Generalization of a counterexample to Derksen's theorem in characteristic two

Keisuke Hakuta¹

Received: 30 August 2017 / Accepted: 14 January 2019 / Published online: 12 February 2019
© The Author(s) 2019

Abstract

The Derksen group is the subgroup of the tame subgroup which is generated by affine automorphisms and one particular non-linear automorphism. It is known that the Derksen group is equal to the entire tame subgroup in characteristic zero and in dimension greater than or equal to three (*Derksen's Theorem*). Recently, Maubach and Willems (Serdica Math J 37:305–322, 2011) proved that if the defining field of polynomial automorphisms is prime field of characteristic two, then Derksen's Theorem does not hold in dimension three. In this paper, we generalize the above counterexample to the case of higher dimension.

Keywords Affine algebraic geometry · Polynomial automorphism · Tame automorphism · Tame subgroup · Derksen group · Finite field

Mathematics Subject Classification 14R10 · 12E20 · 20B25

1 Introduction

Throughout the paper, we use $\mathbb{K}$ to denote a field of characteristic $p := \text{char}(\mathbb{K}) \geq 0$ and $\mathbb{K}[X_1, \dots, X_n]$ denotes the polynomial ring in n indeterminates $X_1, \dots, X_n$ over $\mathbb{K}$. We denote by $\text{GA}_n(\mathbb{K})$ the general automorphism group of $\text{Spec } \mathbb{K}[X_1, \dots, X_n]$ over $\text{Spec } \mathbb{K}$. $\text{GA}_n(\mathbb{K})$ is anti-isomorphic to $\text{Aut}_{\mathbb{K}} \mathbb{K}[X_1, \dots, X_n]$ (the group of $\mathbb{K}$ -automorphisms of $\mathbb{K}[X_1, \dots, X_n]$). We identify $\text{GA}_n(\mathbb{K})$ with $\text{Aut}_{\mathbb{K}} \mathbb{K}[X_1, \dots, X_n]$ via the above correspondence (See van den Essen 2000, Introduction for details). The subgroup of $\text{GA}_n(\mathbb{K})$ of affine automorphisms is denoted by $\text{Aff}_n(\mathbb{K})$, and the subgroup of $\text{GA}_n(\mathbb{K})$ generated by elementary automorphisms is denoted by $\text{EA}_n(\mathbb{K})$. We recall

---

This work was supported by JSPS KAKENHI Grant-in-Aid for Young Scientists (B) 16K16066.

✉ Keisuke Hakuta
hakuta@cis.shimane-u.ac.jp

¹ Institute of Science and Engineering, Academic Assembly, Shimane University, Matsue, Shimane, Japan

that $\text{Aff}_n(\mathbb{K}) \cong \mathbb{K}^n \rtimes \text{GL}_n(\mathbb{K})$. Let $\text{TA}_n(\mathbb{K})$ be the subgroup of $\text{GA}_n(\mathbb{K})$ generated by two subgroups $\text{Aff}_n(\mathbb{K})$ and $\text{EA}_n(\mathbb{K})$. $\text{TA}_n(\mathbb{K})$ is called the tame subgroup.

Set

$$\epsilon := (X_1 + X_2^2, X_2, X_3, \dots, X_n) \in \text{EA}_n(\mathbb{K}),$$

and

$$\text{DA}_n(\mathbb{K}) := \langle \text{Aff}_n(\mathbb{K}), \epsilon \rangle \subset \text{TA}_n(\mathbb{K}).$$

$\text{DA}_n(\mathbb{K})$ is called the *Derksen group*. It is known that if $\mathbb{K}$ is a commutative $\mathbb{Q}$ -algebra, then the equation

$$\text{DA}_n(\mathbb{K}) = \text{TA}_n(\mathbb{K})$$

holds for each $n \geq 3$ (van den Essen 2000, Theorem 5.2.1). The result is due to Harm Derksen and is called *Derksen's Theorem* (See van den Essen 2000, Section 5.2 for details). Bodnarchuk generalized Derksen's Theorem in $\text{char}(\mathbb{K}) = 0$ (Bodnarchuk 2005). On the other hand, Maubach and Willems proved that Derksen's Theorem in positive characteristic is *not* true for a special case (Maubach and Willems 2011). More precisely, they showed that if $\mathbb{K} = \mathbb{F}_2$, then $\text{DA}_3(\mathbb{F}_2) \subsetneq \text{TA}_3(\mathbb{F}_2)$ holds (Maubach and Willems 2011, Lemma 2.2). Here, we denote by $\mathbb{F}_q$ a finite field with q elements ($p = \text{char}(\mathbb{F}_q)$, $q = p^r$, and $r \geq 1$). In this paper, we generalize the above counterexample (Maubach and Willems 2011, Lemma 2.2) to the case of higher dimension, namely, if $\mathbb{K} = \mathbb{F}_2$, then Derksen's Theorem does not hold for each $n \geq 3$.

2 Main results

Throughout the section, the symmetric group on a finite set $\mathcal{A}$ and the alternating group on $\mathcal{A}$ are denoted by $\text{Sym}(\mathcal{A})$ and $\text{Alt}(\mathcal{A})$, respectively. We remark that there exists a natural map

$$\pi_q : \text{GA}_n(\mathbb{F}_q) \rightarrow \text{Sym}(\mathbb{F}_q^n),$$

(see Maubach and Willems 2011, Definition 1.4 for details).

In this section, we prove the main theorem (Theorem 1) of this paper. Before we prove Theorem 1, we recall a result from (Hakuta 2018, Corollary 2) (this is a corollary of Hakuta 2018, Main Theorem 2) which gives a sufficient condition for the containment $\pi_q(\text{Aff}_n(\mathbb{F}_q)) \subset \text{Alt}(\mathbb{F}_q^n)$.

Corollary 1 (Hakuta 2018, Corollary 2) *If $q = 2^m$, $m \geq 2$, and $n \geq 2$, or $q = 2$ and $n \geq 3$ then we have $\pi_q(\text{Aff}_n(\mathbb{F}_q)) \subset \text{Alt}(\mathbb{F}_q^n)$.*

Thanks to (Hakuta 2018, Corollary 2) and (Maubach 2001, Theorem 2.3(ii)), the same arguments as in (Maubach and Willems 2011, Lemma 2.2) can be used to prove the main theorem (Theorem 1).

Theorem 1 (Main Theorem) *If $n \geq 3$ then $\mathrm{DA}_n(\mathbb{F}_2) \subsetneq \mathrm{TA}_n(\mathbb{F}_2)$. In other words, Derksen’s Theorem is not true in the case where $\mathbb{K} = \mathbb{F}_2$ and $n \geq 3$.*

Proof Since $\pi_2(\epsilon) = \pi_2((X_1 + X_2, X_2, \dots, X_n)) \in \pi_2(\mathrm{Aff}_n(\mathbb{F}_2))$, we have $\pi_2(\mathrm{Aff}_n(\mathbb{F}_2)) = \pi_2((\mathrm{Aff}_n(\mathbb{F}_2), \epsilon)) = \pi_2(\mathrm{DA}_n(\mathbb{F}_2))$. Then by (Hakuta 2018, Corollary 2), the containment $\pi_2(\mathrm{Aff}_n(\mathbb{F}_2)) = \pi_2(\mathrm{DA}_n(\mathbb{F}_2)) \subset \mathrm{Alt}(\mathbb{F}_2^n)$ holds for each $n \geq 3$. On the other hand, from (Maubach 2001, Theorem 2.3(ii)), we have $\pi_2(\mathrm{TA}_n(\mathbb{F}_2)) = \mathrm{Sym}(\mathbb{F}_2^n)$. Hence we obtain

$$\pi_2(\mathrm{DA}_n(\mathbb{F}_2)) \subset \mathrm{Alt}(\mathbb{F}_2^n) \subsetneq \mathrm{Sym}(\mathbb{F}_2^n) = \pi_2(\mathrm{TA}_n(\mathbb{F}_2)).$$

Thus we must have $\mathrm{DA}_n(\mathbb{F}_2) \subsetneq \mathrm{TA}_n(\mathbb{F}_2)$. This completes the proof. $\square$

The rest of this section is devoted to the description of Meister’s Linearization problem in $\mathbb{F}_2$. We refer the reader to van den Essen (2000), Maubach and Poloni (2009) and Maubach and Willems (2011) for details of Meister’s Linearization problem.

In (Maubach and Willems 2011, Section 4) Maubach and Willems compared two subgroups of $\mathrm{GA}_n(\mathbb{K})$. These subgroups are $\mathrm{GLIN}_n(\mathbb{K})$ and $\mathrm{GTAM}_n(\mathbb{K})$ (See below for the definition). Here, we reconsider these subgroups.

Definition 1 (Maubach and Willems 2011, Definition 4.1) Let G be a group, and H a subgroup of G . We define $\mathcal{N}(H, G)$ to be the smallest normal subgroup of G that contains H , i.e., $\mathcal{N}(H, G) = \langle g^{-1}hg \mid h \in H, g \in G \rangle$.

Set

$$\mathrm{GLIN}_n(\mathbb{K}) := \mathcal{N}(\mathrm{GL}_n(\mathbb{K}), \mathrm{GA}_n(\mathbb{K})), \quad (2.1)$$

$$\mathrm{GTAM}_n(\mathbb{K}) := \mathcal{N}(\mathrm{TA}_n(\mathbb{K}), \mathrm{GA}_n(\mathbb{K})), \quad (2.2)$$

and

$$\mathrm{TLIN}_n(\mathbb{K}) := \mathcal{N}(\mathrm{GL}_n(\mathbb{K}), \mathrm{TA}_n(\mathbb{K})). \quad (2.3)$$

Maubach and Willems obtained the following result about two subgroups $\mathrm{GLIN}_n(\mathbb{K})$ and $\mathrm{GTAM}_n(\mathbb{K})$.

Theorem 2 (Maubach and Willems 2011, Theorem 4.2) *If $n \geq 2$ and $\mathbb{K} = \mathbb{F}_2$ then $\mathrm{GLIN}_n(\mathbb{F}_2) \subsetneq \mathrm{GTAM}_n(\mathbb{F}_2)$, and hence $\mathrm{TLIN}_n(\mathbb{F}_2) \subsetneq \mathrm{TA}_n(\mathbb{F}_2)$.*

Here, we give an alternative proof of Theorem 2 (Maubach and Willems 2011, Theorem 4.2) in the case where $n \geq 3$ by using the same argument as in the proof of Theorem 1.

Alternative Proof of Theorem 2 (Maubach and Willems 2011, Theorem 4.2).

From the definitions of the above three subgroups (2.1), (2.2), and (2.3), it is easy to see that $\mathrm{GLIN}_n(\mathbb{F}_2) \subset \mathrm{GTAM}_n(\mathbb{F}_2)$ and $\mathrm{TLIN}_n(\mathbb{F}_2) \subset \mathrm{TA}_n(\mathbb{F}_2)$. Suppose that $n \geq 3$. We prove

$$\mathrm{GLIN}_n(\mathbb{F}_2) \subsetneq \mathrm{GTAM}_n(\mathbb{F}_2) \quad \text{and} \quad \mathrm{TLIN}_n(\mathbb{F}_2) \subsetneq \mathrm{TA}_n(\mathbb{F}_2).$$

By (Hakuta 2018, Corollary 2), we have

$$\pi_2(\text{GLIN}_n(\mathbb{F}_2)) \subset \text{Alt}(\mathbb{F}_2^n) \quad \text{and} \quad \pi_2(\text{TLIN}_n(\mathbb{F}_2)) \subset \text{Alt}(\mathbb{F}_2^n).$$

It is obviously true that $\text{TA}_n(\mathbb{F}_2) \subset \text{GTAM}_n(\mathbb{F}_2)$. From (Maubach 2001, Theorem 2.3(ii)), we have $\pi_2(\text{TA}_n(\mathbb{F}_2)) = \pi_2(\text{GTAM}_n(\mathbb{F}_2)) = \text{Sym}(\mathbb{F}_2^n)$. Hence we obtain

$$\pi_2(\text{GLIN}_n(\mathbb{F}_2)) \subset \text{Alt}(\mathbb{F}_2^n) \subsetneq \text{Sym}(\mathbb{F}_2^n) = \pi_2(\text{GTAM}_n(\mathbb{F}_2)),$$

and

$$\pi_2(\text{TLIN}_n(\mathbb{F}_2)) \subset \text{Alt}(\mathbb{F}_2^n) \subsetneq \text{Sym}(\mathbb{F}_2^n) = \pi_2(\text{TA}_n(\mathbb{F}_2)).$$

Thus $\text{GLIN}_n(\mathbb{F}_2) \subsetneq \text{GTAM}_n(\mathbb{F}_2)$ and $\text{TLIN}_n(\mathbb{F}_2) \subsetneq \text{TA}_n(\mathbb{F}_2)$ hold for each $n \geq 3$. $\square$

Acknowledgements The author would like to thank the anonymous reviewers for their helpful comments to improve the quality of the paper.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- Bodnarchuk, Y.: On generators of the tame invertible polynomial maps group. *Int. J. Algebra Comput.* **15**, 851–867 (2005)
- Hakuta, K.: On permutations induced by tame automorphisms over finite fields. *Acta Math. Vietnam.* **43**, 309–324 (2018)
- Maubach, S.: Polynomial automorphisms over finite fields. *Serdica Math. J.* **27**, 343–350 (2001)
- Maubach, S., Poloni, P.-M.: The Nagata automorphism is shifted linearizable. *J. Algebra* **321**, 879–889 (2009)
- Maubach, S., Willems, R.: Polynomial automorphisms over finite fields: Mimicking tame maps by the Derksen group. *Serdica Math. J.* **37**, 305–322 (2011)
- van den Essen, A.: Polynomial Automorphisms and the Jacobian Conjecture. *Progress in Mathematics*, vol. 190. Birkhäuser, Basel (2000)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.