Jordan triple homomorphisms on $\mathcal{T}_\infty(F)$

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Abstract

Let $\mathcal{T}_\infty(F)$ be the algebra of all $\mathbb{N} \times \mathbb{N}$ upper triangular matrices defined over a field F of characteristic different from 2. We consider the Jordan triple homomorphisms of $\mathcal{T}_\infty(F)$, i.e. the additive maps that satisfy the condition $\phi(xyx) = \phi(x)\phi(y)\phi(x)$ for all $x, y \in \mathcal{T}_\infty(F)$. For the case when F is a prime field we find the form of all such maps ϕ . For the general case we present the form of the surjective maps ϕ .

Keywords Jordan triple homomorphism · Triangular matrices · Infinite matrices

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1 Introduction

The maps satisfying the following condition

$$\phi(xyx) = \phi(x)\phi(y)\phi(x) \quad (1)$$

are called Jordan triple maps.

Some interesting properties of such maps are given in Lu (2003) and D’Amour (1991). The form of ϕ given above was found for example for Banach algebras Molnár (2000), von Neumann algebras (Molnár 2006), nest algebras (Lu 2008), some classes of rings (Bresar 1989), unitary groups Molnár (2013) and full matrix algebras (Kuzma 2000; Lešnjak and Sze 2006). Some modifications of this problem were also studied (like in Dobovišek (2008)).

If ϕ is a Jordan triple map and is additive, i.e.

$$\phi(x + y) = \phi(x) + \phi(y), \quad (2)$$

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then it is called a Jordan triple product homomorphism, or shorter a Jordan triple homomorphism.

It was proved that on some conditions the additivity may follow from (1), for instance in Kim and Park (2013) it is established that in the generalized matrix algebras all such bijective maps are additive.

In this paper we will study Jordan triple homomorphisms of $\mathcal{T}_\infty(F)$ —the algebra of all infinite (whose rows and columns are indexed by natural numbers) matrices over a field F . Some particular finite-dimensional cases, i.e. Jordan triple homomorphisms from $\mathcal{T}_n(\mathbb{C})$ to $\mathbb{C}$ and Jordan triple homomorphisms from $\mathbb{C}$ to $\mathcal{T}_n(\mathbb{C})$ were studied in Kokol-Bukovšek and Mojškerc (2018).

2 Notation and statement of results

2.1 Notation

If we write x^t for some invertible t , we mean $t^{-1}xt$.

We will write $T_\infty(F)$ for the multiplicative group consisting of all matrices that are invertible in $\mathcal{T}_\infty(F)$.

We denote by e_∞ and e_n the identity matrices, infinite and $n \times n$, respectively. By e_{ij} we mean a matrix, which may be finite or infinite, with 1 in the position (i, j) and 0 in every other position.

If $x \in \mathcal{T}_\infty(F)$ is a matrix that has nonzero entries in the positions (i, j) with $i \in I$, $j \in J$, then we will use the notation

$$x = \sum_{i \in I, j \in J} x_{ij} e_{ij}.$$

In this sense we mean the sums in equations (3), (4).

We also introduce the following notation. If $x \in \mathcal{T}_\infty(F)$ and $n \in \mathbb{N}$, then by $x^{(n)}$ and $\tilde{x}^{(n)}$ we denote the matrices

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} & 0 & 0 & \cdots \\ & \ddots & \vdots & \vdots & \vdots & \\ & & x_{nn} & 0 & 0 & \cdots \\ & & & 0 & 0 & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ & \ddots & & \vdots & & \vdots & \\ & & 0 & 0 & 0 & 0 & \cdots \\ & & & x_{n+1,n+1} & x_{n+1,n+2} & x_{n+1,n+3} & \cdots \\ & & & & x_{n+2,n+2} & x_{n+2,n+3} & \\ & & & & & x_{n+3,n+3} & \\ & & & & & & \ddots \end{pmatrix}$$

respectively.

2.2 Standard maps

In our results there will appear some standard maps. We point them out below.

- Suppose that f is a homomorphism of F . Then by $\bar{f}$ we denote the map on $\mathcal{T}_\infty(F)$ such that

$$(\bar{f}(x))_{ij} = f(x_{ij}).$$

Thus

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ & x_{22} & x_{23} & \\ & & x_{33} & \\ & & & \ddots \end{pmatrix} \xrightarrow{\bar{f}} \begin{pmatrix} f(x_{11}) & f(x_{12}) & f(x_{13}) & \cdots \\ & f(x_{22}) & f(x_{23}) & \\ & & f(x_{33}) & \\ & & & \ddots \end{pmatrix}.$$

This $\bar{f}$ is called an induced homomorphism of $\mathcal{T}_\infty(F)$. Clearly, $\bar{f}$ is a homomorphism of $\mathcal{T}_\infty(F)$.

- For any invertible t the map $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ such that $\phi(x) = t^{-1}xt$ is called an inner automorphism and will be denoted by $\mathcal{I}nn_t$. Clearly, this is simply a conjugacy of each x with use of matrix t .
- Let $n \in \mathbb{N}$. We will write $\mathcal{U}p_n$ for the map on $\mathcal{T}_\infty(F)$ if $(\mathcal{U}p_n(x))_{ij} = x_{i+n,j+n}$ for all i, j . This map can be ‘graphically’ seen as follows:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ & x_{22} & x_{23} & \\ & & x_{33} & \\ & & & \ddots \end{pmatrix} \xrightarrow{\mathcal{U}p_n} \begin{pmatrix} x_{n+1,n+1} & x_{n+1,n+2} & x_{n+1,n+3} & \cdots \\ & x_{n+2,n+2} & x_{n+2,n+3} & \\ & & x_{n+3,n+3} & \\ & & & \ddots \end{pmatrix}.$$

- For the two types of sets $S \subset \mathbb{N}$ we define the maps denoted by Cut_S .

1. For any $n \in \mathbb{N}$ by $\mathcal{C}ut_{\{1,2,\dots,n\}}$ we define the map on $\mathcal{T}_\infty(F)$ such that

$$(\mathcal{C}ut_{\{1,2,\dots,n\}}(x))_{ij} = \begin{cases} x_{ij} & \text{if } i > n \\ 0 & \text{otherwise,} \end{cases}$$

so

$$x \xrightarrow{\mathcal{C}ut_{\{1,\dots,n\}}} \tilde{x}^{(n)}.$$

2. For any $n \leq m$ we define the map $\mathcal{C}ut_{\mathbb{N} \setminus \{n,n+1,\dots,m\}}$ as follows:

$$x \xrightarrow{\mathcal{C}ut_{\mathbb{N} \setminus \{n,\dots,m\}}} (\mathcal{U}p_{n-1}(x))^{(m-n+1)} = \begin{pmatrix} x_{nn} & \cdots & x_{nm} & 0 & 0 & \cdots \\ & \ddots & \vdots & \vdots & \vdots & \\ & & x_{mm} & 0 & 0 & \cdots \\ & & & 0 & 0 & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}.$$

3. Additionally, for the convenience, we will assume that $\mathcal{C}ut_\emptyset$ is the identity map.

- If we consider a subalgebra of $\mathcal{T}_\infty(F)$ consisting of the matrices that have nonzero entries only in the first n rows and columns, then we can define on it the map $\mathcal{J}$ as follows:

$$(\mathcal{J}(x))_{ij} = \begin{cases} x_{n+1-j,n+1-i} & \text{for } 1 \leq i, j \leq n. \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that $\mu : \mathbb{N} \rightarrow 2^{\mathbb{N}} \setminus \{\emptyset\}$ is a map such that if $n < m$, $i \in \mu(n)$, $j \in \mu(m)$, then we have $i < j$.

Let $\{d_n\}_{n \in \mathbb{N}}$ be a set of elements of F and let $\{c_n\}_{n \in \mathbb{N}}$ be a family of infinite triangular matrices such that $(c_{nm})_{ij} = 0$ as soon as $i \notin \mu(n)$ or $j \notin \mu(m)$. For $m > n$ let $c_{nm} = \prod_{i=n}^{m-1} c_i$ and notice that $(c_{nm})_{ij} = 0$ as soon as $i \notin \mu(n)$ or $j \notin \mu(m)$.

By $Spl_{\mu, \{c_{nm}\}_{n < m}} : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ we denote a map such that

$$Spl_{\mu, \{d_n, c_n\}_n}(x) = \sum_{n \in \mathbb{N}} d_n \sum_{i \in \mu(n)} e_{ii} + \sum_{n < m} x_{nm} c_{nm}. \quad (3)$$

Thus, putting for instance $\mu(n) = \{2n-1, 2n\}$, $d_n = 1$ and

$$c_n = e_{2n-1, 2n+1} + e_{2n-1, 2n+2} + e_{2n, 2n+2},$$

we get that

$$c_{nm} = e_{2n-1, 2m-1} + (m-n)e_{2n-1, 2m} + (m-n)e_{2n, 2m},$$

so

$$(x_{ij}) \xrightarrow{\mathcal{S}pl_{\mu, \{d_n, cn\}_n}} \begin{pmatrix} x_{11} & 0 & x_{12} & x_{12} & x_{13} & 2x_{12} & \cdots \\ & x_{11} & 0 & x_{12} & 0 & x_{13} & \\ & & x_{22} & 0 & x_{23} & x_{23} & \\ & & & x_{22} & 0 & x_{23} & \\ & & & & x_{33} & 0 & \\ & & & & & x_{33} & \\ & & & & & & \ddots \end{pmatrix}$$

2.3 Results

Now we can present our first main result.

Theorem 1 Assume that F is a field such that $\text{char}(F) \neq 2$ and that $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a Jordan triple homomorphism. If ϕ is surjective, then there exist an invertible matrix t , a natural number k and an epimorphism $f : F \rightarrow F$ such that either

$$\phi = \text{Inn}_t \cdot \mathcal{U}p_k \cdot \overline{f}$$

or

$$\phi = -\text{Inn}_t \cdot \mathcal{U}p_k \cdot \overline{f}.$$

Before stating our second main theorem we introduce one more notion.

We will say that $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a *separable sum*, if there exist nonzero maps $\phi_1, \phi_2, \dots, \phi_n$ ($n \geq 2$), or $\phi_1, \phi_2, \phi_3, \dots$, such that

$$\phi(x) = \sum_i \phi_i(x) \quad \text{for all } x \in \mathcal{T}_\infty(F) \quad (4)$$

(where the meaning of the above sum in the case when there are infinitely many ϕ_i will be explained a bit later) and

$$\exists x (\phi_i(x))_{nm} \neq 0 \Rightarrow \forall j \neq i \forall u \forall y (\phi_j(y))_{nu} = (\phi_j(y))_{un} = (\phi_j(y))_{mu} = (\phi_j(y))_{um} = 0.$$

In particular from the latter condition it follows

$$\phi_i(x)\phi_j(y) = \phi_j(y)\phi_i(x) = 0 \quad \text{for all } i, j \text{ and } x, y \in \mathcal{T}_\infty(F).$$

For the case when F is a prime field, i.e. a field without any (proper) subfields we are able to provide the result omitting the surjectivity assumption. Namely, we have

Theorem 2 Suppose that F is a prime field of at least three elements. If $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a Jordan triple homomorphism, then ϕ is a separable sum of the maps of the form

$$\begin{aligned}
& - Spl_{\mu, \{d_n, c_n\}_n} \cdot Cut_S \cdot Inn_t, \\
& - -Spl_{\mu, \{d_n, c_n\}_n} \cdot Cut_S \cdot Inn_t, \\
& - Spl_{\mu, \{d_n, c_n\}_n} \cdot \mathcal{J} \cdot Cut_S \cdot Inn_t, \\
& - -Spl_{\mu, \{d_n, c_n\}_n} \cdot \mathcal{J} \cdot Cut_S \cdot Inn_t
\end{aligned}$$

for some $t \in T_\infty(F)$, $d_n \in F$, $c_n \in T_\infty(F)$, and $S \subset \mathbb{N}$ being of one of the forms $\{1, 2, \dots, n\}$, $\mathbb{N} \setminus \{n, n+1, \dots, m\}$, $\emptyset$.

3 Proofs

This section consists of 4 subsections. In the first one we investigate the properties of Jordan triple homomorphisms on $T_\infty(F)$ that hold in the case when $\text{char}(F) \neq 2$. In the second and third we prove Theorems 1 and 2. In the last subsection we ask some question connected to our problem.

3.1 First results

Now we will give properties of Jordan triple homomorphism on $T_\infty(F)$ that hold for any field F such that $\text{char}(F) \neq 2$. We start with an obvious observation.

Remark 1 Let $\mathcal{R}$ be any ring and let $\phi : \mathcal{R} \rightarrow \mathcal{R}$ be a Jordan triple map. Then ϕ preserves tripotents.

At the beginning of our investigation we will need some more results about diagonalization of tripotents. Hence we cite some earlier results.

Lemma 1 (Słowik (2018), Lemma 2.1) *Suppose F is a field of characteristic different from 2. If $x \in T_\infty(F)$ is tripotent, then there exists an invertible $t \in T_\infty(F)$ such that x^t is a diagonal matrix.*

We have very similar result for idempotents.

Lemma 2 (Słowik (2014), Lemma 2.3) *Let F be any field. If $x \in T_\infty(F)$ is an idempotent, then there exists a matrix $t \in T_\infty(F)$ such that $t^{-1}xt$ is a diagonal matrix.*

Moreover, from the construction of t given in the proof of the above lemma, it follows

Corollary 1 *If for $x \in T_\infty(F)$ from Lemma 2*

1. *for some $i \in \mathbb{N}$ it holds $x_{ip} = 0$ for all $p > i$*
or
2. *for some $j \in \mathbb{N}$ it holds $x_{rj} = 0$ for all $r < j$,*

then for t from the same lemma we have

1. *$t_{ip} = 0$ for all $p > i$*
or

2. $t_{rj} = 0$ for all $r < j$

respectively.

From the properties of tripotents we obtain the following lemma.

Lemma 3 *Let F be an arbitrary field. If $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a Jordan triple homomorphism, then there exist $t \in \mathcal{T}_\infty(F)$ and disjoint sets $N_1, N_2 \subseteq \mathbb{N}$, $N_1 \cap N_2 = \emptyset$, such that ϕ is a separable sum of ϕ_1 and ϕ_2 such that*

$$(\phi_1(e_\infty))^t = \sum_{n \in N_1} e_{nn}, \quad (\phi_2(e_\infty))^t = - \sum_{n \in N_2} e_{nn}.$$

Proof From Remark 1 we know that $\phi(e_\infty)$ is tripotent. We make use of Lemma 1 and get that

$$(\phi(e_\infty))^t = \sum_{n \in N_1} e_{nn} - \sum_{n \in N_2} e_{nn}$$

for some disjoint N_1, N_2 . Let $\phi' = \mathcal{T}nn_t \cdot \phi$ and consider ϕ' .

Clearly, we have

$$\phi(e_\infty)\phi(x)\phi(e_\infty) = \phi(x) \quad \text{for all } x \in \mathcal{T}_\infty(F). \quad (5)$$

From (5) and the form of $\phi'(e_\infty)$ we get

$$(\phi(x))_{ij} = (\phi(e_\infty))_{ii}(\phi(x))_{ij}(\phi(e_\infty))_{jj}. \quad (6)$$

This means that if $(\phi(e_\infty))_{ii} \neq (\phi(e_\infty))_{jj}$, then $(\phi(x))_{ij} = 0$ for all $x \in \mathcal{T}_\infty(F)$, so we can have $(\phi'(x))_{ij} \neq 0$ only in one of the three cases: $i, j \in N_1$, $i, j \in N_2$, or $i, j \notin N_1 \cup N_2$.

Therefore we can write ϕ' as a separable sum of ϕ_1, ϕ_2, ϕ_3 such that

1. $\phi_1(e_\infty) = \sum_{n \in N_1} e_{nn}$,
2. $\phi_2(e_\infty) = - \sum_{n \in N_2} e_{nn}$,
3. $\phi_3(e_\infty) = 0$.

Obviously, in the third case from (5) we obtain that $\phi_3(x) = 0$ for all $x \in \mathcal{T}_\infty(F)$. $\square$

The maps ϕ_2 that appeared in Lemma 3 are connected to the below remark.

Remark 2 Let $\mathcal{R}$ be a ring and let $\phi : \mathcal{R} \rightarrow \mathcal{R}$ be a Jordan triple homomorphism. Then $\psi : \mathcal{R} \rightarrow \mathcal{R}$ given by the rule $\psi(x) = -\phi(x)$ for all $x \in \mathcal{R}$, is a Jordan triple homomorphism as well.

Proof As

$$\psi(xy) = -\phi(xy) = -\phi(x)\phi(y)\phi(x) = (-\phi(x))(-\phi(y))(-\phi(x)) = \psi(x)\psi(y)\psi(x)$$

and

$$\psi(x+y) = -\phi(x+y) = -(\phi(x) + \phi(y)) = -\phi(x) - \phi(y) = \psi(x) + \psi(y),$$

the result follows. $\square$

From now on we will focus on such Jordan triple homomorphisms ϕ that $\phi(e_\infty) = e_\infty$.

Remark 3 Suppose that $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a Jordan triple homomorphism. If $\phi(e_\infty) = e_\infty$, then ϕ preserves idempotents.

Proof If $x^2 = x$, then we have

$$(\phi(x))^2 = \phi(x)e_\infty\phi(x) = \phi(x)\phi(e_\infty)\phi(x) = \phi(xe_\infty x) = \phi(x^2) = \phi(x).$$

$\square$

Proposition 1 Let F be a field such that $\text{char}(F) \neq 2$ and let $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ be a Jordan homomorphism such that $\phi(e_\infty) = e_\infty$. Then there exist $t \in \mathcal{T}_\infty(F)$ and a family of disjoint sets $\{I_n\}_{n \in \mathbb{N}}$ for which we have $(\phi(e_{nn}))^t = \sum_{i \in I_n} e_{ii}$ for each $n \in \mathbb{N}$.

Proof From Remark 3 we know that the matrices $\phi(e_{nn})$ are idempotents.

If for all $n \in \mathbb{N}$ we have $\phi(e_{nn}) = 0$, then we can write that for $t = e_\infty$ we have $(\phi(e_{nn}))^t = \sum_{i \in I_n} e_{ii}$, where $I_n = \emptyset$ for all n . Otherwise, we perform as follows.

Consider the number n_1 defined as follows:

$$k = \min_{n \in \mathbb{N}} \{i \in \mathbb{N} : \phi(e_{nn})_{ii} \neq 0\} \Rightarrow (\phi(e_{n_1 n_1}))_{kk} \neq 0.$$

By Lemma 2 there exist $t_1 \in \mathcal{T}_\infty(F)$ such that $(\phi(e_{n_1 n_1}))^{t_1} = \sum_{i \in I_{n_1}} e_{ii}$ for some I_{n_1} and $t_1 \in \mathcal{T}_\infty(F)$.

Let us put $\phi_1 = \mathcal{I}nn_{t_1} \cdot \phi$. We have $\phi_1(e_{n_1 n_1}) = \sum_{j \in I_{n_1}} e_{jj}$.

Consider now n_2 such that

$$k = \min_{n \in \mathbb{N} \setminus \{n_1\}} \{i \in \mathbb{N} : \phi(e_{nn})_{ii} \neq 0\} \Rightarrow (\phi(e_{n_2 n_2}))_{kk} \neq 0.$$

Let n'_2 be the minimal number k for which we have $(\phi_1(e_{n_2 n_2}))_{kk} \neq 0$.

Consider $\phi_1(e_{n_2 n_2})$. As it is idempotent, we can write

$$\phi_1(e_{22}) := x_2 = \sum_{i \in I_2} e_{ii} + \sum_{i \in I_2, k > i} x_{ik}^{(2)} e_{ik} + \sum_{i \in I_2, l < i} x_{li}^{(2)} e_{li} + \sum_{i < l} x_{il}^{(2)} e_{il}.$$

From $e_{n_1 n_1} \cdot e_{n_2 n_2} \cdot e_{n_1 n_1} = 0$ and (1) we get that $I_1 \cap I_2 = \emptyset$.

Clearly, $\phi(e_{n_1 n_1} + e_{n_2 n_2})$ is also an idempotent. From this and (2) we get that $(\phi_1(e_{n_2 n_2}))_{j_1 l} = (\phi_1(e_{22}))_{k j_1} = 0$ for any $j_1 \in I_1, l > j_1, k < j_1$. By Lemma 2, there

exist t'_1 such that $(x_2)^{t'_1}$ is a diagonal matrix. Moreover, by Corollary 1, t'_1 is such that $(t'_1)_{kj_1} = (t'_1)_{j_1l} = 0$ for all $j_1 \in I_1, k < j_1, l > j_1$, so $(\phi_1(e_{n_1n_1}))^{t'_1}$ is also diagonal. We put then $t_2 = t_1 t'_1$. Notice that the first $n'_2 - 1$ columns of t_2 are the same as first $n'_2 - 1$ columns of t_1 .

Consider now $\phi_2 := \mathcal{I}nn_{t_2} \cdot \phi$ and $x_3 := \phi_2(e_{n_3n_3})$, where n_3 satisfy the condition

$$k = \min_{n \in \mathbb{N} \setminus \{n_1, n_2\}} \{i \in \mathbb{N} : \phi(e_{nn})_{ii} \neq 0\} \Rightarrow (\phi(e_{n_3n_3}))_{kk} \neq 0.$$

Let n'_3 be the minimal number k for which we have $(\phi_2(e_{n_3n_3}))_{kk} \neq 0$.

Again from $(e_{n_1n_1} + e_{n_2n_2})e_{n_3n_3}(e_{n_1n_1} + e_{n_2n_2}) = 0$ and (1) we get that $(I_{n_1} \cup I_{n_2}) \cap I_{n_3} = \emptyset$. From the fact that $e_{n_1n_1} + e_{n_2n_2} + e_{n_3n_3}$ is an idempotent we get that

$$(x_3)_{il} = (x_3)_{ki} = 0 \quad \text{for } i \in I_1 \cup I_2, l > i, k < i. \quad (7)$$

Also for x_3 there exist $t'_2 \in T_\infty(F)$ such that $(x_3)^{t'_2}$ is diagonal. Moreover, as (7) holds, $(\phi_2(e_{n_1n_1}))^{t'_2}$ and $(\phi(e_{n_2n_2}))^{t'_2}$ are diagonal as well. We define then t_3 as $t_2 t'_2$. This time the first $n'_3 - 1$ columns of t_3 are the same as the first $n'_3 - 1$ columns of t_2 .

Performing this way we build a sequence of matrices $t_1, t_2, t_3, \dots$ whose first columns, at least first $n'_2 - 1$ columns, first $n'_3 - 1$ columns, etc., coincide with the columns of the desired t . Thus, we can find all the columns of t which implies that we can also find t . $\square$

From now on every time when some map ϕ and some sets I_n are mentioned, we will mean that $\phi(e_{nn}) = \sum_{i \in I_n} e_{ii}$ for every $n \in \mathbb{N}$. If it will not be clear to which map ϕ the symbol I_n applies, then we will write $I_n(\phi)$ instead of I_n .

Let us observe that an analogous result holds for the case when $\phi(e_\infty)$ is different from e_∞ , but it is somewhat “close” to it – more precisely, it is a diagonal matrix with only a finite number of diagonal entries equal to 1 and all the others entries equal to 0. We present it in the following form.

Proposition 2 *Let F be a field such that $\text{char}(F) \neq 2$ and suppose $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_k(F)$ is a Jordan homomorphism such that $\phi(e_\infty) = e_k$. Then there exist $t \in T_\infty(F)$ and a family of disjoint sets $\{I_n\}_n$ for which we have $(\phi(e_{nn}))^t = \sum_{i \in I_n} e_{ii}$ for each n .*

To prove some facts about $\phi(e_{nn})$ we will need a few more simple remarks.

Remark 4 If $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a map satisfying (1) and $\phi(e_\infty) = e_\infty$, then $\phi(x^2) = (\phi(x))^2$ for any $x \in \mathcal{T}_\infty(F)$.

Proof We have

$$\phi(x^2) = \phi(x \cdot e_\infty \cdot x) = \phi(x)\phi(e_\infty)\phi(x) = \phi(x)e_\infty\phi(x) = (\phi(x))^2.$$

$\square$

Corollary 2 *Suppose that $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ fulfills (1), $\phi(e_\infty) = e_\infty$ and $\phi(0) = 0$. Then $(\phi(e_{nm}))^2 = 0$ for any $n < m$.*

Proof By Remark 4 we have $(\phi(e_{nm}))^2 = \phi(e_{nm}^2) = \phi(0)$, so by the assumption $\phi(0) = 0$, we are done. $\square$

Lemma 4 Let F be a field such that $\text{char}(F) \neq 2$ and let $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ be a Jordan triple homomorphism. If ϕ satisfies the conditions

1. $\phi(e_\infty) = e_\infty$,
2. for any $n \in \mathbb{N}$ there exist $I_n \subseteq \mathbb{N}$ such that $\phi(e_{nn}) = \sum_{i \in I_n} e_{ii}$, and I_n, I_m are disjoint for $n \neq m$,

then for any $n < m$, $\alpha \in F$, the element $\phi(\alpha e_{nm})$ is of one of the following forms

$$\phi(\alpha e_{nm}) = \sum_{i \in I_n, j \in I_m} a_{ij} e_{ij} \quad \text{for some } a_{ij} \in F \quad (8)$$

or

$$\phi(\alpha e_{nm}) = \sum_{i \in I_m, j \in I_n} a_{ij} e_{ij} \quad \text{for some } a_{ij} \in F. \quad (9)$$

Proof From Remark 3 we know that ϕ preserves idempotents. Therefore, as ϕ is additive, $(\phi(e_{nn}) + \phi(\alpha e_{nm}))^2 = \phi(e_{nn}) + \phi(\alpha e_{nm})$. From this and Corollary 2, we get that either

$$\phi(\alpha e_{nm}) = \sum_{i \in I_n, j > i} a_{ij} e_{ij} \quad \text{for some } a_{ij} \in F$$

or

$$\phi(\alpha e_{nm}) = \sum_{j \in I_n, i < j} a_{ij} e_{ij} \quad \text{for some } a_{ij} \in F.$$

Analogously, $\phi(e_{mm}) + \phi(\alpha e_{nm})$ is an idempotent so either

$$\phi(\alpha e_{nm}) = \sum_{i \in I_m, j > i} a_{ij} e_{ij} \quad \text{for some } a_{ij} \in F$$

or

$$\phi(\alpha e_{nm}) = \sum_{j \in I_m, i < j} a_{ij} e_{ij} \quad \text{for some } a_{ij} \in F.$$

Hence, the claim holds. $\square$

Lemma 5 Suppose that $\text{char}(F) \neq 2$ and $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a Jordan triple homomorphism such that $\phi(e_\infty) = e_\infty$ and that for each $n \in \mathbb{N}$ there exist $I_n \subseteq \mathbb{N}$ for which we have $\phi(e_{nn}) = \sum_{i \in I_n} e_{ii}$. Let $n \leq m$. If for all $i, j \in \mathbb{N}$ we have $(\phi(\alpha e_{ij}))_{kl} = 0$ for all $k \leq l$, $k \leq n$, $l \geq m$, then $(\phi(x))_{kl} = 0$ for all $x \in \mathcal{T}_\infty(F)$ and $k \leq n$, $l \geq m$.

Proof As $(\phi(\alpha e_{ij}))_{kl} = 0$ for all $k \leq l, k \leq n, l \geq m$, from additivity of ϕ , it follows that $(\phi(x))_{kl} = 0$ for all x with finite support ($k \leq l, k \leq n, l \geq m$).

Consider now arbitrary $x \in \mathcal{T}_\infty(F) \setminus \{0\}$. From our assumption, it follows that there exist p, r such that $k \in I_p, l \in I_r$ and that we have

$$(\phi(x))_{kl} = (\phi(e_{pp} + e_{rr})\phi(x)\phi(e_{pp} + e_{rr}))_{kl}.$$

However, by (1) the latter is equal to $\phi((e_{pp} + e_{rr})x(e_{pp} + e_{rr}))$. Clearly the matrix $(e_{pp} + e_{rr})x(e_{pp} + e_{rr})$ has finite support, so we must have $(\phi(x))_{kl} = 0$. $\square$

The above results will be useful in the proof of the following proposition.

Proposition 3 *Let F be a field such that $\text{char}(F) \neq 2$ and let $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ be a Jordan triple homomorphism. Suppose that ϕ satisfies the conditions*

1. $\phi(e_\infty) = e_\infty$,
2. for any $n \in \mathbb{N}$ there exist $I_n \subseteq \mathbb{N}$ such that $\phi(e_{nn}) = \sum_{i \in I_n} e_{ii}$, and I_n, I_m are disjoint for $n \neq m$,
3. ϕ is not a separable sum.

If $n < m$ and $i_n \in I_n, i_m \in I_m$, then $i_n < i_m$.

Proof Suppose that for some $n < m$ there exist $i_n \in I_n, i_m \in I_m$ such that $i_n > i_m$. With no loss of generality we will consider the case when there exist only one such pair i_n, i_m . The case when we have more such i_n, i_m for fixed pair n, m is analogous.

Obviously, there exist $k > m$ such that $i_l > i_n, i_m$ for any $i_l \in I_l$, where $l \geq k$. Indeed, almost all numbers k satisfy this relation. Moreover, we can choose n, m, k in such a way that $i_k = i_n + 1$ for some $i_k \in I_k$. Suppose now that we have $(\phi(e_{nm}))_{i_m i_n} \neq 0$. Then, since $\phi(e_{nn}) + \phi(e_{nm}) + \phi(e_{nl})$ is idempotent, $\phi(e_{nm})$ is of form (8), and $\phi(e_{nl})$ ($l \geq k$) is of form (9), we either have

$$(\phi(e_{nm}))_{i_m i_n} = 0 \tag{10}$$

or

$$(\phi(e_{nl}))_{i_n i_l} = 0 \quad \text{for } l \geq k. \tag{11}$$

From (10) and fact that $e_{mm} + e_{nm} + e_{nl} + e_{ml}$ is idempotent, it follows that

$$[\phi(e_{mm}) + \phi(e_{nl}) + \phi(e_{ml})]^2 = \phi(e_{mm}) + \phi(e_{nl}) + \phi(e_{ml}) \quad \text{for } l \geq k.$$

However, the latter is possible only if $\phi(e_{ml}) = 0$.

Since $e_{mm} + e_{nm} + e_{nl} + e_{ml}$ is idempotent and $\phi(e_{ml}) = 0$, we must also have $\phi(e_{nl}) = 0$. Analogously, as for every $i \leq m$ the element $e_{im} + e_{il} + e_{mm} + e_{ml}$ is idempotent, we obtain that $\phi(e_{il}) = 0$ for $l \geq k$. Hence, ϕ is a separable sum of ϕ_1 and ϕ_2 such that $\phi_1(x) = \phi(x^{(k-1)})$ and $\phi_2(x) = \phi(\tilde{x}^{(k-1)})$.

Similarly, if (11) held, from the fact that $e_{mm} + e_{nm} + e_{nl} + e_{ml}$ is idempotent, we get

$$[\phi(e_{mm}) + \phi(e_{nm}) + \phi(e_{ml})]^2 = \phi(e_{mm}) + \phi(e_{nm}) + \phi(e_{ml}).$$

Hence, either $\phi(e_{ml}) = 0$ or $\phi(e_{nm}) = 0$. The first case was discussed in the preceding paragraph, whereas if $\phi(e_{nm}) = 0$, then from the fact that $e_{mm} + e_{nm} + e_{nl} + e_{ml}$ is idempotent, we get (again) that $\phi(e_{ml}) = 0$. This means (again) that ϕ is a separable sum. $\square$

From the preceding proposition we get

Corollary 3 Assume that $\text{char}(F) \neq 2$ and that $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a Jordan triple homomorphism such that $\phi(e_\infty) = e_\infty$. Then ϕ is a separable sum of

1. some maps ψ such that for all $n < m$, $i_n \in I_n$, $i_m \in I_m$ we have $i_n < i_m$,
2. some maps ψ such that for all $n < m$, $i_n \in I_n$, $i_m \in I_m$ we have $i_n > i_m$.

Moreover, for both sorts of these maps ψ we have

$$\psi(e_{nk}) = \psi(e_{nm})\psi(e_{mk}) \quad \text{for } n \leq m \leq k. \quad (12)$$

Proof Forms of ϕ given in points (1) and (2) follow from Proposition 3. Equality (12) is a consequence of the fact that $\phi(e_{mm}) + \phi(e_{nm}) + \phi(e_{nk}) + \phi(e_{mk})$ is idempotent and points (1) and (2). $\square$

Clearly, the maps ψ from point (2) of the above lemma satisfy the condition $\phi(e_{nn}) \neq 0$ only for finite number of e_{nn} .

3.2 Proof of Theorem 1

First we present two properties of surjective Jordan triple homomorphisms.

Lemma 6 Let F be any field. If $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ is a surjective Jordan triple homomorphism, then there exist $t \in \mathcal{T}_\infty(F)$ such that either $(\phi(e_\infty))^t = e_\infty$ or $(\phi(e_\infty))^t = -e_\infty$.

Proof From Lemma 3 we know that for some $t \in \mathcal{T}_\infty(F)$ the matrix $(\phi(e_\infty))^t$ is diagonal. For this t $\text{Inn}_t \cdot \phi$ is a separable sum of $\text{Inn}_t \cdot \phi_1$ and $\text{Inn}_t \cdot \phi_2$ such that $(\phi_1(e_\infty))^t = \sum_{i \in N_1} e_{ii}$, $(\phi_2(e_\infty))^t = \sum_{i \in N_2} e_{ii}$ for some disjoint N_1, N_2 . Then from (6) we obtain that

$$\text{im}(\mathcal{I}nn_t \cdot \phi_1) \subseteq \{x \in \mathcal{T}_\infty(F) : i \notin N_1 \vee j \notin N_1 \Rightarrow x_{ij} = 0\},$$

$$\text{im}(\mathcal{I}nn_t \cdot \phi_2) \subseteq \{x \in \mathcal{T}_\infty(F) : i \notin N_2 \vee j \notin N_2 \Rightarrow x_{ij} = 0\}.$$

Clearly we have

$$\text{im}(\mathcal{I}nn_t \cdot \phi) \subseteq \text{im}(\mathcal{I}nn_t \cdot \phi_1) \cup \text{im}(\mathcal{I}nn_t \cdot \phi_2).$$

This means that if $x \in \mathcal{T}_\infty(F)$ was such that $x_{n_1 n_2} \neq 0$ for some $n_1 \in N_1, n_2 \in N_2$, then x would not be in $\text{im}(\mathcal{I}nn_t \cdot \phi)$, and consequently $x^{t^{-1}}$ would not be in $\text{im}(\phi)$. Thus, ϕ would not be surjective. Therefore we must have either $N_1 = \mathbb{N}$ (and $N_2 = \emptyset$) or $N_2 = \mathbb{N}$ (and $N_1 = \emptyset$). $\square$

Lemma 7 *Let $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ be a Jordan triple map satisfying the condition $\phi(e_\infty) = e_\infty$. Then ϕ preserves invertible matrices.*

Proof Consider some $x \in \mathcal{T}_\infty(F)$. From $\phi(x^{-1}x^2x^{-1}) = \phi(e_\infty) = e_\infty$ and (1) we obtain $\phi(x^{-1})\phi(x^2)\phi(x^{-1}) = e_\infty$. This means that $\phi(x^2)$ and $\phi(x^{-1})$ are both invertible. Since from Remark 3 we know that $\phi(x^2) = (\phi(x))^2$, we conclude that $\phi(x)$ is invertible as well. $\square$

Now we can prove our first main result.

Proof of Theorem 1: Let ϕ be a map satisfying our assumptions. By Lemma 7 there exist $t \in \mathcal{T}_\infty(F)$ such that $(\phi(e_\infty))^t = e_\infty$. By the arguments given in the proof of that lemma and by Corollary 3 the map $\phi_1 := \mathcal{I} \setminus \setminus_t \cdot \phi$ is a map such that for every $n < m, n \in I_n, m \in I_m$, we have $i_n < i_m$. Moreover, ϕ_1 is not a separable sum of any maps.

Consider the sets I_n ($n \in \mathbb{N}$). We wish to prove that $|I_n| \leq 1$.

Since

$$\phi_1(\alpha e_{nn}) = \phi_1(e_{nn}\alpha e_{nn}e_{nn}) = \phi_1(e_{nn})\phi_1(\alpha e_{nn})\phi(e_{nn}),$$

all the matrices $\phi_1(\alpha e_{nn})$ have nonzero entries only in the positions (i, j) with $i, j \in I_n$. Moreover, for any $i, j \in I_n$ and any $x \in \mathcal{T}_\infty(F)$

$$(\phi_1(x))_{ij} = (\phi_1(e_{nn})\phi_1(x)\phi_1(e_{nn}))_{ij} = (\phi_1(e_{nn}xe_{nn}))_{ij}. \quad (13)$$

This means that the coefficients from the positions $(i, j), i, j \in I_n$, of the matrices in $\text{im}(\phi_1)$ depend only on $\phi_1(\alpha e_{nn})$.

Let $I_n = \{k, k+1, \dots, k+l-1\} \neq \emptyset$. We define the map $\psi : F \rightarrow \mathcal{T}_l(F)$ as follows:

$$\psi(\alpha) = \begin{pmatrix} (\phi_1(\alpha e_{nn}))_{kk} & (\phi_1(\alpha e_{nn}))_{k,k+1} & \cdots & (\phi_1(\alpha e_{nn}))_{k,k+l-1} \\ & (\phi_1(\alpha e_{nn}))_{k+1,k+1} & & \vdots \\ & & \ddots & \\ & & & (\phi_1(\alpha e_{nn}))_{k+l-1,k+l-1} \end{pmatrix}.$$

From the previous paragraph and the assumption that ϕ is a surjection, it follows that ψ should be a surjection as well. One can see that we have $\psi(0) = 0$ and from Lemma 7 we can conclude that for $\alpha \neq 0$ the matrix $\psi(\alpha)$ is invertible. Hence, in $\text{im}(\psi)$ there are no matrices y such that $y_{11} \neq 0$ and $y_{22} = 0$. This yields $l = 1$.

Summing up, for every $n \in \mathbb{N}$ we have $|I_n| \leq 1$. Suppose that for some $n \in \mathbb{N}$ $I_n = \emptyset$. Then, by Lemma 5 for every i, j such that $i \leq n, j \geq n, \alpha \in F$, we would have $\phi_1(\alpha e_{ij}) = 0$. If $\phi_1(e_{kk}) \neq 0$ for some $k < n$, then ϕ_1 would be a separable sum of ϕ'_1 and ϕ''_1 such that $\phi'_1(x) = \phi_1(x^{(n-1)})$, $\phi''_1(x) = \phi_1(\tilde{x}^{(n)})$. However, since ϕ_1 is surjective, it is not a separable sum. Therefore, if $\phi_1(e_{nn}) = 0$, then $\phi_1(e_{kk}) = 0$ for all $k \leq n$, and consequently $\phi_1(e_{ij}) = 0$ for all $i, j, i \leq n$. Thus we can write that $\mathcal{U}p_k \cdot \phi_2 = \phi_1$ for some ϕ_2 satisfying the condition $\phi_2(e_{nn}) \neq 0$ for all $n \in \mathbb{N}$.

Let us focus on ϕ_2 . For this ϕ_2 , for all $n \leq m$ and $\alpha \in F$ we have $\phi_2(\alpha e_{nm}) = \alpha' e_{nm}$ for some $\alpha' \in F$.

For all pairs $n, m, n \leq m$ we define the maps $f_{nm} : F \rightarrow F$ by the rule:

$$f_{nm}(\alpha) = \alpha' \Leftrightarrow \phi_2(\alpha e_{nm}) = \alpha' e_{nm}.$$

Clearly, all the f_{nm} are surjective.

Since $\phi_2(e_{nn}) = e_{nn}$ we have $f_{nn}(1) = 1$.

Observe now that

$$\phi(\alpha e_{nm}) = \phi((\alpha e_{nn} + e_{mm})e_{nm}(\alpha e_{nn} + e_{mm})) = \phi(\alpha e_{nn} + e_{mm})\phi(e_{nm})\phi(\alpha e_{nn} + e_{mm})$$

and

$$\phi(\alpha e_{nm}) = \phi((e_{nn} + \alpha e_{mm})e_{nm}(e_{nn} + \alpha e_{mm})) = \phi(e_{nn} + \alpha e_{mm})\phi(e_{nm})\phi(e_{nn} + \alpha e_{mm})$$

force

$$f_{nm}(\alpha) = f_{nn}(\alpha)f_{nm}(1) = f_{nm}(1)f_{mm}(\alpha). \quad (14)$$

From $f_{nn}(\alpha)f_{nm}(1) = f_{nm}(1)f_{mm}(1)$ we get two possibilities. Either $f_{nm}(1) = 0$ or $f_{nn}(\alpha) = f_{mm}(\alpha)$ for all $\alpha \in F$. If $f_{nm}(1) = 0$, then, by (14), $f_{nm}(\alpha) = 0$ for all $\alpha \in F$, and consequently the image of ϕ_2 would contain only diagonal matrices - a contradiction. Hence, we must have $f_{nn}(\alpha) = f_{mm}(\alpha)$ for all $\alpha \in F$ and all $n, m \in \mathbb{N}$. Denote this map by f .

As $\phi(e_{mm}) + \phi(e_{nm}) + \phi(e_{nk}) + \phi(e_{mk})$ is idempotent for all $n < m < k$, we have $f(1)f_{nk}(1) = f_{nm}(1)f_{mk}(1)$, so from $f(1) = 1$, we get that $f_{nk}(1) = f_{nm}(1)f_{mk}(1)$ for all n, m, k , i.e. we have

$$\begin{aligned} f_{n,n+2}(1) &= f_{n,n+1}(1)f_{n+1,n+2}(1) \\ f_{n,n+3}(1) &= f_{n,n+1}(1)f_{n+1,n+2}(1)f_{n+2,n+3}(1) \\ f_{n,n+4}(1) &= f_{n,n+1}(1)f_{n+1,n+2}(1)f_{n+2,n+3}(1)f_{n+3,n+4}(1) \\ &\dots \end{aligned}$$

Define a diagonal matrix d as follows

$$d_{11} = f_{12}(1), \quad d_{22} = (f_{23}(1))^{-1}, \quad d_{nn} = d_{n-1,n-1}(f_{n-1,n}(1))^{-1} \text{ for } n \geq 3.$$

and consider $\phi_3 := \mathcal{I}nn_d \cdot \phi_2$. For every n, m we have $\phi_3(e_{nm}) = e_{nm}$. If g and g_{nm} are the maps defined analogously to f and f_{nm} but for ϕ_3 , then $g_{nm} = g$ for all $n \leq m$.

Let now $\alpha, \beta \in F$. From

$$[\phi_3((\alpha e_{nn} + e_{nm})\beta e_{nn}(\alpha e_{nn} + e_{nm}))]_{nm} = [\phi_3(\alpha e_{nn} + e_{nm})\phi(\beta e_{nn})\phi(\alpha e_{nn} + e_{nm})]_{nm}$$

we obtain that $g(\alpha\beta) = g(\alpha)g(\beta)$. Hence, g is a homomorphism of F , so ϕ_3 is a homomorphism induced by f . Therefore, we have $\phi = \mathcal{I}nn_{t'} \cdot \mathcal{U}p_k \cdot \mathcal{I}nn_d \cdot \bar{f}$. We define a diagonal matrix d' as follows

$$d'_{ii} = \begin{cases} 1 & \text{for } i \leq k \\ d_{i-k,i-k} & \text{for } i > k \end{cases}$$

and then we have $\phi = \mathcal{I}nn_{t \cdot d'} \cdot \mathcal{U}p_k \cdot \bar{f}$. $\square$

3.3 Proof of Theorem 2

In this subsection we prove our second main result.

Proof Theorem 2: By Lemma 3 and Corollary 3 the map ϕ is a separable sum of maps

- ϕ_1 such that $\phi_1(e_{nn}) = \sum_{i \in I_n} e_{ii}$ and $i_n < i_m$ for any $i_n \in I_n, i_m \in I_m, n < m$;
- ϕ_2 such that $\phi_2(e_{nn}) = -\sum_{i \in I_n} e_{ii}$ and $i_n < i_m$ for any $i_n \in I_n, i_m \in I_m, n < m$;
- ϕ_3 such that $\phi_3(e_{nn}) = \sum_{i \in I_n} e_{ii}$ and $i_n > i_m$ for any $i_n \in I_n, i_m \in I_m, n < m$;
- ϕ_4 such that $\phi_4(e_{nn}) = -\sum_{i \in I_n} e_{ii}$ and $i_n > i_m$ for any $i_n \in I_n, i_m \in I_m, n < m$.

Let us discuss the maps of the form as ϕ_1 . We have

$$\phi_1(e_{nn}) = \sum_{i \in I_n} e_{ii}. \quad (15)$$

Since F is prime it is either $\mathbb{F}_p$ – the p -element field or $\mathbb{Q}$.

First we consider $F = \mathbb{F}_p$ ($p \neq 2$). From the additivity and (15) we obtain that $\phi_1(\alpha e_{nn}) = \alpha \sum_{i \in I_n} e_{ii}$ for all $\alpha \in F, n \in \mathbb{N}$.

Now we move to the case $F = \mathbb{Q}$. Again from additivity $\phi_1(\alpha e_{nn}) = \alpha \sum_{i \in I_n} e_{ii}$ for any $\alpha \in \mathbb{N} \cup \{0\}$. Next we notice that from the latter and $\phi_1(0) = 0$ we get $\phi_1(\alpha e_{nn}) = \alpha \sum_{i \in I_n} e_{ii}$ for any $\alpha \in \mathbb{Z}$. now from

$$\phi_1(e_{nn}) = \phi_1 \left(\underbrace{\frac{1}{k}e_{nn} + \frac{1}{k}e_{nn} + \cdots + \frac{1}{k}e_{nn}}_{k \text{ terms}} \right)$$

we obtain $\phi_1\left(\frac{1}{k}e_{nn}\right) = \frac{1}{k} \sum_{i \in I_n} e_{ii}$ for $k \in \mathbb{N}$.

Thus, regardless of F we have $\phi_1(\alpha e_{nn}) = \alpha \sum_{i \in I_n} e_{ii}$.

From Corollary 3 we know that for all $n < m < k$ we have $\phi_1(e_{nm})\phi_1(e_{mk}) = \phi_1(e_{nk})$. Moreover, as for every n, m it holds

$$\phi_1(\alpha e_{nm}) = \phi_1((\alpha e_{nn} + e_{mm})e_{nm}(\alpha e_{nn} + e_{mm})) = \phi_1(\alpha e_{nn})\phi_1(e_{nm})\phi_1(e_{mm}) = \alpha \phi_1(e_{nm}).$$

Hence, the map ϕ_1 is $\mathcal{S}_{\sqrt{\uparrow}\mu, \{d_n, c_n\}_n}$ for some μ . Moreover, by the argument given in the proof of Theorem 1, if $\phi_1(e_{nn}) = 0$ for some n , then $\phi_1(e_{kk}) = 0$ for all $k < n$. Notice that we can also have $\phi_1(e_{kk}) = 0$ for all $k \geq m$ for some $m \in \mathbb{N}$. Hence, we can write that $\phi_1 = \mathcal{S}_{\sqrt{\uparrow}\mu, \{d_n, c_n\}_n} \cdot \text{Cut}_S \cdot \mathcal{I}nn_t$ for some $t \in T_\infty(F)$ and some set $S \subseteq \mathbb{N}$.

For the maps of the form as ϕ_3 the analogous arguments apply but as we have $i_n > i_m$ for $i_n \in I_n, i_m \in I_m, n < m$, and $\phi_3(e_{nn}) \neq 0$ for only finite number of e_{nn} , we have $\phi_3 = \mathcal{S}_{\sqrt{\uparrow}\mu, \{d_n, c_n\}_n} \cdot \mathcal{J} \cdot \text{Cut}_S \cdot \mathcal{I}nn_t$. For the maps of the form as ϕ_2 and ϕ_4 , it suffices to apply Remark 2. $\square$

3.4 Closing comments and questions

An obvious question that arises is the one about the form of an arbitrary Jordan triple homomorphism ϕ of $\mathcal{T}_\infty(F)$. It can be noticed that this ϕ depends strongly on $\phi(\alpha e_{nn})$. Hence, one may consider the maps $f_n : F \rightarrow \mathcal{T}_{k_n}(F)$ (where k_n is a natural number that may be different for each n). It can be shown that if ϕ is a Jordan triple homomorphism and $\text{char}(F) \neq 2$, then f_n are homomorphisms. Such maps were studied in Omladič et al. (1999) and a similar topic was discussed in Kokol-Bukovšek (1999, 2002), but at the moment we can not find the forms of all maps described there which are additive.

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Declarations

Conflict of interest The author declare that there is no conflict of interest.

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