Tilting modules and exceptional sequences for leaf quotients of type A zig-zag algebras

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Abstract

We classify generalized tilting modules and full exceptional sequences for the family of quasi-hereditary quotients of type A zig-zag algebras and for a related family of algebras. We also give a characterization of these quotients as quasi-hereditary algebras with simple preserving duality that are “close” to self-injective algebras.

Keywords Generalized tilting module · Exceptional sequence · Zig-zag algebra · Extension · Quasi-hereditary algebra

Mathematics Subject Classification 16D10 · 16G20

1 Introduction and description of the results

Tilting theory, which originates from Brenner and Butler (1980), Happel and Ringel (1982), plays an important role in contemporary representation theory of finite-dimensional algebras. At the same time, classification of all tilting modules for a given algebra is usually a hard problem. This problem has been studied in various special cases, for example in Buan and Krause (2004), Meltzer and Unger (1993), Yamaura (2012), and some generalizations of this problem were studied in Adachi (2016). For self-injective algebras this problem is trivial. The generalized tilting modules for the Auslander algebra of $\mathbb{k}[x]/(x^n)$ were classified in Geuenich (2018). This algebra is, in fact, quadratic dual to the main protagonist of the present note. In an earlier work, Brüstle et al. (1999) characterized classical tilting modules for this algebra as the Δ -filtered generalized tilting modules.

In the present note we classify all generalized tilting modules for the quasi-hereditary quotient B_n of the type A zig-zag algebra with n simple modules (cf. Khovanov and Seidel 2002; Huerfano and Khovanov 2001; Ehrig and Tubbenhauer

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2018). These algebras describe, in particular, blocks of Temperley–Lieb algebras, cf. Martin (1991), Belletête et al. (2018), and some blocks of the parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$, cf. Stroppel (2005). These algebras are very close to self-injective algebras (in the sense that all indecomposable projective modules but one are injective) and have finite representation type. Both these factors significantly simplify our arguments. As it turns out, see Theorem 2, all generalized tilting modules for such algebras are obtained by adding a standard or a costandard module (in the terminology of the quasi-hereditary structure) to a basic projective-injective module.

Other interesting, but much less studied, objects are so-called *exceptional sequences* of modules. A recent preprint (Hille and Ploog 2017) provides a classification of full exceptional sequences for the Auslander algebra of $\mathbb{k}[x]/(x^n)$, which is, as mentioned above, the quadratic dual of our algebra B_n . In Theorem 3 we provide a classification of full exceptional sequences for B_n . We show that each exceptional sequence for B_n is uniquely determined by choosing one module in each pair consisting of a standard and the corresponding costandard B_n -module.

Apart from these classification results, in Theorem 1 we characterize B_n as the basic, connected, quasi-hereditary algebra with n simple modules which has a simple preserving duality and for which all indecomposable projective modules but one are injective.

The paper is organized as follows: all preliminaries are collected in Sect. 2. Theorem 1 is proved in Sect. 3. Section 4 contains a technical result on non-vanishing of second self-extensions of certain indecomposable B_n -modules. This result plays an important role both in the proof of Theorem 2, which can be found in Sect. 5, and in the proof of Theorem 3, which can be found in Sect. 6. In the last section, Sect. 7, we also solve similar classification problems for a slightly smaller quotient of type A zig-zag algebras.

2 Zig-zag algebras, their quotients and modules

2.1 Basic setup and notation

We work over an algebraically closed field $\mathbb{k}$. For a finite-dimensional $\mathbb{k}$ -algebra A given by some quiver and (admissible) relations, we denote by $L(v)$ the simple A -module associated to a vertex v of the quiver. We also denote by $P(v)$ the indecomposable projective cover of $L(v)$ and by $I(v)$ the indecomposable injective envelope of $L(v)$. If not stated otherwise, by module we mean a right module.

By a *simple preserving duality* of a module category we mean an involutive contravariant autoequivalence which preserves the isomorphism classes of simple modules. Simple preserving dualities are given by involutive algebra anti-automorphisms which fix pointwise some full set of primitive orthogonal idempotents.

We will often abuse notation and write $X = Y$ instead of $X \cong Y$ for two modules X and Y .

2.2 Zig-zag algebras

Let Q be a finite connected unoriented graph without loops and with at least one edge. Let $\tilde{Q}$ denote the quiver obtained from Q via substituting every edge $i \text{ --- } j$ in Q by two oriented edges $i \rightleftarrows j$. We denote by A_Q the quotient of the path algebra $\mathbb{k}\tilde{Q}$ of $\tilde{Q}$ by the ideal generated by the following relations:

- any path of length three is zero;
- any path of length two which is not a cycle is zero;
- for any vertex v , all length two cycles which start and terminate at v are equal.

The algebra A_Q is usually called the *zig-zag algebra* associated with Q , see Huerfano and Khovanov (2001), Ehrig and Tubbenhauer (2018). Directly from the definition it follows that A_Q is $\mathbb{Z}$ -graded by path length and this grading is positive in the sense that all non-zero homogeneous components have non-negative degree and the degree zero component is semi-simple, see Huerfano and Khovanov (2001).

It follows directly from the definition that, for each vertex v in Q , the projective module $P(v)$ has Loewy length three and is rigid in the sense that the radical and the socle filtrations of $P(v)$ coincide. In particular, $P(v)$ has a unique Loewy filtration. The layers of this unique Loewy filtration are given as follows:

- the top of $P(v)$ is isomorphic to $L(v)$;
- the socle of $P(v)$ is isomorphic to $L(v)$;
- the quotient $\text{Rad}(P(v))/\text{Soc}(P(v))$ is semisimple and, for any vertex w , the multiplicity of $L(w)$ in this quotient equals the number of edges in Q between v and w .

Consequently, $P(v) \cong I(v)$ and A_Q is a self-injective algebra.

2.3 The algebras A_n, B_n and C_n

For $n \in \mathbb{Z}_{>1}$, we denote by A_n the algebra A_Q , where Q is the following Dynkin diagram of type A :

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \dots \text{ --- } n$$

We denote by B_n the quotient of A_n by the additional relation that the length two loop at the vertex n is zero. We denote by C_n the quotient of B_n by the additional relation that the length two loop at the vertex 1 is zero. The vertices 1 and n are both leaves in the Dynkin diagram of type A . We therefore refer to these quotients as *leaf quotients of type A zig-zag algebras*.

Both B_n and C_n inherit a $\mathbb{Z}$ -grading from A_n . It is easy to see that B_n is never self-injective, while C_n is self-injective if and only if $n = 2$.

The algebra A_n has an involutive anti-automorphism given by swapping the arrows in each part $i \rightleftarrows j$ from the definition. This induces involutive anti-automorphisms on B_n and C_n . In particular, module categories over all these algebras have simple preserving dualities.

2.4 Indecomposable modules over A_n , B_n and C_n

By Corollary 1 in Huerfano and Khovanov (2001) the algebra A_n has $n(n+1)$ isomorphism classes of indecomposable modules. It is easily checked that the algebra A_n is special biserial in the sense of Butler and Ringel (1987), Wald and Waschbusch (1985). In particular, all indecomposable A_n -modules split into three families: projective-injective modules, string modules and band modules, see Proposition 2.3 in Wald and Waschbusch (1985). Even more, in case of the algebra A_n there are no band modules. Apart from the projective-injective modules $P(i)$ and the simple modules $L(i)$, where $i = 1, 2, \dots, n$, we only have four finite families of string modules which look as follows:

- The modules $M(i, j)$, where $1 \leq i < j \leq n$ are such that $i \equiv j \pmod{2}$:

- The modules $N(i, j)$, where $1 \leq i < j \leq n$ are such that $i \not\equiv j \pmod{2}$:

- The modules $W(i, j)$, where $1 \leq i < j \leq n$ are such that $i \equiv j \pmod{2}$:

- The modules $S(i, j)$, where $1 \leq i < j \leq n$ are such that $i \not\equiv j \pmod{2}$:

It is clear that all string modules in the four families above are indecomposable and not isomorphic to any projective-injective nor simple module. This completes the classification of the $n(n+1)$ isomorphism classes of indecomposable A_n -modules.

All indecomposable A_n -modules but $P(n)$ are also B_n -modules. All indecomposable B_n -modules but $P(1)$ are also C_n -modules.

2.5 Quasi-hereditary structure on B_n

Let Λ be a finite-dimensional $\mathbb{k}$ -algebra with a fixed order $L_1 < L_2 < \dots < L_k$ on a full set of representatives of simple Λ -modules. For $i = 1, 2, \dots, k$, we denote by P_i the indecomposable projective cover of L_i and by I_i the indecomposable injective envelope of L_i . Recall, see Dlab and Ringel (1989), Cline et al. (1988), that Λ is called *quasi-hereditary* (with respect to the order $<$) provided that there are Λ -modules Δ_i , for $i = 1, 2, \dots, k$, (called *standard modules*) with the properties that

- $P_i \twoheadrightarrow \Delta_i$ and the kernel of this surjection has a filtration with subquotients Δ_j , for $j > i$;
- $\Delta_i \rightarrow L_i$ and the kernel of this surjection has composition subquotients L_j , for $j < i$.

Equivalently, there are Λ -modules ∇_i , for $i = 1, 2, \dots, k$, (called *costandard modules*) with the properties that

- $\nabla_i \hookrightarrow I_i$ and the cokernel of this injection has a filtration with subquotients ∇_j , for $j > i$;
- $L_i \hookrightarrow \nabla_i$ and the cokernel of this injection has composition subquotients L_j , for $j < i$.

Following Ringel (1991), we denote by T_i the indecomposable tilting Λ -module corresponding to i . It is uniquely determined by the properties that $\Delta_i \hookrightarrow T_i$ and the cokernel of this inclusion has a filtration with standard subquotients. Equivalently, $T_i \twoheadrightarrow \nabla_i$ and the kernel of this surjection has a filtration with costandard subquotients.

The terminology here is slightly conflicting. We will refer to tilting modules in the sense of Miyashita (1986) by *generalized tilting modules* and those that are connected to the quasi-hereditary structure, in the sense of Ringel (1991), simply by *tilting modules*.

Consider the algebra $\Lambda = B_n$ and choose the order $L(1) < L(2) < \dots < L(n)$. Set $\Delta(1) = \nabla(1) = L(1)$; $\Delta(i) = S(i-1, i)$, for $i > 1$; and $\nabla(i) = N(i-1, i)$, for $i > 1$. Then, for $i < n$, there are obvious short exact sequences

$$\Delta(i+1) \hookrightarrow P(i) \twoheadrightarrow \Delta(i) \quad \text{and} \quad \nabla(i) \hookrightarrow I(i) \twoheadrightarrow \nabla(i+1).$$

Consequently, B_n is quasi-hereditary. It is easy to see that B_n is *not* quasi-hereditary with respect to any other order on the full set of representatives of isomorphism classes of simple B_n -modules. Set

$$\mathbf{D} := \{\Delta(n), \Delta(n-1), \dots, \Delta(1) = \nabla(1), \nabla(2), \dots, \nabla(n)\}.$$

Proposition 1 *For a quasi-hereditary algebra Λ we have the following vanishing extensions. For all $k > 0$ and for all $i \leq j$, we have*

$$\text{Ext}_{\Lambda}^k(\Delta_j, \Delta_i) = 0 \quad \text{and} \quad \text{Ext}_{\Lambda}^k(\nabla_i, \nabla_j) = 0.$$

For all $k > 0$ and for all i, j , we have

$$\mathrm{Ext}_A^k(\Delta_i, \nabla_j) = 0.$$

In particular, for any $X \in \mathbf{D}$ and $k > 0$, we have

$$\mathrm{Ext}_{B_n}^k(X, X) = 0.$$

Proof Using Lemma 2.2 in Klucznik and König (1999), see also (Cline et al. 1988, Theorem 2.3), in the first claim we may assume that j is maximal, so Δ_j is projective and ∇_j is injective. The claim is then clear. For the second claim, see Proposition 2.1 in Klucznik and König (1999) or Corollary 3 in Ringel (1991). The third claim is a special case of the first one. $\square$

Note that, for the algebra B_n , we have $T(1) = L(1)$ and $T(i) = P(i-1) = I(i-1)$, for $i > 1$.

3 A characterization of the algebra B_n

In this section we propose a characterization of the quasi-hereditary algebras B_n inside the class of all quasi-hereditary algebras. Our result says that the algebras B_n are non-semi-simple quasi-hereditary algebras which are, in some sense, “closest” to self-injective algebras.

Theorem 1 *Let Λ be a basic, connected, quasi-hereditary algebra with respect to some order $L_1 < L_2 < \dots < L_n$, where $n \in \mathbb{Z}_{>0}$. Assume the following:*

- (a) *Exactly $n - 1$ of indecomposable projective Λ -modules are injective.*
- (b) *The algebra Λ has a simple preserving duality.*

Then $n > 1$ and $\Lambda \cong B_n$.

Proof By (b), Λ has a simple preserving duality which we denote by $\star$. Then $P_i^\star = I_i$, for all i .

Our first observation is that the module P_n is not injective. Indeed, assume that $P_n \cong I_j$, for some j . If $j = n$, then P_n has top and socle isomorphic to L_n and this simple module also appears in $P_n = \Delta_n$ with multiplicity 1. Therefore $P_n = I_n = L_n$. This means that P_n is a direct summand of Λ as an algebra. As Λ is assumed to be connected, we obtain that $n = 1$. At the same time, the only basic quasi-hereditary algebra with one isomorphism class of simple modules is $\mathbb{k}$ and this algebra does not satisfy (a) as there is at least one projective-injective module. Therefore $P_n = I_n$ is not possible. As a bonus, we now know that $n > 1$.

Assume that $j < n$. Then $P_n = I_j$ is a tilting module and it contains L_n . Therefore $P_n = T_n$. As $\star$ is a simple preserving duality, we have $T_n^\star = T_n$ and therefore $P_n = T_n = T_n^\star = I_j^\star = P_j$, which is a contradiction. This shows that P_n is not injective.

We prove the claim of the theorem by induction on n . By the above, the basis of the induction is the case $n = 2$. We always have $T_1 = L_1$. As $P_1 = I_1$ is projective-injective, it is a tilting module. If $L_1 = T_1 = P_1 = I_1$, then, by the same argument

as in the second paragraph of the proof we get a contradiction using the fact that Λ is connected. Therefore $P_1 = T_2$. In particular, Δ_2 embeds into $T_2 = I_1$ and hence $\text{soc}(\Delta_2) = L_1$.

So, the module Δ_2 has simple top L_2 and simple socle L_1 . If the composition multiplicity of L_1 in Δ_2 were greater than 1, that would imply existence of a non-split self-extension for the simple module L_1 . But since Λ is assumed to be quasi-hereditary no such self-extension can exist. This means that there is a short exact sequence

$$0 \rightarrow L_1 \rightarrow \Delta_2 \rightarrow L_2 \rightarrow 0.$$

As $P_2 = \Delta_2$, it follows that the quiver of Λ has exactly one arrow from the vertex of L_2 to the vertex of L_1 , call it α . Applying $\star$, we see that there is exactly one arrow in the opposite direction, call it β . As the composition multiplicity of L_2 in $P_2 = \Delta_2$ is 1, we have $\alpha\beta = 0$. At the same time, the composition multiplicity of L_2 in $P_2 = I_2 = T_1$ is at least 2 as P_2 has a copy of L_2 at the top, I_2 has a copy of L_2 in the socle and T_2 has a copy of L_1 somewhere. It follows that $\beta\alpha \neq 0$ and we obtain that $\Lambda \cong A_2$. This completes the basis of our induction.

To prove the induction step, we assume that $n \geq 3$. Similarly to the case $n = 2$, we have that $T_1 = L_1$ is not projective. Therefore $L_1, P_1, P_2, \dots, P_{n-1}$ is a complete list of representatives of isomorphism classes of tilting Λ -modules. In particular, $T_n = P_j$, for some $j < n$. The latter means that $\Delta_n = P_n$ embeds into P_j and hence P_n has simple socle L_j . Also, $\text{Hom}_\Lambda(P_n, P_s) = 0$, for all $s \in \{1, 2, \dots, j-1, j+1, \dots, n-1\}$ since for such s we have $P_s = T_i$ for some $i < n$ and the composition multiplicity of L_n in such tilting modules is zero. This implies that the quiver of Λ contains no arrows from the vertex of L_n to the vertex of such L_s . By applying $\star$, we see that there are no arrows back either. Therefore the only arrows from the vertex of L_n in the quiver of Λ are those going to the vertex of L_j and the only arrows to the vertex of L_n in the quiver of Λ are those going from the vertex of L_j .

The equality $\text{Hom}_\Lambda(P_n, P_s) = \text{Hom}_\Lambda(P_n, I_s) = 0$ also means that P_n has no composition subquotients L_s with s as in the previous paragraph. We claim that this implies that $j = n-1$. Indeed, assume $j < n-1$ and let e be the primitive idempotent of Λ corresponding to L_n . Consider the quasi-hereditary algebra $\Lambda/(\Lambda e \Lambda)$. As L_n does not appear in P_{n-1} , the module P_{n-1} is a module over $\Lambda/(\Lambda e \Lambda)$. As $n-1$ is now maximal and $\Lambda/(\Lambda e \Lambda)$ is quasi-hereditary, the module P_{n-1} must be a standard module for $\Lambda/(\Lambda e \Lambda)$. If $P_{n-1} = I_{n-1} = L_{n-1}$, we obtain the same contradiction with connectedness of Λ as above. If $P_{n-1} = I_{n-1}$ is not simple, then the multiplicity of L_{n-1} in P_{n-1} must be at least two. This is not possible for a standard module. Therefore $j = n-1$.

Similarly to the case $n = 2$, we see that there is exactly one arrow from the vertex of L_n to the vertex of L_{n-1} , call it α . Applying $\star$, we see that there is exactly one arrow in the opposite direction, call it β . As the composition multiplicity of L_n in $P_n = \Delta_n$ is 1, we have $\alpha\beta = 0$. Furthermore the precomposition of α with any other arrow in the quiver of Λ is zero and, similarly, the postcomposition of β with any other arrow in the quiver of Λ is zero.

Now consider the basic quasi-hereditary algebra $\Lambda/(\Lambda e \Lambda)$. Its quiver is obtained from the quiver of Λ by deleting the vertex of L_n and the arrows α and β . Therefore $\Lambda/(\Lambda e \Lambda)$ is connected as Λ was connected and the vertex of L_n only had arrows to and from the vertex of L_{n-1} by the previous paragraph. The simple preserving duality on Λ induces a simple preserving duality on $\Lambda/(\Lambda e \Lambda)$. The projective-injective modules $P_1, P_2, \dots, P_{n-2}$ of Λ are also modules over $\Lambda/(\Lambda e \Lambda)$ and remain projective-injective. Therefore $\Lambda/(\Lambda e \Lambda)$ satisfies all assumptions of our theorem and has one less projective than Λ . Therefore we can now apply the induction assumption.

By induction, we have $\Lambda/(\Lambda e \Lambda) \cong B_{n-1}$. Also, from the above we see that the quiver of Λ is isomorphic to the quiver for B_n . From what we concluded in the induction step we have that there are exact sequences

$$\begin{aligned} 0 \rightarrow \Delta_n \rightarrow P_{n-1} \rightarrow \Delta_{n-1} \rightarrow 0, \\ 0 \rightarrow L_{n-1} \rightarrow \Delta_n \rightarrow L_n \rightarrow 0, \quad 0 \rightarrow L_{n-2} \rightarrow \Delta_{n-1} \rightarrow L_{n-1} \rightarrow 0. \end{aligned}$$

As $P_{n-1} = I_{n-1}$ has simple socle and

$$\text{Ext}_\Lambda^1(L_n, L_{n-2}) = \text{Ext}_\Lambda^1(L_{n-2}, L_n) = 0$$

by the above, we obtain that P_{n-1} has simple top and socle isomorphic to L_{n-1} and $\text{rad}(P_{n-1})/\text{soc}(P_{n-1}) = L_n \oplus L_{n-2}$. This means that in the part

$$\begin{array}{ccccc} L_{n-2} & \xrightarrow{\delta} & L_{n-1} & \xrightarrow{\beta} & L_n \\ & \xleftarrow{\gamma} & & \xleftarrow{\alpha} & \\ & & L_{n-1} & & \end{array}$$

of the quiver of Λ we have the relation $\gamma\delta = c\beta\alpha$, for some non-zero $c \in \mathbb{k}$. Changing β to $\beta' = c\beta$ defines an isomorphism from Λ to B_n . This completes the proof. $\square$

4 Non-vanishing of $\text{Ext}_{B_n}^2$

4.1 The result

The aim of this subsection is to prove the following technical result.

Proposition 2 *Let X be an indecomposable B_n -module. Assume that X is neither projective nor in $\mathbf{D}$. Then $\text{Ext}_{B_n}^2(X, X) \neq 0$.*

Since we have a complete classification of indecomposable B_n -modules, to prove Proposition 2, we need only to consider the cases when X is either a simple module $L(i)$, with $i > 1$, or one of the modules $W(i, j)$, $M(i, j)$, $N(i, j)$, $S(i, j)$ from Sect. 2.4, with $j > i + 1$.

To simplify notation, in the following sections, for a subset $Z \subset \{0, 1, 2, \dots, n\}$, we set

$$P_Z = \bigoplus_{z \in Z} P(z).$$

For two integers $i \leq j$, we denote by $[i, j]$ the set of all integers s satisfying the conditions $i \leq s \leq j$ and $s \equiv i \pmod{2}$.

4.2 Proof of Proposition 2 for modules $W(i, j)$

Let e_k be the idempotent corresponding to $L(k)$ and let $J = B_n e B_n$, where $e = e_{j+1} + \cdots + e_n$. Then, by Lemma 2.2 in Klucznik and König (1999), for any modules $X, Y \in B_n/J - \text{mod}$ we have

$$\text{Ext}_{B_n/J}^k(X, Y) = \text{Ext}_{B_n}^k(X, Y).$$

This allows us to assume $j = n$ without losing any generality.

From the definition of $W(i, n)$, there is a surjection $P_{[i, n]} \twoheadrightarrow W(i, n)$. It is easy to see that the kernel of this surjection is isomorphic to $W(i-1, n-1)$, if $i \neq 1$, and to $N(1, n-1)$, if $i = 1$.

Consider first the case $i = 1$. In this case there is a surjection from $P_{[2, n-1]}$ to $N(1, n-1)$ and it is easy to see that the kernel of this surjection is isomorphic to $N(2, n)$. The obvious injection $N(2, n) \hookrightarrow W(1, n)$ does not factor through $P_{[2, n-1]}$ as the latter maps only to the radical of $W(1, n)$ while the image of the injection $N(2, n) \hookrightarrow W(1, n)$ is not contained in the radical. This implies that the second self-extension of $W(1, n)$ does not vanish.

Now, consider the case $i = 2$. In this case there is a surjection from $P_{[1, n-1]}$ to $W(1, n-1)$ and it is easy to see that the kernel of this surjection is isomorphic to $N(1, n)$. The obvious projection $N(1, n) \twoheadrightarrow W(2, n)$ does not factor through $P_{[1, n-1]}$ as the latter maps only to the radical of $W(2, n)$. This implies that the second self-extension of $W(1, n)$ does not vanish.

Finally, consider the case $i > 2$. In this case there is a surjection from $P_{[i-1, n-1]}$ to $W(i-1, n-1)$ and it is easy to see that the kernel of this surjection is isomorphic to $W(i-2, n)$. The obvious projection $W(i-2, n) \twoheadrightarrow W(i, n)$ does not factor through $P_{[i-1, n-1]}$ as the latter maps only to the radical of $W(i, n)$. This implies that the second self-extension of $W(i, n)$ does not vanish.

4.3 Proof of Proposition 2 for modules $M(i, j)$

Similarly to Sect. 4.2, we may assume $n = j$ without losing any generality.

From the definition of $M(i, n)$, there is a surjection $P_{[i+1, n-1]} \twoheadrightarrow M(i, n)$. It is easy to see that the kernel of this surjection is isomorphic to $M(i+1, n-1)$, if $i < n-2$, and to $L(n-1)$, if $i = n-2$.

Consider first the case $i = n-2$. In this case the kernel of $P(n-1) \twoheadrightarrow L(n-1)$ is isomorphic to $W(n-2, n)$. The dimension of the homomorphism space from $W(n-2, n)$ to $M(n-2, n)$ is two, where one of the homomorphisms factors through $L(n)$ and the other one factors through $L(n-2)$. At the same time, the dimension of the homomorphism space from $P(n-1)$ to $M(n-2, n)$ is one. This implies that the second self-extension of $M(n-2, n)$ does not vanish.

Consider now the case $i = n - 4$. In this case, by recursion, the kernel of the surjection $P(n - 2) \twoheadrightarrow M(n - 3, n - 1)$ is isomorphic to $L(n - 2)$. The obvious injection $L(n - 2) \hookrightarrow M(n - 4, n)$ does not factor through $P(n - 2)$ as the only map from $P(n - 2)$ to $M(n - 4, n)$ sends the radical of $P(n - 2)$ to zero. This implies that the second self-extension of $M(n - 4, n)$ does not vanish.

Finally, consider now the case $i < n - 4$. In this case, by recursion, the kernel of $P_{[i+2, n-2]} \twoheadrightarrow M(i + 1, n - 1)$ is isomorphic to $M(i + 2, n - 2)$. The obvious injection $M(i + 2, n - 2) \hookrightarrow M(i, n)$ does not factor through $P_{[i+2, n-2]}$ as the image of $M(i + 2, n - 2)$ in $P_{[i+2, n-2]}$ covers the socle of $P_{[i+2, n-2]}$ and any map from $P_{[i+2, n-2]}$ to $M(i, n)$ sends the socle of $P_{[i+2, n-2]}$ to zero. This implies that the second self-extension of $M(i, n)$ does not vanish.

4.4 Proof of Proposition 2 for modules $N(i, j)$

Similarly to Sect. 4.2, we may assume $n = j$ without losing any generality. Note that $i < n - 2$ as $N(n - 1, n) \in \mathbf{D}$ and we must have $i \not\equiv n \pmod{2}$.

From the definition of $N(i, n)$, it follows that there is a surjection $P_{[i+1, n]} \twoheadrightarrow N(i, n)$ and it is easy to see that the kernel of this surjection is isomorphic to $N(i + 1, n - 1)$. By recursion, there is a surjection $P_{[i+2, n-1]} \twoheadrightarrow N(i + 1, n - 1)$ and it is easy to see that the kernel of this surjection is isomorphic to $N(i + 2, n)$. The obvious injection $N(i + 2, n) \hookrightarrow N(i, n)$ does not factor through $P_{[i+2, n-1]}$ as the image of $N(i + 2, n)$ in $P_{[i+2, n-1]}$ covers the socle of $P_{[i+2, n-1]}$ and any map from $P_{[i+2, n-1]}$ to $N(i, n)$ sends the socle of $P_{[i+2, n-1]}$ to zero. This implies that the second self-extension of $N(i, n)$ does not vanish.

4.5 Proof of Proposition 2 for modules $S(i, j)$

Similarly to Sect. 4.2, we may assume $n = j$ without losing any generality. Note that $i < n - 1$ as $S(n - 1, n) \in \mathbf{D}$.

From the definition of $S(i, n)$, it follows that there is a surjection $P_{[i, n-1]} \twoheadrightarrow S(i, n)$ and it is easy to see that the kernel of this surjection is isomorphic to $M(1, n - 1)$, if $i = 1$, and to $S(i - 1, n - 1)$, if $i > 1$.

Consider first the case $i = 1$ and $n = 4$. There is a surjection $P(2) \twoheadrightarrow M(1, 3)$ whose kernel is isomorphic to $L(2)$. The obvious injection $L(2) \hookrightarrow S(1, 4)$ does not factor through $P(2)$ as the image of $L(2)$ in $P(2)$ is in the socle and any map from $P(2)$ to $S(1, 4)$ kills the socle of $P(2)$. This implies that the second self-extension of $S(1, 4)$ does not vanish.

Consider now the case $i = 1$ and $n > 4$. There is a surjection $P_{[2, n-2]} \twoheadrightarrow M(1, n - 1)$ with kernel isomorphic to $M(2, n - 2)$. The obvious injection $M(2, n - 2) \hookrightarrow S(1, n)$ does not factor through $P_{[2, n-2]}$ as the image of $M(2, n - 2)$ in $P_{[2, n-2]}$ intersects the socle and any map from $P_{[2, n-2]}$ to $S(i, n)$ kills the socle of $P_{[2, n-2]}$. This implies that the second self-extension of $S(1, n)$ does not vanish.

Consider now the case $i = 2$. In this case there is a surjection $P_{[1, n-2]} \twoheadrightarrow S(1, n - 1)$ whose kernel is isomorphic to $M(1, n - 2)$. The unique up to a scalar non-zero homomorphism $M(1, n - 2) \rightarrow S(2, n)$ does not factor through $P_{[1, n-2]}$ as the image

of $M(1, n-2)$ in $P_{[1, n-2]}$ intersects the socle and any map from $P_{[1, n-2]}$ to $S(2, n)$ kills the socle of $P_{[1, n-2]}$. This implies that the second self-extension of $S(2, n)$ does not vanish.

Finally, consider the case $i > 2$. In this case, by recursion, there is a surjection $P_{[i-1, n-2]} \twoheadrightarrow S(i-1, n-1)$ whose kernel is isomorphic to $S(i-2, n-2)$. The unique up to a scalar non-zero homomorphism $S(i-2, n-2) \rightarrow S(i, n)$ does not factor through $P_{[i-1, n-2]}$ as the image of $S(i-2, n-2)$ in $P_{[i-1, n-2]}$ intersects the socle and any map from $P_{[i-1, n-2]}$ to $S(i, n)$ kills the socle of $P_{[i-1, n-2]}$. This implies that the second self-extension of $S(i, n)$ does not vanish.

4.6 Proof of Proposition 2 for simple modules $L(i)$

Similarly to Sect. 4.2 we can assume that $i = n$ without losing any generality. The kernel of the projective cover $P(n) \twoheadrightarrow L(n)$ is $L(n-1)$ and $L(n)$ appears in the top of the kernel $P(n-1) \twoheadrightarrow L(n-1)$. This implies that the second self-extension of $L(n)$ does not vanish.

5 Generalized tilting modules for B_n

5.1 Generalized tilting modules

Let Λ be a finite-dimensional algebra and T a Λ -module. Recall from Miyashita (1986) that T is called a *generalized tilting* Λ -module provided that

- T has finite projective dimension;
- $\text{Ext}_{\Lambda}^i(T, T) = 0$, for all $i > 0$;
- the module Λ_{Λ} has a finite coresolution by modules in $\text{add}(T)$.

We will say that a Λ -module is *basic* provided that it is isomorphic to a direct sum of pairwise non-isomorphic indecomposable modules.

5.2 Classification of basic generalized tilting B_n -modules

Theorem 2 *Basic generalized tilting B_n -modules are, up to isomorphism, exactly the following modules:*

$$P(1) \oplus P(2) \oplus \cdots \oplus P(n-1) \oplus X, \quad (1)$$

where $X \in \mathbf{D}$.

Proof Let T be a generalized tilting B_n -module. Corollary 1 in Miyashita (1986) states that the number of isomorphism classes of simple modules in B_n and $\text{End}_{B_n}(T)$ are equal. It follows that the number of indecomposable summands of T must be n . From this it also follows that every projective-injective module must be a direct summand of any generalized tilting module. As $P(i)$ is both projective and injective for all $i < n$, we have that each such $P(i)$ is a summand of T . Therefore T is isomorphic to a module of the form (1), for some B_n -module X . As T must be ext-self-orthogonal, from Proposition 2 it follows that $X \in \mathbf{D}$. $\square$

It remains to check that, for each $X \in \mathbf{D}$, the module (1) is a generalized tilting module. As B_n is quasi-hereditary, it has finite global dimension, so the projective dimension of any module is finite. Further, from Proposition 1 and the fact that each $P(i)$, for $i < n$, is both projective and injective, it follows that the module (1) is ext-self-orthogonal. What is left is to show that the module (1) coresolves the right regular B_n -module. If X is projective, the module (1) is equal to the right regular B_n -module. Note that to show that a module coresolves the right regular B_n -module it is enough to show that it coresolves every indecomposable projective B_n -module. If X is not projective, every indecomposable projective B_n -module except $P(n)$ is a direct summand of the module (1). It therefore only remains to show that the module (1) coresolves $P(n)$. The claim follows from Lemma 1 below since all indecomposable projective-injective B_n -modules are direct summands of the module (1).

Lemma 1 *Let $X \in \mathbf{D}$ be non-projective, then X has a projective resolution of the form*

$$0 \rightarrow Q_k \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow X \rightarrow 0.$$

where $Q_k \cong P(n)$ and all other Q_i are projective-injective.

Proof Write the modules in $\mathbf{D} \setminus \{\Delta(n)\}$ in the following order:

$$\Delta(n-1), \Delta(n-2), \dots, \Delta(1) = \nabla(1), \nabla(2), \dots, \nabla(n)$$

and let us prove the claim by induction with respect to this order.

The basis of the induction follows from the short exact sequence

$$0 \rightarrow P(n) \rightarrow P(n-1) \rightarrow \Delta(n-1) \rightarrow 0.$$

The induction steps follow from the short exact sequences

$$0 \rightarrow \Delta(i+1) \rightarrow P(i) \rightarrow \Delta(i) \rightarrow 0, \quad i < n,$$

and

$$0 \rightarrow \nabla(i) \rightarrow P(i) \rightarrow \nabla(i+1) \rightarrow 0, \quad i < n.$$

This completes the proof. $\square$

5.3 A partial order of generalized tilting B_n -modules

Let Λ be an Artin algebra and let $\mathcal{T}_\Lambda$ denote the set of all basic generalized tilting Λ -modules up to isomorphism. Define the *right perpendicular category* of a Λ -module T to be

$$T^\perp := \{X \in \Lambda\text{-mod} \mid \text{Ext}_\Lambda^k(T, X) = 0 \text{ for all } k > 0\}.$$

We can now define a partial order $\leq$ on $\mathcal{T}_\Lambda$ by letting $T \leq T'$ if $T^\perp \subseteq T'^\perp$. In Happel and Unger (2005) the authors prove that, for any Artin algebra Λ , the Hasse diagram for $(\mathcal{T}_\Lambda, \leq)$ is equal to the graph K_Λ which is defined as follows. Let the vertices of K_Λ be the elements of $\mathcal{T}_\Lambda$. There is an edge between T and T' if and only if $T = M \oplus X$, $T' = M \oplus Y$, where X, Y are indecomposable and there exists a short exact sequence

$$0 \rightarrow X \rightarrow \tilde{M} \rightarrow Y \rightarrow 0$$

where $\tilde{M} \in \text{add}(M)$.

By Theorem 2 every basic (generalized) tilting module is of the form

$$P(1) \oplus P(2) \oplus \cdots \oplus P(n-1) \oplus X,$$

for some $X \in \mathbf{D}$. So the set $\mathbf{D}$ is in bijection with the set $\mathcal{T}_{B_n}$. $\mathbf{D}$ therefore inherits a partial order from $\mathcal{T}_{B_n}$, which we will denote by $<$. From the short exact sequences

$$0 \rightarrow \Delta(i+1) \rightarrow P(i) \rightarrow \Delta(i) \rightarrow 0, \quad i < n,$$

and

$$0 \rightarrow \nabla(i) \rightarrow P(i) \rightarrow \nabla(i+1) \rightarrow 0, \quad i < n.$$

we then get the partial order

$$\nabla(n) < \cdots < \nabla(2) < L(1) < \Delta(2) < \cdots < \Delta(n).$$

6 Exceptional sequences for B_n

6.1 Exceptional sequences

Let Λ be a finite-dimensional algebra. Recall, see Bondal (1989), that a Λ -module M is called *exceptional* provided that

- $\text{Hom}_\Lambda(M, M) = \mathbb{k}$;
- $\text{Ext}_\Lambda^i(M, M) = 0$, for all $i > 0$.

A sequence $\mathbf{M} = (M_1, \dots, M_k)$ of Λ -modules is called an *exceptional sequence* provided that

- each M_i is exceptional;
- $\text{Ext}_\Lambda^i(M_t, M_s) = 0$, for all $1 \leq s < t \leq k$ and all $i \geq 0$.

An exceptional sequence is called *full (or complete)* if it generates the derived category.

6.2 Classification of exceptional sequences for B_n

Proposition 3 *A B_n -module M is exceptional if and only if $M \in \mathbf{D}$.*

Proof Each of the modules $P(1), P(2), \dots, P(n-1)$ has a nilpotent endomorphism given by sending the top to the socle. Therefore none of these modules is exceptional. If a B_n -module M is neither projective-injective nor in $\mathbf{D}$, then M is not exceptional by Proposition 2.

On the other hand, every $M \in \mathbf{D}$ has trivial endomorphism algebra and hence is exceptional by Proposition 1. The claim follows. $\square$

Theorem 3 *There are exactly 2^{n-1} full exceptional sequences of B_n -modules and they are all of the form*

$$(\nabla(i_1), \dots, \nabla(i_k), L(1), \Delta(j_1), \dots, \Delta(j_l)), \quad (2)$$

where

- $k + l = n - 1$, where $0 \leq k, l \leq n - 1$;
- $\{i_1, \dots, i_k, j_1, \dots, j_l\} = \{2, \dots, n\}$;
- $i_s > i_t$ and $j_s < j_t$ for $s < t$.

Proof By Proposition 3, any exceptional sequence of B_n -modules consists of modules from $\mathbf{D}$.

For any $i \in \{2, 3, \dots, n\}$, we have $\Delta(i) \not\cong \nabla(i)$ and there are non-zero maps $\Delta(i) \rightarrow \nabla(i)$ and $\nabla(i) \rightarrow \Delta(i)$. Therefore $\Delta(i)$ and $\nabla(i)$ cannot appear in the same exceptional sequence. Consequently, any full exceptional sequence must be of the form (2), up to ordering of elements in this sequence. That this ordering of elements yields an exceptional sequence follows by Proposition 1 and the fact that there are no non-zero maps from $\Delta(i)$ to $\nabla(j)$ for $i \neq j$; from $\Delta(j)$ to $\Delta(i)$ and from $\nabla(i)$ to $\nabla(j)$ for $i < j$. The fact that this ordering is uniquely defined follows from the following lemma.

Lemma 2 *Let $1 \leq i < j \leq n$ and $k = j - i$. Then*

$$\mathrm{Ext}_{B_n}^k(\Delta(i), \Delta(j)) \neq 0 \quad \text{and} \quad \mathrm{Ext}_{B_n}^k(\nabla(j), \nabla(i)) \neq 0.$$

Proof There is an exact sequence:

$$0 \rightarrow \Delta(j) \rightarrow P(j-1) \rightarrow \dots P(i+1) \rightarrow P(i) \rightarrow \Delta(i) \rightarrow 0.$$

This is the first part of the projective resolution of $\Delta(i)$. Note that the only projective modules with non-zero homomorphisms to $\Delta(j)$ are $P(j)$ and $P(j-i)$. Furthermore, the identity map on $\Delta(j)$ does not factor through $P(j-1)$ as the embedding from $\Delta(j) \hookrightarrow P(j-1)$ ends up in the radical of $P(j-1)$. This implies the inequality $H^k(\mathrm{Hom}_{B_n}(P_\bullet, \Delta(j))) = \mathrm{Ext}_{B_n}^k(\Delta(i), \Delta(j)) \neq 0$, and the second inequality $\mathrm{Ext}_{B_n}^k(\nabla(j), \nabla(i)) \neq 0$ follows using the simple preserving duality defined in Sect. 2.3. $\square$

It remains to check that any sequence of the form (2) generates the derived category. For this it is enough to show that all simple modules can be obtained from

modules in (2) using the operations of taking kernels of epimorphisms and cokernels of monomorphisms. Let us prove, by induction on i , that $L(i)$ can be obtained in this way. The basis of the induction is trivial as $L(1)$ is in (2). Now let us prove the induction step. Assume that $L(i - 1)$ can be obtained. Then (2) contains either $\Delta(i)$ or $\nabla(i)$. In the first case, $L(i)$ is the cokernel of the inclusion $L(i - 1) \hookrightarrow \Delta(i)$. In the second case, $L(i)$ is the kernel of the projection $\nabla(i) \twoheadrightarrow L(i - 1)$. The claim of the theorem follows. $\square$

7 Generalized tilting modules and exceptional sequences for C_n

7.1 Self-orthogonal C_n -modules

Proposition 4 *The only ext-self-orthogonal C_n -modules are the projective-injective modules, and also the modules $N(i, i + 1)$ and $S(i, i + 1)$, where $i = 1, 2, \dots, n - 1$.*

Proof That the projective-injective modules are ext-self-orthogonal is clear, this also applies to the projective modules $N(n - 1, n)$ and $S(1, 2)$ and the injective modules $N(1, 2)$ and $S(n - 1, n)$.

For $i < n - 1$, the module $N(i, i + 1)$ has the following projective resolution:

$$0 \rightarrow N(n - 1, n) \rightarrow P(n - 1) \rightarrow \cdots \rightarrow P(i + 2) \rightarrow P(i + 1) \rightarrow N(i, i + 1) \rightarrow 0.$$

The only term in this resolution that has a non-zero homomorphism to $N(i, i + 1)$ is the zero term $P(i + 1)$. This implies that

$$H^k(\text{Hom}_{C_n}(P_\bullet, N(i, i + 1))) = \text{Ext}_{C_n}^k(N(i, i + 1), N(i, i + 1)) = 0$$

for all $k > 0$. Therefore $N(i, i + 1)$ is an ext-self-orthogonal module.

The algebra C_n has the obvious automorphism α that swaps 1 and $n, 2$ and $n - 1$ and so on. Twisting by α swaps modules of type S and modules over type N . Therefore ext-self-orthogonality of $S(i, i + 1)$ follows from ext-self-orthogonality of $N(n - i, n + 1 - i)$ by applying α .

It remains to show that all other C_n -modules are not ext-self-orthogonal. To start with, let us show that all simple C_n -modules are not ext-self-orthogonal. If $i \neq 1, n$, then the kernel of the projective cover $P(i) \twoheadrightarrow L(i)$ is $W(i - 1, i + 1)$. The latter has projective cover $P(i - 1) \oplus P(i + 1) \twoheadrightarrow W(i - 1, i + 1)$ whose kernel has $L(i)$ at the top. Therefore $\text{Ext}_{C_n}^2(L(i), L(i)) \neq 0$. Similarly, the kernel of the projective cover $P(1) \twoheadrightarrow L(1)$ is $L(2)$ and $L(1)$ appears in the top of the kernel $P(2) \twoheadrightarrow L(2)$. Therefore $\text{Ext}_{C_n}^2(L(1), L(1)) \neq 0$. Applying α , we get $\text{Ext}_{C_n}^2(L(n), L(n)) \neq 0$. So, all C_n -simples are not ext-self-orthogonal.

Now, for $X \in \{W, M, S, N\}$, $1 \leq i$ and $i + 1 < j \leq n$, consider the module $X(i, j)$. If $i \geq 4$, then the first two terms of the projective resolution of $X(i, j)$ are exactly the same modules as for B_n and therefore the fact that $X(i, j)$ is not ext-self-orthogonal follows from Proposition 2. Applying α , we obtain that $X(i, j)$ is not ext-self-orthogonal if $j \leq n - 3$. So, it remains to consider the nine modules $X(i, j)$,

where $i = 1, 2, 3$ and $j = n - 2, n - 1, n$. Taking into account the parity of n , we have altogether 18 different cases left.

Consider the case n is odd and $X(i, j) = W(1, n)$. Then the kernel of the projective cover $P_{[1,n]} \twoheadrightarrow W(1, n)$ is isomorphic to $W(2, n - 1)$, if $n > 3$, and $L(2)$, if $n = 3$. The kernels of the projective covers $P_{[2,n-1]} \twoheadrightarrow W(2, n - 1)$ and $P_2 \twoheadrightarrow L(2)$ are isomorphic to $W(1, n)$ in both cases. The identity homomorphism of $W(1, n)$ does not factor through $P_{[2,n-1]}$ and hence defines a non-zero element in $\text{Ext}_{C_n}^2(W(1, n), W(1, n))$. Therefore $W(1, n)$ is not ext-self-orthogonal.

Consider the case n is even and $X(i, j) = S(1, n)$. Then the kernel of the projective cover $P_{[1,2]} \twoheadrightarrow S(1, n)$ is isomorphic to $S(2, n - 1)$. The kernel of the projective cover $P_{[2,n-2]} \twoheadrightarrow S(2, n - 1)$ is isomorphic to $S(1, n - 2)$. The obvious inclusion of $S(1, n - 2)$ to $S(1, n)$ does not factor through $P_{[2,n-2]}$ and hence defines a non-zero element in $\text{Ext}_{C_n}^2(S(1, n), S(1, n))$. Therefore $S(1, n)$ is not ext-self-orthogonal.

The remaining cases are similar and left to the reader. The claim of the proposition follows. $\square$

7.2 Generalized tilting C_n -modules

In this subsection we assume $n > 2$ as the algebra C_2 is self-injective.

Theorem 4 *Basic generalized tilting C_n -modules are, up to isomorphism, exactly the following modules:*

$$P(2) \oplus P(3) \oplus \cdots \oplus P(n - 1) \oplus S(i, i + 1) \oplus N(j, j + 1), \quad (3)$$

where $i, j \in \{1, 2, \dots, n - 1\}$.

Proof As mentioned in the proof of Theorem 2 the basic projective-injective module $P(2) \oplus P(3) \oplus \cdots \oplus P(n - 1)$ must be a summand of any generalized tilting module and the total number of summands must be n . This leaves us with two undetermined summands which, by Proposition 4, must be of the form $N(i, i + 1)$ or $S(i, i + 1)$, where $i = 1, 2, \dots, n - 1$. For $1 \leq s < t \leq n - 1$, we have the exact sequence

$$0 \rightarrow N(t, t + 1) \rightarrow P(t) \rightarrow P(t - 1) \rightarrow \cdots \rightarrow P(s + 1) \rightarrow N(s, s + 1). \quad (4)$$

For $t = n - 1$, this is the projective resolution of $N(s, s + 1)$ since $N(n - 1, n) = P(n)$. For $s < t < n$ the sequence (4) is the first part of the projective resolution of $N(s, s + 1)$. Note that the only projective modules with non-zero homomorphisms to $N(t, t + 1)$ are $P(t)$ and $P(t + 1)$. Furthermore, the identity endomorphism on $N(t, t + 1)$ does not factor through $P(t)$ which implies that $H^{t-s}(\text{Hom}_{C_n}(P_\bullet, N(t, t + 1))) \neq 0$ and hence gives a non-zero extension from $N(s, s + 1)$ to $N(t, t + 1)$ of degree $t - s$. Therefore two modules of type N cannot be summands of the same generalized tilting module.

Applying the automorphism α to the sequence (4), for $1 \leq s < t \leq n - 1$, we get the exact sequence

$$0 \rightarrow S(s, s + 1) \rightarrow P(s + 1) \rightarrow \cdots \rightarrow P(t - 1) \rightarrow P(t) \rightarrow S(t, t + 1). \quad (5)$$

For $s = 1$ this is the projective resolution of $S(t, t + 1)$ since $S(1, 2) = P(1)$. We also obtain a non-zero extension from $S(t, t + 1)$ to $S(s, s + 1)$ of degree $t - s$ by applying α to the corresponding result for modules of type N . Hence two modules of type S cannot be summands of the same generalized tilting module. Therefore from Proposition 4 it follows that any generalized tilting C_n -module is of the form (3).

It remains to prove that every module of the form (3) is a generalized tilting module. From (4) and (5) it follows that any module of the form (3) has finite projective dimension and coresolves the regular C_n -module. More precisely, if the module (3) is projective, it is equal to the regular C_n -module. If the module (3) is not projective, all projective C_n -modules except $P(1)$ and $P(n)$ is a direct summand of the module (3). That the module (3) coresolves these modules follows from the exact sequence (4) with $t = n - 1$, $s = j$ and the exact sequence (5) with $s = 1$, $t = i$.

So, it remains to check that any module of the form (3) is ext-self-orthogonal, more precisely, that, for $k > 0$, we have

$$\text{Ext}_{C_n}^k(S(i, i + 1), N(j, j + 1)) = 0 = \text{Ext}_{C_n}^k(N(j, j + 1), S(i, i + 1)). \quad (6)$$

Using α , the first equality reduces to the second one. Consider the projective resolution of $N(j, j + 1)$ which is given by (4), for $s = j$ and $t = n - 1$. If $i < j$, none of the terms in this resolution has a non-zero homomorphism to $S(i, i + 1)$ and hence all extensions from $N(j, j + 1)$ to $S(i, i + 1)$ vanish in this case.

If $i = j$, then only the term in homological degree zero of the resolution has a non-zero homomorphism to $S(i, i + 1)$ and hence all extensions of non-zero homological degree vanish. If $i > j$, then only terms in homological degrees $i - j$ and $i - j - 1$ have non-zero homomorphisms to $S(i, i + 1)$. Consider the corresponding truncations of the projective resolutions. The first truncation is (4) for $s = j$ and $t = i$. The unique non-zero homomorphism from $N(i, i + 1)$ to $S(i, i + 1)$ does factor through $P(i)$ and hence the extension at this position vanishes. The second truncation is (4) for $s = j$ and $t = i - 1$. In this case there are no non-zero homomorphisms from $N(i - 1, i)$ to $S(i, i + 1)$ and hence there is no extension at this position either. This prove (6) and completes the proof. $\square$

7.3 Exceptional sequences for C_n

Theorem 5 *There are no full exceptional sequences of C_n -modules.*

Proof As projective-injective modules do not have trivial endomorphism algebra, from Proposition 4 it follows that any exceptional sequence of modules must consist of modules of the form $S(i, i + 1)$ and $N(i, i + 1)$. As

$$\text{Hom}_{C_n}(S(i, i + 1), N(i, i + 1)) \neq 0 \quad \text{and} \quad \text{Hom}_{C_n}(N(i, i + 1), S(i, i + 1)) \neq 0,$$

the modules $S(i, i + 1)$ and $N(i, i + 1)$ cannot belong to the same exceptional sequence of C_n -modules. Therefore an exceptional sequence of C_n -modules cannot contain more than $n - 1$ modules and thus cannot be full. $\square$

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