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# Triangles of nearly equal area

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## Abstract

Given any $n$ points in the plane, not all on the same line, there exist two non-collinear triples such that the ratio of the areas of the triangles they determine, differs from 1 by at most $O(\log n/n^2)$. If we furthermore insist that the two triangles have a common edge, then there are two with area ratios differing from 1 by at most $O(1/n)$. This improves some results of Ophir and Pinchasi (Discrete Appl. Math. **174** (2014), 122–127). We also give some constructions for these and related problems.

**Keywords** Sidon set · Triangle area · Golden ratio · Plastic number · Morphic number

**Mathematics Subject Classification** 52C10

Consider $n$ points in the plane, not all on a line. We want to find two triangles determined by the points with area ratio as close as possible to 1. Ophir and Pinchasi (2014) showed that in any set of $n$ points in the plane with no three on a line, there are two triples $\{a, b, c\}$ and $\{a', b', c'\}$ of points such that the triangles $\triangle abc$ and $\triangle a'b'c'$ have almost the same area in the precise sense that

$$\left|\frac{\triangle abc}{\triangle a'b'c'} - 1\right| < \frac{60\log^{1/3} n}{n^{2/3}}.$$

We present the following two improvements of this result.

**Theorem 1** *Given a set $S$ of $n$ non-collinear points in the plane, there exist distinct points $a, b, c, d \in S$ such that $c$ and $d$ are both not on the line through $a$ and $b$, and*

$$\frac{1}{r} \le \frac{\triangle abd}{\triangle abc} \le r$$

*where* $r = 3^{3/(n-3)} = 1 + \frac{3\ln 3}{n} + O(1/n^2)$.

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**Theorem 2** *Given a set $S$ of $n$ non-collinear points in the plane, there exist non-collinear triples of points $\{a, b, c\}$ and $\{a', b', c'\}$ from $S$ such that*

$$\frac{1}{r} \leq \frac{\triangle abc}{\triangle a'b'c'} \leq r$$

*where $r = 1 + O(\frac{\log n}{n^2})$.*

The proof of Theorem 1 is a simple pigeon-hole argument, that can be generalised as follows to higher dimensions.

**Corollary 3** *Given a set $S$ of $n$ points that span $d$-dimensional Euclidean space, there exist $d+2$ distinct points $a_1, a_2, \dots, a_d, b, c \in S$ such that $a_1, \dots, a_d$ span a hyperplane not containing $b$ and $c$, and the ratio between the volume of the simplices with vertex sets $\{a_1, \dots, a_d, b\}$ and $\{a_1, \dots, a_d, c\}$ lies in $[1/r, r]$, where $r = 1+O_d(\frac{\log n}{n^2})$.*

Theorem 1 is best possible in the sense that we cannot guarantee two triangles with only one vertex in common to have almost the same area.

**Proposition 4** *There exists a set of $n$ points $p_1, \dots, p_n$ in the plane such that whenever $\frac{1}{14} \leq \frac{\triangle p_i p_j p_k}{\triangle p_{i'} p_{j'} p_{k'}} \leq 14$, then $\{i, j, k\}$ and $\{i', j', k'\}$ have their two largest elements in common.*

On the other hand, we do not know if Theorem 2 can be improved. Its proof depends on the following result of Ophir and Pinchasi (2014), for which it is also not known whether it is asymptotically tight.

*Given any set $S$ of $n$ elements of $\mathbb{R}$, there exist two distinct pairs $\{a, b\}$ and $\{a', b'\}$ of points from $S$ such that*

$$\left| \frac{|a-b|}{|a'-b'|} - 1 \right| \leq \frac{9 \log n}{n^2}.$$

This result in fact holds for any $n$-element metric space, where it is best possible up to the constant factor of 9. Ophir and Pinchasi conjecture that for $n$ points in $\mathbb{R}$, there are always two pairs of ratio $1 + c/n^2$. We give the following lower bound for triangle areas, showing that the ratio in Theorem 2 cannot be improved beyond $1 + O(1/n^2)$ either.

**Proposition 5** *There exists a set $S$ of $n$ real numbers such that the ratio between the area of any two triangles with vertices from the set $\{(s, n^{5s}) | s \in S\}$ is $\gtrsim 1 + 1/n^2$.*

We say that a set $\{a_1, a_2, \dots, a_n\}$ of $n$ integers is a *Sidon set* if the sums $a_i + a_j$, $i \leq j$, are all different. Ophir and Pinchasi noted that the example of Erdős and Turán (1941) of a Sidon set of $n$ integers from $\{1, 2, \dots, n^2 + O(n)\}$ is also an example of $n$ points in $\mathbb{R}$ for which the ratio of the distance between any two distinct pairs differ from 1 by at least $1/n^2$. We next observe that there is a simple construction of $n$ points in $\mathbb{R}$ with a slightly better lower bound of $4/n^2$. This construction additionally has a ratio of $\Theta(n)$ between the minimum and maximum distance in the set, where a Sidon set has ratio $\Theta(n^2)$.

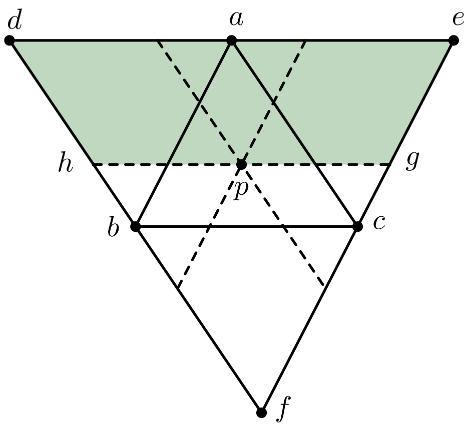


**Fig. 1** Apply an affine transformation so that $\triangle abc$ is equilateral with side length 1

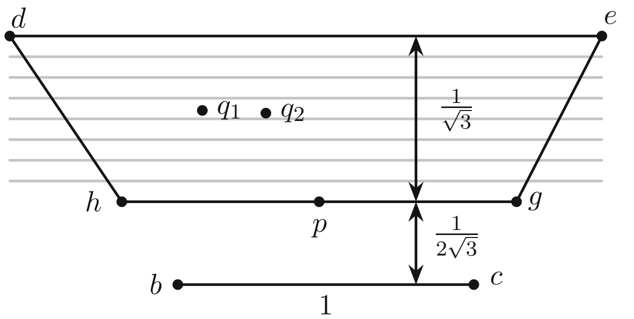


**Fig. 2** Pigeon-hole principle inside a trapezium

**Proposition 6** *There exists a set of n points on the real line such that for any two distinct pairs* $\{a, b\}$ *and* $\{c, d\}$ *from the set with* $|a - b| \geq |c - d|$, *we have*

$$1 + \frac{4}{n^2} + O(1/n^3) \leq \left|\frac{a-b}{c-d}\right| \leq O(n).$$

## 1 Proofs

***Proof of Theorem 1*** Choose $a, b, c \in S$ such that $\triangle abc$ has maximum area among all triples of points from $S$. Without loss of generality we may apply an affine transformation so that $\triangle abc$ becomes an equilateral triangle of side length 1, as in Figure 1.

Let $\triangle def$ be the triangle with sides parallel to the sides of $\triangle abc$ and such that $a, b, c$ are midpoints of the edges of $\triangle def$. Then all $n$ points are inside $\triangle def$. Let $p$ be the centroid of $\triangle abc$ (and $\triangle def$). Consider the three lines through $p$ parallel to the three sides of $\triangle abc$. At least $n/3$ points must lie on the side of one of these lines that is opposite to the parallel side of $\triangle abc$. Without loss of generality the trapezium $degh$ contains at least $n/3$ of the points. Let $k = \lceil n/3 \rceil - 1$. Subdivide the trapezium using $k$ parallel lines of height $\frac{1}{2\sqrt{3}}r^i$, $i = 0, 1, \ldots, k-1$, above the line $bc$, where $r$ is chosen such that $\frac{1}{2\sqrt{3}}r^k = \sqrt{3}/2$ (Figure 2). Since $k < n/3$, there are two points in at least one of the regions, say $q_1$ and $q_2$. Then $\frac{1}{r} \leq \triangle bcq_1/\triangle bcq_2 \leq r$, and since $k \geq n/3 - 1$,

$$r = 3^{1/k} \leq 3^{3/(n-3)} = 1 + 3\ln 3/n + O(1/n^2). \qquad \square$$

***Proof of Proposition 4*** Let $p_i = (2^{2^i}, 2^{2^{i+1}})$, $i = 1, \ldots, n$. If $i < j < k$, then the area of $\triangle p_i p_j p_k$ is

$$\triangle p_i p_j p_k = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2^{2^i} & 2^{2^j} & 2^{2^k} \\ 2^{2^{i+1}} & 2^{2^{j+1}} & 2^{2^{k+1}} \end{vmatrix} = \frac{1}{2}(2^{2^k} - 2^{2^j})(2^{2^k} - 2^{2^i})(2^{2^j} - 2^{2^i}).$$

Thus $\triangle p_i p_j p_k \leq \frac{1}{2}2^{2^{k+1}+2^j}$ and

$$\begin{aligned}\triangle p_i p_j p_k &\geq \frac{1}{2}(2^{2^k} - 2^{2^2})(2^{2^k} - 2^{2^1})(2^{2^j} - 2^{2^1}) \\ &= \frac{1}{2}2^{2^{k+1}+2^j}(1 - 2^{4-2^k})(1 - 2^{2-2^k})(1 - 2^{2-2^j})\end{aligned}$$

We now consider two distinct triples $\{i < j < k\}$ and $\{i' < j' < k'\}$, where without loss of generality, $k \geq k'$. If $k > k'$ then

$$\begin{aligned}\frac{\triangle p_i p_j p_k}{\triangle p_{i'} p_{j'} p_{k'}} &\geq 2^{2^{k+1}-2^k+2^j-2^{j'}}(1 - 2^{4-2^4})(1 - 2^{2-2^4})(1 - 2^{2-2^2}) \\ &\geq 2^{2^k+2^2-2^{k-2}}(1 - 2^{4-2^4})(1 - 2^{2-2^4})(1 - 2^{2-2^2}) \\ &\geq 2^{2^4+2^2-2^{4-2}}(1 - 2^{4-2^4})(1 - 2^{2-2^4})(1 - 2^{2-2^2}) > 2^{15}.\end{aligned}$$

If $k = k'$ then without loss of generality, $j \geq j'$. If $j > j'$, then

$$\begin{aligned}\frac{\triangle p_i p_j p_k}{\triangle p_{i'} p_{j'} p_{k'}} &\geq 2^{2^j-2^{j-1}}(1 - 2^{4-2^3})(1 - 2^{2-2^3})(1 - 2^{2-2^3}) \\ &\geq 2^4(1 - 2^{4-2^3})(1 - 2^{2-2^3})(1 - 2^{2-2^3}) > 14.\end{aligned}$$

Therefore, if the ratio is at most 14, then $j = j'$ and $k = k'$. $\square$

***Proof of Theorem 2*** Without loss of generality, the maximum-area triangle $\triangle abc$ is equilateral with area 1. Then its height is 2, the distance between its centroid and any side is 2/3, and its side length is $4/\sqrt{3}$. Thus $S$ is contained in $\triangle def$ of Figure 1, so any two points are at distance $\leq 8/\sqrt{3}$.

Assume that for any two distinct triangles $\triangle xyz$ and $\triangle x'y'z'$,

$$\max\left\{\frac{\triangle xyz}{\triangle x'y'z'}, \frac{\triangle x'y'z'}{\triangle xyz}\right\} \geq 1 + \frac{6\log n}{n^2}.$$

We next show that the distance between any two points $p, p' \in S$ is $\gtrsim 4\log n/n^2$. Since the perpendicular distance from $p$ to some edge of $\triangle abc$, say $ab$, is $\geq 2/3$, we obtain

$$\frac{\triangle abp'}{\triangle abp} \leq 1 + \frac{pp'}{2/3}.$$

Similarly, since $p'$ is at perpendicular distance $\geq 2/3 - pp'$ from $ab$, we obtain

$$\frac{\triangle abp}{\triangle abp'} \leq 1 + \frac{pp'}{2/3 - pp'}.$$

It follows that

$$1 + \frac{6\log n}{n^2} \leq 1 + \frac{pp'}{2/3 - pp'},$$

from which $pp' \gtrsim 4\log n/n^2$ follows.

Among all $\binom{n}{3}$ triples of points, the $\binom{n}{3} - n^2$ smallest areas are all $\leq (1 + \frac{6\log n}{n^2})^{-n^2} \sim n^{-6}$. Suppose that each pair of points belongs to more than 6 triangles of area $\gtrsim n^{-6}$. Then there are at least $7\binom{n}{2}/3 > n^2$ triangles of area $\gtrsim n^{-6}$, a contradiction.

Therefore, some pair of points $\{p, q\}$ belongs to at most 6 triangles of area $\gtrsim n^{-6}$, hence to at least $n - 8$ triangles $\triangle pqp_i$, $i = 1, \ldots, n-8$, each of area $\lesssim n^{-6}$. Since the distance between $p$ and $q$ is $\gtrsim 4\log n/n^2$, the perpendicular distance of any $p_i$ to the line $\ell$ through $p$ and $q$ is $\lesssim 1/(2n^4\log n)$.

We now choose coordinates so that $\ell$ becomes the $x$-axis. Then each $p_i = (x_i, \varepsilon_i)$, where $|\varepsilon_i| \lesssim 1/(2n^4\log n)$. Since $\triangle abc$ has width 2, one of its three vertices, say $a = (x, h)$, is at distance $|h| \geq 1$ from $\ell$. By the result of Ophir and Pinchasi applied to $x_1, x_2, \ldots, x_{n-8}$, there are two pairs $\{i, j\}$ and $\{s, t\}$ such that

$$\frac{|x_i - x_j|}{|x_s - x_t|} = 1 + O(\log n/n^2).$$

We next show that the ratio between the areas of $\triangle ap_ip_j$ and $\triangle ap_sp_t$ is asymptotically the same as $|x_i - x_j|/|x_s - x_t|$.

We claim that $\triangle ap_ip_j = |h(x_i - x_j)|(1 + o(\log n/n^2))$. Indeed,

$$\pm 2\triangle ap_ip_j = \begin{vmatrix} 1 & 1 & 1 \\ x & x_i & x_j \\ h & \varepsilon_i & \varepsilon_j \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x_i - x & x_j - x \\ h & \varepsilon_i & \varepsilon_j \end{vmatrix}$$
$$= h(x_j - x_i)\left(1 - \frac{\varepsilon_i(x_j - x)}{h(x_j - x_i)} + \frac{\varepsilon_j(x_i - x)}{h(x_j - x_i)}\right).$$

Since $|x_j - x| \le p_j p \le 8/\sqrt{3}$, $|h| \ge 1$, and

$$|x_j - x_i| \ge p_i p_j - |\varepsilon_i| - |\varepsilon_j| \gtrsim \frac{4 \log n}{n^2} - \frac{1}{n^4 \log n} \gtrsim \frac{4 \log n}{n^2},$$

we obtain

$$\left| \frac{\varepsilon_i (x_j - x)}{h(x_j - x_i)} \right| = O\left( \frac{1}{n^2 \log^2 n} \right).$$

Similarly,

$$\left| \frac{\varepsilon_j (x_i - x)}{h(x_j - x_i)} \right| = O\left( \frac{1}{n^2 \log^2 n} \right),$$

and it follows that

$$2\triangle a p_i p_j = |h(x_i - x_j)|(1 + O(1/(n^2 \log^2 n))).$$

Similarly,

$$2\triangle a p_s p_t = |h(x_s - x_t)|(1 + O(1/(n^2 \log^2 n))),$$

and we conclude that

$$\frac{\triangle a p_i p_j}{\triangle a p_s p_t} = \frac{|x_i - x_j|}{|x_s - x_t|}(1 + O(1/(n^2 \log n))) = 1 + O(\log n / n^2).$$ □

***Proof of Proposition 5*** Let $S$ be a Sidon set of $n$ elements from $\{1, 2, \ldots, N\}$ where $N = n^2 + O(n)$. Write $p_s = (s, n^{5s})$ and $q_s = (s, 0)$ for each $s \in S$. Consider three $s, t, u \in S$ with $s < t < u$.

Then

$$2\triangle p_s p_t p_u = \begin{vmatrix} 1 & 1 & 1 \\ s & t & u \\ n^{5s} & n^{5t} & n^{5u} \end{vmatrix} = n^{5u}(t - s) + n^{5t}(s - u) + n^{5s}(u - t)$$

and

$$2\triangle q_s q_t p_u = \begin{vmatrix} 1 & 1 & 1 \\ s & t & u \\ 0 & 0 & n^{5u} \end{vmatrix} = n^{5u}(t - s).$$

Since the ratio between these two areas is close to 1, we can replace $\triangle p_s p_t p_u$ with $\triangle q_s q_t p_u$ in our calculations. Specifically,

$$\begin{aligned}\left|\frac{\triangle p_s p_t p_u}{\triangle q_s q_t p_u} - 1\right| &= \left|\frac{n^{5t}(s-u) + n^{5s}(u-t)}{n^{5u}(t-s)}\right| \\ &\leq n^{5(t-u)}\left|\frac{s-u}{t-s}\right| + n^{5(s-u)}\left|\frac{u-t}{t-s}\right| \\ &< n^{-5}N + n^{-10}N \sim \frac{1}{n^3}.\end{aligned}$$

Consider distinct triples $\{a < b < c\}$ and $\{d < e < f\}$ of elements from $S$, where we assume without loss of generality that $c \geq f$. Then

$$\frac{\triangle p_a p_b p_c}{\triangle p_d p_e p_f} \gtrsim \left(1 - \frac{1}{n^3}\right)^2 \frac{\triangle q_a q_b p_c}{\triangle q_d q_e p_f} = \left(1 - \frac{1}{n^3}\right)^2 \frac{n^{5c}(b-a)}{n^{5f}(e-d)}.$$

If $c = f$ then $\{a, b\} \neq \{d, e\}$, and we assume without loss of generality that $b - a > e - d$. We obtain

$$\frac{\triangle p_a p_b p_c}{\triangle p_d p_e p_f} \geq \left(1 - \frac{1}{n^3}\right)^2 \frac{b-a}{e-d} \geq \left(1 - \frac{1}{n^3}\right)^2 \frac{N}{N-1} \gtrsim 1 + \frac{1}{n^2}.$$

On the other hand, if $c > f$ then the ratio is even larger:

$$\frac{\triangle p_a p_b p_c}{\triangle p_d p_e p_f} \geq \left(1 - \frac{1}{n^3}\right)^2 n^5 \frac{1}{N} \gtrsim n^3.$$

□

***Proof of Proposition 6*** Fix $\varepsilon > 0$, and let $p_i = (1+\varepsilon)^i - 1$, $i = 0, 1, \ldots, n-1$. Then for any integer $a, b$ with $0 \leq a < b \leq n-1$,

$$p_b - p_a = (1+\varepsilon)^b - (1+\varepsilon)^a.$$

Take any $a, b, c, d \in \{0, \ldots, n-1\}$ with $a < b$, $c < d$, $\{a, b\} \neq \{c, d\}$ and without loss of generality, $b - a \leq d - c$. If $b - a = d - c$ then without loss of generality, $a < c$, and

$$\frac{p_d - p_c}{p_b - p_a} = \frac{(1+\varepsilon)^d - (1+\varepsilon)^c}{(1+\varepsilon)^b - (1+\varepsilon)^a} = (1+\varepsilon)^{c-a}\frac{(1+\varepsilon)^{d-c} - 1}{(1+\varepsilon)^{b-a} - 1} = (1+\varepsilon)^{c-a} \geq 1 + \varepsilon.$$

If $b - a < d - c$, then, setting $b - a = k \in \{1, 2, \ldots, n-2\}$,

$$\begin{aligned}\frac{p_d - p_c}{p_b - p_a} &\geq \frac{p_d - p_{d-b+a-1}}{p_b - p_a} = (1+\varepsilon)^{(d-b+a-1)+(n-1-b)}\frac{p_{k+1} - p_0}{p_{n-1} - p_{n-1-k}} \\ &\geq \frac{(1+\varepsilon)^{k+1} - 1}{(1+\varepsilon)^{n-1} - (1+\varepsilon)^{n-1-k}}.\end{aligned}$$

This last expression will be $\geq 1+\varepsilon$ if and only if $(1+\varepsilon)^{k+1}+(1+\varepsilon)^{n-k} \geq (1+\varepsilon)^n+1$. If we use the Binomial Theorem to expand this up to second order, we obtain

$$\begin{aligned} 1 + (k+1)\varepsilon + \binom{k+1}{2}\varepsilon^2 + O(k^3\varepsilon^3) \\ + 1 + \varepsilon(n-k) + \binom{n-k}{2}\varepsilon^2 + O((n-k)^3\varepsilon^3) \\ \geq 1 + n\varepsilon + \binom{n}{2}\varepsilon^2 + O(n^3\varepsilon^3) + 1, \end{aligned}$$

which is equivalent to $\varepsilon \geq \left(\binom{n}{2} - \binom{k+1}{2} - \binom{n-k}{2}\right)\varepsilon^2 + O(n^3\varepsilon^3)$, that is, we need the inequality $1 \geq k(n-k-1)\varepsilon + O(n^3\varepsilon^2)$ to hold for all $k = 1, 2, \ldots, n-2$. Since $k(n-k-1) \leq \left(\frac{n-1}{2}\right)^2$, we obtain that we need $\varepsilon \leq \frac{4}{n^2} + O(\frac{1}{n^3}) + O(n\varepsilon^2)$. Thus we can take $\varepsilon = \frac{4}{n^2} + O(\frac{1}{n^3})$.

This shows that we obtain a minimum ratio of $1 + \frac{4}{n^2} + O(\frac{1}{n^3})$. □

Instead of using points where the successive distances $p_{i+1} - p_i$ form a geometric progression, as in the above proof, we can also use an arithmetic progression. If we take the $n$ points $p_0 = 0$, $p_i = \sum_{j=0}^{i-1}(1 + j\varepsilon)$, $i = 1, 2, 3, \ldots, n-1$, then a calculation shows that we obtain the same optimal asymptotics of $\varepsilon = \frac{4}{n^2} + O(\frac{1}{n^3})$.

## 2 Final remarks

We did not touch on the problem of Ophir and Pinchasi on whether there exist in any set of $n$ elements of $\mathbb{R}$ two pairs with ratio better than $1 + O(\log n/n^2)$, but we did find the sets of points for which the smallest ratio $> 1$ is a maximum when $n \leq 4$. Thus consider a set $S \subset \mathbb{R}$ of $n$ points that maximizes

$$\min\left\{\frac{|a-b|}{|c-d|} : a, b, c, d \in S, |a-b| \geq |c-d| > 0\right\}$$

among all sets of $n$ points in $\mathbb{R}$.

If $n = 3$, it is easy to see that there is a unique extremal set up to similarity, namely $S = \{a < b < c\}$ such that $\frac{c-b}{b-a}$ equals the golden ratio $(1 + \sqrt{5})/2$.

For $n = 4$ the problem is already non-trivial, as there are 6 different distances. Using a case analysis, we can show that up to similarity there are two extremal sets. One of them is the above geometric progression construction $\{0, 1, 1+r, 1+r+r^2\}$, where $r$ is the unique real root of the cubic polynomial $r^3 - r - 1$. The other configuration is $\{0, 1, r, r^2\}$. The number

$$r = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} = 1.3247179572\ldots$$

is known as the *plastic number* of van der Laan (1960), which is closely related to the golden ratio (Aarts et al. 2001; Rush 2012; Stewart 1996).



## References

Aarts, J., Fokkink, R., Kruijtzer, G.: Morphic numbers. Nieuw Arch. Wiskd. **2**(5), 56–58 (2001)

Erdős, P., Turán, P.: On a problem of Sidon in additive number theory, and some related problems. J. London Math. Soc. **16**, 212–215 (1941)

Ophir, A., Pinchasi, R.: Nearly equal distances in metric spaces. Discrete Appl. Math. **174**, 122–127 (2014)

Rush, D.E.: Degree *n* relatives of the golden ratio and resultants of the corresponding polynomials. Fibonacci Quart. **50**, 313–325 (2012)

Stewart, I.: Tales of a neglected number. Math. Recreat. SciAm 274(6), 102–103 (1996)

van der Laan, H.: Le nombre plastique. Brill, Leiden (1960)