Beitr Algebra Geom (2018) 59:645–663
https://doi.org/10.1007/s13366-018-0388-4

ORIGINAL PAPER

Spaces of sums of powers and real rank boundaries

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Received: 12 May 2017 / Accepted: 21 February 2018 / Published online: 6 March 2018

Abstract We investigate properties of Waring decompositions of real homogeneous forms. We study the moduli of real decompositions, so-called Space of Sums of Powers, naturally included in the Variety of Sums of Powers. Explicit results are obtained for quaternary quadrics, relating the algebraic boundary of SSP to various loci in the Hilbert scheme of four points in $\mathbb{P}^3$. Further, we study the locus of general real forms whose real rank coincides with the complex rank. In case of quaternary cubics the boundary of this locus is a degree forty hypersurface $J(\sigma_3(v_3(\mathbb{P}^3)), \tau(v_3(\mathbb{P}^3)))$.

Keywords Space of sums of powers · Waring decomposition · Real rank · Variety of sums of powers

Mathematics Subject Classification 14P10 · 51N35

1 Introduction

Let $f \in K[x_1, \ldots, x_n]_d = S^d(V^*)$ be a homogeneous form of degree d in n variables over a field K. We study *Waring decompositions*:

Michałek was supported by IP Grant 0301/IP 3/2015/73 of the Polish Ministry of Science.

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$$f=\sum_{i=1}^{r}\lambda_i\left(\sum_{j=1}^{n}a_{i,j}x_i\right)^d, \quad \text{where } \lambda_i, a_{i,j}\in K.$$

The smallest possible $r=\mathrm{rk}_K(f)$ is called the *Waring rank* of f.

Waring rank and decompositions have attracted attention of many mathematicians. On the one hand, their study is motivated by applications e.g. in computer sciences. On the other hand these problems are related to beautiful mathematics: geometry (through secant varieties) (Zak 2005), representation theory (through homogeneous varieties) (Landsberg and Manivel 2004), algebra (through apolar ideals and resolutions) (Ranestad and Schreyer 2000), moduli spaces (Mukai 1992a) and many more (Landsberg 2012).

The main questions that motivated us are:

1. How to find a Waring decomposition for a general form f?
2. What is the geometry (moduli) of all decompositions?
3. How does the answer to first two questions depend on $K=\mathbb{R}$ or $K=\mathbb{C}$?
4. When a real form admits a real Waring decomposition with $r=\mathrm{rk}_{\mathbb{C}}(f)$?

To investigate the geometry of the Waring decompositions one defines the Variety of Sums of Powers (Ranestad and Schreyer 2000):

$$\mathrm{VSP}(f):=\overline{\{(\ell_1,\dots,\ell_r): f=\sum\lambda_i\ell_i^d, [\ell_i]\in\mathbb{P}(V^*)\}}.$$

Here, the closure is taken in the (smoothable component of the) Hilbert scheme of r points in $\mathbb{P}(V^*)$ and is usually considered for $K=\mathbb{C}$. Seminal works of Mukai, Ranestad, Schreyer Mukai (1992a, b, 2004); Ranestad and Schreyer (2000, 2013) and others provide descriptions of VSP's in special cases. Following Michałek et al. (2017), we investigate a semialgebraic subset of the real locus of VSP, Space of Sums of Powers, corresponding to those decompositions in which all ℓ_i are real. We note that a real point of VSP does *not* have to correspond to fully real decomposition. The geometry and topology of the real locus of VSP is another very interesting subject, studied e.g. in Kollár and Schreyer (2004).

Definition 1.1 The Space of Sums of Powers, $\mathrm{SSP}(f)$, is the semialgebraic subset of the real locus of $\mathrm{VSP}(f)$, where all linear ℓ_i forms are real. We also define its algebraic boundary $\partial_{alg}\mathrm{SSP}(f)$—the Zariski closure of difference of the Euclidean closure of $\mathrm{SSP}(f)$ minus the interior of the closure.

The first part of the paper is devoted to the study of the algebraic boundary $\partial_{alg}\mathrm{SSP}(f)$. We start by defining the anti-polar form $\Omega(f)$ for a general form of even degree in Definition 2.1. This is a generalization of the dual quadric. As we show, in several cases $\Omega(f)$ governs the nonreduced structure of apolar schemes—cf. Proposition 2.6. This allows us to provide an explicit description of $\partial_{alg}\mathrm{SSP}(f)$, when f is a quaternary quadric 2.11, extending the results obtained for ternary quadrics [Michałek et al. 2017, Section 2]. In the ternary case, due to results on Gorenstein resolutions, one could provide a description of $\partial_{alg}\mathrm{SSP}(f)$ in terms of hyperdeterminants. We did not find a straightforward generalization to quaternary case, however our methods may be applied to ternary case. They provide a simpler, more explicit (but less intrinsic) description 2.1.1. Further, we study the geometry of $\partial_{alg}\mathrm{SSP}(f)$ in detail, over $\mathbb{Q}$ and over $\mathbb{C}$, relating it to the geometry of the Hilbert scheme. In particular, we discuss how the VSP intersects various loci of the Hilbert scheme.

We note that our approach towards $\partial_{alg}\mathrm{SSP}(f)$ allows a description of intersection with subvarieties of $\mathrm{VSP}(f)$. This is useful, when the $\mathrm{VSP}(f)$ itself is a large variety. We apply it for quinary quadrics 2.1.3.

In the second part of the article we study the locus of real general forms for which the real rank equals the complex rank. In general, the complex rank is known by Alexander-Hrischowitz theorem (Alexander and Hirschowitz 1995). It follows that there exists a Zariski dense semialgebraic set $\mathcal{R}_{n,d}$ of real forms with such real rank. The *real rank boundary* $\partial_{alg}\mathcal{R}_{n,d}$ is defined as the Zariski closure of the topological boundary of $\mathcal{R}_{n,d}$.

We present an implemented, fast, deterministic algorithm, that for a general quaternary cubic f returns its unique Waring decomposition. This is an interesting example, one of the only two identifiable that is not binary (Galuppi and Mella 2016), studied both classically (Clebsch 1861) and in modern context (Oeding and Ottaviani 2013). Our algorithm can be used not only in applications, but working parametrically over the field $K(t)$ it allows to provide a description of $\partial_{alg}\mathcal{R}_{4,3}$. We obtain Proposition 3.4;

The join variety $J(\sigma_3(v_3(\mathbb{P}^3)), \tau(v_3(\mathbb{P}^3)))$ of the third secant variety of the third Veronese of $\mathbb{P}^3$ and the tangential variety is an irreducible hypersurface of degree 40 in $\mathbb{P}^{19}$. It equals $\partial_{alg}(\mathcal{R}_{4,3})$.

The geometry of $\partial_{alg}\mathcal{R}_{n,d}$ is also related to the classical work of Hilbert on cones of sums of squares and nonnegative polynomials (Hilbert 1888). This allows us to provide a description of one component of $\partial_{alg}\mathcal{R}_{4,4}$ as the dual of a variety of quartic symmetroids in Proposition 3.5.

In the Appendix 4 we present several algorithms used to:

- prove that a given variety is irreducible (or reducible) over $\mathbb{C}$,
- compute the real locus of a variety,
- describe a shape of the resolution of a generic member of a family of ideals.

2 Boundaries of spaces of sums of powers

Definition 2.1 (*The anti-polar* $\Omega(f)$) Consider a homogeneous polynomial $f \in S^{2d}(V^*)$ of degree $2d$. It induces, through the middle catalecticant, a linear map $A_f : S^d(V) \to S^d(V^*)$. Suppose that A_f is an isomorphism (which holds for generic f). The inverse map A_f^{-1} defines a polynomial $\Omega(f) \in S^{2d}(V)$ as follows. For $x \in V^*$ we define $\Omega(f)(x) :=< x^d, A_f^{-1}(x^d) >$, where $< \cdot, \cdot >$ denotes the perfect paring of $S^d(V)$ and $S^d(V^*) = S^d(V)^*$.

Remark 2.2 Explicitly for a fixed base on V, to evaluate $\Omega(f)$ on x one multiplies the inverse (or adjoint up to scalar) of the middle catalecticant from left and from right by the vector that evaluates all monomials of degree d on x.

The following proposition explains the name of $\Omega(f)$, relating it to the anti-polar quartic defined in [Michałek et al. 2017, Theorem 4.1].

Proposition 2.3 *For a homogeneous polynomial $f \in S^{2d}(V^*)$ let us denote by $C(f)$ its middle catalecticant. If $C(f)$ is invertible, then (up to scalar) we have:*

$$\Omega(f)(l) \ := \ \det\bigl(C(f+\ell^{2d})\bigr) - \det\bigl(C(f)\bigr),$$

where $l \in V^*$.

Proof As the Definition 2.1 and the one in the proposition are intrinsic, we may fix a basis and assume l is a basis element. Then $C(l^{2d})$ is simply given by a matrix with one nonzero entry on the diagonal. Hence, $\det\bigl(C(f+l^{2d})\bigr) - \det\bigl(C(f)\bigr)$ equals the complimentary minor to the nonzero entry. This exactly agrees with Definition 2.1 by Remark 2.2. □

Example 2.4 If f is a quadric of full rank, than $\Omega(f)$ is simply the dual quadric.

Remark 2.5 It is important to note that neither the inverse nor the adjoint of A_f is the middle catalecticant of $\Omega(f)$—cf. Dolgachev (2004) and [Dolgachev 2012, Remark 1.4.1], contrary to the case of quadrics. In fact, it is often the case that A_f^{-1} is not a middle catalecticant of any equation of degree $2d$—see e.g. [Michałek et al. 2016, Proposition 7.1].

The following proposition is based on results from Ranestad and Schreyer (2013).

Proposition 2.6 *Let $f \in S^{2d}(V^*)$ be such that the middle catalecticant A_f is nondegenerate. Suppose $S \subset \mathbb{P}(V)$ of length equal to the rank of A_f is apolar to f. Then S has a nonreduced structure at a point $l \in S$ if and only if $\Omega(f)(l) = 0$.*

Proof Consider a new scheme $\tilde{S}$ defined by:

$$I_{\tilde{S}} := I_S : l^{\perp},$$

where $l^{\perp}$ is the maximal ideal defining l. As A_f is nondegenerate we know that $(I_S)_d = (0)$. On the other hand, by degree count we know that there exists $g \in (I_{\tilde{S}})_d$. We may determine the g as follows:

$$g \in I_{\tilde{S}} \Leftrightarrow gl^{\perp} \subset I_S \Rightarrow g(l^{\perp})_d \subset I_S \Rightarrow (g(l^{\perp})_d)(f) = 0 \Leftrightarrow g((l^{\perp})_d(f)) = 0.$$

But $(l^{\perp})_d(f) = A_f(l_d^{\perp})$, so $g \in I_{\tilde{S}} \Rightarrow g(A_f(l_d^{\perp})) = 0$. As A_f is nondegenerate and $l_d^{\perp}$ is a hypersurface in $S^d V^*$ this determines g uniquely up to scalar and the above implication is an equivalence. We obtain that S is reduced at l if and only if $v_d(l) \notin A_f(l_d^{\perp})$ if and only if $A_f^{-1}(v_d(l)) \notin (l^{\perp})_d$, where v_d is the d-th Veronese (evaluating g as a polynomial on l is the same as evaluating g as a linear map on $v_d(l)$). Hence, S is nonreduced at l if and only if $< v_d(l), A_f^{-1}(v_d(l)) >= 0$, where $< \cdot, \cdot >$ is the usual dual pairing. □

The following definitions will be useful in the study of SSP.

Definition 2.7 ($VNSP$, $\mathfrak{F}$) For a given form $f \in S^d(V^*)$ we define the Variety of Nonreduced Sums of Powers as a subscheme with the reduced structure of VSP(f) corresponding to the locus of nonreduced schemes. In other words:

$$\mathrm{VNSP}(f) = \mathrm{VSP}(f) \setminus \{(\ell_1, \ldots, \ell_r) : f = \sum \lambda_i \ell_i^d, [\ell_i] \in \mathbb{P}(V^*)\},$$

or equivalently it parametrizes smoothable, nonsmooth apolar schemes.

As a moduli space VSP comes with a universal family $\pi : \mathrm{VSP}(f) \times \mathbb{P}(V^*) \supset \mathfrak{F} \to \mathrm{VSP}(f)$, where the fiber over a given point of VSP(f) equals the corresponding apolar scheme.

Let us describe how Proposition 2.6 may be used to find the boundary of SSP inside the VSP in special cases. Let $f \in S^{2d}(V^*)$ have a nondegenerate middle catalecticant. Further assume that the rank of the catalecticant equals the generic rank in $S^{2d}(V^*)$.

1. Assume we are given the universal family $\pi : \mathrm{VSP}(f) \times \mathbb{P}(V^*) \supset \mathfrak{F} \to \mathrm{VSP}(f)$.
2. Let $\mathfrak{B} = \mathrm{VSP}(f) \times V(\Omega(f)) \subset \mathrm{VSP}(f) \times \mathbb{P}(V^*)$.
3. The algebraic boundary of SSP(f) inside VSP(f) is contained in $\pi(\mathfrak{B} \cap \mathfrak{F})$.

We note that set-theoretically $\pi(\mathfrak{B} \cap \mathfrak{F})$ coincides with VNSP(f). The construction above follows from the fact that a real decomposition can change to a complex one only by passing through a nonreduced scheme. As a consequence we obtain the following lemma that will allow us to find defining equations of $\partial_{alg}\mathrm{SSP}(f)$.

Lemma 2.8 *Suppose that* SSP(f) $\neq \emptyset$. *If the top dimensional component of* $\pi(\mathfrak{B} \cap \mathfrak{F})$ *is irreducible, then its reduced structure coincides with* $\partial_{alg}\mathrm{SSP}(f)$.

In principal, this method could provide the description for the boundary of the SSP inside the VSP for quaternary quartics. Unfortunately, in this case the VSP, which is a 5-fold eludes an explicit description. Ranestad communicated to the authors that this is one of the most interesting outstanding problems on VSP's.

2.1 Quadrics

We now apply Proposition 2.6 to explicitly obtain the boundary of SSP for quadrics f in up to $n \leq 5$ variables, i.e. in all cases when VSP is smooth. For $n = 2, 3$ this was achieved in Michałek et al. (2017). It that case, as the codimension of the apolar ideal $f^\perp$ was at most three, one could apply the classical results of Buchsbaum-Eisenbud on resolutions of Gorenstein schemes (Buchsbaum and Eisenbud 1977) and define the boundary by an appropriate hyperdeterminant. In case $n > 3$ we could still take the resolution of $f^\perp$, however explicit results using this technique seem much harder. Instead we follow Proposition 2.6.

Lemma 2.9 *Fix two real forms* f_1, f_2, *both with nonempty* SSP*'s. Suppose* f_1 *can be obtained from* f_2 *by a* complex *change of coordinates. If the top dimensional component of* VNSP(f_1) $\subset$ VSP(f_1) *is irreducible, then* $\partial_{alg}\mathrm{SSP}(f_1)$ *is isomorphic (as a complex algebraic variety) to* $\partial_{alg}\mathrm{SSP}(f_2)$.

Proof The isomorphism between f_1 and f_2 provides an isomorphism between their VSP's. We show that this isomorphism is also an isomorphisms of the algebraic boundaries. By the assumption the boundary is nonempty in both cases, hence of codimension one. However, it has to be contained in VNSP, which is irreducible and preserved by the isomorphism. □

Remark 2.10 Note that we need the assumption that SSP's are nonempty. We have $\mathrm{SSP}(x^2 + y^2) = \mathrm{VSP}(x^2 + y^2)$, hence the algebraic boundary is empty. On the other hand, a quadric with a different signature (but of course isomorphic over $\mathbb{C}$) satisfies $\mathrm{SSP}(x^2 - y^2) \subsetneq \mathrm{VSP}(x^2 - y^2)$ and the algebraic boundary consists of two points.

We will apply Lemma 2.9 to quadrics of different signature. Our aim is to describe the algebraic boundary of the SSP.

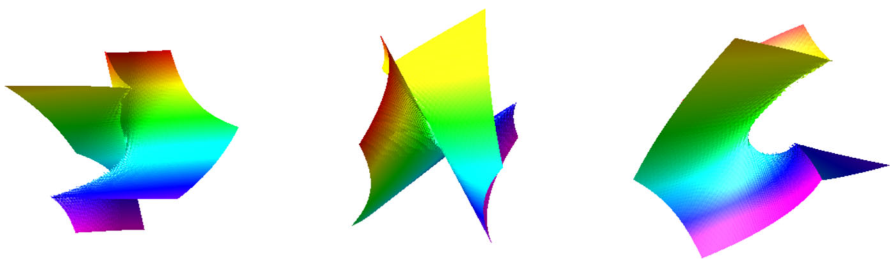
2.1.1 Ternary quadrics

The case of ternary quadrics is well-understood [Michałek et al. 2017, Section 2]. We use it as a warm-up. Fix $f = x_1x_3 + x_2^2$, which up to real isomorphism is the only indefinite quadric. The $\mathrm{VSP}(f)$ is a smooth Fano 3-fold V_5—quintic del Pezzo threefold—admitting a realization as an intersection $G(3, 5) \cap \mathbb{P}^6$. The boundary $\partial_{alg}\mathrm{SSP}(f)$ is given by a special hyperdeterminant, which turns out to be a degree 20 polynomial in 6 variables with 13956 terms.

We show how this polynomial simplifies, if we work in a local affine patch of $\mathrm{VSP}(f)$ described in Ranestad and Schreyer (2013). Indeed, computing $\pi(\mathfrak{B} \cap \mathfrak{F})$ in Appendix 4.1 we obtain a quartic surface:

$$27a^2 - 32b^3 + 36abc - 4b^2c^2 + 4ac^3.$$

Its singular locus is a smooth curve—a complete intersection of a quadric and cubic surface. It corresponds to the locus where the apolar scheme is local, i.e. all three points coming together. As the quartic is irreducible it follows that it coincides with $\partial_{alg}\mathrm{SSP}(f)$.



Three pictures of the affine patch of $\partial_{alg}\mathrm{SSP}(f = x_1x_3 + x_2^2)$. The three-dimensional ambient affine space represents the quintic Fano threefold—moduli of schemes apolar to f of length three. The surface corresponds to those schemes that are supported at two points: one smooth, one with nonreduced structure

Spec $\mathbb{C}[x]/(x^2)$. The curve in the singular locus represents apolar schemes isomorphic to Spec $\mathbb{C}[x]/(x^3)$. The surface divides the threefold into two regions: one is the $\mathrm{SSP}(f)$ corresponding to fully real decomposition, the other corresponds to decompositions where one linear form is real and the other two are conjugate.

2.1.2 Quaternary quadrics

Let $f = x_1x_4 + x_2^2 + x_3^2$. The following theorem computes the boundary of the SSP relating its geometry to the Hilbert scheme and the geometry of the apolar schemes. Ranestad and Schreyer Ranestad and Schreyer (2000) provide an explicit local description of the variety $\mathrm{VSP}(f)$. We will be working locally on such affine patches.

Theorem 2.11

1. *The variety* $\mathfrak{Y} = \partial_{alg}\mathrm{SSP}(f) \subset \mathrm{VSP}(f)$ *is irreducible over* $\mathbb{C}$, *of dimension* 5 *and equals* $\pi(\mathfrak{B} \cap \mathfrak{F})$.
2. *The singular locus of* $\mathfrak{Y}$ *has two four dimensional components over* $\mathbb{Q}$: $\mathfrak{Y}_{3,1}$ *and* $\mathfrak{Y}_{2,2}$. *The general point of* $\mathfrak{Y}_{3,1}$ *(resp.* $\mathfrak{Y}_{2,2}$*) corresponds to a nonreduced scheme with two support points; one of which has multiplicity* 3 *(resp.* 2*), the other has multiplicity* 1 *(resp.* 2*).*
3. *Over* $\mathbb{C}$, $\mathfrak{Y}_{2,2}$ *however has two irreducible components;* $\mathfrak{Y}'_{2,2}$ *and* $\mathfrak{Y}''_{2,2}$ *intersecting along surface (identified later with* $\mathfrak{Y}_{4,sing}$*).*
4. *The two components* $\mathfrak{Y}_{3,1}$, $\mathfrak{Y}_{2,2}$ *intersect along a three dimensional threefold* $\mathfrak{Y}_4$, *irreducible over* $\mathbb{Q}$. *It is the singular locus of* $\mathfrak{Y}_{3,1}$.
5. *Over* $\mathbb{C}$, $\mathfrak{Y}_4$ *has two components, corresponding to intersections of* $\mathfrak{Y}'_{2,2}$ *and* $\mathfrak{Y}''_{2,2}$ *with* $\mathfrak{Y}_{3,1}$.
6. *The general point of* $\mathfrak{Y}_4$ *corresponds to a nonreduced, local scheme of length four, isomorphic to* Spec$\mathbb{C}[x]/(x^4)$.
7. *The singular locus* $\mathfrak{Y}_{4,sing}$ *of* $\mathfrak{Y}_4$ *coincides with the singular locus of* $\mathfrak{Y}_{2,2}$. *It is a two dimensional smooth surface. The general point of* $\mathfrak{Y}_{4,sing}$ *corresponds to a nonreduced, local scheme isomorphic to*

$$\mathrm{Spec}\,\mathbb{C}[x, y]/(x^2 - y^2, xy)$$

.

8. *The locus of real points of* $\mathfrak{Y}_4$ *and* $\mathfrak{Y}_{4,sing}$ *coincide.*

By Lemma 2.9 the same result holds for all indefinite, full rank quadrics.

Proof By [Ranestad and Schreyer 2013, Theorem 1.1] $\mathrm{VSP}(f)$ is a smooth, six dimensional variety. Ranestad and Schreyer provide an explicit local description of this variety and the universal family $\mathfrak{F}$ in a Macaulay 2 package VarietyOfPolarSimplices.m2 (http://www.math.uni-sb.de/ag/schreyer/home/computeralgebra.htm). We extend their defining equations by $\Omega(f)$, obtaining $\mathfrak{B} \cap \mathfrak{F} \subset \mathrm{VSP}(f) \times \mathbb{P}^3$. Eliminating the variables corresponding to $\mathbb{P}^3$ automatically is not possible (due to a large ambient dimension of the VSP). However, one may find automorphisms of the (local) description of the VSP that reduce the number of variables. Afterwards we perform elimination, obtaining explicitly the defining equation of $\pi(\mathfrak{B} \cap \mathfrak{F})$ (defined over $\mathbb{Q}$)-see Appendix 4.2. We check that it defines a prime ideal, over $\mathbb{C}$, as follows—details of implementation are presented in Appendix 4.2.1. We fix four linear forms defined over $\mathbb{Q}$ and intersect them with the $\pi(\mathfrak{B} \cap \mathfrak{F})$ obtaining a curve C. It is enough to show that C is irreducible over $\mathbb{C}$, as $\pi(\mathfrak{B} \cap \mathfrak{F})$ was a hypersurface—in particular equidimensional. We project, not changing the degree, until C becomes a plane curve, that is defined by g_C. We prove it is irreducible as follows. We consider all possible factorizations of g_C into a product of degrees $4+4$, $3+5$, $2+6$, $1+7$ with coefficients given by variables. Comparing all coefficients we obtain ideals, that equal the whole ring. By the previous discussion this proves the statement 1.

The computation of the singular locus of $\mathfrak{Y}$ and its decomposition over $\mathbb{Q}$ is now straightforward. The two components $\mathfrak{Y}_{3,1}$ and $\mathfrak{Y}_{2,2}$ are distinguished by the dimension of their singular locus, which is respectively 3 and 2. There are several ways to prove that $\mathfrak{Y}_{3,1}$ corresponds to schemes of type (3, 1) and $\mathfrak{Y}_{2,2}$ of type (2, 2). One can restrict the family $\mathfrak{F}$ to $\mathfrak{Y}_{3,1}$ and intersect it with $\Omega(f)$. By Proposition 2.6 in case of schemes of type (3, 1) these yields a family of local (i.e. supported at one point) schemes (as only one of the points was nonreduced), while the schemes of type (2, 2) provide two distinct support points. Statements 2 and 4 follow.

To prove the statement 3 we project $\mathfrak{Y}_{2,2}$ obtaining a hypersurface H of degree 4. If $\mathfrak{Y}_{2,2}$ were irreducible, then H would have to be irreducible. However, the equation defining H decomposes as a product of two quadrics.

It may seem not clear why $\mathfrak{Y}_4$—corresponding to all four points coming together—allows further degeneration. The reason is the geometry of the punctual Hilbert scheme of schemes of length four. The smoothable component is irreducible and consists of alignable schemes with a general point corresponding to the aligned scheme $\operatorname{Spec} \mathbb{C}[x]/(x^4)$. However, not all schemes are aligned. The scheme $\operatorname{Spec} \mathbb{C}[x, y]/(x^2 - y^2, xy)$ is local, Gorenstein, but has a two dimensional tangent space. A very explicit degeneration of four points to this scheme is presented e.g. in [Landsberg and Michałek 2017, p. 11]. The main difference between $\mathbb{C}[x]/(x^4)$ and $\mathbb{C}[x, y]/(x^2 - y^2, xy)$ is that in the first case the four points degenerate on a line, while in the second they come from distinct, linearly independent directions.

More explicitly, for a scheme corresponding to a point of $\mathfrak{Y}_4$ one can find an explicit projection to a line that preserves its degree—a feature possible only in case of $\operatorname{Spec} \mathbb{C}[x]/(x^4)$. Further, one can find explicitly points of $\mathfrak{Y}_{4,sing}$ and directly prove that the associated schemes are isomorphic to $\operatorname{Spec} \mathbb{C}[x, y]/(x^2 - y^2, xy)$.

The last statement follows by computation presented in Appendix 4.2.2. Precisely, we show that one equation in the ideal of $\mathfrak{Y}_4$ is of the form $f_1^2 + f_2^2$ and $I(\mathfrak{Y}_4) + (f_1, f_2) = I(\mathfrak{Y}_{4,sing})$. □

Remark 2.12 In the given embedding, $\mathfrak{Y}_{2,2}$ is of degree 4. Thus, $\mathfrak{Y}'_{2,2}$ and $\mathfrak{Y}''_{2,2}$ are of degree 2 each. As they are also of codimension 2, each of them should be defined by a quadric in a hyperplane section. Using arithmetics in a finite field we have checked that they are smooth.

Remark 2.13 As the locus in the Hilbert scheme corresponding to schemes isomorphic to $\mathbb{C}[x]/(x^4)$ is irreducible, it is surprising that $\mathfrak{Y}_4$ is reducible over $\mathbb{C}$. It would be interesting to know if it could happen for generic form f in $n + 1$ variables of rank r that:

- the codimension one subvariety of the smoothable component of the Hilbert scheme of r points in $\mathbb{P}^n$ corresponding to nonreduced, Gorenstein schemes is irreducible,
- the boundary $\partial_{alg}\mathrm{SSP}(f)$ is reducible and positive dimensional.

There is always a divisor in the smoothable component of the Hilbert scheme, which is corresponding to $r - 2$ reduced points and a multiplicity 2 point. In some cases, however, there are other *divisors* corresponding to nonreduced Gorenstein schemes. For more information we refer the reader to [Iarrobino 1984, Theorem 1] and further consequences of this fact are discussed in [Buczyński et al. 2015, Example A.17], [Landsberg and Michałek (2017), p. 7]. However, even in such a case we do not know if $\partial_{alg}\mathrm{SSP}(f)$ could be reducible (although we expect it).

Remark 2.14 All apolar schemes that we obtain must be Gorenstein by [Buczyńska and Buczyński 2014, Lemma 2.3]. Indeed, otherwise there would exist a Gorenstein scheme of smaller length that would be apolar to the form. In this range (e.g. in ambient dimension at most three or if length is at most ten) all Gorenstein schemes are smoothable. Hence, the form would have smaller border rank, i.e. would not be generic.

2.1.3 Quinary quadrics

In this case the VSP is a ten dimensional smooth variety. As before we may work locally in an affine patch described by Ranestad and Schreyer. However, in this case explicit computation of $\pi(\mathfrak{B}\cap\mathfrak{F})$ is much harder, as the elimination of variables is both time and memory consuming. Instead we apply Lemma 2.8. We proceed as follows:

1. Eliminate by hand possibly many variables, using explicit isomorphisms of the ambient affine space of the affine patch of $\mathfrak{B} \cap \mathfrak{F}$.
2. Consider the compactification $\mathfrak{C}$ of the affine patch of $\mathfrak{B} \cap \mathfrak{F}$ in a projective space.
3. Fix a $\mathbb{P}^2$ in a projective space that is a compactification of the affine patch of VSP.
4. Restrict the family $\mathfrak{C}$ to the given $\mathbb{P}^2$ and project obtaining $\mathfrak{C}'$.
5. Check that $\mathfrak{C}'$ is a reduced, irreducible curve of degree 10.

As $\mathfrak{C}'$ is reduced, irreducible and of degree 10, it follows that the maximal dimensional component of $\pi(\mathfrak{B} \cap \mathfrak{F})$ must be irreducible and defined by an unique polynomial of degree 10. A simplified, explicit computation is presented in Appendix 4.3. The code uses finite fields to make the computations fast (performing formal computation takes several hours, but is possible with the same code after changing the field). By Lemma 2.8 we obtain the following proposition.

Proposition 2.15 *In the affine space* $\mathbb{A}^{10} \subset \mathrm{VSP}(f = x_1x_5 + x_2^2 + x_3^2 + x_4^2)$ *the algebraic boundary* $\partial_{alg}\mathrm{SSP}(f)$ *is an irreducible hypersurface of degree* 10.

We further note that the curve $\mathfrak{C}'$ representing $\partial_{alg}\mathrm{SSP}(f)$ has 30 singular points.

3 The real rank boundary

3.1 Quaternary cubics

A general quaternary cubic $f \in \mathbb{R}[x_1, x_2, x_3, x_4]$ has complex rank $R(4, 3) = 5$ and the decomposition is uniquely given by

$$f = \ell_1^3 + \ell_2^3 + \ell_3^3 + \ell_4^3 + \ell_5^3 \tag{1}$$

which was claimed by Sylvester in 1851, and proved by Clebsch in 1861 Clebsch (1861). It is known as Sylvester Pentahedral Theorem. Similarly to [Michałek et al. (2017), Section 5], we propose the following algorithm to compute the five linear forms ℓ_i.

Algorithm 3.1 *Input: A general quaternary cubic f. Output: The decomposition (1).*

1. *Compute the apolar ideal $f^\perp$. It is generated by six quadrics $g_1, g_2, g_3, g_4, g_5, g_6$.*
2. *Compute the syzygies of $f^\perp$. Find the five linear syzygies $l_{ij}, i = 1, \dots, 5, j = 1, \dots, 6$ on the quadrics satisfying*

$$\sum_{j=1}^{6} l_{ij} g_j = 0 \quad \text{for} \quad \text{all} \quad i = 1, \dots, 5$$

3. *Compute a vector $(c_1, c_2, c_3, c_4, c_5, c_6) \in \mathbb{R}^6 \setminus \{0\}$ that satisfies*

$$c_1 l_{i1} + c_2 l_{i2} + c_3 l_{i3} + c_4 l_{i4} + c_5 l_{i5} + c_6 l_{i6} = 0$$

   for all $i = 1 \dots 5$.
4. *Let J be the ideal generated by the quadrics $c_6 g_1 - c_1 g_6$, $c_6 g_2 - c_2 g_6, \dots$, $c_6 g_5 - c_5 g_6$. Compute the variety $V(J)$ in $\mathbb{P}^3$. It consists precisely of the points dual to $\ell_1, \ell_2, \dots, \ell_5$.*

To prove the correctness of this algorithm, we need the following lemma.

Lemma 3.2 *For a general quaternary cubic $f \in \mathbb{R}[x_1, x_2, x_3, x_4]$, the minimal free resolution of its apolar ideal is given by*

$$0 \longrightarrow S \longrightarrow S^6 \xrightarrow{M} S^{10} \xrightarrow{N} S^6 \longrightarrow S \longrightarrow 0. \tag{2}$$

Further, the map M can be represented by 6×10 matrix with two blocks

$$M = [\ L \ \mid \ Q\ ] \tag{3}$$

where L is a 6×5 matrix with linear forms and Q is a 6×5 matrix with quadric forms. Moreover, there exists a row operation that makes the last row of L equal to 0. It distinguishes a unique five dimensional subspace of quadrics in $f^\perp$ with five linear syzygies.

Proof Since the apolar ideal of a quaternary cubic is Gorenstein, codimension four with six quadric generators, as in Reid (2015), the resolution has the form (2). So we only need to show that the matrix representation M has the form (3).

To prove this, consider a family of cubics in $A[x_1, x_2, x_3, x_4]$ where $A = \mathbb{R}[a_0, \ldots, a_3]$;

$$\begin{aligned} f = {} & a_0x_1^3 + a_1x_1^2x_2 + a_2x_1^2x_3 + a_3x_1^2x_4 + 11x_1x_2^2 \\ & -12x_1x_2x_3 + 7x_1x_2x_4 + 32x_1x_3^2 - 28x_1x_3x_4 + 11x_1x_4^2 \\ & +8x_2^3 - 13x_2^2x_3 + 34x_2^2x_4 + 19x_2x_3^2 - 38x_2x_3x_4 \\ & +16x_2x_4^2 + 7x_3^3 - 41x_3^2x_4 + 7x_3x_4^2 + 13x_4^3 \end{aligned}$$

where 16 of its coefficients are fixed. After applying $GL_{\mathbb{R}}(4)$-action on this family, it is dominant on the space of cubics, which we prove using an easy computation of a Jacobian. Hence, it is enough to prove the lemma holds for this family.

Using the code in Appendix 4.4, we can show that the apolar ideal $f^{\perp}$ has six quadric generators $g_1, \ldots, g_6$ in S. Also, there are exactly five independent linear syzygies $l_{ij}, i = 1 \ldots 5$, and a unique nonzero vector $(c_1, \ldots, c_6) \in A^6$, satisfying

$$\sum_{j=1}^{6} c_j l_{ij} = 0, \quad i = 1, \ldots, 5.$$

Hence, the first syzygy matrix M has five linear columns and five quadric columns, i.e.

$$M = [\quad L \quad | \quad Q \quad]$$

where L is a 6×5 matrix with linear forms and Q is a 6×5 matrix with quadric forms. Also, by multiplying M on left by the invertible matrix

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \tag{4}$$

we make the last row of L equal to 0. □

Proposition 3.3 *Algorithm 3.1 computes the unique decomposition of general cubic f.*

Proof By Lemma 3.2, steps 1 through 3 of the Algorithm 3.1 are well defined. For the step 4, consider the inverse of matrix U defined in (4):

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -c_1/c_6 & -c_2/c_6 & -c_3/c_6 & -c_4/c_6 & -c_5/c_6 & 1/c_6 \end{bmatrix}.$$

By applying the column operation given by the right multiplication of this matrix to the generators $[\, g_1 \dots g_6 \,]$, we get the new generators

$$[\, c_6 g_1 - c_1 g_6 \; c_6 g_2 - c_2 g_6 \dots c_6 g_5 - c_5 g_6, g_6 \,].$$

The first five of them have five linear syzygies given by the five columns of UL. Moreover, choosing right basis as in Reid (2015) and applying Lemma 3.2, the second syzygy matrix N also has the form

$$N^t = [\quad Q' \quad | \quad L' \quad]$$

and the first row of L' is 0.

From this, the ideal J generated by the first five generators $c_6 g_1 - c_1 g_6$, $c_6 g_2 - c_2 g_6$,..., $c_6 g_5 - c_5 g_6$ has the minimal free resolution of the form

$$0 \longrightarrow S \longrightarrow S^5 \xrightarrow{P} S^5 \longrightarrow S \longrightarrow 0 \tag{5}$$

where P is given by the 5×5 upper submatrix of L as we seen above.

Since the resolution has the form (5), the ideal J has dimension 0 and degree 5, and it is also contained in the apolar ideal $f^\perp$. Further, by Lemma 3.2, the five chosen quadrics span the unique subspace contained in $f^\perp$ with this resolution.

Since there is the unique decomposition (1) for the given general cubic f by Clebsch (1861), the linear forms in the decomposition define an ideal $K \subset f^\perp$ of five points in $\mathbb{P}^3$. Since K is Gorenstein of codimension 3, by the Buchsbaum-Eisenbud structure theorem, it has the free resolution of the form

$$0 \longrightarrow S \longrightarrow S^5 \xrightarrow{Q} S^5 \longrightarrow S \longrightarrow 0. \tag{6}$$

It means that the ideal J we obtained by the Algorithm 3.1 is actually same as the ideal K, which defines the five linear forms in (1), as we wanted. □

Using this algorithm, we can easily check whether the given quaternary cubic has real rank 5 or not. Namely, one computes the unique decomposition (1) and checks whether it is real. The real rank boundary can be obtained as in the following proposition. This proposition confirms [Michałek et al. 2017, Conjecture 5.5] for quaternary cubics.

Proposition 3.4 *The real rank boundary $\partial_{alg}(\mathcal{R}_{4,3})$ equals the join $J(\sigma_3(v_3(\mathbb{P}^3)),$ $\tau(v_3(\mathbb{P}^3)))$ of third secant of the third Veronese embedding of $\mathbb{P}^3$ and its tangential* *variety. It is the irreducible hypersurface of degree* 40 *in the* $\mathbb{P}(S^3(\mathbb{C}^4)))$ *with parametric representation*

$$g = \ell_1^3 + \ell_2^3 + \ell_3^3 + \ell_4^2\ell_5, \qquad where \quad \ell_1, \dots, \ell_5 \in \mathbb{R}[x, y, z, w]_1. \tag{7}$$

Proof The parametrization defines a unirational variety Y in $\mathbb{P}^{19}$. The Jacobian of this parametrization is found to have corank 1. This means that Y has codimension 1 in $\mathbb{P}^{19}$. Hence Y is an irreducible hypersurface, defined by a unique (up to sign) irreducible homogeneous polynomial Φ in 20 unknowns with rational coefficients.

Let g be a real cubic with the form (7) that is a general point in Y. As ϵ goes to 0, the real cubics $(\ell_4 + \epsilon\ell_5)^3 - \ell_4^3$ and $(i\ell_4 + \epsilon\ell_5)^3 + (-i\ell_4 + \epsilon\ell_5)^3$ converge to the cubic $\ell_4^2\ell_5$ in $\mathbb{P}^{19}$. It means that any small neighborhood of g in $\mathbb{P}^{19}$ contains cubics of real rank 5 and cubics of real rank > 5. This implies that Y lies in the real rank boundary $\partial_{\text{alg}}\mathcal{R}_{4,3}$. Since Y is irreducible and codimension 1, it follows that $\partial_{\text{alg}}\mathcal{R}_{4,3}$ exists and has Y as an irreducible component.

Using the algorithm 3.1, we can exactly compute the degree of the hypersurface $\partial_{\text{alg}}\mathcal{R}_{4,3}$ which is 40. This is done as follows. First, fix the field $K = \mathbb{Q}(t)$ with a new variable t. We fix two cubic f_1 and f_2 in $\mathbb{Q}[x_1, x_2, x_3, x_4]_3$, and run the algorithm 3.1 for $f_1 + tf_2 \in K[x_1, x_2, x_3, x_4]$. Step 4 returns an ideal J in $K[x_1, x_2, x_3, x_4]$ that defines 5 points in $\mathbb{P}^3$ over the algebraic closure of K. By eliminating each of two variables of $\{x_1, x_2, x_3, x_4\}$, we obtain six binary forms of degree 5 such that their coefficients are degree 15 polynomials in t. The discriminant of each binary form is a polynomial in $\mathbb{Q}[t]$ of degree $120 = 8 * 15$. The greatest common divisor of these discriminants is a polynomial $\Psi(t)$ of degree 40. We can check that $\Psi(t)$ is irreducible in $\mathbb{Q}[t]$.

By definition, Φ is a homogeneous polynomial with integer coefficients, irreducible over $\mathbb{Q}$, in the 20 coefficients of a general cubic f. Its specialization $\Phi(f_1 + tf_2)$ is a non-constant polynomial in $\mathbb{Q}[t]$, of degree $\deg(X)$ in t. That polynomial divides $\Psi(t)$ because Y lies in the real rank boundary. Since the latter is also irreducible over $\mathbb{Q}$, we conclude that $\Phi(f_1 + tf_2) = \gamma \cdot \Psi(t)$, where γ is a nonzero rational number. Hence Φ has degree 40. We conclude that $\deg(Y) = 40$, and therefore $Y = \partial_{\text{alg}}(\mathcal{R}_{4,3})$. □

3.2 Quaternary quartics

Proposition 3.5 *One component of the boundary* $\partial_{alg}\mathcal{R}_{4,4}$ *equals the dual of a* 24 *dimensional variety of quartic symmetroids in* $\mathbb{P}^3$*, that is, the surfaces whose defining polynomial is the determinant of a symmetric* 4×4*-matrix of linear forms.*

Proof By Hilbert's classification we know that for quaternary quartics the cone of positive forms $P_{4,4}$ *strictly* contains the cone of sums of squares $\Sigma_{4,4}$. Hence, the cone $Q_{4,4}$ of sums of (arbitrary many—c.f. Remark 3.7) 4-th powers of linear forms, which is the convex hull of the Veronese and equals the dual $P^*_{4,4}$, is strictly contained in the cone $\Sigma^*_{4,4}$ of forms with psd catalecticant.

The boundary of $Q_{4,4}$ has two components. One (that is not interesting for our purposes) is the boundary of $\Sigma^*_{4,4}$ given by the determinant of the catalecticant. The other one, which we denote by B, is the dual of the Zariski closure of the extremal rays of $P_{4,4}\backslash\Sigma_{4,4}$—the variety of quartic symmetroids in $\mathbb{P}^3$ by [Blekherman et al. (2012), Theorem 3].

The forms in $\Sigma^*_{4,4}\backslash Q_{4,4}$ are obviously of real rank greater than 10, as they are not sums of powers by definition, and any other presentation would contradict the signature of the catalecticant. Further a generic point of B has real rank at most 10 by [Blekherman et al. (2012), Proposition 7]. Changing the linear forms of the Waring decomposition of quartics in B we obtain a Zariski dense set of forms of real rank at most 10, intersecting B in a relatively open set, which proves the proposition. □

The following lemma is well-known to experts. We present a sketch of a proof based on [Blekherman et al. 2013, Lemma 4.18].

Lemma 3.6 *For all positive integer k, the following set*

$$C_{k,2d} := \left\{ \sum_{i=1}^{k} \lambda_i \ell_i^{2d} \in S^{2d}(V^*) \mid \lambda_i \in \mathbb{R}_{\geq 0},\ \ell_i \in V^* \right\}$$

is a closed cone. In other words, the cone of forms whose positive rank is less than or equal to k is closed.

Proof Let $\Delta = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k_{\geq 0} \mid \sum_{i=1}^k \lambda_i = 1\}$ be a simplex and let O be the unit sphere in V^*. Consider the image C of the map:

$$\Delta \times O^k \ni (\lambda_1, \dots, \lambda_k, \ell_1, \dots, \ell_k) \to \sum_{i=1}^{k} \lambda_i \ell_i^{2d} \in S^{2d}(V^*).$$

Clearly, C is compact and $0 \notin C$. Thus, the cone over C is closed and it coincides with $C_{k,2d}$. □

Remark 3.7 There exists a nonempty open set inside $Q_{4,4}$ whose elements have real rank strictly greater than 10. Indeed, the quartic $(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$ has real rank 11 by [Reznick (1992), Proposition 9.26] and hence belongs to $Q_{4,4}\backslash C_{10,4}$. The latter is open by Lemma 3.6. Further, every element of this set has real rank strictly greater than 10. Indeed, since every element of $Q_{4,4}$ has definite 10×10 catalecticant matrix, it cannot have decompositions whose coefficients have distinct signs.

One can easily prove that quaternary quartics f of signature (9, 1) have rank at least 11 if $\Omega(f)$ has no real points. This and Remark 3.7 motivate the following conjecture.

Conjecture 3.8 *The real rank boundary $\partial_{alg}\mathcal{R}_{4,4}$ is reducible. The discriminant of $\Omega(f)$ is one of its components. Further, at least one more component comes from (the components of) the algebraic boundary of $C_{10,4}$.*

Acknowledgements Open access funding provided by Max Planck Society. We would like to thank Joachim Jelisiejew for interesting discussions about the geometry of the Hilbert scheme. We thank Grigoriy Blekherman and Rainer Sinn for pointing us towards the results on positive real rank. Michałek was supported by the Foundation for Polish Science (FNP) and is a member of AGATES group. Part of the research was realized during research visits at FU Berlin and RIMS in Kyoto—we express our gratitude to Klaus Altmann, Takayuki Hibi and Hiraku Nakajima for their hospitality.

Appendix

Ternary quadrics

```
loadPackage("VarietyOfPolarSimplices")
p=0, n=3, (R,A,I)=unfoldingEquations(p,n)
J=flatteningRelations(R,A,I), S=ring I, X=(entries
 vars(R))_0
q=2*X_0*X_(n-1)+sum for i from 1 to n-2 list X_i^2
F=sub(I,S)+sub(J,S), B=sub(q,S), FintB=sub(F+B,sub(X_0,S)
 =>1)
P1=eliminate({sub(X_1,S),sub(X_2,S)}, FintB)
for i from 0 to 2 do
P2=sub(P1,sub(temp=leadMonomial(P1_i),S)=>sub(temp-
 1/leadCoefficient(P1_i)*P1_i,S));
Boundary=(gens P2)_3
```

Algebraic boundary $\partial_{alg}\mathrm{SSP}(xt + y^2 + z^2)$

```
loadPackage("VarietyOfPolarSimplices")
p=0, n=4, (R,A,I)=unfoldingEquations(p,n)
J=flatteningRelations(R,A,I), S=ring I, X=(entries
 vars(R))_0
q=2*X_0*X_(n-1)+sum for i from 1 to n-2 list X_i^2
F=sub(I,S)+sub(J,S), B=sub(q,S), FintB=sub(F+B,sub(X_0,S)
 =>1)

K1=sub(FintB,sub(X_(n-1),S)=>sub(-1/2*(sum for i from
 1 to n-2 list X_i^2),S))
KK1=K1
for i from 6 to 17 do
K1=sub(K1,sub(temp=leadMonomial(K1_i),S)=>sub(temp-1/
 leadCoefficient(K1_i)*K1_i,S))

K2=ideal(for i from 0 to 5 list K1_i)
Boundary=eliminate(K2,{sub(X_1,S),sub(X_2,S)})
```

$\partial_{alg}\mathrm{SSP}(xt + y^2 + z^2)$ *is prime.*

```
va=(entries sub(vars(A),S))_0
```

```
aa=for i from 6 to 17 list leadMonomial(KK1_i)
AA=QQ[toList(set(va)-set(aa)),t]
nB=homogenize(sub(Boundary,AA),t)
cB=ideal(random(1,AA), random(1,AA),random
 (1,AA),random(1,AA))+nB
eB=eliminate(cB,{(vars AA)_0_0,(vars AA)_1_0,(vars
 AA)_2_0,(vars AA)_3_0})
MM=QQ[a,b,t,c_0..c_40]
I=ideal(a,b,t)
gC=sub((gens gb eB)_0_0,matrix{{0,0,0,0,a,b,t}})
for i from 1 to 4 do
(
vars1=sub((i+2)*(i+1)/2,ZZ);
vars2=sub((8-i+2)*(8-i+1)/2,ZZ);
Am1=matrix{{c_0..c_(vars1-1)}};
Am2=matrix{{c_(vars1)..c_(vars1+vars2-1)}};
ff1=Am1*(transpose gens I^(i));
ff2=Am2*(transpose gens I^(8-i));
zer=sub(ff1*ff2-gC,{c_0=>1});
JJ=ideal(diff(gens I^8 , zer));
print degree JJ )
```

Real points of $\mathfrak{Y}_4$ *and* $\mathfrak{Y}_{4,sing}$

```
sl=radical ideal singularLocus Boundary
y31=(primaryDecomposition(sl))_0, y22
 =(primaryDecomposition(sl))_1
y4=radical ideal singularLocus y31, y4sing=radical ideal
 singularLocus y22
f1=va_12-3*va_16,f2=va_17-3*va_14
y4_0==f1^2+f2^2
y4+ideal(f1,f2)==y4sing
```

Quinary quadric

```
loadPackage("VarietyOfPolarSimplices")
p=1009, n=5, (R,A,I)=unfoldingEquations(p,n)
J=flatteningRelations(R,A,I), S=ring I
X=(entries vars(R))_0
q=2*X_0*X_(n-1)+sum for i from 1 to n-2 list X_i^2
F=sub(I,S)+sub(J,S), B=sub(q,S)
FintB=sub(F+B,sub(X_0,S)=>1);
K2=sub(FintB,sub(X_(n-1),S)=>sub(-1/2*(sum for i from 1
 to n-2 list X_i^2),S));
K3=K2;
```

```
for i from 10 to 39 do
K3=sub(K3,sub(temp=leadMonomial(K3_i),S)=>sub(temp-1/
 leadCoefficient(K3_i)*K3_i,S))
K4=ideal(for i from 0 to 9 list K3_i);
va=(entries sub(vars(A),S))_0, aa=for i from 10 to 39
 list leadMonomial(K2_i)
SS=ZZ/1009[toList(set(va)-set(aa)),X,t]
K4=sub(K4,SS);
K4=ideal(homogenize(gens K4,t));
for i from 0 to 7 do
K4=sub(K4,{SS_i=>random(ZZ/1009)*SS_8+random(ZZ/1009)
 *SS_9+random(ZZ/1009)*t});
L=eliminate(K4,{sub(X_1,SS),sub(X_2,SS),sub(X_3,SS)})
L=saturate(L,t)
degree L
```

Quaternary cubic

```
A=QQ[a_0..a_3]
B=A[b_(0,0)..b_(5,3)]
R=B[x_1,x_2,x_3,x_4]
bs:=d->(entries basis(d,R))_0;
f=a_0*x_1^3+a_1*x_1^2*x_2+a_2*x_1^2*x_3+a_3*x_1^2*
 x_4+11*x_1*x_2^2-12*x_1*x_2*x_3+
7*x_1*x_2*x_4+32*x_1*x_3^2-28*x_1*x_3*x_4+11*x_1*x_4^2
 +8*x_2^3-13*x_2^2*x_3+
34*x_2^2*x_4+19*x_2*x_3^2-38*x_2*x_3*x_4+16*x_2*x_4^2
 +7*x_3^3-41*x_3^2*x_4+
7*x_3*x_4^2+13*x_4^3;
cat=sub(diff(transpose basis(1,R)*basis(2,R),f),A)
kn=kernel cat
gsI=for i from 0 to 5 list sum for j from 0 to #(bs(2))-1
 list kn_i_j*(bs(2))_j
I=ideal gsI
candSyz=for i from 0 to 5 list sum for j from 0 to 3
 list b_(i,j)*R_j
allsum=sum for i from 0 to 5 list candSyz_i*gsI_i
bb=flatten for i from 0 to 5 list for j from 0 to 3
 list b_(i,j)
MM=transpose matrix for j from 0 to #bb-1 list
for i from 0 to #(bs(3))-1 list coefficient(bb_j,
 coefficient((bs(3))_i,allsum))
rank MM
for i from 1 to 4 list rank submatrix(MM,0..19,i..19+i)
rank submatrix(MM,0..19,0..0|2..20)
kn1to4=for i from 1 to 4 list(
```

```
    temp=kernel submatrix(MM,0..19,i..19+i);
    transpose matrix{for j from 0 to 23 list
    if(j<i) then 0 else if(i<=j and j<20+i) then
     temp_0_(j-i) else 0}
    );
knMatrix=kn1to4_0;
for i from 1 to 3 do knMatrix=knMatrix|kn1to4_i;
temp=kernel (sm=submatrix(MM,0..19,0..0|2..20));
kn5=transpose matrix{for j from 0 to 23 list
if(j==0) then temp_0_0 else if(j==1) then 0
else if(j>1 and j<=20) then temp_0_(j-1) else 0};
knMatrix=knMatrix|kn5;
rank knMatrix
linSyz=for k from 0 to 4 list for i from 0 to 5 list
sum for j from 0 to 3 list knMatrix_k_(i*4+j)*R_j;
L=transpose matrix{linSyz_0};
for i from 1 to 4 do L=L|transpose matrix{linSyz_i};

C=A[c_0..c_5]
S=C[x_1,..,x_4]
LC=sub(L,S);
linSum=for j from 0 to 4 list sum for i from 0 to 5
 list c_i*LC_j_i;
knlist=for k from 0 to #linSum-1 list kernel transpose
 matrix for j from 0 to 5 list
for i from 0 to 3 list coefficient(c_j,coefficient(S_i,
 linSum_k));
clist=intersect knlist;
rank clist
J=ideal(for i from 0 to 4 list gsI_i*clist_0_(i+1)
 -gsI_(i+1)*clist_0_i);
```

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