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ORIGINAL PAPER

Convexity limit angles for isoptics

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Abstract
Given an oval C in the plane, the α-isoptic C_α of C is the plane curve composed of the points from which C can be seen under the angle $\pi - \alpha$. We consider isoptics of ovals parametrized with the support function $p(t) = a + \cos nt, n \in \mathbb{N}$, and present an example of an oval such that when α increases, the α-isoptics begin to be convex, then lose their convexity and finally are convex again along a curve intersecting the isoptics orthogonally. Next we give an example of a curve from the same family, for which the curvature of the isoptics changes its sign three times. These changes occur on the symmetry axes of the oval C and coincide with the orthogonal trajectories which start at the points with extremal curvature. Finally, we formulate the hypothesis concerning the general case where we expect $n - 1$ convexity limit angles for the isoptics of an oval parametrized by $p(t) = a + \cos nt$.

Keywords Isoptic curve · Convex curve · Limit angle · Support function

Mathematics Subject Classification 53A04 · 53C44 · 53A25

1 Introduction

Isoptic curves were defined in 1704 by Philipe de La Hire, as mentioned in Lawrence (1972) p. 58. These curves remain interesting, as can be seen for example in Kunkli et al. (2013) and Dana-Picard (2020). Isoptics also have applications in mechanics and optics (see Weiss and Martini 2000; Wunderlich 1971).

Isoptics of convex curves are not necessarily convex. For isoptics of an ellipse, in Miernowski and Mozgawa (1997) was introduced the notion of the limit angle, described below.

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The inspiration to study this problem came about when we noticed certain irregularities. These were on graphs of the dual curves to isoptics of the oval, with support function $p(t) = 10 + \cos 3t$, see Skrzypiec (2018).

The results presented below have application to optics and architecture (i.e., lighting and shades).

2 Isoptics with two limit angles

Theorem 1 *Fix $8 < a < 8\sqrt{2}$ and let C be the oval described by the support function $p(t) = a + \cos 3t$. For $\alpha \in (0, \pi)$, let C_α be the α-isoptic of C. Then there exist angles $0 < \alpha_1 < \alpha_2 < \pi$ such that the point of C_α which intersects the positive X-axis has a positive curvature if $\alpha \in (0, \alpha_1) \cup (\alpha_2, \pi)$ and a negative curvature if $\alpha \in (\alpha_1, \alpha_2)$.*

Remark 1 An angle α_0 for which C_α is convex for $\alpha < \alpha_0$ and concave for $\alpha_0 < \alpha$ (α near α_0) is called a limit angle (Miernowski and Mozgawa 1997). For the sake of clarity and generality, we call it a convexity limit angle, and extend its use also in the case where the sign of the curvature is considered only along a curve intersecting the isoptic orthogonally.

Proof Using the parametrization

$$z_\alpha(t) = p(t)e^{it} + \left\{ -p(t)\cot\alpha + \frac{1}{\sin\alpha} p(t+\alpha) \right\} ie^{it}, \ \ t \in \mathbb{R}, \tag{1}$$

of isoptics from Cieślak et al. (1991), we prove that the curve C given by

$$z(t) = p(t)e^{it} + p'(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi), \tag{2}$$

with the support function $p(t) = a + \cos 3t$, where $a > 8$ for some values of a, is the example needed in our theorem.

Let us notice that this curve has three axes of symmetry. One of them is the X-axis. We obtain the others by rotations by the angles $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$. Each of these axes of symmetry passes through points of extremal curvature of C. Isoptics C_α are symmetric with respect to the same axes of symmetry as C. That is why it is enough to study them in the neighbourhood of one of these symmetry axes. Without loss of generality, we choose the X-axis.

Let us recall that the curve, which in each point forms a right angle with a curve from a given family of curves, is called the orthogonal trajectory. It is easy to see that orthogonal trajectories of isoptics of the curve with the support function $p(t) = a + \cos 3t$, starting with the points of greatest and smallest curvature, coincide with the parts of the symmetry axes mentioned above. Using the parametrization (1) of isoptics, a straightforward computation shows that the points $z_\alpha\left(-\frac{\alpha}{2}\right)$ lie on the positive X-axis. Tangent vectors of isoptics at these points are equal to

$$z'_\alpha\left(-\frac{\alpha}{2}\right) = i\,\frac{2\left(p\left(\frac{\alpha}{2}\right)\sin\frac{\alpha}{2} - p'\left(\frac{\alpha}{2}\right)\sin\frac{\alpha}{2}\right)}{\sin\alpha}.$$

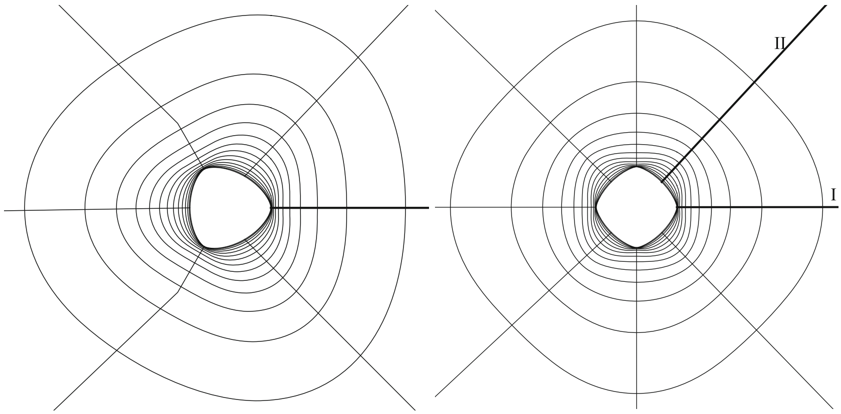

Fig. 1 Isoptics of curves with the support function $p(t) = 10 + \cos 3t$ (left) and $p(t) = 16 + \cos 4t$ (right), along with some of their orthogonal trajectories

Note that they are vertical. Therefore the set of points $\{z_\alpha\left(-\frac{\alpha}{2}\right),\ \alpha \in (0, \pi)\}$ forms the orthogonal trajectory of isoptics, starting at $z(0)$. Similarly, the points $z_\alpha\left(\pi - \frac{\alpha}{2}\right)$ lying on the negative X-axis form the orthogonal trajectory, starting at $z(\pi)$. The first derivative of the curvature of isoptics (see Cieślak et al. 1991)

$$k_\alpha(t) = \frac{[z'_\alpha(t), z''_\alpha(t)]}{|z'_\alpha(t)|^3} = \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)| - [q(t), q'(t)]\right) \tag{3}$$

is zero along these trajectories.

We now compute the number of sign changes of the curvature of isoptics, along these trajectories. Let us consider the curvature of the isoptic of the curve with support function $p(t) = a + \cos 3t$. It is given by the formula

$$k_\alpha(t) = \frac{n_\alpha(t)}{d_\alpha(t)},$$

where

$$\begin{aligned} n_\alpha(t) = {} & 24 + a^2 + 20\cos\alpha - 8\cos 2\alpha - 4\cos 3\alpha + a\cos 3t + a\cos 3(t+\alpha) \\ & + 8\cos 3(2t+\alpha) - 9a\cos(3t+\alpha) + 8\cos 2(3t+\alpha) - 9a\cos(3t+2\alpha) \\ & + 4\cos(6t+\alpha) + 8\cos(6t+4\alpha) + 4\cos(6t+5\alpha) \end{aligned}$$

and

$$\begin{aligned} d_\alpha(t) = {} & \frac{1}{\cos\frac{\alpha}{2}}\big(12 + a^2 + 14\cos\alpha + 4\cos 2\alpha + 2\cos 3\alpha - 2a\cos 3t - 2a\cos 3(t+\alpha) \\ & + 8\cos 3(2t+\alpha) - 6a\cos(3t+\alpha) + 8\cos 2(3t+\alpha) - 6a\cos(3t+2\alpha) \\ & + 4\cos(6t+\alpha) + 8\cos(6t+4\alpha) + 4\cos(6t+5\alpha)\big). \end{aligned}$$

We notice that the denominator of this curvature is always positive since the tangent vector to the isoptic is always non-zero, and for $\beta \in \left(0, \frac{\pi}{2}\right)$ we have $\cos\beta > 0$.

Along the positive X-axis the numerator of the curvature of the isoptic C_α is given by

$$n_\alpha\left(-\frac{\alpha}{2}\right) = -\left(-a + 6\cos\frac{\alpha}{2} + 2\cos\frac{3\alpha}{2}\right)\left(a - 12\cos\frac{\alpha}{2} + 4\cos\frac{3\alpha}{2}\right).$$

Hence our problem simplifies to finding roots of functions

$$f_1(a, \alpha) = -a + 6\cos\frac{\alpha}{2} + 2\cos\frac{3\alpha}{2} \tag{4}$$

and

$$f_2(a, \alpha) = a - 12\cos\frac{\alpha}{2} + 4\cos\frac{3\alpha}{2}. \tag{5}$$

The denominator of the curvature of C_α at $t = -\frac{\alpha}{2}$ can be written as

$$d_\alpha\left(-\frac{\alpha}{2}\right) = \frac{f_1(a, \alpha)^3}{\cos\frac{\alpha}{2}},$$

so the function f_1 has no roots. Moreover, $f_1(a, \alpha) < 0$ for $a > 8$ and $\alpha \in (0, \pi)$ since it has no roots and $f_1(a, 0) < 0$. For the function f_2 let us substitute $x = \cos\frac{\alpha}{2}$. Since $\alpha \in (0, \pi)$, we consider the resulting polynomial

$$v_2(a, x) = 16x^3 - 24x + a \tag{6}$$

for $x \in (0, 1]$ with the parameter $a > 8$. To find the number of distinct roots of polynomial v_2 we apply the Sturm theorem (see Serret 1866; Sturm 1829) on the interval $(0, 1]$. Constructing the Sturm sequence

$$\begin{aligned}
X_0(a, x) &= v_2(a, x) = 16x^3 - 24x + a,\\
X_1(a, x) &= 48x^2 - 24,\\
X_2(a, x) &= 16x - a,\\
X_3(a, x) &= -3ax + 24,\\
X_4(a, x) &= -\frac{128}{a} + a = \frac{(8\sqrt{2} - a)(8\sqrt{2} + a)}{a},\\
X_5(a, x) &= 0
\end{aligned}$$

the signs of their values for $a \in (8, 8\sqrt{2})$ are presented in Table 1.

Hence for $a \in (8, 8\sqrt{2})$ we have $Z(0) - Z(1) = 3 - 1 = 2$. Therefore, based on the Sturm theorem we conclude that the polynomial v_2 has two zeroes in the interval $(0, 1]$ if the parameter $a \in (8, 8\sqrt{2})$.

Table 1 The signs of values of the Sturm sequence of $v_2 = 16x^3 - 24x + a$ for $x \in (0, 1]$ with the parameter $a \in (8, 8\sqrt{2})$

i	0	1	2	3	4	5	
sign of $X_i(a, 0)$	+	−	−	+	−	0	$Z(0) = 3$
sign of $X_i(a, 1)$	+	+	+	−	−	0	$Z(1) = 1$

Similar computations as those above yield that along the negative X-axis the curvature of isoptics of the given oval is always positive.

We conclude that if $a \in (8, 8\sqrt{2})$, then isoptics of the curve with support function $p(t) = a + \cos 3t$ have two convexity limit angles. For example for $a = 10$ those convexity limit angles are $\alpha_1 = 2\arccos\left(\frac{\sqrt{21}-1}{4}\right)$ and $\alpha_2 = \frac{2}{3}\pi$. □

3 Isoptics with three limit angles

Theorem 2 *Fix $15 < a < 17$ and let C be the oval parametrized by the support function $p(t) = a + \cos 4t$. For $\alpha \in (0, \pi)$, let C_α be the α-isoptic of C. Then there exist angles $0 < \alpha_1 < \alpha_2 < \alpha_3 < \pi$ such that the point of C_α which intersects the coordinate axes has a positive curvature if $\alpha \in (0, \alpha_1) \cup (\alpha_2, \pi)$ and a negative curvature if $\alpha \in (\alpha_1, \alpha_2)$. Moreover, when C_α intersects the lines $y = x$ and $y = -x$, C_α has a positive curvature if $\alpha \in (0, \alpha_3)$ and a negative curvature if $\alpha \in (\alpha_3, \pi)$.*

Proof We prove that the curve C with support function $p(t) = a + \cos 4t, a \in (15, 17)$, is the example needed in our theorem.

We note that this curve has four axes of symmetry. Two of them, the X and Y axes, pass through the points of maximal curvature of C. The other two are the lines $y = x$ and $y = -x$, passing through the points of minimal curvature of C. Isoptics C_α are symmetric with respect to the same axes of symmetry as C. Moreover, the first derivative of the curvature of isoptics is zero along these symmetry axes. Similar to the development above for $p(t) = a + \cos 3t$, we can show that orthogonal trajectories which start at $z\left(k\frac{\pi}{4}\right)$ are straight lines and lie on the axes of symmetry of the oval C. That is why we will study them along two orthogonal trajectories (see Fig. 1). The first of them starts at $z(0)$ and lies on the positive X-axis. It consists of the points $z\left(\frac{\pi}{4}\right)$ The second trajectory starts at $z(\frac{\pi}{4})$ and lies on the line $y = x$. It is the set of the points $\left\{z_\alpha\left(\frac{\pi}{4} - \frac{\alpha}{2}\right),\ \alpha \in (0, \pi)\right\}$.

Along the positive X-axis the curvature of the isoptic $C_\alpha = \{z_\alpha(t), \quad t \in (0, 2\pi)\}$ is given by

$$k_\alpha\left(-\frac{\alpha}{2}\right) = \frac{-\cos\frac{\alpha}{2}(4 - a + 8\cos\alpha + 3\cos 2\alpha)(-8 + a - 16\cos\alpha + 9\cos 2\alpha)}{(4 - a + 8\cos\alpha + 3\cos 2\alpha)^3}.$$

Let us denote the following two functions:

$$f_1(a, \alpha) = 4 - a + 8\cos\alpha + 3\cos 2\alpha \tag{7}$$

Table 2 The signs of the values of the Sturm sequence of the polynomial $v_2 = 18x^2 - 16x - 17 + a$ for $x \in (-1, 1]$ with the parameter $a > 15$

i	0	1	2	3	4
sign of $X_i(a, -1)$	$+$	$-$	$+$	0	$Z(-1) = 2$ if $a < 20.$
$\overline{5}$	$+$	$-$	$-$	0	$Z(-1) = 1$ if $a > 20.$
$\overline{5}$ sign of $X_i(a, 1)$	$+$	$+$	$+$	0	$Z(1) = 0$ if $a < 20.$
$\overline{5}$	$+$	$-$	$-$	0	$Z(1) = 1$ if $a > 20.$
$\overline{5}$					

and

$$f_2(a, \alpha) = -8 + a - 16\cos\alpha + 9\cos 2\alpha. \tag{8}$$

Since the curvature function of isoptics is well defined for $\alpha \in (0, \pi)$, the function f_1 has no roots. Hence

$$k_\alpha\left(-\frac{\alpha}{2}\right) = 0 \iff f_2(a, \alpha) = 0.$$

Let us substitute $x = \cos\alpha$. Since $\alpha \in (0, \pi)$ we consider the resulting polynomial

$$v_2(a, x) = 18x^2 - 16x - 17 + a \tag{9}$$

for $x \in (-1, 1]$ with the parameter $a > 15$. Its Sturm sequence is

$$\begin{aligned} X_0(a, x) &= 18x^2 - 16x - 17 + a, \\ X_1(a, x) &= 36x - 16, \\ X_2(a, x) &= 17 + \frac{32}{9} - a, \\ X_3(a, x) &= 0. \end{aligned}$$

The signs of the values of the Sturm sequence are presented in Table 2.

Hence for $a \in (15, 20.\overline{5})$ we have $Z(-1) - Z(1) = 2$ and using the Sturm theorem we conclude that the polynomial v_2 has two roots in the interval $(-1, 1]$. If $a > 20.\overline{5}$ then $Z(-1) - Z(1) = 0$ and we conclude that v_2 has no roots between -1 and 1.

Now let us study the behaviour of the curvature of C_α along the orthogonal trajectory, starting at the point of C with maximal curvature, for example, along the half-line $l(x) = z\left(\frac{\pi}{4}\right) + x \cdot (1, 1)$, where $x > 0$.

Along the half-line l the curvature of the isoptic is given by

$$k_\alpha\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{-\cos\frac{\alpha}{2}(4 + a + 8\cos\alpha + 3\cos 2\alpha)(-8 - a - 16\cos\alpha + 9\cos 2\alpha)}{(4 + a + 8\cos\alpha + 3\cos 2\alpha)^3}.$$

Table 3 The signs of values of the Sturm sequence of $w_2 = 18x^2 - 16x - 17 - a$ for $x \in (-1, 1]$ with the parameter $a > 15$

i	0	1	2	3	
sign of $X_i(a, -1)$	+	−	+	0	$Z(-1) = 2$ if $a < 17$
	−	−	+	0	$Z(-1) = 1$ if $a > 17$
sign of $X_i(a, 1)$	−	+	+	0	$Z(1) = 1$

Similarly to the development above, we can consider the functions

$$g_1(a, \alpha) = 4 + a + 8\cos\alpha + 3\cos 2\alpha \tag{10}$$

and

$$g_2(a, \alpha) = -8 - a - 16\cos\alpha + 9\cos 2\alpha \tag{11}$$

and substitute $x = \cos\alpha$. Then we can consider the polynomials

$$w_1(a, x) = 6x^2 + 8x + 1 + a \tag{12}$$

and

$$w_2(a, x) = 18x^2 - 16x - 17 - a \tag{13}$$

with the parameter $a > 15$ for $x \in (-1, 1]$. The function g_1 has no roots.

Let us construct the Sturm sequence of w_2

$$\begin{aligned}
X_0(a, x) &= 18x^2 - 16x - 17 - a,\\
X_1(a, x) &= 36x - 16,\\
X_2(a, x) &= 17 + \frac{32}{9} + a,\\
X_3(a, x) &= 0.
\end{aligned}$$

The signs of its values are presented in Table 3.

Hence for $a \in (15, 17)$ we have $Z(-1) - Z(1) = 1$ and based on the Sturm theorem we conclude that the polynomial w_2 has one root in the interval $(-1, 1]$. If $a > 17$ then $Z(-1) - Z(1) = 0$ which implies that w_2 has no roots between -1 and 1.

Combining the results of the two trajectories we conclude that isoptics of the oval parametrized by the support function $p(t) = a + \cos 4t$ have three convexity limit angles if $a \in (15, 17)$. We can describe them by the functions

$$\alpha_1(a) = \arccos \frac{8 + \sqrt{2}\sqrt{185 - 9a}}{18},$$
$$\alpha_2(a) = \arccos \frac{8 - \sqrt{2}\sqrt{185 - 9a}}{18},$$
$$\alpha_3(a) = \arccos \frac{8 - \sqrt{2}\sqrt{185 + 9a}}{18}$$

of the variable a.

We note that for each $a \in (15, 17)$ we have $\alpha_1(a) < \alpha_2(a) < \alpha_3(a)$. So for $\alpha \in (0, \alpha_1(a)) \cup (\alpha_2(a), \alpha_3(a))$, the isoptics of the oval C are convex and for $\alpha \in (\alpha_1(a), \alpha_2(a)) \cup (\alpha_3(a), \pi)$ they are nonconvex.

□

4 General case for $p(t) = a + \cos nt$

For $a > n^2 - 1$ the curve C parametrized by the support function $p(t) = a + \cos nt$ is an oval and has n axes of symmetry. Isoptics of such ovals also have n symmetry axes, passing through the points of extremal curvature of C and coinciding with the orthogonal trajectories of the evolutions of the ovals. Those isoptics are invariant with respect to rotations about the origin by the angles $\frac{2k\pi}{n}$, where $k = 1, \dots, n$. That is why we consider the curvature of the isoptics of the oval with the support function $p(t) = a + \cos nt$, where $a > n^2 - 1$ along two half-lines.

- The first half-line starts at $z(0)$ and lies on the positive X-axis. It starts on the given oval, at the point in which it has maximal curvature. It is a set of the points $\left\{z_\alpha\left(-\frac{\alpha}{2}\right),\ \alpha \in (0, \pi)\right\}$ and we denote it **I**.
- The second half-line starts at $z\left(\frac{\pi}{n}\right)$and passes though the points $z_\alpha\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)$ for $\alpha \in (0, \pi)$. It starts on the given oval, at the point in which it has minimal curvature. We denote it **II**.

For the curvature

$$k_\alpha(t) = \frac{[z'_\alpha(t), z''_\alpha(t)]}{|z'_\alpha(t)|^3}$$

let us denote

$$n_\alpha(t) = [z'_\alpha(t), z''_\alpha(t)] \sin^2 \alpha, \qquad d_\alpha(t) = |z'_\alpha(t)|^3 \sin^2 \alpha.$$

Along the trajectory **I** we obtain

$$d_\alpha\left(-\frac{\alpha}{2}\right) = \frac{1}{\sin\alpha}\left(\left(a + \cos\frac{ns}{2}\right)\sin\frac{\alpha}{2} - n\cos\frac{\alpha}{2}\sin\frac{n\alpha}{2}\right)^3$$

and

$$n_\alpha\left(-\frac{\alpha}{2}\right) = -f_1(\alpha, a, n) \cdot f_2(\alpha, a, n),$$

where

$$f_1(\alpha, a, n) = -2a \sin\frac{\alpha}{2} + (n+1)\sin\frac{(n-1)\alpha}{2} + (n-1)\sin\frac{(n+1)\alpha}{2} \tag{14}$$

and

$$f_2(\alpha, a, n) = 2a \sin\frac{\alpha}{2} - (n+1)^2\sin\frac{(n-1)\alpha}{2} + (n-1)^2\sin\frac{(n+1)\alpha}{2}. \tag{15}$$

We need to calculate the number of roots of the functions $f_1(\alpha, a, n)$ and $f_2(\alpha, a, n)$. The functions vary with α, and have parameters a and n. From the form of the denominator we observe that f_1 has no roots. We can also check it applying the same procedure for f_1 as we will use for f_2, below. We perform a change of variables in f_2, in order to obtain a polynomial of a real variable. We will do this for two cases: n odd and n even.

For n odd let us substitute $u = \frac{\alpha}{2}$ where $u \in (0, \frac{\pi}{2})$. Then

$$f_2(u, a, n) = 2a \sin u - (n+1)^2 \sin(n-1)u + (n-1)^2 \sin(n+1)u.$$

Using the trigonometric identity for $\sin nx$ we get

$$\begin{aligned} f_2(u, a, n) = \sin u \Bigg(& 2a - (n+1)^2 \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} (-1)^i \cdot \binom{n-1}{2i+1} \cos^{n-2i-2} u \sin^{2i} u \\ &+ (n-1)^2 \sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor} (-1)^i \cdot \binom{n+1}{2i+1} \cos^{n-2i} u \sin^{2i} u \Bigg). \end{aligned}$$

Since $\sin u > 0$ for $u \in (0, \frac{\pi}{2})$, using the Pythagorean trigonometric identity and the substitution $x = \cos u$, with $x \in (0, 1)$, for the function $\frac{f_2}{\sin u}$ we obtain the polynomial

$$\begin{aligned} v_2(x, a, n) = 2a &- (n+1)^2 \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} (-1)^i \cdot \binom{n-1}{2i+1} x^{n-2i-2}(1-x^2)^i \\ &+ (n-1)^2 \sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor} (-1)^i \cdot \binom{n+1}{2i+1} x^{n-2i}(1-x^2)^i. \end{aligned} \tag{16}$$

We now need to calculate the number of zeroes for $x \in (0, 1)$.

We can repeat the above algorithm for even n. However, if we do it in a slightly different way we will obtain polynomials of lower degrees. Noticing that $n = 2k$ for $k \in \mathbb{Z}$, let us consider the function

$$f_2(\alpha, a, k) = 2\sin\frac{\alpha}{2}\Bigg(a + (4k^2+1)\cos k\alpha - 4k(\cos\alpha + 1)\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i \cdot \binom{k}{2i+1}\cos^{k-2i-1}\alpha\sin^{2i}\alpha\Bigg).$$

Substituting $x = \cos\alpha$ where $x \in (-1, 1)$ and using the trigonometric identity for $\cos nx$, the function $\frac{f_2}{2\sin\frac{\alpha}{2}}$ becomes the polynomial

$$\begin{aligned} v_2(x, a, n, k) = a + (4k^2+1)\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i \cdot \binom{k}{2i}x^{k-2i}(1-x^2)^i \\ - 4k(x+1)\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i \cdot \binom{k}{2i+1}x^{k-2i-1}(1-x^2)^i. \end{aligned} \tag{17}$$

Our approach would be similar to the previous cases, applying the Sturm theorem for finding the number of roots. However, since the polynomial $v_2(x, a, n, k)$ depends on the degree n, we need to generalize this theorem.

Along the trajectory **II**, after similar computations we get the following formulae

$$n_\alpha\left(\frac{\pi}{n} - \frac{\alpha}{2}\right) = -g_1(\alpha, a, n) \cdot g_2(\alpha, a, n)$$

and

$$d_\alpha\left(\frac{\pi}{n} - \frac{\alpha}{2}\right) = \frac{(g_1(\alpha, a, n))^3}{\sin\alpha},$$

where

$$g_1(\alpha, a, n) = 2a\sin\frac{\alpha}{2} + (n+1)\sin\frac{(n-1)\alpha}{2} + (n-1)\sin\frac{(n+1)\alpha}{2} \tag{18}$$

and

$$g_2(\alpha, a, n) = -2a\sin\frac{\alpha}{2} - (n+1)^2\sin\frac{(n-1)\alpha}{2} + (n-1)^2\sin\frac{(n+1)\alpha}{2}. \tag{19}$$

The function $g_1(\alpha, a, n)$ has no roots. We want to transform the function $g_2(\alpha, a, n)$ into a real variable polynomial. For n odd using the substitution $x = \cos\frac{\alpha}{2}$ we obtain the following polynomial

$$\begin{aligned} w_2(x,a,n) = -2a - (n+1)^2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \cdot \binom{n-1}{2i+1} x^{n-2i-2}(1-x^2)^i \\ + (n-1)^2 \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \cdot \binom{n+1}{2i+1} x^{n-2i}(1-x^2)^i, \end{aligned} \tag{20}$$

for the interval (0, 1). For n even substituting $x = \cos u$ we obtain

$$\begin{aligned} w_2(x,a,n,k) = a + (4k^2+1) \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \cdot \binom{k}{2i} x^{n-2i}(1-x^2)^i \\ - 4k(x+1) \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \cdot \binom{k}{2i+1} x^{k-2i-1}(1-x^2)^i, \end{aligned} \tag{21}$$

where $n = 2k$ and for which we are looking for the number of roots in the interval $(-1, 1)$. This number of zeroes depends on the parameter a and we need to find such $a_1(n)$, that for $a \in \left(n^2-1, a_1(n)\right)$ the polynomials v_2 have the largest possible number of zeros in the appropriate intervals.

Whereas results for arbitrary n are as yet unknown, those for some fixed values of n are. We also put forward a hypothesis concerning the general case.

For $n = 5$ the polynomial

$$v_2(x,a,5) = a + 256x^5 - 400x^3 + 120x$$

has two roots for $a \in \left(24, \sqrt{\frac{6975}{8} + \frac{129^{3/2}}{8}}\right) \approx (24, 32.48)$ and the polynomial

$$w_2(x,a,5) = -a + 256x^5 - 400x^3 + 120x$$

has two roots for $a \in \left(24, \frac{1}{2}\sqrt{\frac{3}{2}(2325 - 43\sqrt{129})}\right) \approx (24, 26.24)$. Hence we have four convexity limit angles if $a \in \left(24, \frac{1}{2}\sqrt{\frac{3}{2}(2325 - 43\sqrt{129})}\right) \approx (24, 26.24)$. Those limit angles satisfy the inequality $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2} < \alpha_3 < \alpha_4 < \pi$. Moreover, α_1 and α_2 lie on trajectory **I** and α_3 and α_4 lie on trajectory **II**.

For $n = 6$ the polynomial

$$v_2(x,a,6,3) = a + 100x^3 - 48x^2 - 99x + 12$$

has three roots for $a \in (35, 37)$ and the polynomial

$$w_2(x,a,6,3) = -a + 100x^3 - 48x^2 - 99x + 12$$

Table 4 The maximal number of convexity limit angles along orthogonal trajectories **I** and **II**

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
I	1	2	2	2	3	4	4	4	5	6	6	6	7	8	8	8	9
II	0	0	1	2	2	2	3	4	4	4	5	6	6	6	7	8	8

has two roots for $a \in \left(35, \frac{7}{625}\left(127\sqrt{889} - 416\right)\right) \approx (35, 37.75)$. Hence we have five convexity limit angles if $a \in (35, 37)$. Those limit angles satisfy the inequality $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2} < \alpha_3 < \alpha_4 < \frac{5}{6}\pi < \alpha_5 < \pi$. Moreover, α_1, α_2 and α_5 lie on trajectory **I** and α_3 and α_4 lie on trajectory **II**.

The maximal number of convexity limit angles along the orthogonal trajectories starting at the points of C of the largest (**I**) and the smallest (**II**) curvature, are obtained from computer experimentation, and are presented in Table 4.

These observations led us to formulate the following hypothesis

Hypothesis *For the isoptics of the curve with support function $p(t) = a + \cos nt$, $a > n^2 - 1$, for $a \in \left(n^2 - 1, a_1(n)\right)$, where $a_1(n) > n^2 - 1$, there exist $n - 1$ convexity limit angles. These limit angles can be divided into two groups of sizes $v(n)$ and $w(n)$, where $v(n) + w(n) = n - 1$. The first group of size $v(n)$ appears on the orthogonal trajectory, starting at the point $z\left(\frac{2k\pi}{n}\right)$. The second group of size $w(n)$ is found on the orthogonal trajectory, starting at the point $z\left(\frac{\pi}{n} + \frac{2k\pi}{n}\right)$, where $k = 0, 1, \ldots, n - 1$,*

$$v(n) = \begin{cases} 2k, & n = 4k + 0 \\ 2k, & n = 4k + 1 \\ 2k + 1, & n = 4k + 2 \\ 2(k + 1), & n = 4k + 3 \end{cases}$$

$$w(n) = \begin{cases} 2k - 1, & n = 4k + 0 \\ 2k, & n = 4k + 1 \\ 2k, & n = 4k + 2 \\ 2k, & n = 4k + 3 \end{cases}$$

and $v(n) + w(n) = n - 1$.

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