Correction to: Annihilators of local cohomology modules and simplicity of rings of differential operators

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The author would like to correct the errors in the publication of the original article. The corrected details are given below for your reading.

The Proposition 3.3 should read as:

Proposition 3.3 $D_K(K[[x_1, \dots, x_n]])$ and $D_K(K\{x_1, \dots, x_n\})$ (in this case, K is also a complete valued field) are simple rings.

Proof If K is a finite field, then the simplicity of $D_K(K[[x_1, \dots, x_n]])$ follows from (Smith and Van den Bergh 1997, Theorem 4.2.1), so hereafter we can assume that K is infinite. Let J be a non-zero two-sided ideal of $D_K(R)$, where R is either $K[[x_1, \dots, x_n]]$ or $K\{x_1, \dots, x_n\}$; by Lemma 3.2, J contains a non-zero element of R , namely f . By (Greuel and Pfister 2008, Lemma 6.2.2) (in this step is where we are using that K is infinite) there exists a K -linear automorphism ϕ of R such that $g := \phi(f)$ is y_n -regular (where $y_k := \phi(x_k)$ for any $1 \leq k \leq n$), meaning that g contains a term λy_n^m for some integer $m \geq 1$ and some unit $\lambda \in K$; set S as

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either $K[[y_1, \dots, y_n]]$ or $K\{y_1, \dots, y_n\}$, and also set S' as either $K[[y_1, \dots, y_{n-1}]]$ or $K\{y_1, \dots, y_{n-1}\}$. Now, from the Weierstrass Preparation Theorem (Greuel and Pfister 2008, Corollary 6.2.8) it follows that g can be written in a unique way as $g = up$, where $u \in S$ is a unit, and $p \in S'[y_n]$ can be written as

$$p = y_n^d + s_1(y_1, \dots, y_{n-1})y_n^{d-1} + \dots + s_d(y_1, \dots, y_{n-1}),$$

and $s_i \in (y_1, \dots, y_{n-1})S'$; this shows, after inverting u , that $p \in \phi(J) \cap R$ (notice that $\phi(J)$ is also a two-sided ideal of $D_K(S)$ because ϕ is linear). Then, setting $\Delta := \frac{1}{d!} \frac{\partial^d}{\partial y_n^d}$, it follows that $\Delta(p) \in \phi(J)$ is a unit of S , whence $1 \in \phi(J)$ and so $1 \in J$; the proof is therefore completed. $\square$

The Theorem 3.6 should read as

Theorem 3.6 *Let K be any field, let R be a commutative Noetherian ring, and let $I \subset R$ be any ideal. Moreover, suppose that R is either a regular ring, affine or affinoid over K , or that R is a regular local ring containing K . Then, $H_I^i(R) \neq 0$ if and only if $(0 :_R H_I^i(R)) = 0$.*

Proof On the one hand, the case where R is regular and either affine or affinoid over K is a direct consequence of Theorem 2.4 (with $A = R$, $B = D_K(R)$ and $M = H_I^j(R)$) and Theorem 3.5; on the other hand, assume now that R is a regular local ring containing K . By Lemma 2.2, we can further assume that R is complete. However, in this case R is a formal power series ring over K , and therefore the result follows once again combining Theorem 2.4 (with $A = R$, $B = D_K(R)$ and $M = H_I^j(R)$) and Theorem 3.5. $\square$

As announced in the Introduction of this manuscript, we recover the following criterion which ensures the vanishing of a local cohomology module (cf. [Lyubeznik 1993, Corollary 3.6 (e)]).

The Theorem 4.1 should read as

Theorem 4.1 *Let (V, uV) be a discrete valuation ring of mixed characteristic with uniformizer u , let R be a V -algebra that is either smooth over V , or a formal power series ring over V and let $D_V(R)$ be the ring of V -linear differential operators. Moreover, let I be an ideal of R and fix $i \in \mathbb{N}$. Then:*

- (i) *If the multiplication by u $H_I^i(R) \xrightarrow{\cdot u} H_I^i(R)$ is not injective, then $(0 :_R H_I^i(R)) \subseteq uR$, and equality holds if and only if this multiplication is the zero map.*
- (ii) *If the multiplication by u $H_I^i(R) \xrightarrow{\cdot u} H_I^i(R)$ is not surjective, then $(0 :_R H_I^i(R)) \subseteq uR$.*

Proof Firstly, suppose that the multiplication by u $H_I^i(R) \xrightarrow{\cdot u} H_I^i(R)$ is not injective and set M as its kernel, which is a non-zero left $D_V(R)$ -module. Since M is annihilated by u , M can also be regarded as left module over the ring $D_V(R)/uD_V(R)$; moreover, using Lemma 2.9 it follows that

$$D_V(R)/uD_V(R) \cong D_{V/uV}(R/uR).$$

In addition, one deduces from Theorem 3.6 that $D_{V/uV}(R/uR)$ is a simple ring (indeed, since R is either smooth or a formal power series ring over V , it follows that R/uR is either smooth or a formal power series ring over the field V/uV , because the property of being smooth is stable under base change). Summing up, M is a non-zero left $D_{V/uV}(R/uR)$ -module, hence Theorem 2.4 implies that $(0 :_{R/uR} M) = 0$ and therefore $uR = (0 :_R M)$. Finally, since $(0 :_R H_I^i(R)) \subseteq (0 :_R M)$, one concludes that $(0 :_R H_I^i(R)) \subseteq uR$, hence part (i) holds.

On the other hand, assume that the multiplication by u $H_I^i(R) \xrightarrow{u} H_I^i(R)$ is not surjective, and set C as the cokernel of this multiplication; by the same argument used before with M , one can ensure that $(0 :_R C) = uR$, and therefore $(0 :_R H_I^i(R)) \subseteq (0 :_R C) = uR$, just what we finally wanted to show. $\square$

The following notion will play a crucial role very soon (see Proof of Theorem 4.4).

The Theorem 4.4 should read as

Theorem 4.4 *Let R be either a smooth $\mathbb{Z}$ -algebra, or a formal power series ring over $\mathbb{Z}$, let I be an ideal of R , let p be a prime number, and fix $i \in \mathbb{N}$. Then:*

- (i) *If the multiplication by p $H_I^i(R) \xrightarrow{p} H_I^i(R)$ is not injective, then $(0 :_R H_I^i(R)) \subseteq pR$, and equality holds if and only if this multiplication is the zero map.*
- (ii) *If the multiplication by p $H_I^i(R) \xrightarrow{p} H_I^i(R)$ is not surjective, then $(0 :_R H_I^i(R)) \subseteq pR$.*
- (iii) *If, in addition, R verifies the star assumption, and there are infinitely many prime integers p such that multiplication by p on $H_I^i(R)$ is not surjective, then $H_I^i(R) \neq 0$ if and only if $(0 :_R H_I^i(R)) = 0$.*

Proof Firstly suppose that the multiplication by p $H_I^i(R) \xrightarrow{p} H_I^i(R)$ is not injective and set M as its kernel, which is a non-zero left $D_{\mathbb{Z}}(R)$ -module. Since M is annihilated by p , M can also be regarded as left module over the ring $D_{\mathbb{Z}}(R)/pD_{\mathbb{Z}}(R)$; moreover, using Lemma 2.9 it follows that

$$D_{\mathbb{Z}}(R)/pD_{\mathbb{Z}}(R) \cong D_{\mathbb{F}_p}(R/pR).$$

In addition, one deduces from Theorem 3.6 that $D_{\mathbb{F}_p}(R/pR)$ is a simple ring (indeed, since R is either smooth or a formal power series ring over $\mathbb{Z}$, it follows that R/pR is either smooth or a formal power series ring over the field $\mathbb{F}_p$, because the property of being smooth is stable under base change). Summing up, M is a non-zero left $D_{\mathbb{F}_p}(R/pR)$ -module, hence Theorem 2.4 implies that $(0 :_{R/pR} M) = 0$ and therefore $pR = (0 :_R M)$. Finally, since $(0 :_R H_I^i(R)) \subseteq (0 :_R M)$, one concludes that $(0 :_R H_I^i(R)) \subseteq pR$, hence part (i) holds.

The Lemmas 4.5 and 4.6 should read as:

Lemma 4.5 *Let R be a commutative Noetherian ring, let I be an ideal of R , let p be a prime number which is a nonzerodivisor on R , and fix $i \in \mathbb{N}$; moreover, assume that multiplication by p on $H_I^{i+1}(R)$ is injective. Then, one has that*

$$\text{Coker}(H_I^i(R) \xrightarrow{p} H_I^i(R)) = H_I^i(R/pR).$$

Proof The result follows directly from the exactness of

$$H_I^i(R) \xrightarrow{\cdot p} H_I^i(R) \longrightarrow H_I^i(R/pR) \longrightarrow H_I^{i+1}(R) \xrightarrow{\cdot p} H_I^{i+1}(R),$$

which is just the one induced by the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot p} R \longrightarrow R/pR \longrightarrow 0. \quad \square$$

Lemma 4.6 *Let R be either a smooth $\mathbb{Z}$ -algebra, or a formal power series ring over $\mathbb{Z}$, let I be an ideal of R , let p be a prime number, and fix $i \in \mathbb{N}$. Then, all but finitely many prime integers satisfy that $\text{Coker}(H_I^i(R) \xrightarrow{\cdot p} H_I^i(R)) = H_I^i(R/pR)$.*

Proof By Bhatt et al. (2014, Theorem 3.1 (2)), all but finitely many prime integers are nonzerodivisors on $H_I^{i+1}(R)$, and therefore the result follows immediately from Lemma 4.5. $\square$

Now, we are finally in position to show that the assumptions required in part (iii) of Theorem 4.4 are satisfied in some specific situations.

The original article has been corrected.