The complete characterization of tangram pentagons

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Abstract

The old Chinese puzzle tangram gives rise to serious mathematical problems when one asks for all tangram figures that satisfy particular geometric properties. All 13 convex tangram figures are known since 1942. They include the only triangular and all six quadrangular tangram figures. The families of all n -gonal tangram figures with $n \geq 6$ are either infinite or empty. Here we characterize all 53 pentagonal tangram figures, including 51 non-convex pentagons and 31 pentagons whose vertices are not contained in the same orthogonal lattice.

Keywords Tangram · Dissection · Tiling · Pentagon · Lattice

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1 Introduction

The tangram, known as a Chinese puzzle (Goodrich 1817), is a collection of seven polygons, called *tans*: five isosceles right triangles, two with legs of length 1, one with $\sqrt{2}$ and two with 2, a square with sides of length 1 and a parallelogram with sides of length 1 and $\sqrt{2}$ and an angle of $\frac{\pi}{4}$. These seven pieces are arranged, using Euclidean isometries, to form dissections of prescribed or unknown polygons, as in Fig. 1. A *dissection (or tiling) of a polygon P into pieces (or tiles) $P_1, \dots, P_k$* is given if P is the union of all pieces P_i , $1 \leq i \leq k$, and if no two pieces have interior points

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Fig. 1 The seven tans along with their lattices, dissecting a convex and a non-convex pentagon

Fig. 2 All 13 convex tangrams

in common. A polygon T is called a *tangram* if T can be dissected into (isometric images of) the seven tans.

Tangram puzzles usually ask to find dissections of prescribed polygons (Elffers 1978; Goodrich 1817; Read 1985; Slocum et al. 2004). We do not address aspects of the tangram related to craft, art and design. Fruitful mathematical problems appear when one aims to detect systematically all tangrams satisfying particular geometric properties. Such questions have been posed and studied in several books (e.g. Elffers 1978; Müller 2013, Chapter 7; 2020; Read 1985; Slocum et al. 2004)), papers in mathematical journals (e.g. Gardner 1974a; Graber et al. 2016; Heinert 1998a, b; Read 2004; Wang and Hsiung 1942), private publications (Müller 1997–2014) and contributions to mathematical competitions (Brunner 2014, 2015; Heinert 1996; Pohl 2018, 2019, 2020). The most prominent result of that kind is the following one by Wang and Hsiung (cf. Fig. 2).

Theorem 1 (Wang and Hsiung 1942) *There exist, up to isometry, exactly 13 convex tangrams: one triangle, six quadrangles, two pentagons and four hexagons.*

This motivates the question for other natural classes of tangrams. When asking for all simple n -gons for fixed n , the cases of triangles and quadrangles appear trivial or simple, see Sect. 2. Already for hexagons one gets uncountably many incongruent tangrams: for example, one may shift the right part of the dissection on the right-hand

side of Fig. 1 slightly up, this way obtaining a continuum of hexagons. Similarly, one finds uncountably many simple n -gons for all $n = 7, \dots, 23$. The number of vertices of any tangram is at most 23, which is the total number of vertices of all seven tans.

Here we characterize all simple pentagonal tangrams, this way answering a question that seems to have been posed by Lindgren in 1968 (cf. Gardner 1974a). Most of them are non-convex, see e.g. Fig. 1. In the case of pentagons we observe a technical problem concerning the respective position of the tans: In the present paper a *lattice* always means an isometric image of $\mathbb{Z}^2$. Every tan induces a unique lattice that contains all its vertices. If a tangram admits a dissection such that all tans induce the same lattice, we call it a *lattice tangram*. Otherwise we call it a *non-lattice tangram*. Figure 1 illustrates both situations. Simple polygons that are lattice tangrams have been called *snug tangrams* by Read (Gardner 1974a; Read 1985, 2004).

All convex tangrams from Theorem 1 are lattice tangrams. We shall obtain the following counterpart on pentagons.

Theorem 2 *There exist, up to isometry, exactly 53 simple pentagons that are tangrams: two convex ones, 20 non-convex lattice ones and 31 non-convex non-lattice ones.*

These pentagons are given in detail in Sect. 4. Before that, we comment briefly the situation for triangular and quadrangular tangrams (Sect. 2), and we provide a topological tool that permits a systematic approach to pentagons (Sect. 3).

The present paper is an extended version of Pohl (2019).

2 Triangular and quadrangular tangrams

Proposition 3 *There exist, up to isometry, only one triangular and six quadrangular tangrams. All of them are convex.*

Before proving Proposition 3, we note two obvious properties of tangrams.

Lemma 4 (i) *The area of every tangram is 8.*

(ii) *The sizes of the inner angles of every tangram are integer multiples of $\frac{\pi}{4}$.*

Proof of Proposition 3 By Lemma 4, only the isosceles right triangle with legs of length 4 can be a triangular tangram. This triangle is indeed a tangram (cf. Wang and Hsiung 1942 or Fig. 2).

All six convex quadrangular tangrams are known from Wang and Hsiung (1942), see Fig. 2. It remains to show that there are no non-convex quadrangular tangrams. For that, assume that we are given such a quadrangle Q . Since its inner angles sum up to 2π and satisfy Lemma 4(ii), their sizes must be $\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Let $\xi, \eta \in \mathbb{R}$ be the lengths of the sides not emanating from the non-convex vertex, see Fig. 3.

We have, w.l.o.g.,

$$\xi \geq \eta > \frac{\xi}{\sqrt{2}}. \quad (1)$$

Fig. 3 A potential non-convex quadrangular tangram

Lemma 4(i) together with (1) gives

$$8 = \frac{1}{2} \left(\frac{\xi}{\sqrt{2}} \right)^2 + \frac{1}{2} \left(\eta - \frac{\xi}{\sqrt{2}} \right)^2 \begin{cases} > \frac{1}{2} \left(\frac{\xi}{\sqrt{2}} \right)^2 = \frac{1}{4} \xi^2, \\ \leq \frac{1}{2} \left(\frac{\xi}{\sqrt{2}} \right)^2 + \frac{1}{2} \left(\xi - \frac{\xi}{\sqrt{2}} \right)^2 = \left(\frac{2 - \sqrt{2}}{2} \right) \xi^2. \end{cases} \quad (2)$$

The two inequalities give

$$5.226 \dots = \sqrt{8(2 + \sqrt{2})} \leq \xi < 4\sqrt{2} = 5.656 \dots \quad (3)$$

Since ξ and η are sums of side lengths of tans, which are integers or integer multiples of $\sqrt{2}$, we obtain

$$\xi, \eta \in \left\{ k + l\sqrt{2} : k, l \in \{0, 1, \dots\} \right\}. \quad (4)$$

Then (3) implies

$$\xi \in \left\{ 4 + \sqrt{2}, 1 + 3\sqrt{2} \right\}.$$

For any of the two choices of ξ , we see that the left-hand equation from (2) does not have a solution η that satisfies (4). $\square$

3 A topological lemma

Now we work in a slightly generalized setting. An isosceles right triangle with legs of length 1 is called a *basic triangle*. A polygon admitting a dissection into finitely many basic triangles is called a *generalized tangram*. Of course, every tangram is a generalized tangram, since every tan can be subdivided into basic triangles (see the left-hand part of Fig. 4). Although that subdivision is not unique (namely, for the square tan as well as for the large triangular tans), the lattice associated to every basic

Fig. 4 A tangram with a subdivision into basic triangles (left) and concepts for the proof of Lemma 5 (right)

triangle coincides with that of the original tan. So the concepts of lattice and non-lattice tangrams extend naturally to generalized tangrams.

In the remainder of this section we shall prove the following lemma. It will be a crucial tool for the classification of all pentagonal tangrams, since these are either convex or possess exactly one non-convex vertex (cf. Sect. 4).

Lemma 5 *Let T be a generalized tangram that is a simple polygon having exactly one non-convex vertex v_1 . Then one of the following is satisfied.*

- (I) *All vertices of T belong to the same lattice $\Lambda = \varphi(\mathbb{Z}^2)$, where φ is a Euclidean isometry. Every side of T is parallel to one of the segments $\varphi((0, 0)(1, 0))$, $\varphi((0, 0)(0, 1))$, $\varphi((0, 0)(1, 1))$ or $\varphi((0, 0)(1, -1))$.*
- (II) *One of the straight lines defined by a side of T emanating from v_1 dissects T into two convex generalized tangrams T_1 and T_2 with corresponding lattices Λ_1 and Λ_2 as in (I). The lattice Λ_2 is the image of Λ_1 under some rotation by $\frac{\pi}{4}$.*

3.1 Preparations

Suppose now that we are given a generalized tangram T satisfying the assumptions of Lemma 5, along with a dissection into basic triangles. Two basic triangles are called equally oriented if their associated lattices differ at most by a translation. Since T is a simple polygon, there are at most two different orientations. Let P be the union of one class of equally oriented basic triangles, and let Q be the union of the remaining basic triangles. Then Q is either empty or the lattices associated to the pieces of Q are obtained from those of P by rotations of $\frac{\pi}{4}$. Two basic triangles of P are considered equivalent if their lattices agree. Let $P_1, \dots, P_m$ be the respective unions of all classes of equivalent basic triangles of P . Similarly, Q splits into unions $Q_1, \dots, Q_n$ of equivalent triangles. This way we obtain the dissection

$$T = (P_1 \cup \dots \cup P_m) \cup (Q_1 \cup \dots \cup Q_n). \quad (5)$$

(Figure 4 illustrates an eight-angled tangram with corresponding dissections into tans as well as into basic triangles, on the left, and the resulting dissection introduced above, on the right.)

We cannot assume the polygons P_i and Q_j to be simple or even connected. In that setting we call an element v of the boundary $\text{bd}(R)$ of a polygon R a *vertex* of R if there is no circular disc D centered at v such that $D \cap R$ is a half-disc of D . When

Fig. 5 The vertices of T and the arc Δ

speaking of a *side* of R , we mean a maximal line segment in $\text{bd}(R)$ that does not contain one of the vertices of R in its relative interior.

Fact 6 Let $1 \leq i < i' \leq m$ and $1 \leq j < j' \leq n$. Then

- (i) P_i and $P_{i'}$ (and, similarly, Q_j and $Q_{j'}$) do not have a common vertex,
- (ii) P_i and Q_j have at most one common vertex.

Proof (i): A common vertex of P_i and $P_{i'}$ would belong to both the lattices associated to P_i and $P_{i'}$, respectively. But this intersection is empty, because these lattices do not agree and are translates of each other.

(ii): The lattices associated to P_i and Q_j are images of each other under a rotation by $\frac{\pi}{4}$. Thus they have at most one point in common, the only possible joint vertex of P_i and Q_j . $\square$

The *skeleton* S of the dissection in (5) is the union of the boundaries of its pieces, i.e.,

$$S = \text{bd}(P_1) \cup \dots \cup \text{bd}(P_m) \cup \text{bd}(Q_1) \cup \dots \cup \text{bd}(Q_n).$$

We shall deal with arcs contained in S . We say that an arc starts (or ends) in P_i if a segment of positive length of the beginning (or the end) of that arc is contained in $\text{bd}(P_i)$.

Fact 7 Let $\Theta \subseteq \text{bd}(T) \cap P$ be an arc that does not contain the non-convex vertex v_1 in its relative interior. If Θ starts in P_i , $1 \leq i \leq n$, then $\Theta \subseteq P_i$. (The analogue applies to arcs in $\text{bd}(T) \cap Q$.)

Proof Assume that $\Theta \not\subseteq P_i$. Then there is a point $x_0 \in \Theta$ where Θ switches from P_i into some $P_{i'}$ with $i' \neq i$. Since Θ does not pass through the only non-convex vertex v_1 of T , the point x_0 must be a common vertex of P_i and $P_{i'}$. This contradicts Fact 6(i). $\square$

Let $v_1, \dots, v_k$ be the vertices of T appearing along its boundary in counter-clockwise direction. These are illustrated in the left-hand part of Fig. 5.

Fact 8 *From now on we can assume that*

- (i) *the arc $v_1 v_2 \dots v_k$ starts in P_1 and v_1 is a vertex of P_1 ,*
- (ii) *$\text{bd}(T) \not\subseteq P$ and $\text{bd}(T) \not\subseteq Q$.*

Proof (i): There exists one of the pieces P_i , $1 \leq i \leq m$, or Q_j , $1 \leq j \leq n$, such that v_1 is a vertex of that piece and one of the segments $v_1 v_2$ or $v_1 v_k$ starts in that piece. By possibly renaming the pieces in (5) we can assume that that piece is P_1 . By possibly reversing the order of vertices of T we can assume that $v_1 v_2$ starts in P_1 , w.l.o.g.

(ii): If $\text{bd}(T) \subseteq P$, then application of Fact 7 to the arc $\Theta = v_1 v_2 \dots v_k$, which starts in P_1 by (i), yields $\Theta \subseteq P_1$. Consequently, all vertices $v_1, \dots, v_k$ of T are vertices of P_1 , which belong to the same lattice according to the definition of P_1 . This way we reach claim (I) of Lemma 5, and the proof is complete if $\text{bd}(T) \subseteq P$. Thus the case $\text{bd}(T) \subseteq P$ does not require further consideration.

The situation $\text{bd}(T) \subseteq Q$ does not appear, since $v_1 v_2$ starts in P_1 by (i). $\square$

3.2 The arc Δ

We call an interior point x of T a *T-vertex* of the dissection (5) if x is a joint vertex of two pieces P_i and Q_j and belongs to the relative interior of a side of some $P_{i'}$, $i' \neq i$, or $Q_{j'}$, $j' \neq j$. That is, a side of $P_{i'}$ or $Q_{j'}$ passes through x and at least one joint segment of P_i and Q_j emanates from x and cannot be continued over x within the skeleton S . Then we say that P_i and Q_j are *below* the T-vertex x and $P_{i'}$ (or $Q_{j'}$, respectively) is *above* x . (Note that the name T-vertex is motivated by the shape of the capital letter T, but not by the name T of our polygon.)

A directed line segment $xy \subseteq S$ is called a *primal segment* if there exist i and j such that $xy \subseteq P_i \cap Q_j$, x is a joint vertex of P_i and Q_j , y is a vertex of one of P_i or Q_j , and xy does not contain further vertices of P_i or Q_j . Figure 4 illustrates all T-vertices as small circles and all primal segments as arrows.

Fact 9 *Let $xy \subseteq P_i \cap Q_j$ be a primal segment. Then either $y = v_1$ and v_1 is in the relative interior of some side of one of P_i or Q_j or y is a T-vertex having one of P_i or Q_j above.*

Proof Suppose that y is a vertex of P_i , w.l.o.g. By Fact 6(ii), y is no vertex of Q_j . So y is in the relative interior of a side of Q_j .

If y is in the interior of T , Fact 6 implies that y is a vertex of some $Q_{j'}$, $j' \neq j$. Then y is a T-vertex with P_i and $Q_{j'}$ below and Q_j above.

Now let $y \in \text{bd}(T)$. If $y \neq v_1$ then y had to be a vertex of both P_i and Q_j , since v_1 is the only non-convex vertex of T . This contradiction shows that $y = v_1$. $\square$

Next we define a particular arc $\Delta = a_0 a_1 \dots a_l$ in the skeleton S as a union of primal segments (see Fig. 5). We know from Fact 8 that, when starting in v_1 and following $\text{bd}(T)$ counter-clockwise, we meet a first point $a_0 \in \text{bd}(T) \setminus \{v_1\}$ where we switch from P to Q . There exist (by Fact 6(i) unique) $1 \leq i_1 \leq m$ and $1 \leq j_1 \leq n$ such that a_0 is a vertex of both P_{i_1} and Q_{j_1} . There is (at least) one primal segment $a_0 a_1 \subseteq P_{i_1} \cap Q_{j_1}$ such that P_{i_1} and Q_{j_1} are left and right beside $a_0 a_1$, respectively. This segment is the beginning of Δ .

Now we continue the definition of Δ recursively. Suppose that $a_{r-1}a_r \subseteq P_{i_r} \cap Q_{j_r}$ is the last chosen segment in Δ . By Fact 9, we can continue as follows.

- If $a_r = v_1$, we put $l = r$ and Δ is complete.
- If a_r is a T-vertex with P_{i_r} above, there is a (by Fact 6(i) unique) $1 \leq i_{r+1} \leq m$ such that $P_{i_{r+1}}$ is below a_r . We put $j_{r+1} = j_r$ and continue Δ with a next primal segment $a_r a_{r+1} \subseteq P_{i_{r+1}} \cap Q_{j_{r+1}}$ having $P_{i_{r+1}}$ on the left and $Q_{j_{r+1}}$ on the right.
- If a_r is a T-vertex with Q_{j_r} above, we proceed analogously and obtain a next primal segment $a_r a_{r+1} \subseteq P_{i_{r+1}} \cap Q_{j_{r+1}}$ having $P_{i_{r+1}}$ on the left and $Q_{j_{r+1}}$ on the right, where $i_{r+1} = i_r$.

Note that this definition terminates after finitely many steps, since it produces no loops. Indeed, suppose that x_0 would be a first point of self-intersection. Then x_0 cannot be in the relative interior of a primal segment, because these do not contain vertices. Hence x_0 is a T-vertex with $x_0 = a_r = a_{r'}$, $r < r'$, and the primal segments $a_{r-1}a_r$ and $a_{r'-1}a_{r'}$ arrive at x_0 from opposite directions. This is impossible, because both segments have P on the left and Q on the right.

In the following we use the notation $\widehat{xy}$, $x, y \in \text{bd}(T)$, for the counter-clockwise arc in $\text{bd}(T)$ that begins in x and ends in y .

Fact 10 *When following $\text{bd}(T)$ counter-clockwise, a_0 is the only point where one switches from P to Q . Accordingly, there exists exactly one point $b_0 \in \widehat{a_0 v_1} \setminus \{a_0\}$ where one switches back from Q to P .*

Proof First note that we can recover Δ uniquely by the following backwards procedure: Applying Fact 9 to the last segment $a_{l-1}a_l = a_{l-1}v_1$ of Δ , we see that v_1 is in the relative interior of a side of one of P_{i_l} or Q_{j_l} . Since there is at most one of the $m+n$ tiles from (5) having v_1 in the relative interior of one of its sides, this determines that very tile uniquely. Moreover, since P is on the left and Q is on the right of Δ , this recovers at least a little segment at the end of Δ . Now we find a_{l-1} by following that direction until we end at a T-vertex (or at $\text{bd}(T)$, where we meet a_0). The polygon above that T-vertex a_{l-1} is $P_{i_{l-1}}$ or $Q_{j_{l-1}}$. Again by using the lateral position of P and Q relative to Δ we determine the direction of the segment $a_{l-2}a_{l-1}$. Following that direction until the next T-vertex (or until $\text{bd}(T)$, where we find a_0) we find a_{l-2} . Continuing this way we reproduce Δ uniquely.

Now, for proving the fact, let $a'_0 \in \text{bd}(T)$ be a point where one switches from P to Q . Starting from a'_0 , we define an arc $\Delta' \subseteq S$ from a'_0 to v_1 by the same rules as we did with Δ from a_0 . We can reproduce Δ' beginning at its end v_1 , as we did with Δ . But this reproduction gives the same arc $\Delta' = \Delta$, so that $a'_0 = a_0$. $\square$

Fact 11 *Let R_0 be the piece among $P_1, \dots, P_m, Q_1, \dots, Q_n$ such that the segment $v_k v_1$ ends in R_0 . Then v_1 is in the relative interior of a side of R_0 .*

Proof Assume that our claim fails. Then v_1 is a vertex of R_0 .

Case I: $R_0 = P_{i^*}$ for some $1 \leq i^* \leq m$. Fact 6(i) yields $i^* = 1$, since v_1 is a vertex of both P_1 and P_{i^*} . Applying Fact 7 to the arcs $\widehat{v_1 a_0}, \widehat{b_0 v_1} \subseteq P$, we see that a_0 and b_0 are vertices of P_1 . Similarly, applying Fact 7 to $\widehat{a_0 b_0} \subseteq Q$, we see that both a_0 and b_0 are vertices of some Q_j . This contradicts Fact 6(ii).

Fig. 6 The situation after Fact 12

Case 2: $R_0 = Q_{j^*}$ for some $1 \leq j^* \leq n$. Now $b_0 = v_1$, since we switch from Q_{j^*} to P_1 in v_1 . Applications of Fact 7 to $\widehat{v_1 a_0} \subseteq P$ and $\widehat{a_0 v_1} \subseteq Q$ shows that v_1 and a_0 are joint vertices of both P_1 and Q_{j^*} , again contradicting Fact 6(ii). $\square$

Now we learn more on the end of $\Delta = a_0 a_1 \dots a_l$.

Fact 12 *The points a_{l-1} , $a_l = v_1$ and v_k are collinear. The vertex v_1 is in the relative interior of a side of Q_{j_l} . In particular, $b_0 = v_1$.*

Proof There is at most one piece among $P_1, \dots, P_m, Q_1, \dots, Q_n$ having v_1 in the relative interior of a side. By Fact 11, such a piece R_0 exists and $v_1 v_k$ represents the direction of that side. On the other hand, application of Fact 9 to the last primal segment $a_{l-1} a_l = a_{l-1} v_1$ of Δ shows that one of P_{i_l} or Q_{j_l} has v_1 in the relative interior of a side. Consequently, $R_0 \in \{P_{i_l}, Q_{j_l}\}$ and $a_{l-1}, a_l = v_1$ and v_k are collinear. Since P_{i_l} is on the left of $a_{l-1} a_l$ and since $v_k v_1$ ends in R_0 , we obtain $R_0 = Q_{j_l}$. Finally, since $v_k v_1$ ends in $R_0 = Q_{j_l}$, Fact 10 yields $b_0 = v_1$. $\square$

Figure 6 gives a scheme of we have reached so far.

3.3 Conclusion

The counter-clockwise arcs $\widehat{v_1 a_0}, \widehat{a_0 v_1} \subseteq \text{bd}(T)$ are contained in P and Q , respectively. Thus there are $1 \leq \hat{i} \leq m$, $1 \leq \hat{j} \leq n$ such that $\widehat{v_1 a_0}$ ends in $P_{\hat{i}}$ and $\widehat{a_0 v_1}$ begins in $Q_{\hat{j}}$. Since a_0 is a vertex of $P_{\hat{i}}, P_{i_1}, Q_{\hat{j}}$ and Q_{j_1} , Fact 6(i) yields $P_{\hat{i}} = P_{i_1}$ and $Q_{\hat{j}} = Q_{j_1}$. Similarly, $P_{i_l} = P_1$, because of the joint vertex $a_l = v_1$. Moreover, we apply Fact 7 to $\widehat{v_1 a_0} \subseteq P$ and $\widehat{a_0 v_1} \subseteq Q$ and obtain

$$\widehat{v_1 a_0} \subseteq P_{i_1} = P_{\hat{i}} = P_1 = P_{i_l}, \quad \widehat{a_0 v_1} \subseteq Q_{j_1} = Q_{\hat{j}} = Q_{j_l}. \quad (6)$$

In particular, both a_0 and a_{l-1} are joint vertices of $P_{i_1} = P_{i_l}$ and $Q_{j_1} = Q_{j_l}$. Then Fact 6(ii) implies $a_0 = a_{l-1}$, and in turn $l = 1$. That is, the arc Δ consists of the single segment $\Delta = a_0 a_1 = a_0 v_1$.

We see that Δ dissects T along the straight line spanned by $v_1 v_k$ into two convex generalized tangrams. By (6), all vertices of one of them are vertices of P_{i_1} and all vertices of the other one are vertices of Q_{j_1} . This completes the proof of Lemma 5.

4 The complete list of pentagonal tangrams

In this section we prove Theorem 2. We distinguish the cases of convex and non-convex tangrams, the latter case being split into the two alternatives according to Lemma 5.

4.1 Convex pentagons

Proposition 13 *There exist, up to isometry, exactly two convex pentagonal tangrams.*

This follows from Theorem 1, see Fig. 2 for an illustration.

4.2 Non-convex lattice pentagons

Let a simple non-convex pentagon T be a tangram. Since the sizes of its inner angles are integer multiples of $\frac{\pi}{4}$ (see Lemma 4(ii)) and add up to 3π , T has only one non-convex vertex. Thus T satisfies one of the situations (I) or (II) from Lemma 5. Here in Sect. 4.2 we assume that (I) applies.

The following discussion seems to go back to Read (published in Gardner 1974a) and, independently, to Heinert (Heinert 1996). We reproduce it here, since the publication (Gardner 1974a) misses six of the 20 solutions (see (Gardner 1974b, p. 125)) and since Heinert's manuscript (Heinert 1996) is almost inaccessible. Note that Read and Heinert assumed their tangrams a priori to be lattice tangrams. We use only the situation of Lemma 5(I), i.e., that the boundary of T is arranged along a lattice.

The inner angles of T have sizes $\alpha_i = k_i \frac{\pi}{4}$ with $k_1 \in \{5, 6, 7\}$ (non-convex vertex) and $k_i \in \{1, 2, 3\}$, $2 \leq i \leq 5$ (convex vertices). Their sum is $\alpha_1 + \dots + \alpha_5 = 3\pi$, i.e., $k_1 + \dots + k_5 = 12$. Hence the multi-set $\{k_1, \dots, k_5\}$ is one of $\{5, 3, 2, 1, 1\}$, $\{5, 2, 2, 2, 1\}$, $\{6, 3, 1, 1, 1\}$, $\{6, 2, 2, 1, 1\}$ or $\{7, 2, 1, 1, 1\}$.

For fixed $\{k_1, \dots, k_5\}$, the permutations $k_{\varrho(1)} \dots k_{\varrho(5)}$ give the successive order of the angles along the boundary of T . Cyclic shifts and reversions can be considered equal, since they correspond to isometric images of T . Thus it remains to consider the following 16 orders of inner angles: 53211, 53121, 53112, 52311, 52131, 51321, 52221, 52212, 63111, 61311, 62211, 62121, 62112, 61221, 72111 and 71211.

Next, for each of the 16 possible orders $k_{\varrho(1)} \dots k_{\varrho(5)}$, one generates all corresponding pentagons that satisfy condition (I) from Lemma 5 and whose area is 8 according to Lemma 4(i). This can be done in the lattice $\mathbb{Z}^2$. One fixes a first vertex and the direction of the first side of T : either $(1, 0)$ or $(1, 1)$, both have to be considered separately. Then the order $k_{\varrho(1)} \dots k_{\varrho(5)}$ determines T up to three integer parameters, since three side lengths fix T up to isometry and the length of the i th side is either s_i or $s_i\sqrt{2}$ with $s_i \in \{1, 2, \dots\}$. Now one can easily find the choices of side lengths that give polygons T of area 8.

(Alternatively, one could find these non-convex pentagons T of area 8 as follows: One generates all convex generalized lattice tangrams of areas less than 8, as it has been done in Wang and Hsiung (1942) for those of area 8. Then one examines for all pairs of total area 8 if and how they can be put together to form a non-convex lattice pentagon.)

Fig. 7 All 20 non-convex pentagonal lattice tangrams (Heinert 1996)

Finally, one checks for all these resulting pentagons T of area 8 if they admit a dissection into the seven tans. In fact, it turns out that such a candidate T cannot be dissected only if the two large tans cannot be packed simultaneously inside T .

This search results in the 20 tangrams depicted in Fig. 7.

Proposition 14 (Heinert 1996) *There exist, up to isometry, exactly 20 non-convex simple pentagons that are lattice tangrams.*

4.3 Non-convex non-lattice pentagons

Now we consider non-convex pentagonal tangrams T that are in the situation of Lemma 5(II). Then the straight line spanned by one of the sides emanating from the non-convex vertex dissects T into two generalized tangrams T_1 and T_2 . The end-point of the dissecting segment is a joint vertex of T_1 and T_2 , and it is their only joint vertex, since the lattices associated to the vertices of T_1 and T_2 share at most one point. Hence the number of vertices of T is the sum of those of T_1 and T_2 minus 1 if that joint vertex is a vertex of T , too, or that sum minus 2 if the joint vertex is in the relative interior of a side of T . We obtain the following.

Fact 15 *W.l.o.g., each side of any tan in T has one of the directions $(1, 0)$, $(1, 1)$, $(0, 1)$ or $(-1, 1)$. The pentagon T (together with its dissection into tans) splits into two convex generalized tangrams T_1 and T_2 . The vertices of T_1 belong to the lattice*

Fig. 8 All triangular generalized lattice tangrams of area less than 8 with particular dissections into tans

Fig. 9 All convex quadrangular generalized lattice tangrams of areas $\frac{7}{2}$, 4, 6, 7 or $\frac{15}{2}$ that can be dissected into a subfamily of all seven tans

$\mathbb{Z}^2$, the vertices of T_2 belong to an image of $\mathbb{Z}^2$ under a rotation by $\frac{\pi}{4}$. The polygons T_1 and T_2 have exactly one vertex v_0 in common. If their angles at v_0 add up to π then T_1 is a quadrangle and T_2 a triangle. Otherwise the sum of that angles is smaller than π and both T_1 and T_2 are triangles.

The area of the triangle T_2 is larger than 0 and smaller than 8. All possible generalized tangrams with that property are illustrated in Fig. 8. There they are arranged within the lattice $\mathbb{Z}^2$. In square brackets we give the number of basic triangles that form the respective polygon, i.e., twice the area of that polygon. Figure 8 depicts only those dissections into tans that will be used in the sequel.

Since the areas of T_1 and T_2 sum up to 8, the area of T_1 must be one of $\frac{7}{2}$, 4, 6, 7 or $\frac{15}{2}$. In other words, that area is represented by 7, 8, 12, 14 or 15 basic triangles. Thus the only possible triangle representing T_1 can be D1 (or D2) from Fig. 8.

Next we obtain all quadrangular candidates for T_1 . The sizes $k_1 \frac{\pi}{4}$, $k_2 \frac{\pi}{4}$, $k_3 \frac{\pi}{4}$ and $k_4 \frac{\pi}{4}$ of the successive inner angles of T_1 are given by the string $k_1 k_2 k_3 k_4 \in$

Fig. 10 All 31 non-convex pentagonal non-lattice tangrams

{3311, 3131, 3221, 3212, 2222} w.l.o.g. (cf. Sect. 4.2). Since the vertices and sides of T_1 are arranged along $\mathbb{Z}^2$, it is easy to find all possible candidates that are composed by 7, 8, 12, 14 or 15 basic triangles. Figure 9 depicts only those of them who can be tiled by a subfamily of all seven tans, because the others cannot represent T_1 . Quadrangles with dotted dissections will not give rise to pentagonal tangrams in the end.

Finally, we consider every remaining candidate for T_1 , which is either triangle D1 (or D2) from Fig. 8 or a quadrangle from Fig. 9. We pick a triangle of the respective size from Fig. 8 as a candidate T'_2 for the shape of T_2 . We check if both T_1 and T'_2 can be tiled simultaneously by all seven tans. This excludes the quadrangles H, I, N, O, S, U, Y and Z. If the tiling is possible, we choose all images T_2 of T'_2 under a rotation by an odd multiple of $\frac{\pi}{4}$ such that T_1 together with T_2 forms a dissection of a pentagon T . The last is impossible for the quadrangle X. This way we obtain the 31 mutually incongruent pentagons T from Fig. 10. Respective dissections can be adopted from Figs. 8 and 9. The dissections are not unique in general.

Proposition 16 *There exist, up to isometry, exactly 31 non-convex simple pentagons that are non-lattice tangrams.*

The characterization of all simple pentagonal tangrams is complete.

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