively, in UX have finite index in X_1 and X_2 , respectively. Suppose F is a finitely generated subgroup of G such that $G = \langle U_1, U_2, F \rangle$, and choose a natural number j and an element $x''_1 \in G$ such that $(x''_1)^j = x'_1$ and $F \leq \langle x''_1, x''_2 \rangle$, where $x''_2 = \varphi(x''_1)$. The j -th roots of x' and y' are $(x''_1)^q (x''_2)^r$ and $(x''_1)^m (x''_2)^l$, respectively. Therefore x''_1 and x''_2 are contained in UX , so F is contained in UX and hence $G = UX = U \times X$, as required. $\square$

The following result shows that in the statement of Lemma 3.2 we cannot ask for n to be infinite without falling back into trivial cases.

Lemma 3.3 *Let G be a non-trivial torsion-free abelian group whose pure subgroups are complemented. If G is the direct product of infinitely many copies of the same subgroup of $\mathbb{Q}$, then G is divisible.*

Proof Suppose by contradiction that G is not divisible, so that $G^p < G$ for some prime number p . In order to derive a contradiction, we may assume G is the direct product of countably infinite many isomorphic locally cyclic groups X_i with $i \in \mathbb{N}$; let $\varphi_i : X_1 \rightarrow X_i$ be an isomorphism for all $i \in \{2, 3, \dots\}$, choose $1 \neq x_1 \in X_1$ and put $x_i = \varphi_i(x_1)$ for all $i \in \{2, 3, \dots\}$. Let X be the pure subgroup generated by the elements $x_i x_{i+1}^{-p}$ with $i \in \mathbb{N}$, and notice that $x_1 \notin X$, so $X < G$. Now, there is a subgroup Y of G such that $G = X \times Y$ and it is clear that $Y \simeq G/X$ is locally cyclic and p -divisible, so G contains elements of infinite p -height. On the other hand, $G^p < G$ shows that X_1 contains no such element. It follows that G does not contain any element of infinite p -height. This contradiction completes the proof of the lemma. $\square$

Lemma 3.2 shows that all pure subgroups of a torsion-free abelian group G are complemented whenever G is a direct product of finitely many isomorphic copies of the same subgroup of the additive group of rational numbers. The same is “trivially”

true for torsion-free divisible groups and the next theorem of Černikov shows that torsion-free abelian groups in which all pure subgroups are complemented are essentially combinations of these two extremal cases.

Theorem 3.4 *Let G be a torsion-free abelian group. All pure subgroups of G are complemented if and only if there is a non-negative integer n such that*

$$G = X_0 \times X_1 \times \cdots \times X_n,$$

where X_0 is divisible and there is a subgroup E of $\mathbb{Q}$ such that $X_i \simeq X_{i+1} \simeq E$ for all $1 \leq i < n$.

Proof We begin proving the sufficiency of the condition. Our previous remark shows that $n > 0$ and $X_0 \neq 1$. Let Y be any pure subgroup of G and put $X = X_1 \times \cdots \times X_n$, $Y_1 = X_0 \cap Y$, $Y_2 = \{b \in X : \exists a \in X_0, ab \in Y\}$. Notice that Y_1 is the intersection of two pure subgroups of the torsion-free abelian group G and as such is pure in G . Thus, since X_0 is divisible, there is a subgroup Y_3 such that $X_0 = Y_1 \times Y_3$.

Suppose $x^n \in Y_2$ for some positive integer n and $x \in X$. Then there is an element $x_0 \in X_0$ such that $x_0 x^n$ belongs to Y . Since X_0 is divisible, we may find $x'_0 \in X_0$ with $(x'_0)^n = x_0$. Thus $(x'_0 x)^n = x_0 x^n \in Y$ and so $x'_0 x \in Y$, which means that x lies in Y_2 . Therefore Y_2 is a pure subgroup of X and there exists a subgroup Y_4 such that $X = Y_2 \times Y_4$.

Now, $G = X_0 X = Y Y_3 Y_4$ because $Y Y_3 \geq Y_2$. Moreover, $Y \cap Y_3 = 1 = Y Y_3 \cap Y_4$ and so $G = Y \times Y_3 \times Y_4$ showing that the arbitrary pure subgroup Y is a direct factor of G and so it is complemented. The sufficiency is proved.

Conversely, suppose all pure subgroups of G are complemented and let D be the largest divisible subgroup of G . Then $G = D \times R$ where R is reduced. It follows that R satisfies the properties of the statement; thus we may assume G is reduced and prove that it is the direct product of finitely many isomorphic copies of the same subgroup of $\mathbb{Q}$.

Let

$$1 = G_0 < G_1 < \cdots < G_\alpha < G_{\alpha+1} < \cdots < G_\mu = G$$

be an ascending series of G such that $G_{\alpha+1}/G_\alpha$ is torsion-free and locally cyclic for each ordinal $\alpha < \mu$; in particular, each G_α is pure in G for all $\alpha \leq \mu$. Thus for each ordinal $\alpha \leq \mu$ there exists a subgroup H_α such that $G = G_\alpha \times H_\alpha$.

If $\mu \geq \omega$, then $G = G_\omega \times H_\omega$ and clearly $G_\omega \simeq G_0 \times G_1/G_0 \times \cdots \times G_{n+1}/G_n \times \cdots$. Now, since all direct factors inherit the hypotheses, it follows from Lemma 3.1 that $G_{i+1}/G_i \simeq G_{i+2}/G_{i+1}$ for all $i \in \mathbb{N}_0$ and so Lemma 3.3 yields that G_ω is complete. Thus $G_\omega = 1$, a contradiction showing that μ is a finite ordinal number. In this case, an application of Lemma 3.1 completes the proof of the theorem. $\square$

As for periodic abelian groups, a result similar to the previous one has been proved by Prüfer (1923).

Theorem 3.5 *Let G be a periodic abelian group. All pure subgroups of G are complemented if and only if $G = H \times K$, where H is divisible and for each $p \in \pi(K)$ the Sylow p -subgroup of K has finite exponent.*

Proof Of course, it is enough to prove the statement for a p -group G , p being a prime number.

Suppose first $G = H \times K$, where H is divisible and K has finite exponent. Let X be a pure subgroup of G and consider its divisible part D . Then $G = D \times L$ for some subgroup L of G and $D \times (L \cap X) = X$; of course, $L \cap X$ is pure in G , being a pure subgroup of X . Now, it easily follows that $L \cap X \cap H = 1$ and hence $L \cap X$ has finite exponent. Finally, 4.3.8 of Robinson (1995) yields that $L \cap X$ is a direct factor of G .

Suppose conversely that all pure subgroups of G are complemented and let D be the largest divisible subgroup of G . Then $G = D \times R$ where R is reduced. Since R satisfies the hypothesis, we may assume that G is reduced and prove that it of finite exponent. Assume by way of contradiction that the exponent of G is infinite and let B be a basic subgroup of G ; since B is pure and G/B is divisible, we have that $G = B$. It is clearly even possible to assume that

$$G = X_1 \times X_2 \times \cdots \times X_n \times \cdots,$$

where $X_i = \langle x_i \rangle$ is a cyclic group of order p^i . It is easy to see that the subgroup $Y = \langle x_1 x_2^{-p}, \dots, x_i x_{i+1}^{-p}, \dots \rangle$ is pure in G and G/Y is divisible and non-trivial. By hypotheses G/Y is isomorphic to a direct factor of G and this is a contradiction. The statement is proved. $\square$

Theorems 3.4 and 3.5 can be combined with the following result to obtain a complete description of abelian groups whose pure subgroups are complemented.

Theorem 3.6 *Let G be an abelian group. All pure subgroups of G are complemented if and only if G is the direct product of a periodic group and a torsion-free group, and in both of them the pure subgroups are complemented.*

Proof Since the torsion-part of an abelian group is always pure, the necessity of the condition follows immediately. Suppose conversely that $G = T \times H$, where T is periodic, H is torsion-free and in both T and H the pure subgroups are complemented. Let X be any pure subgroup of G , put $T_1 = X \cap T$ and $H_1 = \{h \in H : \exists g \in T, gh \in X\}$. Notice that T_1 is pure in X (being its torsion-part) and X is pure in G , so that T_1 is pure in G ; thus there is a subgroup T_2 such that $T = T_1 \times T_2$.

Let $1 \neq h_1 \in H_1$ and suppose $h^n = h_1$ for some $h \in H$ and a positive integer n . There is $g \in T$ such that gh_1 belongs to X ; let m be the order of g . Then $(gh)^{nm} = h_1^m = (gh_1)^m$, so there is $x \in X$ such that $x^{nm} = (gh_1)^m$. If $x = g_1 h_2$ with $g_1 \in T_1$ and $h_2 \in H_1$, then $h_2^{nm} = h_1^m$ and hence $h_2^n = h_1$ since H is torsion-free. This proves that H_1 is pure in H , so there is a subgroup H_2 such that $H = H_1 \times H_2$.

Now, $G = TH = XT_2H_2$ since $XT_2 \geq H_1$. Moreover, $X \cap T_2 = 1 = XT_2 \cap H_2$ and so $G = X \times T_2 \times H_2$ showing that the pure subgroup X is a direct factor of G and so it is complemented. The sufficiency is proved. $\square$

Let π be a set of prime. We say that a subgroup X of an abelian group A is π -pure in G if $nG \cap A = nA$ for all π -numbers n .

Corollary 3.7 *Let π be a proper subset of $\mathbb{P}$ and let G be an abelian group. All π -pure subgroups of G are complemented if and only if $G = G_\pi \times G_{\pi'}$ where*

- (i) G_π is a π -group whose pure subgroups are complemented, and
- (ii) $G_{\pi'}$ is a direct product of cyclic π' -groups of prime order.

Proof The sufficiency of the condition follows from Theorem 3.6 and the easy observation that all π -pure subgroups of G are pure.

Suppose therefore that all π -pure subgroups of G are complemented. Since all pure subgroups of G are also π -pure and the torsion part T of G is pure, we have that $G = T \times H$ for some torsion-free subgroup H which, by Theorem 3.4, is a direct product of locally cyclic subgroups. On the other hand, since π is a proper subset of $\mathbb{P}$, we have that any non-trivial torsion-free locally cyclic group contains a proper π -pure subgroup, so that $H = 1$ and hence G is periodic.

Now, $G = G_\pi \times G_{\pi'}$ where G_π and $G_{\pi'}$ are its π and π' -components, respectively. Of course, every subgroup of $G_{\pi'}$ is π -pure, so all its subgroups are complemented and hence it is a direct product of elementary abelian p -groups with $p \in \pi'$. Finally, we observe that any π -pure subgroup of G_π is $\mathbb{P}$ -pure and so pure. The statement is proved. $\square$

We remark that the property of being a pure subgroup of an abelian group can be described in lattice-theoretic terms (see Ferrara and Trombetti 2021a), which means that if there is a projectivity between two abelian groups, then one of these groups has all its pure subgroups complemented if and only if the other one has such a property. Although p -purity cannot be described only in lattice-theoretic terms for any prime p , it easily follows from Theorem 1.6.5 of Schmidt (1994), Corollary 3.7 and our previous remark that at least the following statement is true.

Theorem 3.8 *Let π be a set of primes and let φ be a projectivity between two abelian groups G_1 and G_2 . If all π -pure subgroups of G_1 are complemented, then there is a set of primes π_1 such that $|\pi_1| = |\pi|$, $|\pi'_1| = |\pi'|$ and all π_1 -pure subgroups of G_2 are complemented.*

3.1 Pure decompositions

Let G be an abelian group which is the direct product of a family of (non-trivial) locally cyclic groups $\{X_i\}_{i \in I}$ such that:

1. each X_i is either torsion-free or a p -group for some prime p ;
2. there is at most one isomorphism type $\mathcal{H}$ of locally cyclic torsion-free groups which are not divisible;
3. in case $\mathcal{H}$ exists, there are finitely many $i \in I$ such that $X_i \in \mathcal{H}$;
4. for each prime p there is a non-negative integer $n(p)$ such that X_i has order at most $p^{n(p)}$ whenever X_i is a cyclic p -group.

In this case we say that G has the *pure decomposition* provided by the family $\{X_i\}_{i \in I}$. The name is justified by the following immediate consequence of Theorems 3.4, 3.5 and 3.6.

Theorem 3.9 *Let G be an abelian group. Then G admits a pure decomposition if and only if all pure subgroups of G are complemented.*

It should be clear from the last part of the proof of Lemma 3.2 that if we have a pure subgroup X of G , although X has a complement in G , it does not necessarily mean that such a complement is a product of members of the family $\{X_i\}_{i \in I}$. The following result exemplifies this situation.

Lemma 3.10 *Let G be a torsion-free abelian group which is the direct product of two non-trivial locally cyclic groups H and K . If H is not p -divisible and K is not q -divisible for two distinct prime numbers p and q , then G contains a pure subgroup which is not complemented by H or K .*

Proof It follows from Lemma 3.1 that $H \simeq K$. Let $1 \neq h \in H$ and $1 \neq k \in K$ be both elements of p and q -height 0. Let X be the pure subgroup generated by $h^p k^q$. Since $k \notin HX$ and $h \notin KX$, the statement is proved. $\square$

On the other hand, periodic groups with a pure decomposition do not exhibit such a bad behaviour.

Theorem 3.11 *Let G be a periodic abelian group having the pure decomposition provided by the family $\{X_i\}_{i \in I}$. If H is a pure subgroup of G , then there is $J \subseteq I$ such that G is the direct product of H and all members of the family $\{X_j\}_{j \in J}$.*

Proof Obviously, we may suppose $1 \neq H \neq G$ and that G is a p -group for some prime number p . Using Zorn's lemma we find a subset J (maybe empty) of I maximal with respect to satisfying these properties:

1. $\langle H, X_j : j \in J \rangle = H \times \text{Dr}_{j \in J} X_j$;
2. $\langle H, X_j : j \in J \rangle$ is pure in G .

In order to prove that $\langle H, X_j : j \in J \rangle$ equals G , we consider an arbitrary pure proper subgroup A of G and we show that there is $i \in I$ such that $\langle A, X_i \rangle = A \times X_i$ is a pure subgroup of G . Let J_1 be the subset of I consisting of all $i \in I$ such that $A \cap X_i \neq 1$. Choose $j \in J_1$ and let x_j be a generator of the socle of X_j . If X_j is finite, then there is a (pure) cyclic subgroup A_j of A containing x_j and having the same order as X_j . If X_j is infinite, then A contains an infinite locally cyclic subgroup A_j such that x_j lies in A_j (recall that there is a bound on the orders of X_i 's which are finite). It is almost immediate to check that $\langle A_j, X_i : j \in J_1, i \in I \setminus J_1 \rangle = B \times X$ where

$$B = \text{Dr}_{j \in J_1} A_j \quad \text{and} \quad X = \text{Dr}_{i \in I \setminus J_1} X_i.$$

Moreover, it is also easy to prove that $B \times X$ is pure in G , and since $B \times X$ and G share the same socle they must be equal by Theorem 3.9. Now, $A = B \times (A \cap X)$,

so $A \cap X$ is pure in X and has trivial intersection with any X_i , $i \in I \setminus J_1$; choose $j' \in I \setminus J_1$ such that $X_{j'}$ has maximal order (even infinite if possible). Theorem 3.9 yields that $X = (A \cap X) \times K$ for some subgroup K and the projection D of $X_{j'}$ onto K is a locally cyclic subgroup of maximal order of K , so it is pure in K and hence there exists a subgroup L of K such that $K = D \times L$. Of course,

$$G = B \times X = B \times (A \cap X) \times X_{j'} \times L = A \times X_{j'} \times L,$$

which means that $A \times X_{j'}$ is pure in G . The statement is proved. $\square$

Corollary 3.12 *Let G be an abelian group of 0-rank ≤ 1 having the pure decomposition provided by the family $\{X_i\}_{i \in I}$. If H is a pure subgroup of G , then there is $J \subseteq I$ such that G is the direct product of H and all the members of family $\{X_j\}_{j \in J}$.*

Proof Since G may be supposed non-periodic, we may find a unique $n \in I$ such that X_n is torsion-free. Let T_1 be the torsion-part of H and observe that the torsion-part T of G is the direct product of all members of the family $\{X_i\}_{i \in I \setminus \{n\}}$. By Theorem 3.11 there exists $J \subseteq I \setminus \{n\}$ such that $T = T_1 \times X_J$, where X_J is the direct product all members of the family $\{X_i\}_{i \in J}$. If $X_n \cap H = 1$, then $G = X_n \times H \times X_J$. If $X_n \cap H \neq 1$, the arguments used in the proof of Theorem 3.6 yield that $G = H \times X_J$. $\square$

The following result partially deals with the case of 0-rank > 1 .

Theorem 3.13 *Let G be an abelian group having the pure decomposition provided by the family $\{X_i\}_{i \in I}$ and suppose there is a prime number p such that all torsion-free groups in $\{X_i\}_{i \in I}$ are isomorphic to $\mathbb{Q}_{p'}$. If H is a pure subgroup of G , then there is $J \subseteq I$ such that G is the direct product of H and all the members of family $\{X_j\}_{j \in J}$.*

Proof Let I^* be the subset of I made by all i 's such that X_i is isomorphic to $\mathbb{Q}_{p'}$ and let $n = |I^*|$. We will work by induction on n . If $n \leq 1$, the result follows from Corollary 3.12. Suppose the result holds for $n \geq 1$ and assume $|I^*| = n + 1$. Let H be a pure subgroup of G . If H is periodic, then the result follows from Theorem 3.11; thus H is non-periodic. It follows from Theorem 3.9 that $G = H \times K$ for some subgroup K . From this we can see that all pure subgroups of H are complemented in H , so H admits a pure decomposition and in particular $H = L \times M$ for certain subgroups L and M such that L is torsion-free and locally cyclic. It is not difficult to see that $L \simeq \mathbb{Q}_{p'}$ (use Lemma 3.1). Let $h \in L$ of p -height 0 and let p^m be the exponent of the subgroup generated by the subgroups of the family $\{X_i\}_{i \in I}$ which are finite p -groups. If the projection of h on $X^* = \langle X_i : i \in I^* \rangle$ has p -height > 0 , then h^{p^m} has p -height $\geq p^{m+1}$, a contradiction since h^{p^m} has p -height p^m . Thus the projection of h on X^* has p -height 0; choose $i^* \in I^*$ such that the projection of h onto X_{i^*} has p -height 0. It is easy to see that $G = L \times N$ where $N = \langle X_i : i \in I \setminus \{i^*\} \rangle$. Since $L \leq H$, it follows that $H = L \times (H \cap N)$. Moreover, $H \cap N$ is pure in H which is pure in G , so $H \cap N$ is pure in N . By induction hypothesis, there exists a subset J of $I \setminus \{i^*\}$ such that $X = (H \cap X) \times X_J$, where $X_J = \langle X_j : j \in J \rangle$. Finally, $G = H \times X_J$ and the statement is proved. $\square$

Next lemma is an essential ingredient to classify which abelian groups with a pure decomposition have the pure subgroups complemented by subgroups generated by subfamilies of that decomposition.

Lemma 3.14 *Let G be an abelian group having the pure decomposition provided by the family $\{X_i\}_{i \in I}$. If H is a pure subgroup of G having non-trivial intersection with each divisible X_i , then H contains the divisible part of G .*

Proof Suppose by contradiction the result is false and take $j \in I$ such that the divisible subgroup $X = X_j$ is not contained in H . Then there is a prime number p and $1 \neq x \in X \cap H$ such that x has p -height 0 in $X \cap H$. By the very definition of pure decomposition we know that there is a bound p^m on the orders of the cyclic p -subgroups in $\{X_i\}_{i \in I}$. Since X is divisible, there is an element $x_1 \in X$ such that $x_1^{p^{m+1}} = x$; but H is pure in G and so there exists an element $h \in H$ such that $h^{p^{m+1}} = x$. Write $h = x_2 x_3$ for some $x_2 \in X$ and $x_3 \in \langle X_i : i \in I \setminus \{j\} \rangle$; in particular, x_3 has order at most p^{m+1} . If $\text{o}(x_3) \leq p^m$, then $h_1 = h^{p^m}$ would be an element of $X \cap H$ such that $h_1^p = x$, a contradiction. Therefore the order of x_3 is p^{m+1} and so x_3 is contained in the subgroup generated by all divisible X_i with $i \in I \setminus \{j\}$. Now, $h^{p^m} = x_2^{p^m} x_3^{p^m}$ belongs to H and the hypothesis yields that even $x_3^{p^m}$ belongs to H . Thus $x_2^{p^m}$ is in $H \cap X$, the final contradiction. $\square$

Theorem 3.15 *Let G be an abelian group having the pure decomposition provided by the family $\{X_i\}_{i \in I}$ and let n be the number of $i \in I$ such that X_i is non-divisible and torsion-free. Then for each pure subgroup H of G there is a subset I_H of I such that*

$$G = H \times \langle X_i : i \in I_H \rangle$$

if and only if either $n \leq 1$, or $n \geq 2$ and there is a prime p such that all non-divisible torsion-free groups in $\{X_i\}_{i \in I}$ are isomorphic to $\mathbb{Q}_{p'}$.

Proof The necessity of the conditions easily follows from Lemma 3.10.

Suppose now G satisfies one of the conditions in the statement and let H be a pure subgroup of G . Let $\mathcal{J}$ be the set of all subsets I_1 of I such that

- (i) X_i is divisible for each $i \in I_1$ and
- (ii) $H \cap \langle X_i : i \in I_1 \rangle = 1$.

By Zorn's lemma, $\mathcal{J}$ contains a maximal element J^* , say; let $K = H \times \langle X_j : j \in J^* \rangle$. It is easy to see that K is a pure subgroup of G which has non-trivial intersection with all divisible groups in the family $\{X_i\}_{i \in I}$. Thus Lemma 3.14 shows that K contains the divisible component D of G ; clearly, D is the direct product of all divisible groups in $\{X_i\}_{i \in I}$, and $G = D \times X$, where X is the direct product of all X_i 's which are non-divisible. Now, Corollary 3.12 and Theorem 3.13 show that there exists $J \subseteq I$ such that $X = (K \cap X) \times X_J$, where $X_J = \langle X_i : i \in J \rangle$. Since $K = D \times (K \cap X)$, it follows that

$$G = K \times X_J = H \times \langle X_i : i \in J^* \cup J \rangle$$

completing the proof of the theorem. $\square$

3.2 Pure decompositions in arbitrary groups

Theorem 3.16 *Let A be a non-trivial abelian normal subgroup of the group G . If all pure subgroups of A are permutably complemented in G then A admits a pure decomposition provided by a family of G -invariant subgroups.*

Proof By hypothesis there is a subgroup X of G such that $G = X \ltimes A$. Moreover, if B is any pure subgroup of A , then it admits a permutable complement X_B in G , so that $G = X_B B$ with $X_B \cap B = 1$. Thus $A = A \cap G = (A \cap X_B) \times B$ and in particular A admits a pure decomposition by Theorem 3.9. Moreover, since $A \cap X_B$ is G -invariant we see that for each pure subgroup of A there is a G -invariant complement in A .

Let $\mathfrak{L}$ be the set made by the pair $(\{1\}, \emptyset)$ and by all pairs $(U, \mathcal{U})$ where U is a pure subgroup of A and $\mathcal{U}$ is a family of G -invariant subgroups of U providing a pure decomposition of U ; we also say that $(U, \mathcal{U}) \leq (V, \mathcal{V})$ if and only if $U \subseteq V$. Let $\mathfrak{M}$ be any chain of elements of $\mathfrak{L}$, we need to show that it admits an upper bound. First, let M be the union of the first components of the pairs in $\mathfrak{M}$; then M is a pure subgroup of A , so it is a direct factor of A and hence M admits a pure decomposition by Theorem 3.9 (we can assume $M \neq \{1\}$). Let $\mathcal{M}$ be the union of the second components of the elements of $\mathfrak{M}$; then M is the direct product of the subgroups in $\mathcal{M}$. Now, since the pure subgroups of M are complemented (again by Theorem 3.9), it follows from Lemmas 3.1 and 3.3 that $\mathcal{M}$ must provide a pure decomposition of M . Therefore the pair $(M, \mathcal{M})$ is an upper bound for the chain $\mathfrak{M}$. It is thus possible to apply Zorn's lemma and get a maximal element $(C, \mathcal{C})$ of $\mathfrak{L}$.

Since C is a pure subgroup of A , then it admits a G -invariant complement D in A . Now, if $D \neq 1$, then D admits a pure decomposition; let $D = E \times F$, where E is any member of such a pure decomposition. As we showed above the subgroup F admits a G -invariant complement K in D and (obviously) $K \simeq E$. Since $C \times K$ is a direct factor of A , it is a pure subgroup of A , so Lemma 3.1 and Theorem 3.9 yield that $(C \times K, \{K\} \cup \mathcal{C})$ belongs to $\mathfrak{L}$, contradicting the maximality of $(C, \mathcal{C})$. Thus $D = 1$ and $A = C$ has the required property. $\square$

Corollary 3.17 *Let G be a group with an ascending normal series with divisible abelian factors. If all divisible subgroups of G are permutably complemented, then G is abelian.*

Proof It is clearly enough to consider the case in which G contains a divisible normal subgroup N such that G/N is divisible; in particular, by hypothesis, there is a complement C to N in G . Notice now that all pure subgroups of N are actually divisible and so they admit permutable complements in G . Thus Theorem 3.16 applies to show that the subgroup N is a direct product of normal divisible subgroups which are locally cyclic; moreover, since the automorphism groups of such groups is residually finite, it follows that all these subgroups and so N lie in the centre of G . Therefore $G = NC$ is abelian and the statement is proved. $\square$

Let G be a group, let A be an abelian normal subgroup (of G) having the pure decomposition provided by the family $\{X_i\}_{i \in I}$ and let n be the number of $i \in I$ such that X_i is non-divisible. Suppose that (1) A is complemented in G , (2) for each

$i \in I$, the subgroup X_i is G -invariant and (3) if $n \geq 2$, there is a prime p such that all non-divisible X_i are isomorphic to $\mathbb{Q}_{p'}$. In such circumstances, it is easy to see using Theorem 3.15 that every pure subgroup of A admits a permutable complement in G .

Notice also that if (3) is not satisfied, it may very well be the case that a pure subgroup of A does not have any complement in G . In fact, consider the semidirect product $G = \langle x \rangle \ltimes (\langle a \rangle \times \langle b \rangle)$, where $\langle a \rangle \simeq \mathbb{Z} \simeq \langle b \rangle$, $x^2 = 1$, $[a, x] = 1$ and $b^x = b^{-1}$. Clearly, the subgroup $\langle a^2 b^3 \rangle$ is a pure subgroup of $\langle a, b \rangle$ having no complement in G . The previous example is very characteristic of such a situation, as the following lemma shows.

Lemma 3.18 *Let G be a group and let A be a torsion-free abelian subgroup of G such that:*

- (i) *there are two isomorphic G -invariant locally cyclic subgroups A_1 and A_2 such that $A = A_1 \times A_2$ (let φ be an isomorphism between A_1 and A_2);*
- (ii) *there are two primes $p \neq q$ such that A_1, A_2 are not p and q -divisible;*
- (iii) *there is $g \in G$ and $a_1 \in A_1$ such that $\varphi(a_1^g) \neq \varphi(a_1)^g$.*

Then A contains a pure subgroup which is not permutably complemented in G .

Proof It follows from Lemma 3.10 that A contains a pure subgroup X which is not complemented by A_1 or A_2 . Suppose by contradiction X admits a complement in G , so it admits a G -invariant complement Y in A . Now, Y is locally cyclic and $\neq A_1, A_2$; in particular, there is $y = y_1 y_2 \in Y$, where $1 \neq y_1 \in Y_1$ and $1 \neq y_2 \in Y_2$. The hypothesis yields the existence of integers l, m, s, t such that $m, t \neq 0$, $(l, m) = 1 = (s, t)$, $lt \neq sm$, $a_1^g = a_1^{l/m}$ and $\varphi(a_1)^g = \varphi(a_1)^{s/t}$. Since Y is G -invariant, we have that y^g belongs to Y and so even $(y^g)^{mt} \in Y$. However, $(y^g)^{mt} = y_1^{tl} y_2^{sm}$ belongs to the locally cyclic group Y if and only if $tl = sm$, a contradiction. The proof is complete. $\square$

On the other hand, if the action of the group is the same on each torsion-free, non-divisible subgroup of a pure decomposition, the situation turns out to be the best possible.

Theorem 3.19 *Let G be a group and let A be an abelian normal subgroup of G having the pure decomposition provided by the family $\{X_i\}_{i \in I}$ of G -invariant subgroups. Suppose that for each $g \in G$ there are relatively prime integers m_g, n_g with $n_g \neq 0$ such that $x^g = x^{m_g/n_g}$ for each x contained in the subgroup X generated by all X_i 's which are torsion-free and non-divisible. If A is complemented in G , then all pure subgroups of A are permutably complemented in G as well.*

Proof Let n be the number of torsion-free non-divisible subgroups in $\{X_i\}_{i \in I}$; of course, by the remark preceding Lemma 3.18 we may assume $n > 1$. The hypotheses of the statement easily yield that each direct factor of X is G -invariant. This fact together with Theorem 3.15 can be employed in the proofs of Theorems 3.4 and 3.6 to show that any pure subgroup B of A admits a G -invariant complement H in A . If K is a complement to A in G , then HK is a permutable complement to B in G . The statement is proved. $\square$

The following result is a consequence of Theorems 3.16, 3.19 and the fact that an abelian K -group is a direct product of cyclic groups of prime orders (so in particular all its subgroups are pure).

Corollary 3.20 *Let A be an abelian normal subgroup of G . Then all subgroups of A are permutably complemented in G if and only if G splits over A and A admits a pure decomposition provided by a family of G -invariant subgroups of prime orders.*

4 Normal subgroups

In this section we deal with the structure of groups with many normal subgroups admitting a complement. This is still connected to the work of Černikov in Černikov (1954); for more results concerning complementation of normal subgroups in finite groups see Christensen (1964). Since it is clear that all normal subgroups of any simple group are permutably complemented, in general, some further condition must be required to obtain any interesting result. This is perfectly exemplified by the following fundamental result of Napolitani (1967) and Emaldi (1970), for the proof of which we refer to Theorem 3.1.14 of Schmidt (1994).

Theorem 4.1 *Let G be a soluble group. Then all subgroups of G are complemented if and only if all its normal subgroups are complemented.*

Recall that a group is *hyperabelian* if it admits an ascending normal series with abelian factors. Being this class a natural generalization of the class of soluble groups it is natural to ask the following question.

(a) Open question

Is it possible to generalize Theorem 4.1 to the case of hyperabelian groups?

On another side, if we require a group G , whose normal subgroups are complemented, to have many central factors, we immediately get the abelianity of G . Notice that the next couple of results provide generalizations to the final results of Sect. 2.

Lemma 4.2 *Let G be a group admitting a central series whose terms are complemented in G . Then G is abelian.*

Proof Suppose by contradiction G' contains a non-trivial element g . By hypothesis, there exist normal subgroups H and K such that $g \in H \setminus K$ and H/K lies in the centre of G/K . If C is any complement to H in G , then CK/K is a complement to H/K in G/K . Since H/K is central, it follows that CK/K is even normal in G/K and clearly $G/CK \simeq H/K$ is abelian. Thus $G' \leq CK \subseteq G \setminus \{g\}$, the desired contradiction. $\square$

Corollary 4.3 *Let G be a locally nilpotent group whose normal subgroups are complemented in G . Then G is abelian.*

Corollary 4.4 *Let G be a locally finite p -group for some prime p . If all normal subgroups of G are complemented in G , then G is abelian.*

If we only ask the group to have a commutator subgroup with many G -invariant central factors, then the situation is slightly less trivial.

Theorem 4.5 *Let G be group admitting a normal series $\mathcal{S}$ with locally cyclic factors. If all normal subgroups of G are complemented in G , then G is a C-group.*

Proof Since the automorphism group of any locally cyclic group is abelian, it follows that the intersections of all members of the series $\mathcal{S}$ with G' give rises to a central series $\mathcal{S}'$ of G' with G -invariant terms. Thus, all terms of the series $\mathcal{S}'$ admit a complement in G' and so Lemma 4.2 yields that G' is abelian; in particular, G is a K-group by Theorem 4.1.

It follows from Lemma 3.1.7 of Schmidt (1994) that G' is the direct product of minimal normal subgroups of G . Since G admits a normal series with locally cyclic factors, we have that G' is actually the direct product of minimal normal subgroups of G which are locally cyclic, and so in particular either of prime order or isomorphic with $\mathbb{Q}$.

Let N be a minimal normal subgroup of G which is contained in G' and isomorphic with $\mathbb{Q}$. Since G/G' is an abelian K-group, it follows that $G/C_G(N)$ is cyclic of order ≤ 2 , so all subgroups of N are G -invariant, which is impossible. Thus G' is periodic and so G as well.

Since G splits over G' , an easy application of Theorem 3.2.5 of Schmidt (1994) completes the proof of the statement. $\square$

The above theorem was proved by Černikov using results coming from his study of complementation of pure subgroups: here we have chosen to avoid this connection.

5 Finite subgroups and subgroups of prime order

The aim of this section is to put under a new light ideas and results from Gorchakov (1960) and Gorchakov (1962). As collateral results we will obtain simpler proofs of results in de Giovanni and Franciosi (1984).

It is well known that every finite symmetric group is a K-group (see Schmidt 1994, Theorem 3.1.2) and the first statement of the section shows that something more happens in finite symmetric groups of prime degree.

Lemma 5.1 *Let p be a prime. The Sylow p -subgroups of $\text{Sym}(p)$ are permutably complemented.*

Proof Let P be any Sylow p -subgroup of $G = \text{Sym}(p)$; clearly, P has order p . Let K be any subgroup of G isomorphic with $\text{Sym}(p-1)$. By order considerations, we have that $|G| = |K|p = |K||P|$ and so $G = KP$ since $K \cap P = 1$. This shows that K is actually a permutable complement and completes the proof of the lemma. $\square$

Theorem 5.2 *Let p be a prime number and let G be a group. Then all subgroups of order p of G are permutably complemented in G if and only if there is a normal subgroup N containing no element of order p and such that G/N is embeddable into a cartesian product of copies of the symmetric group on p elements.*

Proof Suppose first that all subgroups of order p of G are permutably complemented in G . If any, let g be any element of G having order p and let K be a permutable complement to $\langle g \rangle$ in G . Then K has index p in G and, if we set $N_g = K_G$, then G/N_g is naturally isomorphic to a subgroup of $\text{Sym}(p)$. The intersection N of G and of all N_g 's where g ranges in the set of all elements of G having order p is a normal subgroup of G which does not contain any element of order p and G/N is isomorphic to a subgroup of the cartesian product of the groups G/N_g 's.

In order to prove the converse we may assume that G is a subgroup of a cartesian product of copies of the symmetric group on p elements, or, which is the same, that G is residually a subgroup of $\text{Sym}(p)$. However, in such circumstances it is obvious from Lemma 5.1 that all elements of order p of G are permutably complemented in G . $\square$

Let π be a set of primes. The “only if” part of the above theorem can easily be generalized to show that if all π -subgroups of prime order are permutably complemented, then there is a normal subgroup N containing no π -element and such that G/N is embeddable into a cartesian product of copies of the symmetric groups on $p \in \pi$ elements. On the other hand, the “if” part of the statement does not admit a similar generalization. In fact, since the quaternion group Q_8 (in which the subgroups of order 2 are not complemented) is a subgroup of $\text{Sym}(11)$, it follows that Q_8 is trivially an extension of a group not containing any element of order 2 or 11 by a subgroup of $\text{Sym}(11)$.

We employ Theorem 5.2 in the following example proving that, even for locally finite groups, if one requires all subgroups of prime order to be permutably complemented it is still possible to have “large” subgroups which are not (permutable) complemented.

Example 5.3 Let G be the cartesian product of countably infinite many isomorphic copies of $\text{Sym}(3)$.

- (i) G is locally a C-group, but not a K-group.
- (ii) All subgroups of G of prime order are permutably complemented.

Proof Assume by contradiction that G is a K-group. Then G' is a direct product of normal cyclic subgroups of order 3 (see Schmidt 1994, Lemma 3.1.7). This immediately implies that all elements of order 3 have finitely many conjugates, a clear contradiction. On the other hand, Theorem 5.2 shows that all subgroups of order 2 or 3 are permutably complemented in G .

Finally, it is easy to show (and also well known) that any finitely generated subgroup of G can be embedded into the direct product of finitely many copies of the C-group $\text{Sym}(3)$, so it is a C-group itself. It follows that G is locally a C-group. $\square$

Theorem 5.4 Let p be a prime and let G be a periodic group. All subgroups of prime order $\leq p$ are permutably complemented in G if and only if for any such subgroup H there is a normal subgroup N_H of G such that $N_H \cap H = 1$ and G/N_H is a C-group.

Proof Assume all subgroups of prime order $\leq p$ are permutably complemented in G and let H be any subgroup of order p . It easily follows from Theorem 5.2 that there

is a subgroup N not containing elements of order $< p$ and such that G/N does not contain elements of order p ; in particular, $H \leq N$. By hypothesis H is permutably complemented in G , let K be a permutable complement to H in G . Then $K \cap N$ is a permutable complement to H in N . Thus $|N : K \cap N| = p$ and so $K \cap N$ is normal in N . Since $K \cap N$ is trivially normal in K and $G = NK$, then $K \cap N$ is normal in G .

Let $C/K \cap N = C_{K/K \cap N}(N/K \cap N)$ and notice that $\pi(G/C) \subseteq \{q : q \leq p\}$. Since G does not contain elements of q^2 for any prime $q \leq p$, it easily follows that G/C is a C-group. It is clear that the same argument applies when we deal with any prime $q < p$, so the necessity of the condition is proved.

Since the sufficiency of the condition is quite obvious, the theorem follows. $\square$

The next corollary should be put in relation to Example 5.3 and Theorem 2.1.

Corollary 5.5 *Let G be a periodic group. Then all subgroups of prime order are permutably complemented in G if and only if G is residually a (finite) C-group.*

The following example shows that Theorem 5.4 and Corollary 5.5 do not hold when the periodicity assumption is removed.

Example 5.6 There is a non-periodic group G containing an element of order 13 and satisfying the following properties.¹

- (i) All subgroups of prime order are permutably complemented.
- (ii) G admits no homomorphic image which is a C-group and contains an element of order 13.
- (iii) G is residually finite.

Proof Let $G = Q \ltimes N$, where $Q \simeq \mathbb{Q}_{13}$, $N \simeq \mathbb{Z}_{13}$ and Q induces an automorphism of order 4 on N . Since N is the only non-trivial periodic subgroup of G , then clearly all subgroups of prime order are permutably complemented in G . Suppose by contradiction G admits a normal subgroup M such that G/M is a C-group and $M \cap N = 1$. Thus $[M, N] = 1$ and so $M \leq C_G(N) = Z(G)N$. However, it is clear that $G/C_G(N)$ is not a C-group since it contains an element of order 4. $\square$

Example 5.6 shows that the property of having all subgroups of prime order permutably complemented is not preserved by homomorphic images, a fact that could have also been trivially observed by consideration of $\mathbb{Z}$.

Corollary 5.7 *Let G be a periodic group whose subgroups of prime order are permutably complemented. Then G' is abelian and in particular G is locally finite.*

Proof This follows immediately from Corollary 5.5 and Theorem 3.2.5 of Schmidt (1994). $\square$

The following lemma shows that for an arbitrary group it is equivalent to require that all finite subgroups are permutably complemented and that only all subgroups of prime order are such.

¹ It should be noticed that in Gorchakov (1960) there is a very detailed construction of such an example.

Lemma 5.8 *Let H be a finite subgroup of a group G . If all subgroups of prime order of H are permutably complemented in G , then H is permutably complemented in G .*

Proof We proceed by induction on the order of H being the result trivially true in case $|H|$ is 1. Suppose H is non-trivial. Then there is a subgroup $H_1 \leq H$ of prime order which admits a permutable complement K_1 in G . Now, $H \cap K_1$ is a finite subgroup of G having order $< |H|$, so it admits a permutable complement K in K_1 . Since $H = H_1(H \cap K_1)$ and $K_1 = (H \cap K_1)K$, it follows that K is a permutable complement to H in G . $\square$

Corollary 5.9 *Let G be a group. Then all finite subgroups are permutably complemented if and only if all subgroups of prime order are such.*

It easily follows from Lemma 3.1.4 of Schmidt (1994) that the direct product of finitely many groups whose finite subgroups are complemented is itself a group in which all finite subgroups are complemented. This fact, in turn, shows that any group which is residually a group whose finite subgroups are complemented is itself a group in which the finite subgroups are complemented. As for “permutable complementability”, the analogous statement is an easy consequence of Corollary 5.9.

Corollary 5.10 *Let G be a finite group. Then all subgroups of G are permutably complemented if and only if its subgroups of prime order are such.*

The above corollary shows that if we are asked for an example of a K-group whose subgroups of prime order are permutably complemented we only need to provide an example of a finite K-group which is not a C-group.

It is well known that a finitely generated group has finitely many subgroups of a given finite index. Thus a finite subgroup of a finitely generated group admits at most finitely many permutable complements; on the other hand, the consideration of any *Tarski monster* (i.e., any infinite group whose proper non-trivial subgroups have prime order) shows that there may be infinitely many complements. We will employ this fact in the proof of the next theorem.

Theorem 5.11 *Let π be a set of primes and let G be a group. If all finitely generated subgroups of G have the π -subgroups of prime order permutably complemented in them, then all π -subgroups of prime order of G are permutably complemented in G .*

Proof Let X be a π -subgroup of prime order of G and let $\mathcal{L}$ be the set of all finitely generated subgroups of G containing X . By hypothesis and the above remark, it follows that for any $L \in \mathcal{L}$ the subgroup X admits at least one and only finitely many permutable complements in L . Now, a standard argument using a well known theorem of Kuroš on the existence of the inverse limit of finite sets yields that X admits a permutable complement in G . $\square$

Although the class of C-groups is not local (see Example 5.3), the above theorem yields that this is the case for the class of groups whose finite subgroups are permutably complemented.

Corollary 5.12 *Let G be a group. If every finitely generated subgroup F has the finite subgroups permutably complemented in F , then all finite subgroups of G are permutably complemented in G .*

Proof This follows immediately from Theorem 5.11 when π is chosen to be the set of all prime numbers and Corollary 5.9. $\square$

Notice that, since C-groups are locally finite, it follows that the local closure of the class of C-groups is the class of all locally finite² groups whose finite subgroups are permutably complemented. Since the class of C-groups admits a lattice-theoretic characterization (see Schmidt 1994, p. 124), also the class of local C-groups admits a lattice-theoretic characterization, and so even the class of periodic groups whose finite subgroups are permutably complemented admits a lattice-theoretic characterization.³

(b) Open question

Is it possible to provide a lattice-theoretic characterization of the class of all groups whose finite subgroups are permutably complemented?

Observe also that a group G whose abelian subgroups are permutably complemented is periodic and locally a C-group by Corollary 5.9. Of course, G splits over its commutator subgroup (which is abelian) and Lemma 3.1.7 of Schmidt (1994) easily yields that G' is a direct product of cyclic G -invariant subgroups of prime order. Thus Theorem 3.2.5 of Schmidt (1994) shows that G is actually a C-group. We can summarize this in the following result.

Theorem 5.13 *Let G be a group. Then G is a C-group if and only if all its abelian subgroups are permutably complemented.*

The consideration of *torsion-free Tarski monsters* (i.e. infinite groups whose proper non-trivial subgroups are infinite cyclic) shows that the above theorem cannot be extended to complementability (a more trivial example could be $\text{Alt}(4)$).

Clearly, $\mathbb{Z}$ is residually a C-group but it admits finite homomorphic images which are not (residually) C-groups. On the other hand, in the universe of periodic groups, the class of residually C-groups coincides with that of groups whose finite subgroups are permutably complemented (Corollaries 5.9 and 5.5) and this coincides with that of locally C-groups (Corollaries 5.7 and 5.12), a class which is clearly closed with respect to forming homomorphic images. We state this as the following corollary.

Corollary 5.14 *The class of periodic groups which are residually C-groups is closed with respect to forming homomorphic images.*

Let π be a set of primes and let G be a periodic group. Suppose that for any π -subgroup H of prime order of G there is a normal subgroup N_H such that $N_H \cap H = 1$ and G/N_H is a C-group (this is equivalent to require that G is the extension of a normal π' -subgroup by a group which is residually a C-group). Using Corollary 5.14 we easily see that the class of such groups is closed with respect to forming homomorphic images. This fact should be seen in relation to Theorem 5.4.

² Here, using Corollary 5.7, it is possible to replace “locally finite” by “periodic”.

³ We point out that such a fact was already proved in de Giovanni and Franciosi (1984) with a different and longer proof.

Theorem 5.15 *Let G be a periodic group. Then all finite subgroups of G are permutably complemented in G if and only if G has elementary abelian Sylow primary subgroups and it admits a normal series with cyclic factors.*

Proof The necessity of the condition easily follows from Corollary 5.7 and the fact that a primary group which is locally a C-group is elementary abelian; notice in particular that in this case any chief factor must be cyclic.

Now, to the sufficiency. It is easy to see that the commutator subgroup G' of G admits a central series, so Theorem 6.14 of Robinson (1972) jointly with the fact that all Sylow primary subgroups of G are elementary abelian yield that G is locally finite. By Corollary 5.12, we may therefore assume G is finite, supersoluble and with elementary abelian Sylow primary subgroups.

We will work by induction on the order of G , so we may certainly assume G has only one cyclic normal subgroup (necessarily of prime order) by Theorem 3.2.4 of Schmidt (1994). Now, G' is abelian (being nilpotent) and so it must be a p -group. Thus G admits a unique Sylow p -subgroup P and there is an abelian p' -subgroup K such that $G = PK$. Since $|P|$ and $|K|$ are coprime, we may write $P = C_P(G) \times [G, P]$. It follows that either $P = C_P(G)$ or $P = [G, P]$. In the former case, G is abelian and so a C-group. In the latter case, $P = [G, P]$ is cyclic by Maschke's theorem and G is again a C-group by Theorem 3.2.5 of Schmidt (1994). $\square$

It clearly follows from Theorem 5.15 that for a group G condition (b) in Theorem 3.3.10 of Schmidt (1994) is equivalent to the requirement for G to be a periodic T-group (that is, a group in which normality is a transitive relation) whose finite subgroups are permutably complemented. Thus Theorem 3.3.10 characterizes soluble groups whose proper subgroups are intersections of maximal ones as T-groups which are locally C-groups.

We notice also that Theorem 5.15 cannot be generalized to arbitrary groups, a counterexample being any non-abelian central extension of a finite elementary abelian group by a free abelian group of rank 2.

Corollary 5.16 *Let G be an abelian group. All finite subgroups of G are complemented if and only if the torsion-part T of G is a C-group.*

Proof The necessity of the condition follows at once from Theorem 5.15. Suppose the torsion-part T of G is a C-group, so it is a direct product of elementary abelian groups. Let X be a finite subgroup of G . Then X is contained in the $\pi(X)$ -component K of G , which is a pure subgroup of G of bounded exponent, so $G = K \times H$ for some subgroup H of G . Moreover, since K is a direct product of elementary abelian groups, we may write $K = X \times L$ for some subgroup L . The subgroup $L \times H$ is the required complement. $\square$

Since every abelian group splits over its pure, bounded subgroups, it follows that if in Corollary 5.16 we had $|\pi(G)| < \infty$, then G would have split over T . The following example shows that this is not always the case whenever $\pi(G)$ is infinite.

Example 5.17 Let π be an infinite set of primes and let $C_p = \langle c_p \rangle$ be a cyclic group of order p for each $p \in \pi$. Then cartesian product C of all C_p with $p \in \pi$ does not split over its torsion-part.

Proof Let y be the element of C whose non-zero components are the c_p 's for $p \in \pi$; clearly, y is aperiodic and so $y \notin T$. Now, $y + T$ is a non-zero element of C admitting non-zero p -height for each $p \in \pi$. Since C has no such elements, T cannot be a direct factor of C . $\square$

Now, we go deeper into the results of de Giovanni and Franciosi (1984) concerning the structure of (locally) supersoluble groups whose finite subgroups are complemented in some sense. We do this using mainly the results obtained above. We start analyzing the behaviour of maximal subgroups.

Lemma 5.18 *Let G be a group whose finite subgroups are complemented. Then $\text{Frat}(G)$ is aperiodic.*

Proof Let x be any non-identity periodic element of G . Then $\langle x \rangle$ admits a complement K in G ; in particular, $G = \langle x, K \rangle$. If x lies in $\text{Frat}(G)$, then $G = K$, in contradiction to $K \cap \langle x \rangle = 1$. $\square$

In connection to the above lemma we notice that a locally supersoluble group whose finite subgroups are complemented is finite provided that it satisfies the minimal condition on subgroups: in fact, it is certainly a Černikov group whose divisible part is contained in any maximal subgroup and so in the Frattini subgroup (which is torsion-free by Lemma 5.18).

Corollary 5.19 *Let G be a locally nilpotent group. Then all periodic subgroups of G are permutably complemented if and only if $\text{Frat}(G)$ is aperiodic.*

Proof It follows from Lemma 5.18 that we only need to consider the sufficiency of the condition. However, in such a case the result follows from the obvious observation that $G/\text{Frat}(G)$ is a direct product of elementary abelian groups (it is a C-group), so that any periodic subgroup admits a (permutable) complement modulo the aperiodic subgroup $\text{Frat}(G)$. $\square$

Lemma 5.20 *Let G be a group whose maximal subgroups have finite index and whose finite homomorphic images are C-groups. Then all finite subgroups of G are complemented if and only if they are permutably complemented.*

Proof We may certainly assume that all finite subgroups of G are complemented, so it follows from Lemma 5.18 that $\text{Frat}(G)$ is aperiodic. However, any maximal subgroup of G has finite index and so there is a normal subgroup $N \leq \text{Frat}(G)$ such that G/N is residually a C-group. This shows that any cyclic subgroup of prime order is permutably complemented in G , so that such are all the finite subgroups of G . $\square$

Since in a locally supersoluble group the maximal subgroups have finite index, the next statement is an immediate consequence of our Lemma 5.20 and Theorem 3.2.4 of Schmidt (1994).

Corollary 5.21 *Let G be a locally supersoluble group whose finite homomorphic images are K-groups. Then all finite subgroups of G are complemented if and only if they are permutably complemented.*

Let H be a perfect, locally nilpotent, torsion-free group (see for instance Theorem 6.21 of Robinson 1972). The direct product G of H and any elementary abelian group is an example of a non-periodic, locally supersoluble group whose finite homomorphic images are C -groups and whose finite subgroups are permutably complemented. The locally dihedral group $\mathbb{Z}_2 \rtimes \mathbb{Q}$ is a similar example which is hypercyclic.

In the supersoluble case, something more can be obtained.

Theorem 5.22 *Let G be a supersoluble group. Then all finite subgroups of G are complemented if and only if they are permutably complemented.*

Proof Suppose all finite subgroups are complemented in G . In order to prove that these subgroups are also permutably complemented in G we only need to prove that this is the case for the subgroups of prime orders (see Corollary 5.9).

Let H be subgroup of prime order of G . Then there exists a subgroup X of G such that $G = \langle H, X \rangle$ and $H \cap X = \{1\}$. Clearly, H is not contained in the Frattini subgroup of G and hence there exists a maximal subgroup M of G such that $H \cap M = \{1\}$. Since the group G is supersoluble, the subgroup M has prime index in G and so $|G : M| = |H|$. But then $G = HM$ and thus all subgroups of prime orders (and hence all finite subgroups) are permutably complemented in G , as desired. $\square$

Using Corollary 5.9 and employing an argument similar to the one used in the proof of Theorem 5.22 we easily obtain the following characterization. In what follows, $F(G)$ denotes the Fitting subgroup of the group G .

Theorem 5.23 *Let G be a supersoluble group. All finite subgroups of G are complemented if and only if $G = K \rtimes N$, where N is a C -group whose finite G -invariant subgroups are complemented in G , $F(K)$ is aperiodic and $K/F(K)$ is an elementary abelian 2-group.*

Notice that the infinite dihedral group is a mixed supersoluble group whose finite subgroups are permutably complemented, but which admits no torsion-part. The previous theorem is generalized for locally supersoluble groups in the next result, but first recall that in a locally supersoluble group G the set $G(p)$, for any prime p , of all periodic elements of G whose order is not divided by a prime $q \leq p$ is actually a subgroup; for instance, $G(2)$ is the subgroup made by all elements of odd order.

Theorem 5.24 *Let G be a locally supersoluble group. Then all finite subgroups of G are permutably complemented in G if and only if there is a normal subgroup $N \geq G(2)$ such that*

- (i) $N/G(2)$ is torsion-free,
- (ii) G/N is an elementary abelian 2-group, and
- (iii) (here, p_i is the i -th prime number) for each n , every cyclic subgroup $H/G(p_{n+1})$ of prime order p_{n+1} of $G(p_n)/G(p_{n+1})$ admits a G -invariant complement $K/G(p_{n+1})$ in $G(p_n)/G(p_{n+1})$ such that HL/L is a direct factor of $C_{N/L}(HL/L)$, where L is the p'_{n+1} -component of $G(2) \cap C_G(HK/K)$.

In particular, each $G(p_n)/G(p_{n+1})$ is an elementary abelian p_{n+1} -group.

Proof Let's deal with the necessity of the conditions. Let g be any element of prime order p . Then there is a proper subgroup K of G such that $G = \langle g \rangle K$. Clearly, $|G : K_G|$ is finite and not divided by any prime $q > p$, so $G(p)$ is actually contained in K_G and hence K/K_G is a permutable complement to $\langle g \rangle K_G/K_G$ in G/K_G . Therefore the factor group $G/G(p)$ has all its finite subgroups permutably complemented.

Now, let $N/G(2)$ be the intersection of all (normal) subgroups of $G/G(2)$ of index 2. Of course, the periodic subgroups of $G/G(2)$ are 2-groups and every element of order 2 admits a normal complement. Thus $N/G(2)$ is torsion-free and G/N is clearly an elementary abelian 2-group.

Finally, (iii) easily follows from the fact that $G/G(p)$ has all its finite subgroups permutably complemented. The necessity is proved.

Suppose G has the structure described in the statement. As usual, we only need to check that the subgroups of prime orders have permutable complements. Let H be a subgroup of prime order p . If $p = 2$, it is immediate to see that H is permutably complemented modulo N and so even in G . Assume $p > 2$, so $H \leq G(2)$ and we may actually assume $G(p) = 1$ and that H is the only Sylow p -subgroup of G ; moreover, Theorem 5.11 yields that we may even assume G/N finite. Let $C = C_N(H)$, so that $p \notin \pi(G/C)$. Condition (iii) yields that $C = H \times L$ for some subgroup L and C/L_G is a finite elementary abelian p -group. To simplify the notation we may assume $L_G = 1$. Therefore H is a G -invariant subgroup of C . Clearly, G splits over C by the Schur-Zassenhaus' theorem and H admits a G -invariant complement by Maschke's theorem. All in all, H admits a permutable complement in G . $\square$

In the above theorem we may replace (iii) by the condition (iv): all finite subgroups of $G(p_n)/G(p_{n+1})$ are permutably complemented in $G/G(p_{n+1})$ for each n . However, (iv) is stronger than (iii), and in the periodic case the condition on the G -invariant complement in (iii) can be neglected.

Lemma 5.25 *Let G be a hypercyclic group whose finite subgroups are complemented and let N be a normal p -subgroup of G for some prime p . Then any finite subgroup of N admits a G -invariant complement in N .*

Proof Since G is hypercyclic, it follows that $\text{Frat}(N) \leq \text{Frat}(G)$, so Lemma 5.18 yields that N is elementary abelian. Let A be the intersection of all G -invariant subgroups of N of index p in N . If $A \neq 1$, then it contains a non-trivial normal cyclic subgroup B and there is a subgroup K complementing B in G . But then $K \cap N$ is a complement to B in N and hence $|N : K \cap N| = p$. Thus $K \cap N \geq A \geq B$, a contradiction. Therefore $A = 1$ and so any finite subgroup F of N admits a G -invariant complement in N . $\square$

As a consequence of Lemma 5.25 and Theorem 5.24 (replacing condition (iii) by (iv)), we obtain the following result which links complements and permutable complements of finite subgroups in periodic hypercyclic groups.

Corollary 5.26 *Let G be a periodic hypercyclic group. Then all finite subgroups of G are permutably complemented in G if and only if, for all primes p , the finite subgroups of $G/G(p)$ are complemented.*

5.1 A relevant class of examples

The following examples show in particular that the classes of C-groups and that of K-groups are not countably recognizable. Recall here that a class of groups $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$ -group (see for instance Robinson 1972; Ferrara and Trombetti 2017), where a more recent bibliography on the subject can be found).

Example 5.27 [see also (Gorchakov 1960)]

- (i) There is a periodic group whose finite subgroups are permutably complemented but containing a normal Sylow 3-subgroup which is not complemented.
- (ii) There is a periodic group whose primary subgroups are permutably complemented but containing a normal subgroup which is not complemented. Moreover, all its countable subgroups are C-groups.

Proof Let $\{p_{ij}\}_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ be a double indexed set of odd primes and let L_{ij} be the dihedral group of order $2p_{ij}$. Choose two distinct elements a_{ij}, b_{ij} of order 2 in each L_{ij} and let $\bar{L}$ be the cartesian product (indexed by $\mathbb{N} \times \mathbb{N}$) of all the L_{ij} 's. For each $n \in \mathbb{N}$ define $x_n := (x_{ij}) \in \bar{L}$ such that $x_{nj} = a_{nj}$ and $x_{ij} = 1$ for $i \neq n$.

By Zorn's lemma we may find a subset $\mathcal{M}$ of $\mathbb{N}^{\mathbb{N}}$ satisfying the following two properties:

- (i) for all $f, g \in \mathcal{M}$ the set $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite;
- (ii) $\mathcal{M}$ is maximal in $(\mathbb{N}^{\mathbb{N}}, \subseteq)$ with respect to property (i).

For each $\varphi \in \mathcal{M}$, define $y_\varphi := (y_{ij}) \in \bar{L}$ such that $y_{n, \varphi(n)} = b_{n, \varphi(n)}$ for each $n \in \mathbb{N}$ and $y_{ij} = 1$ for all other pairs (i, j) .

It is clear that the subgroups X and Y generated by the x_n 's and the y_φ 's, respectively, are abelian. Let L be the direct product of the L_{ij} 's and put $G = \langle X, Y, L \rangle$. It is easy to see that $G' = L'$ is a $2'$ -group.

Suppose by contradiction that there is a Sylow 2-subgroup K of G such that $G = K G'$. Then for each $n \in \mathbb{N}$ we can find elements $k(n) = (k_{i, j}(n)) \in K$ and $g(n) \in G'$ such that $x_n = k(n) \cdot g(n)$. Since $g(n) \in G'$ there is $s_n \in \mathbb{N}$ such that $x_{n, s_n} = k_{n, s_n}(n)$.

By maximality, we may find $\varphi \in \mathcal{M}$ such that $\varphi(n) = s_n$ for infinitely many values of $n \in \mathbb{N}$. Now, we can write $y_\varphi = k \cdot g$ for some $k = (k_{ij}) \in K$ and $g \in G'$, so that there are infinitely many $n \in \mathbb{N}$ such that $k_{n, s_n} = y_{n, s_n}$. For each of such n 's we have $[x_{n, s_n}, y_{n, s_n}] \neq 1$. Thus the 2-subgroup K contains a $2'$ -element, a contradiction showing that G is not a C-group.

Since $G \leq \bar{L}$ is residually a C-group and it is periodic, it follows from Corollaries 5.5 and 5.9 that all its finite subgroups are permutably complemented.

If $p_{ij} = 3$ for all $i, j \in \mathbb{N}$, we obtain (i). If $p_{ij} \neq p_{uv}$ whenever $(i, j) \neq (u, v)$ we obtain (ii). In fact, as concern the first part of (ii) we just observe that G/G' is an elementary abelian 2-group, so any 2-subgroup of G admits a permutable complement (modulo G'). As for the second part of (ii), it follows from Theorem 3.2.5 of Schmidt (1994) and the countable version of the Schur–Zassenhaus theorem. $\square$

5.2 Primary subgroups

Theorem 5.28 [see also (Gorchakov 1960)] *Let G be a periodic group whose finite subgroups are permutably complemented in G . Then $G' \cap Z(G) = 1$ and all subgroups of $Z(G)$ are complemented in G .*

Moreover, if p is a prime number and all p -subgroups of G' are permutably complemented in G , then all p -subgroups of G are permutably complemented in G .

Proof Being G locally a C-group, it follows from Lemma 2.2 that $G' \cap Z(G) = 1$. Let Z be any subgroup of $Z(G)$, then ZG'/G' admits a complement K/G' in G/G' (notice that G/G' is a direct product of elementary abelian subgroups) and certainly K is even a permutable complement to Z in G .

Let p be a prime number and suppose that all p -subgroups of G' are permutably complemented in G . Let P be a p -subgroup of G , so $P \cap G'$ admits a permutable complement K in G by hypothesis. Clearly, $P = (P \cap G')(P \cap K)$ by Dedekind's modular identity, and $P \cap K \cap G' = 1$. Since G/G' is a C-group, it follows that $P \cap K$ is permutably complemented in G (modulo G'), put L be a permutable complement to $P \cap K$ in G . Now, we easily see that $L \cap K$ is a permutable complement to P in G . $\square$

In correlation to the previous theorem Example 5.3 shows that we cannot expect to have “ p ” replaced by a set of primes (even of cardinality 2).

The next theorem shows that the class of periodic groups whose primary subgroups are permutably complemented is closed with respect to forming homomorphic images.

Lemma 5.29 [see also (Gorchakov 1960)] *Let G be a periodic group and let p be a prime number. Suppose further that all finite subgroups and all p -subgroups of G are permutably complemented in G . If N is a normal subgroup of G , then all finite subgroups and all p -subgroups of G/N are permutably complemented in G/N .*

Proof Since the class of periodic groups whose finite subgroups are permutably complemented coincides with that of groups which are locally C-groups, it follows that all finite subgroups of G/N are permutably complemented.

Let's move on to prove that all p -subgroups of G/N are permutably complemented in G/N . By Theorem 5.28, we only need to prove that an arbitrary p -subgroup P/N of G'/N is permutably complemented in G . To this aim recall that G' is abelian (Corollary 5.7), so that, if N is a p' -subgroup of G' , then $P = NP_1$ for some p -subgroup P_1 of G' . In these circumstances, P_1 admits a permutable complement K in G ; since $G' = P_1(G' \cap K)$, it follows that N lies in $G' \cap K$ and hence that

$$P/N \cap K/N = P_1N/N \cap K/N = N.$$

Therefore in case N is a p' -subgroup of G' , K/N is a permutable complement to P/N in G/N .

Suppose $N \leq G'$ is not a p' -subgroup. By what we have already proved, it is possible to assume that N is actually p -group. Thus P is a p -subgroup of G' , so it admits a permutable complement K in G and it is clear that KN/N is also a permutable complement to P/N in G/N .

Now, we turn to the general case in which N is not necessarily a subgroup of G' . By what we have already proved, it is clear that all p -subgroups of $\overline{G} = G/N \cap G'$ are permutably complemented. Since $\overline{N}$ has trivial intersection with $\overline{G}'$, it is contained in the centre of $\overline{G}$ and Theorem 5.28 shows that it is complemented in $\overline{G}$; let $\overline{K}$ be a complement to $\overline{N}$ in $\overline{G}$. Then all p -subgroups of $\overline{K}$ are permutably complemented in $\overline{K}$ and since G/N is isomorphic to $\overline{K}$ the result is proved. $\square$

The following statement residually characterizes periodic groups whose primary subgroups have permutable complements and should be put in relation to Corollary 5.5.

Corollary 5.30 *Let p be a prime and let G be a periodic group whose finite subgroups are permutably complemented in G . Let N be the intersection of all normal subgroups M such that G/M is a C-group and $G'M/M$ is a p -group.*

Then all p -subgroups of G are permutably complemented in G if and only if G/N is a C-group.

Proof Let's first deal with the necessity of the condition. Now, all p -subgroups of G are permutably complemented and Lemma 5.29 yields that such is the case for G/N . Let X/N be any subgroup of G/N . Since $G'N/N$ is a p -group, it follows that $X/N \cap G'N/N$ admits a permutable complement K/N in G/N ; of course, $X/N = (X/N \cap G'N/N)((X \cap K)/N)$ and $(X \cap K)/N$ has trivial intersection with $G'N/N$. Therefore $(X \cap K)/N$ admits a permutable complement H/N in G/N and we easily see that $(H \cap K)/N$ is also a permutable complement to X/N in G/N .

Now, to the sufficiency. Let P be any p -subgroup of G' . By Corollary 5.5 the group G is residually a C-group. Let $x \in P$. Then there is a normal subgroup N_x of G such that G/N_x is a C-group and x is a non-trivial element of $G'N_x/N_x$. Factoring out the p' -component of $G'N_x/N_x$ we see that there is a normal subgroup M_x such that G/M_x is a C-group and x is a non-trivial element of p -subgroup $G'M_x/M_x$. The net result is that $P \cap N = 1$.

Now, PN/N admits a permutable complement L/N in G/N and clearly L is a permutable complement to P in G . $\square$

We remark that (essentially with the same proof) the above corollary can be proved in relation to a given family of normal subgroups $\{M\}_{M \in \mathcal{M}}$ such that G/M is a C-group with primary commutator subgroup for each $M \in \mathcal{M}$ and the intersection of all these subgroups is 1.

6 Infinite subgroups

In this section we consider groups whose infinite subgroups are complemented in some sense. Groups whose infinite subgroups are permutably complemented have been investigated by Černikov (1967) in the universe of groups with a finite normal series. Here some of his results are extended to the universe of locally graded groups; recall that a group is called *locally graded* if all its non-trivial finitely generated subgroups have proper subgroups of finite index. In this respect, notice the following easy lemma.

Lemma 6.1 *Let G be a group admitting a normal series with finite factors. Then G is locally graded.*

Proof Let F be a finitely generated subgroup of G , so F inherits from G a normal series with finite factors. Let N be a term of such a series which is a maximal subgroup of F . Then F/N is finite and G is locally graded. $\square$

Emaldi (1971) studied groups G whose infinite subgroups are complemented under the additional hypothesis that all homomorphic images of G have a trivial Frattini subgroup; notice here that, dually to Lemma 5.18, *the Frattini subgroup of any group whose infinite subgroups are complemented is periodic*. Of course, the property of having the infinite subgroups (permutably) complemented is inherited by homomorphic images, but, having the infinite subgroups complemented is in general not inherited by subgroups as shown by the consideration of suitable torsion-free Tarski monsters. On the other hand, the property of having the infinite subgroups permutably complemented is inherited by subgroups and so groups with this property must be periodic. Although the consideration of Tarski monsters shows that we cannot hope for “local finiteness” in general, this is possible if we require the group to be locally graded.

Lemma 6.2 *Let G be a locally graded group whose infinite subgroups are permutably complemented. Then G is locally finite.*

Proof Let F be a finitely generated subgroup of G . Since G is periodic, then F/F'' is finite and so F'' is finitely generated. If F'' is infinite, then it contains a proper F -invariant subgroup N of finite index. Thus N is infinite and F/N is a C-group. But C-groups are metabelian, so that $F'' \leq N$, a contradiction. The lemma follows. $\square$

Moving from locally graded groups to soluble groups we can still obtain the same conclusion as Lemma 6.2 but only requiring that the infinite subgroups admit complements. This generalizes the well known fact that a soluble K-group is periodic (see Schmidt 1994, Exercise 3.1.2).

Lemma 6.3 *Let G be an infinite group whose infinite normal subgroups are complemented. If A is a non-trivial abelian normal subgroup of G such that $A \cap \text{Frat}(G) = 1$, then A is complemented in G and it is generated by minimal G -invariant subgroups.*

Proof Let's first show that A contains a minimal G -invariant subgroup. Since $A \cap \text{Frat}(G) = 1$, there is a maximal subgroup M such that $A \not\leq M$ and, of course, $G = AM$. If $A \cap M = 1$, then A is the required minimal G -invariant subgroup. If $A \cap M$ is finite and non-trivial, then it contains a minimal G -invariant subgroup. If $A \cap M$ is infinite, then $A \cap M$ admits a G -invariant complement N in A . Obviously, N is the required minimal G -invariant subgroup. Thus any non-trivial normal subgroup of A contains minimal G -invariant subgroups.

Let T be the subgroup generated by all minimal G -invariant subgroups of A . If T is infinite, then it admits a complement K in G and $A = T \times (K \cap A) = T$, since if $K \cap A \neq 1$, then it contains a minimal G -invariant subgroup. If T is finite, the fact that $A \cap \text{Frat}(G) = 1$ yields that any maximal subgroup of G not containing T has finite index in G and so there is a normal subgroup N of finite index such that $T \cap N = 1$;

in particular, TN/N admits a complement K/N in G/N . Now, $TK = TNK = G$ and $T \cap K \leq N \cap T = 1$, so K is a complement to T in G . As in the previous case, this yields that $T = A$. The statement is proved. $\square$

Theorem 6.4 *Let G be a soluble group whose infinite subgroups are complemented. Then G is locally finite.*

Proof It is certainly possible to assume G is infinite and to work by induction on the derived length n of G , being the result true for $n = 1$ by Lemma 6.2. Let A be the last non-trivial term of the derived series of G . By induction we have that G/A is locally finite, so we may assume A is even torsion-free. As we remarked above, the Frattini subgroup $F = \text{Frat}(G)$ of G is periodic, so that $F \cap A = 1$. It follows from Lemma 6.3 that A is generated by minimal G/A -invariant subgroups. However, a well known theorem of Baer (see Baer 1964) shows that in this case A is periodic, giving $A = 1$ and completing the proof of the statement. $\square$

The following theorem provides a description of abelian groups whose infinite subgroups are complemented.

Theorem 6.5 *Let G be an infinite abelian group whose infinite subgroups are complemented. Then G either is a C-group or is the direct product of a finite abelian C-group and a Prüfer group.*

Proof If $G = X \times Y$, where X and Y are infinite, then $G/X \simeq Y$ and $G/Y \simeq X$ are C-groups and hence G is a C-group. Thus we may assume that G cannot be written as the direct product of two infinite subgroups.

Since G is periodic, it follows that it has even finite rank and so it can be written as the product of finitely many locally cyclic groups only one of which P is infinite. Now, $G = P \times Q$ for some finite subgroup Q and of course $G/P \simeq Q$ is a C-group. The statement is proved. $\square$

We remark that Theorem 6.5 has been recently extended to locally nilpotent groups in the following way.

Theorem 6.6 [see (Ferrara and Trombetti 2021b)] *Let G be a locally nilpotent group whose infinite subgroups are complemented. Then G is either abelian or Černikov.*

Now, we can prove that the condition considered by Černikov is in fact equivalent to (local) solubility.

Lemma 6.7 *Let G be an infinite group whose infinite subgroups are permutably complemented. The following statements are equivalent.*

- (i) G admits a normal series with finite factors.
- (ii) G is locally soluble.
- (iii) G is soluble of derived length at most 3.

Proof Suppose first G admits a normal series $\mathcal{N}$ with finite factors. If N is any infinite term of the series $\mathcal{N}$, then G/N is a C-group and so $G'' \leq N$. Let M be the intersection

of all these infinite terms. Then G/M is metabelian and M admits an ascending G -invariant series with finite terms.

If N is any finite normal subgroup of G , then $G'' \leq C_G(N)$; in particular, $G'' \cap M$ is abelian. Since $G'' \leq M$, then G'' is abelian and G is soluble of derived length at most 3.

Assume now G is locally soluble and let H/K be a chief factor of G . If K is infinite, then G/K is a C-group and so it is even soluble yielding that H/K is finite. If K is finite, then we may put $K = 1$ and assume that H is an infinite abelian subgroup of G . It follows from Theorem 6.5 that H is a direct product of elementary abelian groups. Let A be a proper subgroup of H of finite index. Then there is a permutable complement B to A in G and so $B \cap H$ is a non-trivial finite G -invariant subgroup, a contradiction. It follows that all chief factors of G are finite and so G admits a normal series with finite factors. $\square$

Lemma 6.8 *Let N be an infinite abelian normal subgroup of a group G whose infinite subgroups are permutably complemented. If N is not Černikov, then it is generated by cyclic G -invariant subgroups.*

Proof It follows from Theorem 6.5 that N is a direct product of elementary abelian groups. Let H be a subgroup of N which is maximal with respect to being generated by G -invariant cyclic subgroups. Suppose $H < N$. Then there is a proper subgroup U of N containing H and having prime index in N . Let V be a permutable complement to U in G . Then $V \cap N$ is a cyclic G -invariant subgroup of N and HV is generated by G -invariant cyclic subgroups, contradicting the maximality of H . $\square$

Theorem 6.9 *Let G be an infinite soluble group whose infinite subgroups are permutably complemented. If G contains no Prüfer subgroup, then G is a C-group. If G contains a Prüfer subgroup P , then P has finite index in G .*

Proof It follows from Lemma 6.7 that G is soluble of length at most 3; let L be the smallest infinite subgroup among G , G' and G'' .

Suppose first G contains a Prüfer subgroup P . Then G/L is a C-group, so it does not contain any Prüfer subgroup; in particular, $P \leq L$ and $L'P/L'$ is the finite residual of L/L' by Theorem 6.5, so $L'P$ is G -invariant. Since P is the finite residual of $L'P$, it follows that P is normal in G . Now, it follows from Lemma 6.2 that G is periodic and so $G/C_G(P)$ is finite. Let K be a complement to P in G . Then $C_G(P) = P \times C_K(P)$. If $C_K(P)$ is infinite, then $C_G(P)/C_K(P)$ is a C-group and as such does not contain Prüfer subgroups. This yields that $C_K(P)$ is finite and hence that P has finite index in G .

Suppose now G does not contain any Prüfer subgroup. Let F be a finite subgroup of L , so FL' is L -invariant and finite. It follows from Theorem 3.43 of Robinson (1972) that L contains an infinite abelian subgroup A , which must be a C-group by Theorem 6.5. Of course, A has finite index in AF and so A contains an F -invariant subgroup B of finite index in AF such that $B \cap F = 1$. Now, $F \simeq BF/B$ is a C-group and so F' is abelian. The arbitrariness of F yields that L' is abelian.

A similar argument applied in the infinite nilpotent group $C = C_L(L')$ to any of its finite subgroups yields that C is even abelian. Now, let E be any finite subgroup

of G . Another application of the previous arguments (this time in EC) yields that E' is abelian and so G' is abelian.

If G' is finite, then $L = G$ and $C = C_G(G')$ is abelian of finite index in G . Now, let K be a complement either to C in G (if G' is finite) or to G' in G (if G' is infinite); in particular, $K \simeq G/C$ is an abelian C-group. It follows from Lemma 6.8 that C is generated by G -invariant cyclic subgroups, so that G is a C-group by Theorem 3.2.5 of Schmidt (1994). $\square$

Corollary 6.10 *Let G be an infinite soluble group whose infinite subgroups are permutably complemented. If G is uncountable (or it is not a Černikov group), then G is a C-group.*

Notice that Example 5.27 shows that the class of soluble groups in which all infinite subgroups are permutably complemented is not countably recognizable.

Corollary 6.11 *Let G be an infinite locally nilpotent group whose infinite subgroups are permutably complemented. If G is abelian, it is either a C-group or the direct product of a Prüfer group and a finite C-group.*

If G is non-abelian, then it is the direct product of the locally dihedral 2-group and a finite abelian C-group.

Proof It follows from Lemma 6.7 that G is soluble. If G is a C-group, then it is abelian (since it is locally nilpotent). Suppose G does not have the C-property. Then Theorem 6.9 shows that G is a finite extension of a Prüfer p -group P for some prime p . Then G/P is an abelian C-group. Let D/P be the p -component of G/P . Clearly, $G = D \times K$ for some finite abelian C-group K . If D is abelian, the result is proved, so we may assume D is non-abelian; the only possibility is therefore $p = 2$. Of course, we may write $D = L \rtimes P$, where L is a finite elementary abelian 2-group. Now, $L = C_L(P) \times \langle x \rangle$ and $D = C_L(P) \times H$, where $H = \langle x, P \rangle$ is a locally dihedral 2-group. The statement is proved. $\square$

In the above corollary, it is certainly possible to replace local nilpotency with the admittance of a central series.

In connection to Lemma 6.7, we have the following question, a negative answer to which automatically extends all results in this section to the universe of locally graded groups (see also the subsequent lemma).

(c) Open question

Is there an infinite locally finite simple group whose infinite subgroups are permutably complemented?

Lemma 6.12 *Let G be an infinite locally graded group with no infinite simple sections. If all infinite subgroups of G are permutably complemented, then G is soluble.*

Proof Assume by contradiction that G is not soluble. If G contains a finite normal subgroup N such that G/N is soluble, then $G'' \leq C_G(N)$ and G'' is soluble, a contradiction. We may therefore assume that G contains no finite normal subgroup N such that G/N is soluble.

If G'' contains an infinite proper normal subgroup N , then $G^{(\text{iv})} \leq N$ and $G^{(\text{iv})}$ is infinite. Since $G/G^{(\text{iv})}$ is a C-group, it is metabelian and hence $G'' = G^{(\text{iv})} = N$, a contradiction. Thus G'' does not contain infinite proper normal subgroups.

On the other hand, G'' is infinite, so the hypothesis yields that it is generated by its finite normal subgroups; but any such a finite normal subgroup is trivially central in G'' and hence G'' is abelian, the final contradiction. $\square$

Finally, notice the following modification of Theorem 6.9.

Theorem 6.13 *Let G be an infinite soluble group whose infinite subgroups are complemented. If every homomorphic image of G has a trivial Frattini subgroup, then G is a K-group.*

Proof This easily follows by induction on the derived length of G using our Lemma 6.3 and Lemma 3.1.9 of Schmidt (1994). $\square$

7 Subgroups of infinite rank

This section is devoted to provide a proof of a special case of Theorem B of Camp-Mora and Monetta (2020). Indeed, in Step 5 of the proof of Theorem B of Camp-Mora and Monetta (2020) it is implicitly assumed that G is a semidirect product of two abelian groups, a fact which is not true in general. It seems that the problem (also conditioning Theorem A) comes from the wrong assumption that a metabelian group whose Sylow subgroups have prime exponent must be splitting over the commutator subgroup (see the proof of their Lemma 2), but Example 5.17 is certainly a counterexample to this statement.

The following proof is fundamentally based on results of Sect. 5. Recall that a group G is said of *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements and r is the least positive integer with this property; if such an integer does not exist, the group G is said to have *infinite rank*.

Theorem 7.1 *Let G be a periodic locally soluble group with G' of infinite rank. If every subgroup of G of infinite rank is permutably complemented in G , then G is a C-group.*

Proof Let F be a finite subgroup of G . By Theorem 2 of Hartley (1979) there is an F -invariant abelian subgroup A of infinite rank such that $F \cap A = 1$. Thus $F \simeq FA/A$ is a C-group and G is locally a C-group; in particular, all finite subgroups of G are permutably complemented in G (see Corollary 5.12) and G is metabelian (see Corollary 5.7). Moreover, G' and G/G' are easily seen to be products of elementary abelian groups (see also Theorem 6.9).

Clearly, if P is any normal primary subgroup of G , it is elementary abelian (having a trivial Frattini subgroup) and both its finite and infinite subgroups are permutably complemented in G since any infinite subgroup of P has then infinite rank. It follows from Lemma 3.17 of Schmidt (1994) that P is a direct product of minimal normal subgroups of G and hence P is even a direct product of cyclic G -invariant subgroups

of prime order (recall that the intersection of a permutable complement of a subgroup of P with P itself is G -invariant). Since G splits over G' , it follows from Theorem 3.2.5 of Schmidt (1994) that G is a C -group. $\square$

(d) Open question

Let G be a periodic locally soluble group of infinite rank whose subgroups of infinite rank are permutably complemented. Is G a C -group?

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