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ORIGINAL PAPER

On Anderson-Badawi conjectures

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Abstract Let R be a commutative ring with identity. Badawi (Bull Aust Math Soc 75(3), 417–429, 2007) introduced a generalization of prime ideals called 2-absorbing ideals, and this idea is further generalized in a paper by Anderson and Badawi (Commun Algebra 39(5), 1646–1672, 2011) to a concept called n-absorbing ideals. A proper ideal I of R is said to be an n-absorbing ideal if whenever $x_1 \dots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$ then there are n of the x_i's whose product is in I. It was conjectured by Anderson and Badawi (Commun Algebra 39(5), 1646–1672, 2011) that if I is an n-absorbing ideal of R then I is strongly n-absorbing (**Conjecture 1**) and $Rad(I)^n \subseteq I$ (**Conjecture 2**). In Cahen et al. (in: Fontana et al., Commutative rings. Integer-valued polynomials, and polynomial function, Springer, New York, 2014, Problem 30c), it was conjectured also that $I[X]$ is an n-absorbing ideal of the polynomial ring $R[X]$ for each n-absorbing ideal of the ring R (**Conjecture 3**). In this paper we give an answer to (**Conjecture 2**) for $n = 3$, $n = 4$ and $n = 5$ and we prove that (**Conjecture 1**) and (**Conjecture 3**) hold in various classes of rings.

Keywords 2-Absorbing ideal · n-Absorbing ideal · Strongly n-absorbing · U-ring · Prüfer · PVD

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1 Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. In this paper, we study Anderson–Badawi conjectures. The concept of 2-absorbing ideals was introduced and investigated in Badawi (2007). Recall that a proper ideal I of R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. More generally, let n be a positive integer, a proper ideal I of R is said to be an n-absorbing ideal if whenever $x_1 \ldots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ then there are n of the x_i's whose product is in I. And I is said to be a strongly n-absorbing ideal if whenever $I_1 \ldots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then the product of some n of the I_i's is in I. Anderson and Badawi (2011) conjectured that every n-absorbing ideal of R is strongly n-absorbing (**Conjecture 1**) and $Rad(I)^n \subseteq I$, where $Rad(I)$ denotes the radical ideal of I (**Conjecture 2**).

In Sect. 2, we give an answer to (**Conjecture 2**) in the case where $n = 3, n = 4$ and $n = 5$. After that, we give some equivalent characterizations of n-absorbing ideals and we prove that (**Conjecture 1**) is true in the class of U-rings. Recall that a commutative ring R is said to be a U-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals.

An ideal I of a ring R is an SFT (strong finite type) ideal if there exists an ideal F of finite type with $F \subseteq I$ and an integer n such that for any $a \in I$, $a^n \in F$. A ring R is an SFT-ring if every ideal of R is SFT, which is equivalent to each prime ideal of R is SFT (Arnold 1973). We prove that if every nonzero proper ideal of a ring R is a 2-absorbing ideal of R then R is an SFT ring.

Finally, we prove that if n is an integer with $n \geq 3$, then I is an n-absorbing ideal of R if and only if $I[X]$ (respectively $I[[X]]$) is an n-absorbing ideal of $R[X]$ (**Conjecture 3**) (respectively $R[[X]]$), if the ring R is a Gaussian ring (respectively Noetherian Gaussian ring) or the ring R is a pseudo-valuation domain (PVD).

We start by recalling some background material.

An integral domain R is said to be a valuation domain if $x|y$ (in R) or $y|x$ (in R) for every nonzero $x, y \in R$. An integral domain R is called a Prüfer domain if R_P is a valuation domain for each prime ideal P of R.

The content of a polynomial (respectively a power series) f over a commutative ring R is the ideal $C(f)$ of R generated by all the coefficients of f. A commutative ring R is said to be a Gaussian (respectively P-Gaussian) ring if $C(fg) = C(f)C(g)$ for every f and g in $R[X]$ (respectively f and g in $R[[X]]$).

Let R be an integral domain with quotient field K. A prime ideal P of R is called strongly prime if whenever $x, y \in K$ and $xy \in P$ then $x \in P$ or $y \in P$. A domain R is called a pseudo-valuation domain if P is a strongly prime ideal for each prime ideal P of R.

A prime ideal P of a ring R is said to be a divided prime ideal if $P \subset xR$ for every $x \in R \backslash P$; thus a divided prime ideal is comparable to every ideal of R. An integral domain R is said to be a divided domain if every prime ideal of R is a divided prime ideal.

Let R be a ring, $Spec(R)$ denotes the set of prime ideals of R and $Nil(R)$ denotes the ideal of nilpotent elements of R. If I is a proper ideal of R, then $Min_R(I)$ denotes the set of prime ideals of R minimal over I.

2 On the Anderson–Badawi conjectures

Let R be a commutative ring. Anderson and Badawi (2011) conjectured that every n-absorbing ideal of R is strongly n-absorbing (**Conjecture 1**) and $Rad(I)^n \subseteq I$ (**Conjecture 2**). As observed in Anderson and Badawi (2011), it is easy to see that **Conjecture 1** implies **Conjecture 2**. **Conjecture 1** was proved for $n = 2$, see Anderson and Badawi (2011, Theorem 2.13). It was also verified for arbitrary n when R is a Prüfer domain (Anderson and Badawi 2011, Corollary 6.9). Darani (2013, Theorem 4.2) proved that **Conjecture 1** is true for all commutative rings with torsion-free additive group. Donadze (2016) gives answers for the two conjectures in special cases. Moreover, **Conjecture 2** is true in the case where I is an n-absorbing ideal with exactly n minimal prime ideals $\{P_1, \ldots, P_n\}$. In fact, by Anderson and Badawi (2011, Theorem 2.14) we have $P_1 \ldots P_n \subseteq I$. Since $Rad(I) = \cap_{P_i \in Min_R(I)} P_i \subseteq P_j$ for each $1 \leq j \leq n$, we have $Rad(I)^n \subseteq I$. If in addition, the P_i's are comaximal, then $I = P_1 \cap \cdots \cap P_n$, (Anderson and Badawi 2011, Corollary 2.15) so $I[X] = P_1[X] \cap \cdots \cap P_n[X]$ (respectively $I[[X]] = P_1[[X]] \cap \cdots \cap P_n[[X]]$), which implies, by Anderson and Badawi (2011, Theorem 2.1), that $I[X]$ (respectively $I[[X]]$) is an n-absorbing ideal of $R[X]$ (respectively $R[[X]]$).

Theorem 2.1 *Let I be a 3-absorbing ideal of R. Then $Rad(I)^3 \subseteq I$.*

Proof Let $x, y, z \in Rad(I)$. First observe that $x^2y^2 \in I$. In fact, we have $x^3 \in I$ for all $x \in Rad(I)$, by Anderson and Badawi (2011, Theorem 2.1). Since $x^2y^2(x+y) = xxy^2(x+y) \in I$ and I is a 3-absorbing ideal, we conclude that either $xy^2(x+y) \in I$ or $x^2(x+y) \in I$ or $x^2y^2 \in I$, thus $x^2y^2 \in I$. Now, we prove that $x^2y \in I$. Since $x^2y(x^2+y) = xxy(x^2+y) \in I$ we have that $xy(x^2+y) \in I$ or $x^2(x^2+y) \in I$ or $x^2y \in I$. So $x^2y \in I$ or $xy^2 \in I$. If $xy^2 \in I$, since $x^2y(x+y) = xxy(x+y) \in I$, we conclude that $x^2y \in I$. Finally, since $xyz(x+y+z) \in I$ we have $xyz \in I$. □

Theorem 2.2 *Let I be a 4-absorbing ideal of R. Then $Rad(I)^4 \subseteq I$.*

Proof By Anderson and Badawi (2011, Theorem 2.1), $x^4 \in I$ for each $x \in Rad(I)$. Now following these steps we get the result:

- Let $x_1, x_2 \in Rad(I)$ then $x_1^3x_2^3 \in I$. In fact, we have $x_1^3(x_1+x_2)x_2^3 \in I$ and I is a 4-absorbing ideal.
- Let $x_1, x_2 \in Rad(I)$ then $x_1^3x_2^2 \in I$. In fact, by the last step, as $x_1^3x_2^3 \in I$, then either $x_1^3x_2^2 \in I$ or $x_1^2x_2^3$. If $x_1^2x_2^3 \in I$ and since $x_1^3x_2^2(x_1+x_2) \in I$, we have the result.
- Let $x_1, x_2 \in Rad(I)$ then $x_1^2x_2^2 \in I$ and $x_1^3x_2 \in I$. In fact, we have $x_1^3x_2^2 \in I$ (and $x_1^2x_2^3 \in I$), then either $x_1^2x_2^2 \in I$ or $x_1^3x_2 \in I$. If $x_1^2x_2^2 \in I$, since $x_1^3x_2(x_1+x_2) \in I$, we conclude that $x_1^3x_2 \in I$. If $x_1^3x_2 \in I$, since $x_1^2(x_1+x_2)x_2^2 \in I$, we conclude that $x_1^2(x_1+x_2)x_2 \in I$ or $x_1^2x_2^2 \in I$ or $x_1(x_1+x_2)x_2^2 \in I$. In the first and second cases, we get $x_1^2x_2^2 \in I$. In the last case, since $x_1^2x_2^3 \in I$ we have the result.
- Let $x_1, x_2, x_3 \in Rad(I)$ then $x_1^2x_2^2x_3^2 \in I$. In fact, it suffices to remark that $x_1^2x_2^2x_3^2(x_1+x_2+x_3) \in I$.
- Let $x_1, x_2, x_3 \in Rad(I)$ then $x_1^2x_2x_3 \in I$, since $x_1^2x_2x_3(x_2+x_3) \in I$.

- Let $x_1, x_2, x_3, x_4 \in Rad(I)$ then $x_1x_2x_3x_4 \in I$. In fact, we have $x_1x_2x_3x_4(x_1 + x_2 + x_3 + x_4) \in I$ and since I is a 4-absorbing ideal, the result is clear.

□

Theorem 2.3 *Let I be a 5-absorbing ideal of R. Then $Rad(I)^5 \subseteq I$.*

Proof By Anderson and Badawi (2011, Theorem 2.1), $x^5 \in I$ for each $x \in Rad(I)$.

- Let $x_1, x_2 \in Rad(I)$ then $x_1^4x_2^4 \in I$, since $x_1^4(x_1 + x_2)x_2^4 \in I$.
- Let $x_1, x_2 \in Rad(I)$ then $x_1^4x_2^3 \in I$. In fact, we have $x_1^4xx_2^4 \in I$. Hence, either $x_1^4x_2 \in I$ or $x_1^4x_2^3 \in I$ or $x_1^3x_2^4 \in I$. If $x_1^3x_2^4 \in I$, we have either $x_1^3x_2^3 \in I$ or $x_1^2x_2^4 \in I$. Suppose that $x_1^2x_2^4 \in I$, then either $x_1x_2^4 \in I$ or $x_1x_2^3 \in I$. If $x_1x_2^4 \in I$ and since $x_1^4x_2^3(x_1 + x_2) \in I$, then we get the result.
- Let $x_1, x_2 \in Rad(I)$ then $x_1^3x_2^3 \in I$ and $x_1^4x_2^2 \in I$. In fact, since $x_1^4x_2^3 \in I$ and I is a 5-absorbing ideal we have either $x_1^4x_2^2 \in I$ or $x_1^3x_2^3 \in I$. Suppose that $x_1^4x_2^2 \in I$, since $x_1^3(x_1 + x_2)x_2^3 \in I$ and $x_1^3x_2^4 \in I$, we prove that $x_1^3x_2^3 \in I$. Suppose that $x_1^3x_2^3 \in I$ and since $x_1^4x_2^2(x_1 + x_2) \in I$, we conclude that $x_1^4x_2^2 \in I$.
- Let $x_1, x_2 \in Rad(I)$ then $x_1^3x_2^2 \in I$ and $x_1^4x_2 \in I$. In fact, we have $x_1^4x_2^2 \in I$ so either $x_1^3x_2^2 \in I$ or $x_1^4x_2 \in I$. If $x_1^4x_2 \in I$ we prove that $x_1^3x_2^2 \in I$ since $x_1^3x_2^3 \in I$ and $x_1^3(x_1 + x_2)x_2^2 \in I$. If $x_1^3x_2^2 \in I$, we prove that $x_1^4x_2 \in I$ since $x_1^4(x_1 + x_2)x_2 \in I$.
- Let $x_1, x_2, x_3 \in Rad(I)$ then $(x_1x_2x_3)^2 \in I$. It suffices to remark that $x_1^2x_2^2x_3^2(x_1 + x_2 + x_3) \in I$.
- Let $x_1, x_2, x_3 \in Rad(I)$ then $x_1^3x_2x_3 \in I$. In fact, it is clear since $x_1^3x_2x_3(x_2+x_3) \in I$ and $x_1^3x_2x_3(x_1 + x_2 + x_3) \in I$.
- Let $x_1, x_2, x_3 \in Rad(I)$ then $x_1^2x_2^2x_3 \in I$ since $x_1^2x_2^2x_3(x_1 + x_2 + x_3) \in I$, then either $x_1^2x_2^2x_3+x_1x_2^2x_3^2 \in I$ (1') or $x_1^2x_2^2x_3+x_1^2x_2x_3^2 \in I$ (2') or $x_1^2x_2^2x_3 \in I$. If (1') is true, since $x_1^2x_2^2x_3^2 \in I$ then either $x_1x_2^2x_3^2 \in I$ or $x_1^2x_2x_3^2 \in I$ or $x_1^2x_2^2x_3 \in I$. If $x_1x_2^2x_3^2 \in I$, we get the result. If $x_1^2x_2x_3^2 \in I$, since $x_1x_2^2x_3^2(x_1 + x_2 + x_3^2) \in I$, we conclude.
  If (2') is true, since $x_1^2x_2^2x_3^2 \in I$ then either $x_1x_2^2x_3^2 \in I$ or $x_1^2x_2x_3^2 \in I$ or $x_1^2x_2^2x_3 \in I$. If $x_1^2x_2x_3^2 \in I$, we get the result. If $x_1x_2^2x_3^2 \in I$, since $x_1^2x_2x_3^2(x_1+x_2+x_3^2) \in I$, we conclude.
- Let $x_1, x_2, x_3, x_4 \in Rad(I)$ then $x_1^2x_2x_3x_4 \in I$. It is clear since $x_1^2x_2x_3x_4(x_1 + x_2 + x_3 + x_4) \in I$ and $x_1^2x_2x_3x_4(x_2 + x_3 + x_4) \in I$.
- Let $x_1, x_2, x_3, x_4 \in Rad(I)$ then $x_1x_2x_3x_4 \in I$. In fact, remark that $x_1x_2x_3x_4(x_1 + x_2 + x_3 + x_4) \in I$.

□

Notation (Anderson and Badawi 2011) If I is an n-absorbing ideal of R for some positive integer n, then define $\omega_R(I) = min\{n|\ I\ is\ an\ n-absorbing\ ideal\ of\ R\}$. Applying Anderson and Badawi (2011, Theorem 6.3), we obtain the following result:

Corollary 2.1 *Let P be a prime ideal of a ring R and $n \in \{3, 4, 5\}$.*

(1) *If P^n is a P-primary ideal of R and $P^n \subset P^{n-1}$, then $\omega_R(P^n) = n$.*
(2) *If P is a maximal ideal of R and $P^n \subset P^{n-1}$, then $\omega_R(P^n) = n$.*

(3) *Let I be a P-primary ideal of a ring R. If $P^n \subseteq I$ and $P^{n-1} \not\subseteq I$, then $\omega_R(I) = n$.*

Remark that in the case where $n \geq 6$, we can prove the following results:

(1) Let $x_1, x_2 \in Rad(I)$ then $x_1x_2^{n-1} \in I$. In fact, since $x_1(x_1^{n-1} + x_2)x_2^{n-1} \in I$, we conclude that either $x_1x_2^{n-1} \in I$ or $x_1^{n-1}x_2^{n-1} \in I$.
Now, for each $1 \leq k \leq n-1$, we suppose that $x_1^{n-k}x_2^{n-1} \in I$ and we prove that $x_1^{n-k-1}x_2^{n-1} \in I$.
Since $x_1(x_1^{n-k-1} + x_2)x_2^{n-1} \in I$, we conclude that either $x_1^{n-k-1}x_2^{n-1} \in I$ or $x_1x_2^{n-1} \in I$ or $x_1^{n-k}x_2^{n-2} + x_1x_2^{n-1} \in I$. As $x_1^{n-k}x_2^{n-1} \in I$, then either $x_1^{n-k-1}x_2^{n-1} \in I$ or $x_1^{n-k}x_2^{n-2} \in I$. So the result is clear.
(2) Let $x_1, x_2 \in Rad(I)$ then $x_1^{n-2}x_2^{n-2} \in I$. In fact, it is clear since $x_1^{n-2}(x_1 + x_2)x_2^{n-2}$.
(3) Let $x_1, x_2 \in Rad(I)$ then $x_1^{n-2}x_2^{n-3} \in I$. In fact, it is clear since $x_1^{n-2}(x_1 + x_2)x_2^{n-3} \in I$ and $x_1^{n-2}x_2^{n-2} \in I$.

In the next step, we prove that **Conjecture 1** holds for U-rings.

Definition 2.1 Let R be a commutative ring, I, J two ideals of R and $a \in R$. We define:

(1) $(I : J) = \{x \in R \mid xJ \subseteq I\}$.
(2) $(I : a) = \{x \in R \mid ax \in I\}$.

Notation Let R be a commutative ring, $n \in \mathbb{N}^*$, $x_1, \ldots, x_n \in R$ and $I_1, \ldots, I_n$ be n ideals of R. For $i \in \{1, \ldots, n\}$, we denote by:

- $\hat{x_i}$ the product $x_1 \ldots x_{i-1}x_{i+1} \ldots x_n$.
- $\hat{I_i}$ the product $I_1 \ldots I_{i-1}I_{i+1} \ldots I_n$.

Proposition 2.1 *Let I be a proper ideal of a commutative ring R and $n \in \mathbb{N}^*$. The following conditions are equivalent:*

(1) *I is an n-absorbing ideal of R.*
(2) *For every elements $x_1, \ldots, x_n \in R$ with $x_1 \ldots x_n \notin I$, $(I : x_1 \ldots x_n) \subseteq \cup_{1\leq i\leq n}(I : \hat{x_i})$*

Proof "1) $\Rightarrow$ 2)" Let $a \in (I : x_1 \ldots x_n)$ then $ax_1 \ldots x_n \in I$. Since I is an n-absorbing ideal and $x_1 \ldots x_n \notin I$, we conclude that $a\hat{x_i} \in I$ for some i with $1 \leq i \leq n$. Thus $a \in \cup_{1\leq i\leq n}(I : \hat{x_i})$.
"2) $\Rightarrow$ 1)" Let $x_1, \ldots, x_{n+1} \in R$ such that $x_1 \ldots x_{n+1} \in I$, then $x_1 \in (I : x_2 \ldots x_{n+1})$. If $x_2 \ldots x_{n+1} \in I$ then we are done. Hence we may assume that $x_2 \ldots x_{n+1} \notin I$ and so by (1), $(I : x_2 \ldots x_{n+1}) \subseteq \cup_{2\leq i\leq n+1}(I : \hat{x_i})$. So $x_1 \in (I : \hat{x_i})$ for some i with $2 \leq i \leq n+1$. □

Definition 2.2 (Quartararo and Butts 1975) A commutative ring R is said to be a U-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals.

Example 2.1 (1) Every Prüfer domain is a U-ring (Quartararo and Butts 1975, Corollary 1.6).

(2) Let D be an integral domain with quotient field K. If D is a U-ring and $D \subseteq R \subseteq K$, then R is a U-domain. If D/P is finite for all maximal ideals P of D, then D is a U-domain if and only if D is a Prüfer domain (Quartararo and Butts 1975).

Recall that a proper ideal I of a ring R is a strongly n-absorbing ideal if whenever $I_1 \dots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R, then the product of some n of the I_i's is contained in I.

Theorem 2.4 *Let R be a U-ring and $n \geq 3$. The following conditions are equivalent:*

(1) *I is a strongly n-absorbing ideal.*
(2) *I is an n-absorbing ideal.*
(3) *For every $x_1, x_2, \dots, x_n \in R$ such that $x_1 \dots x_n \notin I$, $(I : x_1 \dots x_n) = (I : \hat{x_i})$ for some $1 \leq i \leq n$.*
(4) *For every t ideals $I_1, \dots, I_t$, $1 \leq t \leq n-1$, and for every elements $x_1, \dots, x_{n-t}$ such that $x_1 \dots x_{n-t} I_1 \dots I_t \nsubseteq I$, $(I : x_1 \dots x_{n-t} I_1 \dots I_t) = (I : \hat{x_i} I_1 \dots I_t)$ for some $1 \leq i \leq n-t$ or $(I : x_1 \dots x_{n-t} I_1 \dots I_t) = (I : x_1 \dots x_{n-t} \hat{I_j})$ for some $1 \leq j \leq t$.*
(5) *For every ideals $I_1, \dots I_n$ of R with $I_1 \dots I_n \nsubseteq I$, $(I : I_1 \dots I_n) = (I : \hat{I_i})$, for some $1 \leq i \leq n$.*

Proof 1) $\Rightarrow$ 2) It is clear.

2) $\Rightarrow$ 3) This follows from the last proposition, since R is a U-ring.

3) $\Rightarrow$ 4) We prove the result by induction on $t \in \{1, \dots, n-1\}$. For $t = 1$ consider $x_1, \dots, x_{n-1} \in R$ and an ideal I_1 of R such that $x_1 \dots x_{n-1} I_1 \nsubseteq I$.

Let $a \in (I : x_1 \dots x_{n-1} I_1)$. Then $I_1 \subseteq (I : a x_1 \dots x_{n-1})$. If $a x_1 \dots x_{n-1} \in I$, then $a \in (I : x_1 \dots x_{n-1})$. If $a x_1 \dots x_{n-1} \notin I$, then by 3), either $(I : a x_1 \dots x_{n-1}) = (I : x_1 \dots x_{n-1})$ or $(I : a x_1 \dots x_{n-1}) = (I : a\hat{x_i})$ for some $1 \leq i \leq n-1$. Since $I_1 \nsubseteq (I : x_1 \dots x_{n-1})$, we conclude that $I_1 \subseteq (I : a\hat{x_i})$ for some $1 \leq i \leq n-1$, and thus $a \in (I : \hat{x_i} I_1)$. Hence $(I : x_1 \dots x_{n-1} I_1) \subseteq (I : x_1 \dots x_{n-1}) \cup \bigcup_{1 \leq i \leq n-1} (I : \hat{x_i} I_1)$. Since R is a U-ring, then either $(I : x_1 \dots x_{n-1} I_1) \subseteq (I : x_1 \dots x_{n-1})$ or $(I : x_1 \dots x_{n-1} I_1) \subseteq (I : \hat{x_i} I_1)$. The other inclusions are evident.

Now, suppose that $t > 1$ and assume that the claim holds for $t-1$. Let $x_1, \dots, x_{n-t}$ be elements of R and let $I_1, \dots, I_t$ be ideals of R such that $x_1 \dots x_{n-t} I_1 \dots I_t \nsubseteq I$.

Consider an element $a \in (I : x_1 \dots x_{n-t} I_1 \dots I_t)$. Thus $I_t \subseteq (I : a x_1 \dots x_{n-t} I_1 \dots I_{t-1})$. If $a x_1 \dots x_{n-t} I_1 \dots I_{t-1} \subseteq I$, then $a \in (I : x_1 \dots x_{n-t} I_1 \dots I_{t-1})$. If $a x_1 \dots x_{n-t} I_1 \dots I_{t-1} \nsubseteq I$, then by the induction hypothesis, either $(I : a x_1 \dots x_{n-t} I_1 \dots I_{t-1}) = (I : x_1 \dots x_{n-t} I_1 \dots I_{t-1})$ or $(I : a x_1 \dots x_{n-t} I_1 \dots I_{t-1}) = (I : a\hat{x_i} I_1 \dots I_{t-1})$ for some $1 \leq i \leq n-t$ or $(I : a x_1 \dots x_{n-t} I_1 \dots I_{t-1}) = (I : a x_1 \dots x_{n-t} I_1 \dots I_{j-1} I_{j+1} \dots I_{t-1})$ for some $1 \leq j \leq t-1$.

Since $x_1 \dots x_{n-t} I_1 \dots I_t \nsubseteq I$, then the first case is removed. Consequently, either $(I : a x_1 \dots x_{n-t} I_1 \dots I_{t-1}) = (I : a\hat{x_i} I_1 \dots I_{t-1})$ for some $1 \leq i \leq n-t$ or

$(I : ax_1 \dots x_{n-t} I_1 \dots I_{t-1}) = (I : ax_1 \dots x_{n-t} I_1 \dots I_{j-1} I_{j+1} \dots I_t)$ for some $1 \le j \le t-1$.

Hence $(I : x_1 \dots x_{n-t} I_1 \dots I_t) \subseteq \cup_{1 \le i \le n-1} (I : \hat{x}_i I_1 \dots I_t) \cup \bigcup_{1 \le j \le t} (I : x_1 \dots x_{n-t} \hat{I}_j)$. Now, since R is a U-ring, $(I : x_1 \dots x_{n-t} I_1 \dots I_t)$ is included in $(I : \hat{x}_i I_1 \dots I_t)$ for some $1 \le i \le n-t$ or $(I : x_1 \dots x_{n-t} \hat{I}_j)$ for some $1 \le j \le t$. The other inclusions are evident.

4) $\Rightarrow$ 5) Let $I_1, \dots, I_n$ be ideals of R such that $I_1 \dots I_n \nsubseteq I$. Suppose that $a \in (I : I_1 \dots I_n)$. Then $I_n \subseteq (I : aI_1 \dots I_{n-1})$. If $aI_1 \dots I_{n-1} \subseteq I$, then $a \in (I : I_1 \dots I_{n-1})$. If $aI_1 \dots I_{n-1} \nsubseteq I$, then by 4), we have either $(I : aI_1 \dots I_{n-1}) = (I : a\hat{I}_j)$ for some $1 \le j \le n-1$ or $(I : aI_1 \dots I_{n-1}) = (I : I_1 \dots I_{n-1})$.

By hypothesis, the second case does not hold. The first case implies that $a \in (I : I_1 \dots I_{j-1} I_{j+1} \dots I_n)$ for some $1 \le j \le n-1$. Hence $(I : I_1 \dots I_n) \subseteq (I : I_1 \dots I_{n-1}) \cup \bigcup_{1 \le j \le n-1} (I : \hat{I}_j) = \bigcup_{1 \le i \le n} (I : \hat{I}_j)$. Since R is a U-ring, we conclude that $(I : I_1 \dots I_n) \subseteq (I : \hat{I}_j)$ for some $1 \le j \le n$. The other inclusions are evident.

5) $\Rightarrow$ 1) Let $I_1, \dots, I_{n+1}$ be ideals of R such that $I_1 \dots I_{n+1} \subseteq I$. Then $I_1 \subseteq (I : I_2 \dots I_{n+1})$. If $I_2 \dots I_{n+1} \subseteq I$, that is clear. If $I_2 \dots I_{n+1} \nsubseteq I$, then by 5), $(I : I_2 \dots I_{n+1}) = (I : I_2 \dots I_{j-1} I_{j+1} \dots I_{n+1})$ for some $2 \le j \le n+1$. So $I_1 \hat{I}_j \subseteq I$ for some $2 \le j \le n+1$. □

Example 2.2 Let R be a Prüfer domain, I a proper ideal of R and $n \ge 3$. Using Anderson and Badawi (2011, Theorem 5.7), we conclude that I is a strongly n-absorbing ideal of R if and only if I is a product of prime ideals of R.

Badawi (2007) proved that if I is a 2-absorbing ideal of a commutative ring R, then either $(I : x) \subseteq (I : y)$ or $(I : y) \subseteq (I : x)$ for each $x, y \in Rad(I) \backslash I$. It is natural to ask if this result can be generalized for each $x, y \in R \backslash I$. The answer is given by the next theorem. Recall, from Badawi (2007), that if I a 2-absorbing ideal, then one of the following statements must hold:

(1) $Rad(I) = P$ is a prime ideal of R and $P^2 \subseteq I$.
(2) $Rad(I) = P_1 \cap P_2$, $P_1 P_2 \subseteq I$ and $Rad(I)^2 \subseteq I$ where P_1, P_2 are the only distinct prime ideals of R that are minimal over I.

Theorem 2.5 *Let I be a 2-absorbing ideal of a commutative ring R.*

(1) *If $Rad(I) = P$ is a prime ideal of R, then either $(I : x) \subseteq (I : y)$ or $(I : y) \subseteq (I : x)$, for every $x, y \in R \backslash I$.*
(2) *If $Rad(I) = P_1 \cap P_2$, where P_1, P_2 are the only distinct prime ideals of R that are minimal over I and $I \ne Rad(I)$, then either $(I : x) \subseteq (I : y)$ or $(I : y) \subseteq (I : x)$ for every $x, y \in R \backslash I$ except if $x \in P_1 \backslash P_2$ and $y \in P_2 \backslash P_1$, in which case $(I : x) = P_2$ and $(I : y) = P_1$.*

Proof (1) Let I be a 2-absorbing ideal of R such that $Rad(I) = P$ is a prime ideal of R. First, remark that:

(a) For each $x \in R \backslash P$, $(I : x) \subseteq P$. In fact, let $y \in R$ such that $yx \in I$. Since P is a prime ideal and $x \notin P$ we conclude that $y \in P$.

(b) Let $x, y \in R \backslash P$ then $(I : x)$ and $(I : y)$ are linearly ordered. Otherwise, let $z_1 \in (I : x) \backslash (I : y)$ and $z_2 \in (I : y) \backslash (I : x)$. Then $x(z_1 + z_2)y \in I$. Since I is a 2-absorbing ideal, we have $x(z_1 + z_2) \in I$ or $(z_1 + z_2)y \in I$ or $xy \in I$ which is impossible.

Now, let $x, y \in R \backslash I$.

- If $x, y \in P \backslash I$, it's clear by Badawi (2007 Theorem 2.5).
- If $x, y \in R \backslash P$, it's clear by the last remark.
- if $x \in R \backslash P$ and $y \in P \backslash I$, we have $(I : x) \subset P \subset (I : y)$ by the last remark and Badawi (2007 Theorem 2.5).

(2) Let I be a 2-absorbing ideal such that $Rad(I) = P_1 \cap P_2$ and $x \in R \backslash Rad(I)$. Then $(I : x) \subseteq P_1 \cup P_2$. In fact, let $z \in (I : x)$, so $zx \in I \subseteq P_1 \cap P_2$. Since $x \notin Rad(I)$, we have $x \notin P_1$ or $x \notin P_2$. So we conclude that $z \in P_1$ or $z \in P_2$. Remark that if $x \in P_1 \backslash P_2$, then $(I : x) = P_2$. In fact, let $z \in (I : x)$ then $xz \in I \subseteq P_1 \cap P_2 \subseteq P_2$. As $x \notin P_2$ then $z \in P_2$. So $(I : x) \subseteq P_2$. Conversely, let $z \in P_2$ then $xz \in P_1 P_2 \subseteq I$. So $z \in (I : x)$.
Similarly, if $x \in P_2 \backslash P_1$ then $(I : x) = P_1$.
Now let $x, y \in R \backslash I$.
If $x, y \in Rad(I) \backslash I$, then $(I : x)$ and $(I : y)$ are linearly ordered by Badawi (2007 Theorem 2.6).
If not, we have the following cases:

- If $x \in Rad(I) \backslash I$ and $y \in R \backslash Rad(I)$, we have $(I : y) \subseteq P_1 \cup P_2 \subseteq (I : x)$.
- If $x, y \in R \backslash Rad(I)$:
  - if $x, y \in P_1 \backslash P_2$, we conclude that $(I : x) = (I : y) = P_2$.
  - if $x, y \in P_2 \backslash P_1$, in this case we have $(I : x) = (I : y) = P_1$.
  - if $x, y \in R \backslash (P_1 \cup P_2)$, we assume that $(I : x)$ and $(I : y)$ are not linearly ordered. Then there exist $z_1 \in (I : x) \backslash (I : y)$ and $z_2 \in (I : y) \backslash (I : x)$. So $x(z_1 + z_2)y \in I$ and no product of two elements is in I which is a contradiction.
  - if $x \in P_1 \backslash P_2$ and $y \in P_2 \backslash P_1$, we have $(I : x) = P_2$ and $(I : y) = P_1$ and it is clear that $(I : x)$ and $(I : y)$ are not linearly ordered in this case.

□

Recall that a 2-absorbing ideal is a generalization of a prime ideal and there are many characterization of a commutative ring with their set of prime ideals , so one can ask if we have a similar result for a commutative ring such that every nonzero proper ideal of R is a 2-absorbing ideal. The following proposition gives an answer.

Proposition 2.2 *Let R be a commutative ring. If every nonzero proper ideal of R is a 2-absorbing ideal then R is an SFT ring.*

Proof By Badawi (2007 Theorem 3.4), R is a zero-dimensional ring and we have three cases.

Case 1: R is quasi-local with maximal ideal $M = Nil(R) \neq \{0\}$ such that $M^2 \subseteq xR$ for each nonzero $x \in M$. To prove that R is an SFT ring it suffices to prove that M is an SFT ideal of R. Since $M \neq (0)$, then there is a nonzero element $y \in M$. Thus

$F = (y)$ is a principal ideal of R such that $x^2 \in F$ for each $x \in M$. So we conclude that M is an SFT ideal.

Case 2: R has exactly two distinct maximal ideals, say $\{M_1, M_2\}$. So either R is isomorphic to $D = R/M_1 \oplus R/M_2$ or $Nil(R)^2 = \{0\}$ and $Nil(R) = \omega R$ for each nonzero $\omega \in Nil(R)$. In the first situation, R is isomorphic to an SFT ring so R is an SFT ring. In the second situation, we have $R \cong R/M_1^2 \oplus R/M_2$, by Badawi (2007 Lemma 3.3). The ring R/M_1^2 is SFT. In fact, let J be an ideal of R/M_1^2, then there exists an ideal I of R such that $M_1^2 \subseteq I \subseteq M_1$ and $J = I/M_1^2$. It is easy to see that $J \subseteq Nil(R/M_1^2) = M_1/M_1^2$ and for each $\bar{x} \in J$, we have $\bar{x}^2 = \bar{0}$. Then by Hizem and Benhissi (2011, Proposition 2.1) R/M_1^2 is an SFT ring.

Case 3: We suppose that R is isomorphic to $F_1 \oplus F_2 \oplus F_3$, where F_1, F_2 and F_3 are fields. It is clear in this case that R is an SFT ring. □

Example 2.3 (1) Let $R = \mathbb{Z} + 6X\mathbb{Z}[X]$ and $P = 6X\mathbb{Z}[X]$. First observe that P^2 is not a 2-absorbing ideal of R. In fact, let $f_1 = 6X^2$, $f_2 = 2$ and $f_3 = 3$ in R, then it is clear to see that $f_1 f_2 f_3 \in P^2$ but $f_1 f_2 \notin I$, $f_2 f_3 \notin I$ and also $f_1 f_3 \notin I$. So R is not an SFT ring.

(2) Let D be a valuation domain with Krull dimension $n \geq 1$, K the quotient field of D and X an indeterminate. Set $R = D + XK[[X]]$, by [4, Example 3.12], R is not a 2-absorbing ring so R is not an SFT ring.

Next, we give some classes of rings in which **Conjecture 3** holds. Recall that **Conjecture 3** is true if $n = 2$ and we can easily prove that if I is a 2-absorbing ideal of R then $I[[X]]$ is also a 2-absorbing ideal of the power series ring $R[[X]]$. In fact, we prove that either $Rad(I[[X]]) = P[[X]]$, with P a prime ideal of R or $Rad(I[[X]]) = P_1[[X]] \cap P_2[[X]]$, with P_1 and P_2 are two prime ideals of R. By Badawi (2007 Theorems 2.8 and 2.9), we conclude that $I[[X]]$ is a 2-absorbing ideal since $I[[X]]_f$ is a prime ideal of $R[[X]]$ for each $f \in Rad(I[[X]]) \backslash I[[X]]$.
Nasehpour (2016) proves that for a Prüfer domain R and $n \geq 3$, an ideal I is n-absorbing if and only if $I[X]$ is n-absorbing. In the following, we generalize this result in the case of a Gaussian U-ring.

Remark also that in a Prüfer domain, we can prove the last result in the power series ring. In fact, let I be an n-absorbing ideal then $I[[X]] = P_1^{n_1}[[X]] \ldots P_k^{n_k}[[X]]$, where $P_1, \ldots, P_k$ are the minimal prime ideals over I and $n_1, \ldots, n_k$ positive integer such that $n_1 + \cdots + n_k = n$. By Fields (1971, Corollary 4) and Anderson and Badawi (2011, Theorems 3.1 and 2.1) we conclude that $I[[X]]$ is an n-absorbing ideal of $R[[X]]$.

Recall that a commutative ring R is said to be a *Gaussian ring* (respectively *P-Gaussian*) if $C(fg) = C(f)C(g)$ for every polynomials f and g in $R[X]$ (respectively f and g in $R[[X]]$).

Theorem 2.6 *Let R be a Gaussian ring* (*respectively a Noetherian Gaussian ring*)*. If R is a U-ring, then I is an n-absorbing ideal of R if and only if $I[X]$* (*respectively $I[[X]]$*) *is an n-absorbing ideal of $R[X]$* (*respectively $R[[X]]$*)*. Moreover, $\omega_R(I) = \omega_{R[X]}(I[X])$* (*respectively $\omega_R(I) = \omega_{R[[X]]}(I[[X]])$*)*.*

Proof We prove the result in the case of polynomial rings.

"$\Leftarrow$" It follows from Anderson and Badawi (2011, Corollary 4.3).
"$\Rightarrow$" Suppose that I is an n-absorbing ideal of R and let $f_1, f_2, \ldots, f_{n+1} \in R[X]$ such that $f_1 \ldots f_{n+1} \in I[X]$.
Since R is a Gaussian ring, we conclude that $C(f_1) \cdots C(f_{n+1}) = C(f_1 \cdots f_{n+1}) \subseteq I$. As I is a strongly n-absorbing ideal of R, by Theorem 2.2, hence $C(\hat{f_i}) \subseteq I$ for some $1 \leq i \leq n+1$, thus $\hat{f_i} \in I[X]$.
The same proof works also in the case of power series rings as a Noetherian Gaussian ring is P-Gaussian (Tsang 1965).

Recall that a commutative ring R is said to be a pseudo-valuation domain (PVD) if every prime ideal of R is strongly prime.

Theorem 2.7 *Let R be a pseudo-valuation domain with associated valuation domain V and let I be an ideal of R such that $Rad(I)$ is not maximal. Then I is an n-absorbing ideal of R if and only if $I[X]$ (respectively $I[[X]]$) is an n-absorbing ideal of $R[X]$ (respectively of $R[[X]]$). Moreover, $\omega_R(I) = \omega_{R[X]}(I[X])$ (respectively $\omega_R(I) = \omega_{R[[X]]}(I[[X]])$).*

Proof Let I be an n-absorbing ideal of R. Then there are at most n prime ideal of R minimal over I. Since $Rad(I)$ is the intersection of all the prime ideals minimal over I and the prime ideals are comparable in a PVD, we conclude that $Rad(I) = P$ for some prime ideal minimal over I.

Recall that a PVD is a divided ring, so I is a P-primary ideal of R by Anderson and Badawi (2011, Theorem 3.2). As $Rad(I)$ is not maximal then I is also a P-primary ideal of V by Anderson and Dobbs (1980, Proposition 3.13).

We show that $P^n \subseteq I$. Let $x_1, \ldots, x_n \in P$, then there is an $x \in P$ such that $(x_1, \ldots, x_n)_V = xV$ since V is a valuation domain.

Hence $x_1 \ldots x_n = x^n b$ for some $b \in V$. As $x \in P = Rad(I)$ and I is n-absorbing then $x^n \in I$ and so $x^n b \in I$. Then $I[X]$ is an n-absorbing ideal of $R[X]$ by Anderson and Badawi (2011, Theorem 3.1) (respectively, by Fields (1971, Corollary 4)), $I[[X]]$ is $P[[X]]$-primary since $P^n[[X]] \subseteq I[[X]]$, so $I[[X]]$ is an n-absorbing ideal of $R[[X]]$). □

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