The Poncelet annuli

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Abstract The sufficient and necessary condition for a circular annulus to have the property of Poncelet porism is given. Moreover, the family of annuli with a property of the Poncelet porism and some family of periodic functions are considered. The close relations between these families are given.

Keywords Porism · Annulus

Mathematics Subject Classification (1991) 53A04

1 Introduction

A polygon is called circumscribed in an annulus if it is simultaneously inscribed in the outer curve and circumscribed on the inner curve. Poncelet in [Bos et al. \(1987\)](#) proved a theorem called Poncelet's closure theorem:

If there exists one circumscribed n -gon in a fixed annulus CD (C and D are smooth conics), then any point of the outer curve is the vertex of some circumscribed n -gon.

The paper [Bos et al. \(1987\)](#) very precisely describes Poncelet's porism. The literature on the porism is given in [Bos et al. \(1987\)](#) and [Weisstein and Poncelet's Porism \(2008\)](#), recent papers on this subject are given by [Radić \(2009, 2010\)](#). A family of annuli bounded by ovals were considered in e.g. [Cieślak and Szczygierska \(2008\)](#), [Mozgawa \(2008\)](#), [Cieślak et al. \(2013\)](#). We introduce the following definition:

An annulus with a property of the Poncelet porism will be called a Poncelet annulus.

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Mozgawa (2008) proved that for a given oval C there exists ovals C_{in} and C_{out} , inside and outside of C , such that the annuli CC_{in} and CC_{out} are Poncelet annuli (for almost all natural numbers). In Cieślak and Szczygielska (2008) it is proved that each oval and a natural number $n \geq 3$ generate a Poncelet annulus. The papers Cieślak and Szczygielska (2008) and Mozgawa (2008) show that the family of all Poncelet annuli is very wide. In this paper the assumption of convexity of the outer curve in the annulus is not important.

2 Families $\Phi_{L,n}$ and $\mathcal{B}_L$

In this section we generalize some results of Cieślak (1986). Let us fix a positive number L and a natural number $n \geq 3$.

We denote by $\Phi_{L,n}$ a family of all C^1 -functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions

$$0 < \varphi(0) < L, \quad (2.1)$$

$$\varphi'(t) > 0, \quad (2.2)$$

$$\varphi^{[n]}(t) = t + L \quad (2.3)$$

for $t \in \mathbb{R}$, where

$$\varphi^{[1]} = \varphi, \quad \varphi^{[k]} = \varphi \circ \varphi^{[k-1]} \quad (2.4)$$

for natural $k \geq 2$.

We note that $\varphi(t) + L = \varphi^{[n]}(\varphi(t)) = \varphi(\varphi^{[n]}(t)) = \varphi(t + L)$, i.e.

$$\varphi(t + L) = \varphi(t) + L \quad \text{for all } t \in \mathbb{R}. \quad (2.5)$$

The inverse function φ^{-1} to φ is given by the formula

$$\varphi^{-1}(t) = \varphi^{[n-1]}(t) - L \quad \text{for all } t \in \mathbb{R}. \quad (2.6)$$

For example the function $\varphi(t) = t + \frac{1}{n}L$ for $t \in \mathbb{R}$ is an element of $\Phi_{L,n}$.

We denote by $\mathcal{B}_L$ a family of all continuous functions $b : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions:

$$b(t) > 0, \quad (2.7)$$

$$b(t + L) = b(t) \quad (2.8)$$

for $t \in \mathbb{R}$.

We associate with each function $b \in \mathcal{B}_L$ two functions $B : \mathbb{R} \rightarrow \mathbb{R}$ and $\beta : \mathbb{R} \rightarrow \mathbb{R}$ given by the formulas

$$B(t) = \int_0^t b(u) du, \quad (2.9)$$

$$\beta(t) = B\left(t + \frac{L}{2}\right) - B(t). \quad (2.10)$$

We note that

$$\beta(t + L) = B\left(t + L + \frac{L}{2}\right) - B(t + L) = B(L) + B\left(t + \frac{L}{2}\right) - B(t) - B(L) = \beta(t)$$

i.e.

$$\beta(t + L) = \beta(t) \quad \text{for } t \in \mathbb{R} \quad (2.11)$$

and the function β reaches its minimum. Let

$$\beta_{\min} = \min_{t \in \mathbb{R}} \beta(t). \quad (2.12)$$

Now, we give two relations between the families $\Phi_{L,n}$ and $\mathcal{B}_L$. By simple verification we have

Lemma 2.1 *Let $b \in \mathcal{B}_L$. A function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula*

$$\varphi(t) = B^{-1}\left(B(t) + \frac{1}{n}B(L)\right) \quad (2.13)$$

belongs to $\Phi_{L,n}$ and it satisfies the differential equation

$$\varphi' = \frac{b}{b \circ \varphi} \quad (2.14)$$

with the initial condition

$$\varphi(0) = B^{-1}\left(\frac{1}{n}B(L)\right). \quad (2.15)$$

The above lemma is a generalization of the Lemma from [Cieřlak \(1986\)](#). In view of the conditions (2.2) and (2.5) we have immediately the following result:

Lemma 2.2 *If $\psi \in \Phi_{L,n}$ then a function $b : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula*

$$b = (\psi^{[n-1]})' + (\psi^{[n-2]})' + \dots + \psi' + 1 \quad (2.16)$$

belongs to $\mathcal{B}_L$, $\psi' = \frac{b}{b \circ \psi}$ and $B(L) = nL$.

Lemma 2.3 *If $\varphi \in \Phi_{L,n}$ then*

$$\varphi(t) - t > 0 \quad \text{for all } t \in \mathbb{R}. \quad (2.17)$$

Proof We assume that

$$\varphi(t_0) \leq t_0 \quad \text{for some } t_0 \in \mathbb{R}. \quad (2.18)$$

The function φ is strictly increasing so we have

$$\varphi^{[2]}(t_0) \leq \varphi(t_0). \quad (2.19)$$

The inequalities (2.18) and (2.19) imply that $\varphi^{[2]}(t_0) \leq t_0$. Repeating this procedure we get $\varphi^{[n]}(t_0) \leq t_0$, i.e. $t_0 + L \leq t_0$. The obtained contradiction proves the inequality (2.17). $\square$

3 A family $\mathcal{B}_{L,n}$

Let us fix a natural number $n \geq 3$. We define a subfamily $\mathcal{B}_{L,n}$ of $\mathcal{B}_L$ as follows: a function $b \in \mathcal{B}_L$ belongs to $\mathcal{B}_{L,n}$ if it satisfies the following condition

$$\beta_{\min} > \frac{1}{n}B(L) \quad (3.1)$$

where B and β are defined by (2.9) and (2.10), respectively.

Example 3.1 We consider the family $\mathcal{B}_{2\pi}$. The function $b(t) = \sin t + r$ for $t \in \mathbb{R}$ belongs to $\mathcal{B}_{2\pi}$ for $r > 1$. We have $B(t) = 1 - \cos t + rt$ and $\beta(t) = B(t + \pi) - B(t) = 2 \cos t + r\pi$. Hence we obtain $\min_{t \in \mathbb{R}} \beta(t) = \beta(\pi) = r\pi - 2$ and $B(2\pi) = 2\pi r$. The inequalityeqref2.1 for $n = 3$ implies $r > \frac{6}{\pi}$. The function b belongs to $\mathcal{B}_{2\pi,3}$ if $r \in (\frac{6}{\pi}, +\infty)$.

Now, we give some properties of the family $\mathcal{B}_{L,n}$.

Lemma 3.1 *Let $b \in \mathcal{B}_L$ and $b \notin \mathcal{B}_{L,n}$. Then a function $b_c : \mathbb{R} \rightarrow \mathbb{R}$ defined by the formula*

$$b_c(t) = b(t) + c \quad (3.2)$$

belongs to $\mathcal{B}_{L,n}$ if and only if

$$c > \frac{2n}{L(n-2)} \left(\frac{1}{n}B(L) - \beta_{\min} \right). \quad (3.3)$$

Proof We have immediately that $B_c(t) = B(t) + ct$ and $\beta_c(t) = \beta(t) + \frac{1}{2}Lc$. Thus the minimum of β_c is given by the formula

$$\beta_{c,\min} = \beta_{\min} + \frac{1}{2}Lc.$$

We note that $B_c(L) = B(L) + Lc$ and the inequality (3.1) for the function β_c has the form

$$\beta_{\min} + \frac{1}{2}Lc > \frac{1}{n}B(L) + \frac{1}{n}Lc.$$

Hence we obtain (3.3). $\square$

Lemma 3.2 *Let $b \in \mathcal{B}_L$. The function φ given by (2.13) satisfies the inequality*

$$\varphi(t) - t < \frac{1}{2}L \quad \text{for } t \in \mathbb{R} \quad (3.4)$$

if and only if $b \in \mathcal{B}_{L,n}$.

Proof The inequality (3.4) can be rewritten in the following equivalent forms

$$\begin{aligned} B^{-1}\left(B(t) + \frac{1}{n}B(L)\right) &< t + \frac{1}{2}L, \\ B(t) + \frac{1}{n}B(L) &< B\left(t + \frac{1}{2}L\right), \\ \beta(t) &> \frac{1}{n}B(L). \end{aligned}$$

$\square$

4 Generation of a Poncelet annulus by an element of the family $\mathcal{B}_{2\pi,n}$

We denote by $\mathcal{E}$ a family of all closed strictly convex curves parametrized by support function. The parametric equation of $C \in \mathcal{E}$ is of the form

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it} \quad \text{for } t \in [0, 2\pi], \quad (4.1)$$

see Santalo (1976) and Bonnesen et al. (1948).

We will assume that $p \in C^2$ and

$$R = p + \ddot{p} > 0. \quad (4.2)$$

Theorem 4.1 *Each element of the family $\mathcal{B}_{2\pi,n}$ and a curve $C \in \mathcal{E}$ generates an annulus having a property of the Poncelet porism for a natural number n .*

Proof Let us fix a natural number n and a function $b \in \mathcal{B}_{2\pi,n}$. The function b determines a function φ given the formula (2.13). We consider a curve C_φ formed by points of intersection of tangents to C at $z(t)$ and $z(\varphi(t))$. The condition (2.3) implies that the annulus CC_φ bounded by C and C_φ is a Poncelet annulus. $\square$

5 Generation of an element of $\mathcal{B}_{2\pi,n}$ by a Poncelet annulus

Theorem 5.1 *If $C \in \mathcal{E}$ then for some natural number n each Poncelet annulus CD determines an element of the family $\mathcal{B}_{2\pi,n}$.*

Proof We assume that an annulus CD is a Poncelet one for a natural number n where $C \in \mathcal{E}$ is given by (4.1). Tangent lines to C passing through a point of D intersects C at two points, and these points define a function φ . Let

$$b = (\varphi^{[n-1]})' + (\varphi^{[n-2]})' + \cdots + \varphi' + 1. \quad (5.1)$$

Obviously $b \in \mathcal{B}_{2\pi}$ and $\varphi' = \frac{b}{b \circ \varphi}$. We have

$$B(t) = \varphi^{[n-1]}(t) + \cdots + \varphi(t) + t - \varphi^{[n-1]}(0) - \cdots - \varphi(0). \quad (5.2)$$

We note that

$$\varphi^{[j]}(2\pi) = \varphi^{[j]}(0) + 2\pi \quad \text{for all natural } j. \quad (5.3)$$

The formulas (5.2) and (5.3) imply

$$B(2\pi) = 2\pi n. \quad (5.4)$$

On the other hand from (5.2) we obtain

$$\begin{aligned} B(\varphi(t)) &= \varphi^{[n]}(t) + \varphi^{[n-1]}(t) + \cdots + \varphi(t) - \varphi^{[n-1]}(0) - \cdots - \varphi(0) \\ &= 2\pi + t + \varphi^{[n-1]}(t) + \cdots + \varphi(t) - \varphi^{[n-1]}(0) - \cdots - \varphi(0) \\ &= 2\pi + B(t) \end{aligned}$$

i. e.

$$B \circ \varphi = B + 2\pi. \quad (5.5)$$

The inequality $\varphi(t) < t + \pi$ gives us $B(\varphi(t)) < B(t + \pi)$. Hence we have

$$\beta(t) = B(t + \pi) - B(t) > B(\varphi(t)) - B(t) = 2\pi = \frac{1}{n} B(2\pi)$$

i. e.

$$\beta_{\min} > \frac{1}{n} B(2\pi) \quad \text{and} \quad b \in \mathcal{B}_{2\pi,n}.$$

□

6 Circular annuli

In this section we give the necessary and sufficient condition for a circular annulus to be a Poncelet annulus. We consider two circles C and D defined as follows:

$$\begin{cases} C : x^2 + y^2 = r^2, \\ D : (x - a)^2 + y^2 = R^2, \\ 0 < a < R - r. \end{cases} \quad (6.1)$$

and the circular annulus CD will be parametrized by

$$\begin{cases} C : z(t) = re^{it} \\ D : w(t) = z(t) + \lambda(t)ie^{it} \quad \text{for } t \in [0, 2\pi], \end{cases} \quad (6.2)$$

where a function λ is given by the formula

$$\lambda(t) = \sqrt{R^2 - (r - a \cos t)^2} - a \sin t. \quad (6.3)$$

Let

$$\sigma(t) = \frac{1}{R}(r - a \cos t). \quad (6.4)$$

We note that

$$\lambda' = \frac{\sigma}{\sqrt{1 - \sigma^2}}\lambda - r. \quad (6.5)$$

Theorem 6.1 *A circular annulus bounded by the circles $x^2 + y^2 = r^2$, $(x - a)^2 + y^2 = R^2$, where $0 < a < R - r$ is a Poncelet's annulus if and only if the number*

$$n = \frac{B(2\pi)}{B\left(2 \arctan \frac{1}{r} \sqrt{R^2 - (r - a)^2}\right)} \quad (6.6)$$

is a positive integer, where a function b determining the function B is given by

$$b(t) = \left[1 - \left(\frac{r - a \cos t}{R} \right)^2 \right]^{\frac{-1}{2}} \quad (6.7)$$

and belongs to the class $\mathcal{B}_{2\pi, n}$.

Proof The tangent line to the circle C at a point re^{it} intersects the circle D at a point $w(t) = z(t) + \lambda(t)ie^{it}$. We pass by $w(t)$ the second tangent line to C . It intersects C at a point with a parameter which will be denoted by $\varphi(t)$. It is easy to see that

$$\tan \frac{\varphi(t) - t}{2} = \frac{\lambda(t)}{r} \quad (6.8)$$

Making use of (6.8) and (6.5) we obtain

$$\varphi' = \frac{2r\sigma\lambda - (r^2 - \lambda^2)\sqrt{1 - \sigma^2}}{(r^2 + \lambda^2)\sqrt{1 - \sigma^2}}. \quad (6.9)$$

The equality (6.4) implies immediately

$$\begin{aligned} \sigma \circ \varphi &= \frac{r}{R} - \frac{a}{R} \cos \varphi \\ &= \frac{r}{R} - \frac{a}{R} \left(\frac{r^2 - \lambda^2}{r^2 + \lambda^2} \cos t - \frac{2r\lambda}{r^2 + \lambda^2} \sin t \right). \end{aligned}$$

Hence we have

$$\begin{aligned} R(r^2 + \lambda^2)\sigma \circ \varphi &= r(r^2 + \lambda^2) - (r^2 - \lambda^2)a \cos t + 2r\lambda a \sin t \\ &= r(r^2 + \lambda^2) - (r^2 - \lambda^2)(r - R\sigma) + 2r\lambda R\sigma' \\ &= 2r\lambda^2 + 2r\lambda R\sigma' + (r^2 - \lambda^2)R\sigma. \end{aligned}$$

We note that

$$\lambda = R\sqrt{1 - \sigma^2} - R\sigma'. \quad (6.10)$$

Thus we have

$$\begin{aligned} (r^2 + \lambda^2)R \cdot \sigma \circ \varphi &= 2r\lambda(\lambda + R\sigma') + (r^2 - \lambda^2)R\sigma \\ &= 2r\lambda R\sqrt{1 - \sigma^2} + (r^2 - \lambda^2)R\sigma \end{aligned}$$

i.e.

$$\sigma \circ \varphi = \frac{2r\lambda\sqrt{1 - \sigma^2} + (r^2 - \lambda^2)\sigma}{r^2 + \lambda^2}. \quad (6.11)$$

Similarly we have

$$\begin{aligned} &(r^2 + \lambda^2)^2 [1 - (\sigma \circ \varphi)^2] \\ &= (r^2 + \lambda^2)^2 \left(1 - \left[\frac{2r\lambda\sqrt{1 - \sigma^2} + (r^2 - \lambda^2)\sigma}{r^2 + \lambda^2} \right]^2 \right) \\ &= r^4 + 2r^2\lambda^2 + \lambda^4 - 4r^2\lambda^2(1 - \sigma^2) - 4r\lambda\sqrt{1 - \sigma^2}(r^2 - \lambda^2)\sigma - (r^2 - \lambda^2)^2\sigma^2 \\ &= r^4 + 2r^2\lambda^2 + \lambda^4 - 4r^2\lambda^2 + 4r^2\lambda^2\sigma^2 - 4r\lambda\sqrt{1 - \sigma^2}(r^2 - \lambda^2)\sigma - (r^2 - \lambda^2)^2\sigma^2 \\ &= (r^2 - \lambda^2)^2 - 4r\lambda\sqrt{1 - \sigma^2}(r^2 - \lambda^2)\sigma + 4r^2\lambda^2\sigma^2 - (r^2 - \lambda^2)^2\sigma^2 \end{aligned}$$

$$\begin{aligned}
 &= (1 - \sigma^2)(r^2 - \lambda^2)^2 - 2\sqrt{1 - \sigma^2}(r^2 - \lambda^2) \cdot 2r\lambda\sigma + 4r^2\lambda^2\sigma^2 \\
 &= (2r\lambda\sigma - (r^2 - \lambda^2)\sqrt{1 - \sigma^2})^2
 \end{aligned}$$

and

$$\sqrt{1 - (\sigma \circ \varphi)^2} = \frac{2r\lambda\sigma - (r^2 - \lambda^2)\sqrt{1 - \sigma^2}}{r^2 + \lambda^2}. \quad (6.12)$$

Comparing (6.9) and (6.12) we get

$$\varphi' = \frac{\sqrt{1 - (\sigma \circ \varphi)^2}}{\sqrt{1 - \sigma^2}}. \quad (6.13)$$

The coincidence of the value of $\varphi(0)$ given by formula (2.15) for b given by (6.7) and the value of $\varphi(0)$ resulting from the formula (6.8) lead us to the formula (6.6). $\square$

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