Short chains in circle and in square coverings

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Received: 1 January 2019 / Accepted: 26 June 2019 / Published online: 29 July 2019
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Abstract

The problem of finding short paths avoiding obstacles in a packing was extensively studied. Similar question concerning coverings received less attention. Here we consider locally finite coverings of the plane (i) by discs of radius 1, (ii) by squares of area 1. Given two discs (squares resp.) centered at x and x' one can ask for the least number of discs (squares) needed to connect the given discs (squares resp.). Estimates of type $(C + o(1))d(x, x')$ as the Euclidean distance $d(x, x') \rightarrow \infty$ will be given. For discs we prove $C = \frac{2}{\sqrt{3}} \approx 1.15 \dots$, for squares we prove $C = 2\sqrt{2}$. The latter constant turns out to be the best possible.

Keywords Circle covering · Short path

Mathematics Subject Classification 52C15

1 Introduction

Let $\mathcal{F}$ be a locally finite covering of the plane by congruent copies of a specific centrally symmetric disc D . In this paper D will be a circular disc of radius 1 or a square of area 1. We say that two discs D, D' of the covering $\mathcal{F}$ are *connected by a chain of finitely many discs* $D_0, D_1, \dots, D_n$ of $\mathcal{F}$, if $D_0 = D, D_n = D'$ and consecutive discs D_i, D_{i+1} with $i = 0, \dots, n - 1$ have a common point. The integer n will be called length of the chain.

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Given two of the discs in $\mathcal{F}$, we ask to find a connecting chain with as few discs as possible. If one knows the position of each disc of the covering, then we say that a map is given. Having a map, we might as well say that determining the ‘length’ of the shortest connecting chain is a computational problem. Without a map, one has to describe an efficient strategy to select discs and also has to provide an upper bound for the number of discs in the constructed chain. The discs $D_0, D_1, \dots, D_n$ of the chain will be selected sequentially, satisfying the following four rules:

- $D_0 = D$, and the position of the target disc D' is known to the searcher.
- After the first k discs are selected, D_{k+1} must be selected from those discs whose centers are not further than a given, reasonable small distance from the center of D_k .
- One can not go back and drop previously selected discs.
- $D_n = D'$.

For two points x, x' in the plane, $d(x, x')$ denotes the Euclidean distance between x and x' . Let $X \subset \mathbb{R}^2$ be the set of centers of the discs in $\mathcal{F}$. For each connecting chain S , the *nerve* of S is the polygonal arc connecting the centers of consecutive discs in the chain S . Let $N(S)$ denote the number of edges of the nerve of S . $N(S)$ is also called length of the connecting chain S . It is natural to consider the following definitions:

Definition 1 For a given centrally symmetric disc D , we say that an algorithm achieves a constant C , if in any covering of the plane by congruent copies of D , the algorithm connects discs centered at x and x' with a chain S so that $N(S) \leq (C + o(1)) \cdot d(x, x')$ as $d(x, x') \rightarrow \infty$.

Definition 2 For a given disc D , the chain constant C_{chain} of the disc D is defined as the infimum of those constants which are achieved by chain algorithms.

In this paper we give lower and upper bounds for the chain constant of unit disc and also for the chain constant of unit square.

2 Results and examples

Concerning circle coverings we will prove that

Theorem 1 *The chain constant C_{chain} of the disc of radius 1 satisfies*

$$1 \leq C_{chain} \leq \frac{2}{\sqrt{3}} \approx 1.15 \dots$$

The proof of the upper bound of Theorem 1 will be given in Sect. 3. The following example shows that no algorithm can achieve a ratio better than the constant 1.

Example 1 Start with an edge to edge tiling of the plane by squares of edge length 2. Consider all unit discs centered at the vertices and the centers of the squares of this tiling. This covering is commonly referred to as double square lattice covering. Figure 1 shows a slightly distorted version of this double square lattice covering. The gap between consecutive discs in a row is ϵ and is chosen to be very small.

Fig. 1 This covering shows that the chain constant of the unit disc is at least 1

Fig. 2 The discs form a reduced chain of length $2d(ST) - 3$

Analysis of Example 1 Let S and T be the centers of two discs of the same row far away from each other (Fig. 1). Notice that in order to connect the discs centered at S and T with a chain we have to have at least one disc from each column of discs. Thus, no chain is shorter than the one which is shaded on Fig. 1. Hence, the shortest chain has length $\frac{1}{1+\frac{1}{2}}d(ST)$.

The following circle covering shows that the algorithm which will prove the upper bound in Theorem 1 cannot always use only those discs which overlap the segment ST .

Example 2 Start with a long horizontal segment ST of length $d(ST) = n + \epsilon$. Figure 2 shows a chain of discs of length $2n - 3$. The construction of this chain is best understood by looking at the nerve of the chain of discs. In this nerve long and short segments of equal lengths alternate. The long segments are slightly shorter than 2 and form 60° with the horizontal line, while the short segments are all horizontal.

Analysis of Example 2 Notice that each disc in the above chain intersects the segment ST , together they cover the segment ST . Moreover, the chain is reduced, meaning that each disc in the chain is needed. One can easily complete this set of discs to a plane covering without adding discs which overlap ST . Since this chain has length $2n - 3$, in the proof of Theorem 1 one must take in account discs which do not overlap ST . Concerning square coverings we will prove

Theorem 2 The chain constant C_{chain} of the unit square is $2\sqrt{2}$.

The proof of Theorem 2 will be given in Sect. 3. Next, we show that no algorithm can achieve a ratio better than the constant 2.

Fig. 3 The square covering on left (on right resp.) proves that the chain constant of the unit square is at least 1 (is at least 2, resp.)

Example 3 The edge-to-edge square tiling with unit edges shows that $1 \leq C_{chain}$. Indeed, if we take two squares in the same column far away from each other, then there is no algorithm due to which the two squares can be connected by a shorter chain than the actual distance between the centers of the selected squares. Next, take a horizontal row of the latter tiling and shift a copy of this row down by ϵ and lay it over the first row to form a strip. Then, tile the plane with parallel copies of this strip to make a covering. This covering improves the lower bound to 2 (Fig. 3). Indeed, if we take two squares along a vertical line far away from each other, then there is no algorithm which connects these squares with a chain shorter than twice of the actual distance between the centers of the selected squares.

Next we improve the lower bound of the chain constant of the unit square from 2 to $2\sqrt{2} = 2.82 \dots$

Example 4 Start with a unit square with a vertical diagonal, duplicate it and translate the second copy vertically down by a small distance ϵ (see the two squares in the circle on Fig. 4). Then take the union of these squares and make two rows by tiling them according to Fig. 4. We will refer to these two rows as a single strip of squares. Finally, we use translates of this strip to tile the plane.

Analysis of Example 4 Notice that any chain crossing this strip must contain at least four discs from this strip. Choose S and T far from each other so that the line ST passes through the strip as shown on Figure 4. Then taking ϵ sufficiently small we get that the shortest connecting chain has to have four squares along each segment of length $\simeq \sqrt{2} + 3\epsilon$. Thus, no algorithm can achieve a ratio smaller than $\frac{4}{\sqrt{2}}$.

3 Proofs

The proof of the upper bound of Theorem 1 will be constructive. We describe an algorithm, which constructs a connecting chain. The main idea of our algorithm is rather simple. We give a few instructions for selecting the first one or two discs (note

Fig. 4 This covering proves that the chain constant of unit square is at least $2\sqrt{2}$

that sometimes it is necessary to select two discs instead only one). The instructions will be relative to the direction toward the target disc. Then the ray connecting the center of the last disc and the center of the target disc will play the role of the new target direction. Using this new target direction we repeat the previously used instructions and select the next pair of discs. We will keep setting the new target directions and each time we select subsequent pairs of discs, until we get close to the target disc. Finally, we will complete the chain using another, less effective heuristic. Any heuristic producing a bounded ratio will do the job, since on balance the overall ratio of the length of the chain and the distance between the centers will still be below the desired constant (this is a very simple task and is left to the reader). Some lemmas will be needed for proving that the constructed chain is short.

Before we recall a lemma from (Bezdek 1999) we describe a very simple everyday situation. It is easy to visualize that the contour of a staircase, which we will call *standard stair-path*. It consists of pairs of perpendicular and equal segments. Moreover, the first segments in the pairs are all horizontal (Fig. 5a). Everybody can see that in order to gain a horizontal distance d the stair-path has to have length $2d$.

Lemma 3 in Bezdek (1999) considers a modified family of stair-paths. In this family the contour of a stair-path still consists of pairs of perpendicular and equal segments. But instead of requiring that the first segments in the pairs are all horizontal we require that their extension passes through the target point T (Fig. 5b). In this case, we say that *the stair-path aims at T* . It turns out that if the terminating point of the stair-path is at least distance 2 away from T , and the segments all have length ≤ 1 , then the length of the path is less than $2d + 4 \log d - 2$. More precisely, we have

Lemma 1 *Let $e > 0$ be a given number and let S and T be two points at distance $d > 2e$ from each other. Let p be a stair-path of pairs of equal segments, emanating at S and aiming to T . If all sides of p have length $\leq e$ and the endpoint of p lies at a distance $\geq 2e$ from T , then the length of the path p is less than $2d + 4e \log \frac{d}{e} - 2e$.*

Magazinov (2017) proved a lemma which is essentially equivalent to the combination of Lemma 1 and its application in Bezdek (1999). The lemma of Magazinov was stated in the context of metrics, was proved differently and was used to improve partial results of Fejes Tóth (2013), Bagett and Bezdek (2003) and Roldan-Pensado

Fig. 5 a Standard stair-path, b modified stair-path

Fig. 6 Generalized stair-path

(2013). The lemma of Magazinov as stated is expected to have other future applications. Returning to the stair-path terminology we re-state these lemmas in the following form.

The general stair-path still consists of pairs of perpendicular segments, but this time, among others, they do not have to be equal. Let us denote the length of the first segment of the i -th pair as *progresses* p_i towards target T , and let us refer to the length of the second segment of the i -th pair as *perpendicular derailment* d_i . Also assume that there is a weight w_i assigned to the i -th pair of segments in the stair-path. We claim the following

Lemma 2 *Let p be a general stair-path starting at S and aiming to T . Let d be the distance between S and T . Assume the stair-path consists of n pairs of perpendicular segments. In the i -th pair the progress is p_i and the derailment is d_i . Also assume that a weight w_i is assigned to the i -th pair of segments (Fig. 6). If*

- *there is a common upper bound, say $e < \frac{d}{2}$, for the progresses p_i , $i = 1, \dots, n$,*
- *the terminal point of the stair-path is at least at a distance $2e$ from T ,*
- *there is a common upper bound, say C_1 , for the ratios $\frac{d_i}{p_i}$ of perpendicular derailments and of the progresses toward T ,*
- *there is a common upper bound, say C_2 , for the ratios $\frac{w_i}{p_i}$ of the weights and the progresses toward T ,*

then the sum of the weights w_i is less than $(C_2 + o(1))d(ST)$ as $d(ST) \rightarrow \infty$.

We do not include the proof of Lemma 2 here since it is essentially the same as the proofs of lemmas in Bezdek (1999) and Magazinov (2017).

Proof of Theorem 1 As promised, we will connect the unit disc centered at S to the unit disc centered at T with a chain of discs with at most $(\frac{2}{\sqrt{3}} + o(1))d$ discs,

Fig. 7 a Step 1, b step 2 of the proof of Theorem 1

where d is the distance between S and T . First we construct a stair-path and check whether the conditions in Lemma 2 are satisfied by the constants $e = 3$, $C_1 = 2.1$, $C_2 = \frac{2}{\sqrt{3}} = 1.15 \dots$ and weights 1 or 2. The points S_i will be selected from centers of discs in the covering and the weights will be chosen so that the conclusion of Lemma 2 estimates the length of a connecting chain of discs. It is going to be enough to explain how the first two discs are selected and prove how they satisfy Lemma 2 as the remaining pairs are selected exactly in the same way. For orientation assume ST is horizontal and ST is the x -axis of the coordinate system with origin at S . Let l_1 be the vertical line with equation $x = \frac{\sqrt{3}}{2}$ and l_2 be the vertical line with equation $x = \sqrt{3}$ (Fig. 7).

Selecting S_1 , Step 1 If there is a unit disc of the given covering whose closure has a common point with the closure of the disc centered at S and has a center with x -coordinate $\geq \frac{\sqrt{3}}{2}$, then we choose that center for S_1 . In this case we choose the weight to be 1 (Fig. 7a). For all such centers (i) the ratio of the y and x coordinates is at most $\frac{\sqrt{13}}{2} : \frac{\sqrt{3}}{2} < 2.1$, and (ii) the ratio of the weight and the horizontal progress is at most $\frac{2}{\sqrt{3}}$.

Notice that no disc with center left from line l_1 can cover the point $(2, 0)$ and no disc of the strip between lines l_1 and l_2 and outside of the circle centered at S with radius 2 can cover any point of the x -axis. Thus, there is at least one disc with a center, which has x -coordinate $> \sqrt{3}$, so that it covers the point $(2, 0)$.

Selecting S_1 , Step 2 If none of the discs qualify for the choice described in Step 1, then we consider that disc of the given covering, whose center has x -coordinate $> \sqrt{3}$ and among such discs covers the most left point on the x -axis. Let us denote this point by W (Fig. 7b). The center of this disc will be S_1 . It is easy to check that (i) the ratio of the y and x coordinates is at most $\frac{1}{2} < 2.1$, and (ii) the ratio of the weight and the horizontal progress is at most $\frac{2}{\sqrt{3}}$. W must be covered by an additional disc with center

Fig. 8 a Step 1, b step 2 of the proof of Theorem 2

say S' . This disc overlaps both discs centered at S and S_1 . The latter condition is the reason why we select weight 2 in this case. Finally, Lemma 2 implies Theorem 1.

Proof of Theorem 2: The beginning of the proof is similar to that of Theorem 1. As promised, we will connect the unit square centered at S to the unit square centered at T with a chain containing at most $(2\sqrt{2} + o(1))d$ squares, where d is the distance between S and T . First we construct a stair-path and check whether the conditions in Lemma 2 are satisfied by the constants $e = 2$, $C_1 = 2.7$, $C_2 = 2\sqrt{2}$ and weights 1 or 2. The points S_i will be selected from centers of squares in the covering and the weights will be chosen so that the conclusion of Lemma 2 estimates the length of a connecting chain of squares. It is enough to explain how the first two segments are selected and to prove how they satisfy Lemma 2 as the remaining pairs are selected exactly in the same way. For orientation assume ST is horizontal and ST is the x -axis of the coordinate system with origin at S . Let l_1 be the vertical line with equation $x = \frac{1}{2\sqrt{2}}$ and l_2 be the vertical line with equation $x = \frac{1}{\sqrt{2}}$.

Selecting S_1 , Step 1 If there is a unit square of the given covering whose closure has a common point with the closure of the square centered at S and has a center with x -coordinate $\geq \frac{1}{2\sqrt{2}}$, then we choose that center for S_1 . In this case we choose the weight to be 1. For all such centers the ratio of the y and x coordinates is at most $\sqrt{\frac{7}{8}} : \frac{1}{2\sqrt{2}} < 2.7$ and the ratio of the weight and the progress is $\leq 2\sqrt{2}$.

Notice that no square with center left from line l_1 can reach further than $\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = 1.06\dots$ and no square with center between lines l_1 and l_2 and outside of the unit disc centered at S can cover any point of the x -axis. These two conditions justify the next step.

Selecting S_1 , Step 2 If none of the discs qualify for the choice described in Step 1, then we consider those squares of the given covering, whose center has x -coordinate $< \frac{1}{2\sqrt{2}}$ and cover some point of the x -axis. Among such squares we chose the one, which covers the most right point, say W , on the x -axis. It is easy to see that this square must

have a common point with the square centered at S . W must be covered by another square. The center of the latter square will be denoted by S_1 . As we explained before, the center of this square must have an x -coordinate $> \frac{1}{\sqrt{2}}$. The latter choice is the reason, why we select $\text{weight} = 2$ in this case. The latter coordinate condition implies that the ratio of the weight and the horizontal progress is $\leq \frac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2}$. Moreover, for all such centers the ratio of the y and x coordinates is at most $\frac{\sqrt{2}}{2} : \frac{1}{\sqrt{2}} = 1 < 2.7$ and the ratio of the weight and the progress is $\leq 2\sqrt{2}$. Finally, of Lemma 2 implies Theorem 2 (Fig. 8).

Acknowledgements Open access funding provided by MTA Alfréd Rényi Institute of Mathematics (MTA RAMKI). The author Liping Yuan supported in part by National Natural Foundation of China (no. 11871192).

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