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Gaussian maps for singular curves on Enriques surfaces

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Abstract

A marked Prym curve is a triple (C, α, T_d) where C is a smooth algebraic curve, α is a $2-$torsion line bundle on C, and T_d is a divisor of degree d. We give obstructions—in terms of Gaussian maps—for a marked Prym curve (C, α, T_d) to admit a singular model lying on an Enriques surface with only one ordinary singular point of multiplicity d, such that T_d is the pull-back of the singular point by the normalization map. More precisely, let (S, H) be a polarized Enriques surface and let (C, f) be a smooth curve together with a morphism $f : C \to S$ birational onto its image and such that $f(C) \in |H|$, $f(C)$ has exactly one ordinary singular point of multiplicity d. Let $\alpha = f^*\omega_S$ and T_d be the divisor over the singular point of $f(C)$. We show that if H is sufficiently positive then certain natural Gaussian maps on C, associated with ω_C, α, and T_d are not surjective. On the contrary, we show that for the general triple in the moduli space of marked Prym curves (C, α, T_d), the same Gaussian maps are surjective.

Keywords Curves on surfaces · Gaussian maps · Enriques surfaces · Moduli space of curves

1 Introduction

The article deals with the problem of finding obstructions—in terms of Gaussian maps—for a Prym curve (C, α) to admit a singular model (with prescribed singularities) in a polarized Enriques surface (S, H). Let us briefly introduce the setting. Let X be a smooth complex projective variety, and let L and M be two invertible sheaves on X. Denote by $R(L, M)$ the kernel of the multiplication map $\Phi^0_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$. The first Gaussian map associated with L, M is the map

$$\Phi_{X,L,M} : \mathrm{R}(L, M) \to H^0(X, \Omega^1_X \otimes L \otimes M)$$

locally defined as $\Phi_{X,L,M}(s \otimes t) = sdt - tds$. If $L = M$ one usually writes $\Phi_{X,L}$ and since it vanishes on symmetric tensors, it is equivalent to study its restriction to $\wedge^2 H^0(L)$. Gaussian maps were introduced by Wahl who showed, in [41], that if (S', H') is a polarized $K3$ surface and $C' \in |H'|$, then the Gaussian map (also called Wahl map) Φ_{ω_C} is not surjective (see also [6] for a different proof). On the other hand Ciliberto, Harris and Miranda proved in [10] that the Wahl map is surjective as soon as it is possible by counting dimensions, with the

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exception $g = 10$ i.e. $g \geq 11$. The converse also holds. In [1] Arbarello, Bruno and Sernesi proved that a Brill–Noether–Petri general canonical curve with non-surjective Gaussian map lies in a $K3$ surface or on a limit thereof. See also [9] for a result relating the corank of the Gaussian map and r-extendibility. Analogous problems for Enriques surfaces have also been studied by some authors. Indeed, let (S, H) be a polarized Enriques surface and let $C \in |H|$ be a smooth curve and $\alpha := \omega_{S_{|_C}}$. In [3] it is proven that the Gaussian map $\Phi_{\omega_C, \omega_C \otimes \alpha}$ is not surjective, whereas in [14] it is shown that for the general Prym curve (C, α) of genus $g \geq 12$, $g \neq 13, 19$ the map is surjective. Gaussian maps have been studied and used by many authors, either in relation to extendibility questions, we mention e.g. [3–38] (see also [33] for a complete survey), or in relation to the second fundamental form of Torelli-type immersions, e.g. [16–18, 20, 24]. The result of Wahl was generalized by some authors, e.g. Zak-L'Vovsky, who proved the following theorem, that we are going to use.

Theorem 1.1 ([32]) *Let C be a smooth curve of genus $g > 0$ and let A be a very ample line bundle on C, embedding C in $\mathbb{P}^n$ for $n \geq 3$. If $C \subset \mathbb{P}^n$ is a hyperplane section of a smooth surface $X \subset \mathbb{P}^{n+1}$, then the Gaussian map $\Phi_{\omega_C, A}$ is not surjective.*

Similar questions for singular curves on $K3$ surfaces are discussed and solved by Kemeny in [27]. In the article the author asks whether one can give an obstruction in terms of suitable Gaussian maps for a curve to have a nodal model lying on a $K3$ surface. Following the author notations, denote by $\bar{\mathcal{M}}_{h,2l}$ the stack of smooth curve of genus h with $2l$ marked points and by $\widetilde{\mathcal{M}}_{h,2l} = \bar{\mathcal{M}}_{h,2l}/S_{2l}$ the stack of curves with unordered marking. Let h, l be two positive integers and $[(C, T)] \in \widetilde{\mathcal{M}}_{h,2l}$. The author introduces the marked Wahl map:

$$W_{C,T} : \bigwedge^2 H^0(C, \omega_C(-T)) \to H^0(C, \omega_C^{\otimes 3}(-2T)). \tag{1.1}$$

Then the following theorems are proven.

Theorem 1.2 ([27]) *Fix any integer $l \in \mathbb{Z}$. Then there exists infinitely many integers $h(l)$, such that the general marked $[(C, T)] \in \widetilde{\mathcal{M}}_{h(l),2l}$ has surjective marked Wahl map.*

Now denote by $\mathcal{V}^n_{g,k}$ the stack parametrizing morphisms $[(f : C \to X, L)]$ where (X, L) is a polarized $K3$ surface with $L^2 = 2g-2$, C is a smooth connected curve of genus $p(g,k)-n$ with $p(g,k) := k^2(g-1)+1$, f is birational onto its image and $f(C) \in |kL|$ is nodal.

Theorem 1.3 ([27]) *Assume $g - n \geq 13$ for $k = 1$ or $g \geq 8$ for $k > 1$, and let $n \leq \frac{p(g,k)-2}{5}$. Then there is an irreducible component $I^0 \subseteq \mathcal{V}^n_{g,k}$ such that for a general $[(f : C \to X, L)] \in I^0$ the marked Wahl map $W_{C,T}$ is non surjective, where $T \subseteq C$ is the divisor over the nodes of $f(C)$.*

The same marked Wahl maps have been studied by Fontanari and Sernesi in [34], where they proved, using very different methods from [27], the following theorem.

Theorem 1.4 ([34]) *Fix an integer $g \geq 9$ Let (S, H) be a polarized $K3$ surface with $Pic(S) = \mathbb{Z}H$ and $H^2 = 2g-2$. Let C be a smooth curve of genus $g-1$ endowed with a morphism $f : C \to S$ birational onto its image and such that $f(C) \in |H|$. If $T = P + Q \subseteq C$ is the divisor over the singular point of $f(C)$, then the Gaussian map $\Phi_{\omega_C(-T), \omega_C(-T)}$ is not surjective.*

Our paper deals with a similar problem for singular curves on Enriques surfaces. Let (S, H) be a polarized Enriques surface and C a smooth curve having a morphism $f : C \to S$

birational onto its image and such that $f(C) \in |H|$ has exactly one ordinary singular point of multiplicity d. Denote by T_d the divisor over the singular point and set $\alpha = f^*K_S$. Then (C, α, T_d) is a marked Prym curve. We investigate the behaviour of the following mixed Gaussian-Prym maps:

$$\Phi_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)} : \mathrm{R}(\omega_C(-T_d), \omega_C(-T_d+\alpha)) \to H^0(C, \omega_C^{\otimes 3}(-2T_d+\alpha)) \quad (1.2)$$

and

$$\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)} : \mathrm{R}(\omega_C, \omega_C(-T_d+\alpha)) \to H^0(C, \omega_C^{\otimes 3}(-T_d+\alpha)). \quad (1.3)$$

More precisely we have the following.

Theorem 1.5 *Let (S, H) be a polarized Enriques surface with $H^2 = 2g-2$. Fix an integer $d \geq 2$, and suppose that either*

(i) S is a very general Enriques surface and $\varphi(H) \geq \sqrt{2}(d+2)$, or
(ii) S in unnodal and $\varphi(H) \geq 2(d+1)$.

Set $g' = g - \binom{d}{2}$ and let C be a smooth curve of genus g' having a birational morphism $f : C \to S$ onto its image and such that $f(C) \in |H|$, $f(C)$ has exactly one ordinary singular point of multiplicity d. Set $\alpha = f^*K_{S_{|C}}$ and let $T_d = p_1 + ... + p_d$ be the divisor over the singular point. Then the Gaussian maps $\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)}$ and $\Phi_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)}$ are not surjective.

Here φ is a measure of the positivity of line bundles on the Enriques surface S, and it is defined as:

$$\varphi(H) := \min\{|H \cdot F| : F \in \mathrm{Pic}(S), F^2 = 0, F \not\equiv 0\}.$$

In the statement of Theorem 1.5, with "general", we mean in a non empty Zariski-open subset of the moduli space , with "very general" we mean outside a countable union of proper Zariski-closed subsets. The proof is along the same lines of Theorem 1.4.

On the contrary, when one considers a general marked Prym curve, the aforementioned maps are "tendentially" surjective. Indeed, let S be the following set:

$$S := \{(g_1, d_1, d_2) : g_1 \geq 3, d_2 \geq 4, d_2(g_1 - 2) > d_1 \geq g_1 + 5, d_1 > d_2\}, \quad (1.4)$$

and denote by $R_{g,d}$ the coarse moduli space of $d-$marked Prym curves. We prove the following.

Theorem 1.6 *Let (g_1, d_1, d_2) be in S (1.4), and $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. Let d be an integer such that $2 \leq d \leq d_2$. If (C, α, T_d) is a general point in $R_{g,d}$, then the Gaussian maps*

$$\Phi_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)}$$

and

$$\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)}$$

are surjective.

In case $d = 2, 3$ or $d = 4$ (see Example 1) we obtain the surjectivity for all genera greater than or equal to 76. More generally, for every $d \geq 2$, we obtain infinitely many genera for which the marked Gaussian maps (we are considering) are surjective. We expect our result far from being sharp (see Remark 6.3).

We briefly explain how the paper is organized. In Sect. 2 we recall the definition of Gaussian maps and prove Proposition 2.2, which is a slight generalization of Theorem 8, [34] (see also Theorem 9). This is a result relating the cokernels of Gaussian maps associated with different line bundles. In Sect. 3 we prove Theorem 1.5, following the same strategy of the proof of Theorem 1.4 ([34]). More precisely, one of the main technical tool is an ampleness result for line bundles on the blow-up at a point of an Enriques surface. This is Proposition 3.2. The study of very ample line bundles on the blow-up at a point of an Enriques surface, gives in turn the existence of curves with exactly one ordinary singular point of any given multiplicity. This is Corollary 3.4. In Sect. 4 we prove the surjectivity of the marked Prym-Gaussian maps for a certain class of $d-$ marked Prym curves living in the product of two curves. In Sect. 5 we give a lower bound for the gonality of curves living in the product of a curve with $\mathbb{P}^1$ (see Proposition 5.1), and we prove a lemma about very ample line bundles on curves. The results contained in Sects. 4 and 5 are then used in the proof of Theorem 1.6, which can be found in Sect. 6.

2 Cokernels of Wahl maps

We briefly recall the definition of Gaussian maps, and their different interpretations, which will be used in the sequel. See for example [39] or [40] for the details. Let X be a smooth complex algebraic variety. Let L and M be two line bundles on X. Let $q_i : X \times X \to X$, $i = 1, 2$ be the two projections. Let $I_{\Delta_{X\times X}}$ be the ideal of the diagonal $\Delta_{X\times X}$ in $X \times X$. Consider the short exact sequence given by the inclusion $I^2_{\Delta_{X\times X}} \to I_{\Delta_{X\times X}}$, and tensor it with $q_1^*L \otimes q_2^*M$, which we denote by $L \boxtimes M$.

$$0 \to I^2_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M \to I_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M \to I_{\Delta_{X\times X}}/I^2_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M \to 0. \tag{2.1}$$

The first Gaussian map associated with L and M is defined as the map induced at the level of global sections:

$$\Phi_{L,M} : H^0(X \times X, I_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M) \to H^0(X \times X, I_{\Delta_{X\times X}}/I^2_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M).$$

Now let $\Phi^0_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ be the multiplication map and denote by $R(L, M)$ its kernel. Using standard identifications, $\Phi_{L,M}$ can be thought as a map

$$R(L, M) \to H^0(X, \Omega^1_X \otimes L \otimes M).$$

If $\alpha = \sum l_i \otimes m_i \in \mathrm{Ker}(\Phi_{L,M})$, $l_i = f_i S$, $m_i = s_i T$, where S and T are two local generators of L and M, respectively, $\Phi_{L,M}$ is locally given by $\Phi_{L,M}(\alpha) = \sum (f_i dg_i - g_i df_i) S \otimes T$.

Another useful description of the first Gaussian map associated with two line bundles L and M is obtained when L is a very ample line bundle giving an embedding $\varphi_L : X \hookrightarrow \mathbb{P}^r$. Denote by M_L the kernel of the evaluation map of sections of L, i.e.:

$$0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0.$$

Then $\varphi_L^* \Omega^1_{\mathbb{P}^r}(1) = \Omega^1_{\mathbb{P}^r}(1)_{|X} \simeq M_L$. Consider indeed the Euler sequence

$$0 \to \Omega_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(-1)^{r+1} \to \mathcal{O}_{\mathbb{P}^r} \to 0,$$

and tensor it with $\mathcal{O}_{P^r}(1)$:

$$0 \to \Omega^1_{\mathbb{P}^r}(1) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \otimes \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1) \to 0.$$

Pulling it back by φ_L we obtain

$$0 \to \varphi_L^* \Omega_{\mathbb{P}^r}(1) \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0,$$

and so we conclude. Now consider a twist by $L \otimes M$ of the conormal exact sequence:

$$0 \to N^\vee_{X/\mathbb{P}^r} \otimes L \otimes M \to M_L \otimes M \to \Omega_X \otimes L \otimes M \to 0, \qquad (2.2)$$

One can show that under the aforementioned identification,

$$\Phi_{L,M} : H^0(X, M_L \otimes M) \to H^0(X, \Omega_X \otimes L \otimes M),$$

i.e. $\Phi_{L,M}$ is the map induced at the level of global sections in 2.2. Now we recall a very useful construction of Lazarsfeld.

Proposition 2.1 (Lemma 1.4.1, [36])) *Let $p_1, \ldots, p_n \in X$ be distinct points such that $L(-\sum_{i=1}^n p_i)$ is generated by global sections, and $h^1(L(-\sum_{i=1}^n p_i)) = h^1(L)$. Then one has an exact sequence:*

$$0 \to M_{L(-\sum_{i=1}^n p_i)} \to M_L \to \bigoplus_{i=1}^n \mathcal{O}_C(-p_i) \to 0. \qquad (2.3)$$

We now observe that a slight modification of [34],Theorem 8, gives the following result which relates the cokernels of Gaussian maps in different embeddings. In the following $X = C$ will be a smooth complex algebraic curve.

Proposition 2.2 *Let C be a smooth complex projective algebraic curve. Let $T_n = p_1 + \ldots + p_n$ be an effective divisor of degree n on C with $p_i \neq p_j$ for $i \neq j$. Let L and M be two very ample line bundles such that $L - T_n$ is very ample and $h^1(L) = h^1(L - T_n)$. Then there exists a surjection between the cokernels of the Gaussian maps:*

$$\operatorname{coker}(\Phi_{L-T_n,M}) \to \operatorname{coker}(\Phi_{L,M}).$$

The proof follows the same steps of [34],Theorem 8. We present it for completeness.

Proof Consider the following commutative diagram with exact rows and columns. The first two rows are (2.2) for the line bundles L and $L - T_n$, the second column is (2.3) twisted by M, and the third column is just the restriction modulo the identification $\omega_C \otimes L \otimes \mathcal{O}_T \simeq \bigoplus_{i=1}^n \mathcal{O}_{p_i}(-p_i)$, and then twisted by M.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^\vee_{C/\mathbb{P}^{r-n}} \otimes L(-T_n) \otimes M & \longrightarrow & M_{L(-T_n)} \otimes M & \longrightarrow & \omega_C \otimes L(-T_n) \otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^\vee_{C/\mathbb{P}^r} \otimes L \otimes M & \longrightarrow & M_L \otimes M & \longrightarrow & \omega_C \otimes L \otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{i=1}^n M(-2p_i) & \xrightarrow{g} & \bigoplus_{i=1}^n M(-p_i) & \longrightarrow & \bigoplus_{i=1}^n M_{|p_i}(-p_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Passing to cohomology we obtain

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{coker}(\Phi_{L(-T_n)}, M) & \longrightarrow & H^1(N^\vee_{C/\mathbb{P}^{r-n}} \otimes L \otimes (-T_n) \otimes M) & \longrightarrow & H^1(M_{L(-T_n)}) \otimes M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{coker}(\Phi_{L,M}) & \longrightarrow & H^1(N^\vee_{C/\mathbb{P}^r} \otimes L \otimes M) & \longrightarrow & H^1(M_L \otimes M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(H^1(g)) & \longrightarrow & H^1(\bigoplus M(-2p_i)) & \xrightarrow{H^1(g)} & H^1(\bigoplus M(-p_i)) & \longrightarrow & 0
\end{array}
$$

Being M very ample we have that $h^1(\bigoplus M(-2p_i)) = h^1(\bigoplus M(-p_i))$. Then $\ker(H^1(g)) = 0$. □

3 Non surjectivity

In this section we are going to prove Theorem 1.5. We proceed in a similar way as in [34]: we will obtain the non-surjectivity result applying Theorem (1.1), together with a result about very ampleness of line bundles on the blow-up at a point of an Enriques surface.

Let S be an Enriques surface. First we recall the definition of two measures of the positivity of line bundles on S: the $\varphi-$function and the Seshadri constant of a big and nef line bundle H on S. The first one is defined as

$$\varphi(H) := \min\{|H \cdot F| : F \in \operatorname{Pic}(S),\ F^2 = 0,\ F \not\equiv 0\},$$

where $\equiv$ denotes the numerical equivalence relation. Now set $\epsilon(H, x) := \inf_{x \in C} \frac{H \cdot C}{\operatorname{mult}_x C}$, where the infimum is taken over all curves C passing through x. The Seshadri constant $\epsilon(H)$ of H is defined as

$$\epsilon(H) := \inf_{x \in X} \epsilon(H, x).$$

We have the following inequalities:

$$0 \le \epsilon(H)^2 \le \varphi(H)^2 \le H^2. \tag{3.1}$$

For background and proofs see for example [21]. Now let $\sigma : S' \to S$ be the blow-up at a point p, and let E be the exceptional divisor. We will now give, in terms of φ, sufficient conditions for a line bundle of the form $\sigma^* H - lE$ to be big and nef.

In the following, when we say a "very general" Enriques surface, we mean that as a point in the moduli space of Enriques surfaces, it lives outside a countable union Zariski-closed subsets. We also recall that a nodal Enriques surface is one which contains -2 curves. In the moduli space of Enriques surfaces these correspond to a divisor. An Enriques surface which does not contain any -2 curves is usually called unnodal.

Proposition 3.1 *Let S be an Enriques surface and $l \ge 1$ be an integer. Let H be a big and nef line bundle on S and suppose that one of the following holds:*

i) S is a very general Enriques surface, $\varphi(H) = l$ and H is not of the type $H \equiv \frac{l}{2}(E_1 + E_2)$ with E_i, $i = 1, 2$, effective isotropic divisors such that $E_1 \cdot E_2 = 2$.
ii) S is a very general Enriques surface and $\varphi(H) \ge l + 1$.
iii) S is unnodal and $\varphi(H) \ge 2l$.

Then $\sigma^* H - lE$ is big and nef.

Proof First we show that $\sigma^*H - lE$ is nef. From [37], Proposition 5.1.5., it follows that $\sigma^*(H) - lE$ is nef if and only if $\epsilon(H) \geq l$. In [26], Theorem 1.3, it is shown that if S is a very general Enriques surface then $\varphi(H) = \epsilon(H)$. Then, in case i) or ii) we conclude. Now suppose we are in situation iii). From [26], Corollary 4.5, it follows that $\epsilon(H) \geq \frac{1}{2}\varphi(H) \geq l$ and we immediately conclude.

From 3.1 and the hypothesis $l \geq 1$, in case ii) and iii) we get $H^2 \geq \varphi(H)^2 > l^2$ and hence $\sigma^*H - lE$ is also big. Consider now the situation i) and suppose that $\sigma^*H - lE$ is not big, i.e. $H^2 = l^2$. Then, again by 3.1, we have $H^2 = \varphi(H)^2 = l^2$. By [29], Proposition 1.4, we must have $H \equiv l(E_1 + E_2)$, where E_i, $i = 1, 2$ are isotropic effective divisors such that $E_1 \cdot E_2 = 2$. □

The proof of the next result is a direct application of Reider's Theorem (see [42], Theorem 1, or [21], Theorem 2.4.5).

Proposition 3.2 *Let $l \geq 1$ and let (S, H) be a polarized Enriques surface. Suppose that either*

(i) S is a very general Enriques surface, $l \geq 1$ and $\varphi(H) \geq \sqrt{2}(l+2)$, or
(ii) S in unnodal, $l = 1$ and $\varphi(H) \geq 3\sqrt{2}$, or $l \geq 2$ and $\varphi(H) \geq 2(l+1)$.

Let $\sigma : S' \to S$ be the blow-up at a point and E the exceptional divisor. Then $\sigma^*H - lE$ is a very ample line bundle on S'.

Proof First observe that $\sigma^*H - lE = \sigma^*(H + K_S) - (l+1)E + K_{S'}$. Set $H' = H + K_S$. By Proposition 3.1, $\sigma^*H' - (l+1)E$ is big and nef. Indeed $\varphi(H') = \varphi(H) \geq \sqrt{2}(l+2) \geq l+2$ in case (i) and $\varphi(H') \geq l+2$ in case (ii). Observe that it is also effective. Indeed suppose by contradiction it is not. Then, by Riemann-Roch and Serre duality, $K_{S'} \otimes (\sigma^*H' - (l+1)E)^\vee = -(\sigma^*H - (l+2)E)$ is effective. Now take a nef effective divisor L in S. Since σ^*L is also nef we obtain $0 \leq \sigma^*L \cdot (-(\sigma^*H - (l+2)E)) = -L \cdot H < 0$, where the latter inequality follows from the fact that H is ample and L effective. We conclude that $\sigma^*H' - (l+1)E$ is effective. Now suppose by contradiction that $\sigma^*H - lE$ is not very ample. Since $\sigma^*H' - (l+1)E$ is an effective, big and nef divisor and $H^2 \geq \varphi(H)^2 \geq 9 + (l+1)^2$ in both cases (i) and (ii), we can apply Reider's theorem. Then there exists a non trivial effective divisor D in S' such that either one of the following holds:

(a) $D^2 = 0$ and $(\sigma^*H' - (l+1)E)D \leq 2$;
(b) $D^2 = -1$ and $(\sigma^*H' - (l+1)E)D \leq 1$;
(c) $D^2 = -2$ and $(\sigma^*H' - (l+1)E)D = 0$;
(d) $(\sigma^*H' - (l+1)E)^2 = 9$, $D^2 = 1$ and $(\sigma^*H' - (l+1)E) \equiv 3D$ in $Num(S')$.

Now we show that none of these can happen.

Let $D \sim \sigma^*L - aE$, for some $L \in \mathrm{Pic}(S)$ and $a \in \mathbb{Z}$. Suppose we are in case (a). Then we have $H'L \leq (l+1)a + 2$ and $L^2 = a^2$. Therefore we obtain the following inequalities:

$$\varphi(H')^2 a^2 \leq H'^2 a^2 = H'^2 L^2 \leq (H' \cdot L)^2 \leq ((l+1)a + 2)^2, \tag{3.2}$$

where the second inequality follows by the Hodge index theorem. If $|a| \geq 2$ we obtain

$$\varphi(H') \leq |\frac{(l+1)a + 2}{a}| \leq (l+1) + |\frac{2}{a}| \leq (l+1) + 1,$$

which contradicts the hypothesis. If $|a| = 1$ from 3.2 we get $\varphi(H) = \varphi(H') \leq (l+1) + 2$ which again is not possible.

If $a = 0$ we get $D = \sigma^* L$ with L effective, not numerically trivial and such that $L^2 = 0$ and $H'L \leq 2$. This gives $\varphi(H) \leq 2$ and we conclude.

Suppose now we are in case (b). As before one have $L^2 = a^2 - 1$, $H'L \leq a(l+1) + 1$. Therefore we obtain

$$\varphi(H')^2(a^2-1) \leq H'^2(a^2-1) = H'^2 L^2 \leq (H' \cdot L)^2 \leq (a(l+1)+1)^2.$$

If $|a| \geq 2$ we find $\varphi(H') < \sqrt{2}(l+2)$. If $a = 1$ then L is an effective divisor such that $L^2 = 0$ and $H'L \leq l+2$. Moreover observe that L is not numerically trivial since otherwise $D \equiv -E$, which is not possibile because D is an effective non trivial divisor. Therefore we obtain $\varphi(H') \leq l+2$. $a = -1$ cannot happen if $l \geq 1$ because H' is nef and L is effective and $L \cdot H' = -l$. If $a = 0$ then $L^2 = -1$. This is not possibile for Enriques surfaces.

Suppose now we are in case (c). Then $H'L = a(l+1)$ and $L^2 = a^2 - 2$. Then, as before,

$$\varphi(H')^2(a^2-2) \leq H'^2(a^2-2) = H'^2 L^2 \leq (H' \cdot L)^2 \leq a^2(l+1)^2.$$

Observe that if $|a| \geq 2$ this gives $\varphi(H') \leq \sqrt{2}(l+1)$ and hence we conclude. Observe that $|a| = 1$ cannot happen because otherwise $L^2 = -1$ and this, again, is not possible. If $a = 0$ then L is a effective divisor such that $L^2 = -2$ and $H'L = 0$. This cannot happen because $H' \cdot L = (H + K_S) \cdot L$, H is ample and L is effective.

Suppose we are now in case (d). Then $H'^2 = 9 + (l+1)^2$ which is not possible since $H'^2 \geq \varphi(H)^2 > 9 + (l+1)^2$ by hypothesis. □

Corollary 3.3 *With the same hypothesis of the previous result we have* $\sigma^* H - lE + \sigma^* K_S = \sigma^*(H + K_S) - lE$ *is very ample on* S'.

Proof Apply Proposition 3.2 with $H + K_S$ instead of H. □

We observe that Proposition 3.2 has the following corollary.

Corollary 3.4 *Let* $l \geq 2$ *and let* (S, H) *be a polarized Enriques surface. Suppose that either*

(i) S *is a very general Enriques surface and* $\varphi(H) \geq \sqrt{2}(l+2)$, *or*
(ii) S *is unnodal and* $\varphi(H) \geq 2(l+1)$.

Then there exists a curve C *in the linear system* $|H|$ *with an ordinary singular point of multiplicity* l.

Now we conclude with the proof of Theorem 1.5.

Proof of theorem 1.5 Let $\sigma : S' \to S$ be the blow-up at a point, and E the exceptional divisor. By the universal property of normalization we can suppose that $C \in |\sigma^* H - dE|$ and $\alpha = \sigma^* K_{S_{|C}}$. From Proposition 3.2 it follows that $\mathcal{O}_C(C) = \omega_C(-T_d + \alpha)$ is very ample. Observe that $h^0(C, \mathcal{O}_C(C)) = h^0(S', \mathcal{O}_{S'}(C)) - 1$. Applying Theorem 1.1 we obtain that $\Phi_{\omega_C, \omega_C(-T_d+\alpha)}$ is not surjective.

Now we want to prove that also $\Phi_{\omega_C(-T_d), \omega_C(-T_d+\alpha)}$ is not surjective using Proposition 2.2 with $L = \omega_C$, $M = \omega_C(-T_d + \alpha)$ and $n = d$. Observe that since $\mathcal{O}_{S'}(C + K_{S'}) \simeq \mathcal{O}_{S'}(\sigma^*(H + K_S) - (d-1)E)$, $\omega_C \simeq \mathcal{O}_C(C + K_{S'})$ is very ample by Corollary 3.3. Analogously $\mathcal{O}_C(C + \sigma^* K_S) \simeq \omega_C(-T_d)$ is very ample. It remains to show that $h^1(\omega_C) = h^1(\omega_C(-T_d))$, or equivalently that $h^0(\omega_C(-T_d)) = h^0(\omega_C) - d$. Consider then the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{S'}(K'_S - E) & \longrightarrow & \mathcal{O}_{S'}(K_{S'}) & \longrightarrow & \mathcal{O}_E(K_{S'}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{S'}(C + K_{S'} - E) & \longrightarrow & \mathcal{O}_{S'}(C + K_{S'}) & \longrightarrow & \mathcal{O}_E(C + K_{S'}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega_C(-T_d) & \longrightarrow & \omega_C & \longrightarrow & \bigoplus_{i=1}^d \mathcal{O}_{p_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

and the one induced at the level of global sections:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{S'}(C + K_{S'} - E)) & \longrightarrow & H^0(\mathcal{O}_{S'}(C + K_{S'})) & \longrightarrow & \mathbb{C}^d & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\omega_C(-T_d)) & \longrightarrow & H^0(\omega_C) & \longrightarrow & \mathbb{C}^d & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where we are using that $H^0(\mathcal{O}_{S'}(K_{S'})) \simeq H^0(\mathcal{O}_S(K_S)) = 0$, $E \simeq \mathbb{P}^1$ and $\mathcal{O}_E(K_{S'})$ is a divisor of degree -1 in $E \simeq \mathbb{P}^1$, $h^1(\mathcal{O}_{S'}(C + K_{S'} - E)) = 0$ by Kawamata vanishing theorem since $\mathcal{O}_{S'}(C - E) = \mathcal{O}_{S'}(\sigma^* H - (d+1)E)$ is big and nef. $H^1(K_{S'}) \simeq H^1(\mathcal{O}_{S'}) = 0$ because S is an Enriques surface and S' is a blow-up. Hence we conclude that $h^0(\omega_C(-T_d)) = h^0(\omega_C) - d$. □

4 Surjectivity for special curves

The results contained in this section will be used in the proof of Theorem 1.6. We start with a proposition giving sufficient conditions for the surjectivity of mixed Gaussian maps on a surface X which is the product of two curves. The central idea to study Gaussian maps on X is to relate them with Gaussian maps on the curves. This idea dates back to Wahl ([40], Lemma 4.12). See also Colombo-Frediani ([18], Theorem 3.1) for the second Wahl map."

Proposition 4.1 *Let $X = C_1 \times C_2$. Let $p_i : X = C_1 \times C_2 \to C_i$, $i = 1, 2$ be the projections. Let L_i and M_i be line bundles on C_i, $i = 1, 2$, such that $deg(L_i), deg(M_i) \geq 2g_i + 2$ and $deg(L_i) + deg(M_i) \geq 6g_i + 3$, for $i = 1, 2$. Let $L = p_1^* L_1 \otimes p_2^* L_2$ and $M = p_1^* M_1 \otimes p_2^* M_2$. Then $\Phi_{X,L,M}$ is surjective.*

Proof We want to relate the Gaussian map $\Phi_{X,L,M}$ with some Gaussian maps on C_i, $i = 1, 2$. Let $q_i : X \times X \to X$, $i = 1, 2$, be the two projections. Recall that $\Phi_{X,L,M}$ is given by:

$$\Phi_{X,L,M} : H^0(X \times X, I_{\Delta_{X \times X}} \otimes q_1^* L \otimes q_2^* M) \to H^0(X \times X, I_{\Delta_{X \times X}} / I^2_{\Delta_{X \times X}} \otimes q_1^* L \otimes q_2^* M)$$

Let $q_{i,1} : C_1 \times C_1 \to C_1$, $i = 1, 2$, be the projections and analogously $q_{i,2} : C_2 \times C_2 \to C_2$. Let (φ_1, φ_2) be the isomorphism which exchange factors:

$$X \times X = (C_1 \times C_2) \times (C_1 \times C_2) \xrightarrow{(\varphi_1, \varphi_2)} (C_1 \times C_1) \times (C_2 \times C_2),$$

i.e. $\varphi_i((x_1, x_2), (y_1, y_2) = (x_i, y_i)$. Observe that

$$I_{\Delta_{X\times X}} \simeq \varphi_1^* I_{\Delta_{C_1\times C_1}} + \varphi_2^* I_{\Delta_{C_2\times C_2}},$$

where $\varphi_i^* I_{\Delta_{C_i\times C_i}}$, $i = 1, 2$, are the inverse image ideal sheaves or equivalently the pullbacks sheaves (because projections are flat). Now consider the isomorphism of $\mathcal{O}_X$-modules:

$$\Omega_X^1 \simeq I_{\Delta_{X\times X}} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{\Delta_{X\times X}}.$$

Under this identification the decomposition

$$\Omega_X^1 \simeq p_1^*\Omega^1{}_{C_1} \oplus p_2^*\Omega^1{}_{C_2}$$

can be read as

$$I_{\Delta_{X\times X}} \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{\Delta_{X\times X}} \simeq (\varphi_1^* I_{\Delta_{C_1\times C_1}} \oplus \varphi_2^* I_{\Delta_{C_2\times C_2}}) \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{\Delta_{X\times X}}.$$

So we obtain the following commutative diagram:

$$\begin{array}{ccc}
(\varphi_1^* I_{\Delta_{C_1\times C_1}} \oplus \varphi_2^* I_{\Delta_{C_2\times C_2}}) \otimes q_1^*L \otimes q_2^*M & \longrightarrow & (\varphi_1^* I_{\Delta_{C_1\times C_1}} \oplus \varphi_2^* I_{\Delta_{C_2\times C_2}}) \otimes q_1^*L \otimes q_2^*M \otimes \mathcal{O}_{\Delta_{X\times X}} \\
\downarrow & & \downarrow\simeq \\
I_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M & \longrightarrow & I_{\Delta_{X\times X}}/I^2_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M
\end{array}$$

Taking global sections we obtain

$$\begin{array}{ccc}
H^0((\varphi_1^* I_{\Delta_{C_1\times C_1}} \oplus \varphi_2^* I_{\Delta_{C_2\times C_2}}) \otimes q_1^*L \otimes q_2^*M) & \xrightarrow{\psi} & H^0((\varphi_1^* I_{\Delta_{C_1\times C_1}} \oplus \varphi_2^* I_{\Delta_{C_2\times C_2}}) \otimes q_1^*L \otimes q_2^*M) \otimes \mathcal{O}_{\Delta_{X\times X}}) \\
\downarrow & & \downarrow\simeq \\
H^0(I_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M) & \xrightarrow{\Phi_{X,L,M}} & H^0(I_{\Delta_{X\times X}}/I^2_{\Delta_{X\times X}} \otimes q_1^*L \otimes q_2^*M)
\end{array}$$

In order to show that $\Phi_{X,L,M}$ is surjective we will show the surjectivity of ψ. Clearly ψ is surjective if each of the direct sum map is surjective:

$$\psi_1 : H^0(\varphi_1^* I_{\Delta_{C_1\times C_1}} \otimes q_1^*L \otimes q_2^*M) \to H^0((\varphi_1^* I_{\Delta_{C_1\times C_1}} \otimes q_1^*L \otimes q_2^*M) \otimes \mathcal{O}_{\Delta_{X\times X}})$$

and

$$\psi_2 : H^0(\varphi_2^* I_{\Delta_{C_2\times C_2}} \otimes q_1^*L \otimes q_2^*M) \to H^0((\varphi_2^* I_{\Delta_{C_2\times C_2}} \otimes q_1^*L \otimes q_2^*M) \otimes \mathcal{O}_{\Delta_{X\times X}})$$

Let us deal with the first map. The same argument will apply also to the second one. Observe that

$$p_j \circ q_i = q_{i,j} \circ \varphi_j.$$

Then we can write

$$\begin{aligned}
q_1^*L \otimes q_2^*M &= q_1^*(p_1^*L_1 \otimes p_2^*L_2) \otimes q_2^*(p_1^*M_1 \otimes p_2^*M_2) && (4.1)\\
&= \varphi_1^*(q_{1,1}^*L_1 \otimes q_{2,1}^*M_1) \otimes \varphi_2^*(q_{1,2}^*L_2 \otimes q_{2,2}^*M_2). && (4.2)
\end{aligned}$$

And so we obtain

$$\varphi_1^* I_{\Delta_{C_1\times C_1}} \otimes q_1^*L \otimes q_2^*M \simeq \varphi_1^*(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1)) \otimes \varphi_2^*((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2))$$

Using that $\mathcal{O}_{\Delta_{X\times X}} \simeq \varphi_1^*\mathcal{O}_{\Delta_{C_1\times C_1}} \otimes \varphi_2^*\mathcal{O}_{\Delta_{C_2\times C_2}}$, we also obtain

$$\begin{aligned}
&\varphi_1^* I_{\Delta_{C_1\times C_1}} \otimes q_1^*L \otimes q_2^*M \otimes \mathcal{O}_{\Delta_{X\times X}} \simeq \\
&\simeq \varphi_1^*(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1) \otimes \mathcal{O}_{\Delta_{C_1\times C_1}}) \\
&\otimes \varphi_2^*((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2) \otimes \mathcal{O}_{\Delta_{C_2\times C_2}})
\end{aligned}$$

So ψ_1 becomes a map:

$$H^0(\varphi_1^*(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1)) \otimes \varphi_2^*((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2)))$$
$$\downarrow$$
$$H^0(\varphi_1^*(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1) \otimes \mathcal{O}_{\Delta_{C_1\times C_1}}) \otimes \varphi_2^*((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2) \otimes \mathcal{O}_{\Delta_{C_2\times C_2}}))$$

Now using that $X \times X \xrightarrow{\simeq} (C_1 \times C_1) \times (C_2 \times C_2)$ and Künneth formula we get:

$$\begin{aligned}&H^0(X \times X, \varphi_1^*(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1)) \otimes \varphi_2^*((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2)))\\ &\quad\simeq H^0(C_1 \times C_1, I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1)) \otimes H^0(C_2 \times C_2, (q_{1,2}^*L_2 \otimes q_{2,2}^*M_2)),\end{aligned}$$

and

$$\begin{aligned}&H^0(\varphi_1^*(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1) \otimes \mathcal{O}_{\Delta_{C_1\times C_1}}) \otimes \varphi_2^*((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2)) \otimes \mathcal{O}_{\Delta_{C_2\times C_2}})\\ &\quad\simeq H^0(I_{\Delta_{C_1\times C_1}} \otimes (q_{1,1}^*L_1 \otimes q_{2,1}^*M_1) \otimes \mathcal{O}_{\Delta_{C_1\times C_1}})) \otimes H^0((q_{1,2}^*L_2 \otimes q_{2,2}^*M_2) \otimes \mathcal{O}_{\Delta_{C_2\times C_2}})).\end{aligned}$$

Under these identifications ψ_1 becomes:

$$H^0(I_{\Delta_{C_1\times C_1}} \otimes q_{1,1}^*L_1 \otimes q_{2,1}^*M_1) \otimes H^0(q_{1,2}^*L_2 \otimes q_{2,2}^*M_2)$$
$$\Big\downarrow \psi_1$$
$$H^0(I_{\Delta_{C_1\times C_1}} \otimes q_{1,1}^*L_1 \otimes q_{2,1}^*M_1 \otimes \mathcal{O}_{\Delta_{C_1\times C_1}}) \otimes H^0(q_{1,2}^*L_2 \otimes q_{2,2}^*M_2 \otimes \mathcal{O}_{\Delta_{C_2\times C_2}}),$$

and it is given by the tensor product $\Phi_{C_1,L_1,M_1} \otimes \Phi^0_{C_2,L_2,M_2}$, where

$$\begin{aligned}\Phi_{C_1,L_1,M_1} &: H^0(I_{\Delta_{C_1\times C_1}} \otimes q_{1,1}^*L_1 \otimes q_{2,1}^*M_1)\\ &\to H^0(I_{\Delta_{C_1\times C_1}} \otimes q_{1,1}^*L_1 \otimes q_{2,1}^*M_1 \otimes \mathcal{O}_{\Delta_{C_1\times C_1}})\end{aligned}$$

and

$$\Phi^0_{C_2,L_2,M_2} : H^0(q_{1,2}^*L_2 \otimes q_{2,2}^*M_2) \to H^0(q_{1,2}^*L_2 \otimes q_{2,2}^*M_2 \otimes \mathcal{O}_{\Delta_{C_2\times C_2}}).$$

Analogously one can show that $\psi_2 = \Phi^0_{C_1,L_1,M_1} \otimes \Phi_{C_2L_2,M_2}$. Therefore we obtain

$$\psi = \Phi_{C_1,L_1,M_1} \otimes \Phi^0_{C_2,L_2,M_2} \oplus \Phi^0_{C_1,L_1,M_1} \otimes \Phi_{C_2L_2,M_2}. \tag{4.3}$$

Now observe that if $\deg(L_i), \deg(M_i) \geq 2g_i + 2$ for $i = 1, 2$, then by Theorem 1 of [2], each Gaussian map is surjective. Moreover, by a classical result of Mumford, also the multiplication maps are (since $deg(L_i), deg(M_i) \geq 2g_i + 1$). So we get the surjectivity of ψ. □

Remark 4.2 Let X_1 and X_2 be two smooth varieties of any dimension. Let L_1, M_1 and L_2, M_2 be two line bundles on X_1 and X_2 respectively. Denote by $L = L_1 \boxtimes L_2$ and $M = M_1 \boxtimes M_2$. We observe that a similar proof gives a lifting of $\Phi_{L,M}$ by $\Phi_{X_1,L_1,M_1} \otimes \Phi^0_{X_2,L_2,M_2} \oplus \Phi^0_{X_1,L_1,M_1} \otimes \Phi_{X_2,L_2,M_2}$.

We are now going to prove a surjectivity result for mixed Gaussian maps on curves living in the product of two curves.

Proposition 4.3 *With the same hypothesis and notations of Proposition 4.1, let D_i be an effective divisor of degree d_i on C_i, $i = 1, 2$, and let C be a smooth curve in the linear system $|p_1^* D_1 + p_2^* D_2|$. Denote by l_i and m_i the degree of L_i and M_i respectively. Moreover suppose that*

1. $l_i, m_i \geq 2g_i + 2$ *and* $l_i + m_i \geq 6g_i + 3$;
2. $l_i + m_i > 2g_i - 2 + d_i$ *for* $i = 1, 2$;
3. $d_2(l_1 + m_1 - (2g_1 - 2)) + d_1(l_2 + m_2 - (2g_2 - 2)) - 4d_1 d_2 > 0.$

Then

$$\Phi_{C, L_{|C}, M_{|C}}$$

is surjective.

Proof Consider the following commutative diagram

$$\begin{array}{ccccc}
H^0(X \times X, \mathcal{I}_{\Delta_X} \otimes L \boxtimes M) & \xrightarrow{\Phi_{X,L,M}} & H^0(X, \Omega^1_X \otimes L \otimes M) & & \\
 & & & \searrow^{\pi_1} & \\
\downarrow & & & & H^0(C, \Omega^1_X \otimes L \otimes M_{|C}) \\
 & & & \swarrow^{\pi_2} & \\
H^0(C \times C, \mathcal{I}_{\Delta_C} \otimes L_{|C} \boxtimes M_{|C}) & \xrightarrow{\Phi_{L_{|C}, M_{|C}}} & H^0(C, \omega_C \otimes L_{|C} \otimes M_{|C}). & &
\end{array} \tag{4.4}$$

Observe that the vertical arrow and π_1 are restriction maps, whereas π_2 comes from the conormal bundle sequence

$$0 \to \mathcal{O}_C(-C) \to \Omega^1_{X_{|C}} \to \omega_C \to 0$$

tensored by $\mathcal{O}_C(L + M)$. We prove that $\Phi_{X,L,M}$, π_1, and π_2 are surjective. From this we obtain the desired surjectivity result. The surjectivity of $\Phi_{X,L,M}$ is just Proposition 4.1. The surjectivity of π_1 will follow from the vanishing of $H^1(X, \Omega_X \otimes L \otimes M(-C)) \simeq H^1(X, p_1^* \omega_{C_1} \otimes L \otimes M(-C)) \oplus H^1(X, p_1^* \omega_{C_2} \otimes L \otimes M(-C))$. Consider the first piece. Observe that

$$H^1(X, p_1^* \omega_{C_1} \otimes L \otimes M(-C)) \simeq H^1(X, p_1^*(\omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) \otimes p_2^*(L_2 \otimes M_2(-D_2))).$$

By Künneth this is just

$$\begin{gathered}
H^0(C_1, \omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) \otimes H^1(C_2, L_2 \otimes M_2(-D_2)). \\
\oplus \\
H^1(C_1, \omega_{C_1} \otimes L_1 \otimes M_1(-D_1)) \otimes H^0(C_2, L_2 \otimes M_2(-D_2)).
\end{gathered}$$

Now observe that $h^1(C_2, L_2 \otimes M_2(-D_2)) = 0$ and $h^1(C_1, \omega_{C_1} \otimes L_1 \otimes M_1(-D_1))$ are zero by Serre duality and the hypothesis 2. Analogously $H^1(X, p_1^* \omega_{C_2} \otimes L \otimes M(-C))$ decomposes as

$$\begin{gathered}
H^0(C_1, L_1 \otimes M_1(-D_1)) \otimes H^1(C_2, \omega_{C_2} \otimes L_2 \otimes M_2(-D_2)). \\
\oplus \\
H^1(C_1, L_1 \otimes M_1(-D_1)) \otimes H^0(C_2, \omega_{C_2} \otimes L_2 \otimes M_2(-D_2)).
\end{gathered}$$

Again, $h^1(C_2, \omega_{C_2} \otimes L_2 \otimes M_2(-D_2))$ and $h^1(C_1, L_1 \otimes M_1(-D_1))$ are zero by Serre duality and the hypothesis 2. The surjectivity of π_2 will follow from the vanishing of $H^1(C, (L_{|C} + L_{M|C} - C_{|C})$. By Serre duality it will be enough to show that

$$\deg(K_C + C_{|C} - L_{|C} - M_{|C}) < 0.$$

This is just hypothesis 3. □

Main construction 4.4 In this remark we consider a construction we will use in the following corollary. First observe that if X is a smooth surface, H is an ample divisor on X and $C \in |H|$ is a smooth curve, then the restriction map

$$\mathrm{Pic}^0_X \to \mathrm{Pic}^0_C$$

is injective by Lefschetz hyperlane theorem (see for example [25], Theorem C).

Now let C_1 and C_2 be two curves of genus g_1 and g_2 respectively. Let X be the product $C_1 \times C_2$. Let $p_i : X \to C_i$, $i = 1, 2$ be the two projections and let D_i be effective divisors of degree d_i such that $|p_1^* D_1 + p_2^* D_2|$ is base-point free. Let C be a smooth irreducible curve in the linear system $|p_1^* D_1 + p_2^* D_2|$. In particular observe that the genus of C is equal to $g = d_1(g_2 - 1) + d_2(g_1 - 1) + d_1 d_1 + 1$. Let $\alpha' \in \mathrm{Pic}^0(C_1)$ be a non trivial 2-torsion line bundle (in particular $g_1 \geq 1$). Then $\alpha_1 := p_1^* \alpha'$ is a non trivial 2-torsion line bundle in $\mathrm{Pic}(X)$ and $\alpha := \alpha_{1_{|C}}$ is a non trivial 2-torsion line bundle in $\mathrm{Pic}(C)$. Assume $\mathrm{supp}(D_1) = \{p_{1,1}, ..., p_{1,d_1}\}$ and denote by T_{d_2} the divisor of the d_2 points of intersection between the fiber $p_1^{-1}(p_{1,1})$ and C. We can assume that T_{d_2} consists of distinct points.

Remark 4.5 Take X as in 4.4. We observe that a sufficient condition for $\mathcal{O}_X(p_1^* D_1 + p_2^* D_2)$ to be base-point free is that both $\mathcal{O}_{C_1}(D_1)$ and $\mathcal{O}_{C_2}(D_2)$ are. Observe that if C is any smooth curve of genus $g \geq 1$, a general effective divisor D of degree $d \geq g + 1$ is base-point free. This follows from classical results but we recall it.

Since every divisor of degree $2g$ is base-point free, we can restrict to the case $g \geq 2$ and $g + 1 \leq d \leq 2g - 1$. Consider first the case $d = 2g - 1$. Let D' be a general divisor of degree $2g - 2$ and $p \in C$ be a point. Then, by Riemann-Roch, it immediately follows that $D' + p$ is a base-point free divisor of degree $2g - 1$. Now suppose $g + 1 \leq d \leq 2g - 2$ and consider the Brill-Noether variety W^r_d parametrizing (isomorphism classes of) line bundles of degree d such that the dimension of the space of global sections is greater than or equal to $r + 1$. Since d is greater than $g + 1$, by Riemann-Roch, $Pic^d(C) = W^{d-g}_d$. Hence we have to show that a general element of W^{d-g}_d, with $g + 1 \leq d \leq 2g - 2$, is base-point free. Line bundles with base points are given, inside W^{d-g}_d, by the image of the natural map

$$W^{d-g}_{d-1} \times W^0_1 \to W^{d-g}_d. \tag{4.5}$$

Consider the isomorphism $W^{d-g}_{d-1} \simeq W^0_{2g-1-d}$ given by $L \to \omega_C \otimes L^\vee$. Since $0 \leq 2g - 1 - d \leq g$, the last one is birational to $\mathrm{Sym}^{2g-1-d} C$ and hence has dimension $2g - 1 - d$. Then the image of 4.5 has dimension $2g - d$. On the other hand W^{d-g}_d has dimension greater than or equal to $\rho(g, d - g, d) = g$. We conclude that if $d \geq g + 1$ the image of 4.5 is a proper subvariety. Hence the general element is base-point free.

Corollary 4.6 *Using the construction 4.4, suppose that one of the following holds:*

1. $g_i \geq 2$ *i=1,2,* $d_1 \geq 5$, $d_2 \geq 4$, $d_1 \geq g_1 + 5$, $d_2 \geq g_2 + 4$;
2. $g_1 = 1$, $g_2 \geq 2$, $d_1 \geq 6$, $d_2 \geq 4$, $d_2 \geq g_2 + 4$, $d_1 > \frac{d_2}{g_2 - 1}$;

3. $g_1 \geq 3$, $g_2 = 1$, $d_1 \geq 5$, $d_2 \geq 5$, $d_1 \geq g_1 + 5$;
4. $C_1 = \mathbb{P}^1$, $g_2 \geq 2$, $d_1 \geq 5$, $d_2 \geq 4$, $d_2 \geq g_2 + 4$, $d_1(g_2 - 1) > 2d_2$;
5. $g_1 \geq 3$, $C_2 = \mathbb{P}^1$, $d_1 \geq 5$, $d_2 \geq 4$, $d_2(g_1 - 2) > d_1 \geq g_1 + 5$.

Then

$$\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)},$$

and

$$\Phi_{C,\omega_C,\omega_C(-T_{d_2}+\alpha)}$$

are surjective.

Proof Set $L_1 = \omega_{C_1}+D_1-p_{1,1}$, $L_2 = \omega_{C_2}+D_2$, $M_1 = \omega_{C_1}+D_1-p_{1,1}+\alpha'$, $M_2 = \omega_{C_2}+D_2$ and $L_1' = \omega_{C_1} + D_1$, $L_2' = \omega_{C_2} + D_2$, $M_1' = \omega_{C_1} + D_1 - p_{1,1} + \alpha'$ and $M_2' = \omega_{C_2} + D_2$. Denote by $l_i, m_i,\ i = 1, 2$ and $l_i', m_i',\ i = 1, 2$ their degrees. To prove the surjectivity of the Gaussian maps we want to apply Proposition 4.3 with $L_i, M_i,\ i = 1, 2$ in the first case, and $L_i', M_i',\ i = 1, 2$, in the second. Since $l_i' \geq l_i,\ i = 1, 2$, $m_i' \geq m_i, i = 1, 2$, it is enough to verify the hypothesis of Proposition 4.3 in the first situation. It is easy to see that the conditions become: $d_1 \geq 5$, $d_2 \geq 4$, $d_1 \geq g_1 + 5$, $d_2 \geq g_2 + 4$ and $d_2(g_1 - 2) + d_1(g_2 - 1) > 0$. Then we conclude as in the statement. □

We end this section with a surjectivity result for the related multiplication maps.

Proposition 4.7 *Using the construction 4.4, suppose that $d_2 \geq 3$ and $d_1 \geq 4$, $g_1 \geq 1$, or $g_1 = 1$ and $d_2 \geq 3$. Then*

$$\Phi^0_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)} \tag{4.6}$$

and

$$\Phi^0_{C,\omega_C,\omega_C(-T_{d_2}+\alpha)} \tag{4.7}$$

are surjective.

Proof Consider first $\Phi^0_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)}$ and denote it by Φ^0. Set $L = K_X+C-p_1^*(p_{1,1})$ and $M = K_X + C - p_1^*(p_{1,1}) + \alpha_1$. Then $L_{|C} = \omega_C(-T_{d_2})$, $M_{|C} = \omega_C(-T_{d_2}+\alpha)$. Consider the following commutative diagram

$$\begin{array}{ccc} H^0(X, L) \otimes H^0(X, M) & \xrightarrow{\Phi^0_{X,L,M}} & H^0(X, L \otimes M) \\ \downarrow & & \downarrow p \\ H^0(C, L_{|C}) \otimes H^0(C, M_{|C}) & \xrightarrow{\Phi^0} & H^0(C, L_{|C} \otimes M_{|C}). \end{array} \tag{4.8}$$

where p is the restriction map. Again, in order to prove the surjectivity result, it is sufficient to prove that $\Phi^0_{X,L,M}$ and p are surjective. Using the identifications in 4.1 with $L_1 = \omega_{C_1} + D_1 - p_{1,1}$ and $L_2 = \omega_{C_2} + D_2$, $M_1 = \omega_{C_1} + D_1 - p_{1,1} + \alpha_1$, $M_2 = \omega_{C_2} + D_2$, and Künneth theorem, the multiplication map:

$$H^0(X \times X, q_1^*L \otimes q_2^*M) \xrightarrow{\Phi^0_{X,L,M}} H^0(X \times X, q_1^*L \otimes q_2^*M \otimes \mathcal{O}_{\Delta_{X\times X}})$$

decomposes as the tensor product of the multiplication maps on the curves $C_i\ :\ i = 1, 2$:

$$\Phi^0 = \Phi^0_{C_1,L_1,M_1} \otimes \Phi^0_{C_2,L_2,M_2}.$$

Since $l_i, m_i \geq 2g_i + 1,\ i = 1, 2$, each multiplication map is surjective by a classical result of Mumford. The surjectivity of p will follow from the vanishing of $H^1(X, L \otimes M(-C))$. By Künneth this is isomorphic to

$$H^0(C_1, L_1 \otimes M_1(-D_1)) \otimes H^1(C_2, L_2 \otimes M_2(-D_2))$$
$$\oplus$$
$$H^1(C_1, L_1 \otimes M_1(-D_1)) \otimes H^0(C_2, L_2 \otimes M_2(-D_2)).$$

Now observe that $h^1(C_2, L_2 \otimes M_2(-D_2)) = h^1(C_1, L_1 \otimes M_1(-D_1)) = 0$. This is a consequence of Serre duality together with the fact that $l_i + m_i > 2g_i - 2 + d_i$. This ends the proof of the surjectivity of 4.6. An identical proof, with $L_1 = \omega_{C_1} + D_1$, $L_2 = \omega_{C_2} + D_2$, $M_1 = \omega_{C_1} + D_1 - p_{1,1} + \alpha'$ and $M_2 = \omega_{C_2} + D_2$, gives the surjectivity of 4.7. □

5 Some useful lemmas

In this section we prove some other results we will use in the proof of Theorem 1.6. Let C be a curve. We will need an upper bound on the gonality of curves in the surface $C \times \mathbb{P}^1$, where C is a curve. The proof is very much inspired by [35] (see Lemma 2.8 and Theorem 6.1).

Let $p_1 : C \times \mathbb{P}^1 \to C$, $p_2 : C \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two projections. Let C_0 be the class of a fiber of p_2. Recall that

$$Pic(C \times \mathbb{P}^1) = p_1^*(Pic(C)) \oplus \mathbb{Z}C_0,$$

and that the Néron-Severi is generated by C_0 and the class of a fiber of p_1, which we will call f. We are going to prove the following:

Proposition 5.1 *Let $C' \in |p_1^*(D_1) + d_2C_0|$ be a curve in $C \times \mathbb{P}^1$. Then*

- *if C is hyperelliptic,*
$$\mathrm{gon}(C') \geq min(d_1, 2d_2).$$
- *If C is any curve, $g(C') > 0$ and $d_2 \geq \frac{d_1}{4} + 1 + \frac{1}{d_1}$,*
$$\mathrm{gon}(C') \geq min(d_1, d_2\,\mathrm{gon}(C)).$$

For the proof we will use the following theorem of Serrano (see [43]):

Theorem 5.2 *Let C' be a smooth curve on a smooth surface X. Let $\varphi : C' \to \mathbb{P}^1$ be a surjective morphism of degree d. Suppose that either*

(a) $C'^2 > (d+1)^2$, *or*
(b) $C'^2 > \frac{1}{2}(d+2)^2$ *and K_X is numerically even.*

Then there exists a morphism $\psi : X \to \mathbb{P}^1$ such that $\psi_{|_{C'}} = \varphi$.

Recall that a divisor D is called numerically even if $D \cdot E$ is even for any other divisor E. In our situation $K_{C\times\mathbb{P}^1}$ is numerically even since $K_{C\times\mathbb{P}^1} \equiv -2C_0 + (2g(C) - 2)f$. Before presenting the proof of Proposition 5.1, we will need the folllowing:

Lemma 5.3 *Let $C' \in |p_1^*(D_1) + d_2C_0|$ be a curve in $X = C \times \mathbb{P}^1$. Let $\varphi : C' \to \mathbb{P}^1$ be a morphism, and suppose that there exists $\psi : X \to \mathbb{P}^1$ such that $\psi_{|_{C'}} = \varphi$. Then $deg(\varphi) \geq min(d_2\,\mathrm{gon}(C), d_1)$.*

Proof Let D be a fiber of ψ. Then $D \sim p_1^*B + aC_0$, with $a \in \mathbb{Z}$ and B a divisor in C. Denote by b the degree of B. Numerically: $D \equiv bf + aC_0$. From $f \cdot D \geq 0$, $C_0 \cdot D \geq 0$, and $D^2 = 0$ one finds $a \geq 0$, $b \geq 0$ and $2ab = 0$. Then we have two cases:

(i) $a = 0$. In this case $D \sim p_1^*B$. Then $\deg(\varphi) = \deg(\psi_{|C'}) = C' \cdot D = d_2 b \geq d_2 \operatorname{gon}(C)$, where the latter inequality follows from the observation that the restriction of ψ to a fiber of p_2 gives a morphism $C \to \mathbb{P}^1$ of degree greater than or equal to $C_0 \cdot D = b$. And so $b \geq \operatorname{gon}(C)$.

(ii) $b = 0$. In this situation $D \sim aC_0$ and then $\deg(\varphi) = \deg(\psi_{|C'}) = aC_0 \cdot C' = ad_1 \geq d_1$.

□

Proof of Proposition 5.1 Let $C' \in |p_1^*(D_1) + d_2 C_0|$ be a curve in $C \times \mathbb{P}^1$ as before. Denote by k the gonality of C', and let $\varphi : C' \to \mathbb{P}^1$ a morphism of degree k. If φ is extendable we conclude using Lemma 5.3. Then, assume that φ is not extendable. By contradiction suppose that $k < \min(d_1, d_2 \operatorname{gon}(C))$. Using Theorem 5.2, we get $C'^2 = 2d_1d_2 \leq \frac{1}{2}(k+2)^2 < \frac{1}{2}(d_1+2)^2$. That cannot happen if $d_2 \geq \frac{d_1}{4} + 1 + \frac{1}{d_1}$. Finally observe that from $k < \min(d_1, d_2 \operatorname{gon}(C))$, we get $(k+1)^2 \leq d_1 d_2 \operatorname{gon}(C)$ and so, if C is hyperelliptic, we get $(k+1)^2 \leq 2d_1d_2 = C'^2 \leq \frac{1}{2}(k+2)^2 \implies k = 1$ and $C' \simeq \mathbb{P}^1$. □

We end this section proving a lemma which gives a criterion for a line bundle of the type $\omega_C(-T_m + \alpha)$ to be base-point free or very ample on a curve of genus g. We will use it in Proposition 6.2 and Theorem 1.6. Since we want this lemma to hold for any effective divisor T_m of degree m, we have to suppose $m \leq g - 3$. This condition in fact guarantees that $h^0(C, \omega_C(-T_m + \alpha)) \geq 2$.

Lemma 5.4 *Let C be a smooth irreducible curve of genus g and T_m be an effective divisor of degree $m \leq g - 3$, and α a (non-trivial) 2−torsion line bundle.*

(a) Suppose that $\omega_C(-T_m + \alpha)$ is not base-point free. Then

(i) $h^0(C, T_m + \alpha) = 0$, and there exists a point p such that $\dim(|2(T_m + p)|) \geq 1$, or

(ii) $h^0(C, T_m + \alpha) \geq 1$, and there exists a point p such that $\dim(|T_m + \alpha + p)|) \geq 1$.

(b) Suppose $\omega_C(-T_m + \alpha)$ is not very ample. Then

(i) there exist points p and q such that $h^0(C, T_m + \alpha + p) = 0$, and $\dim(|2(T_m + p + q)|) \geq 1$, or

(ii) there exist points p and q such that $h^0(C, T_m + \alpha + p) \geq 1$, and $\dim(|T_m + \alpha + p + q)|) \geq 1$.

Proof Consider first (a). Suppose that p is a base-point for $\omega_C(-T_m + \alpha)$. Then, by Riemann-Roch, $h^0(T_m + p + \alpha) = h^0(T_m + \alpha) + 1$. If $h^0(T_m + \alpha) \geq 1$ we conclude. If $h^0(T_m + \alpha) = 0$, then $h^0(T_m + \alpha + p) = 1$. Then there exists an effective divisor E such that $E \sim T_m + p + \alpha$. This gives $2E \sim 2(T_m + p)$. Now observe that $h^0(2(T_m + p)) \geq 2$, since otherwise $2E = 2(T_m + p)$ and hence $E = T_m + p$, which gives $\alpha = 0$. Since α is not trivial by hypothesis, this cannot happen.

Now let us deal with (b). Suppose there exist two points p and q such that q is a base-point for $\omega_C(-T_m + \alpha - p)$. Then, by Riemann-Roch, $h^0(T_m + p + q + \alpha) = h^0(T_m + p + \alpha) + 1$. If $h^0(T_m + p + \alpha) \geq 1$ we conclude. If $h^0(T_m + p + \alpha) = 0$, then $h^0(T_m + p + q + \alpha) = 1$. Then there exists an effective divisor E such that $E \sim T_m + p + q + \alpha$. Then $2E \sim 2(T_m + p + q)$. As before, it follows that $h^0(2(T_m + p + q)) \geq 2$. □

6 Surjectivity for the general point

Let $\widetilde{\mathrm{M}}_{g,d}$ be the Deligne–Mumford stack of smooth irreducible curves of genus g with d unordered distinct points. Let R_g be the stack of Prym curves of genus g. We consider the stack of Prym curves of genus g with d unordered distinct points:

$$\mathrm{R}_{g,d} := \mathrm{R}_g \times_{\mathrm{M}_g} \widetilde{\mathrm{M}}_{g,d}. \tag{6.1}$$

We denote by $R_{g,d}$ the coarse moduli space. We want to show that under some assumptions on g, d, the Gaussian maps $\Phi_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)}$, and $\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)}$ are surjective for a general (C, α, T_d) in $R_{g,d}$. For convenience, we introduce the following set:

$$S := \{(g_1, d_1, d_2) : g_1 \geq 3, d_2 \geq 4, d_2(g_1 - 2) > d_1 \geq g_1 + 5, d_1 > d_2\}. \tag{6.2}$$

Fix $(g_1, d_1, d_2) \in S$ and set $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. We are going to prove that for all $0 \leq d \leq d_2$, if $(C, \alpha, T_d) \in R_{g,d}$ is a general point, the Gaussian maps $\Phi_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)}$, and $\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)}$ are surjective.

Notations 6.1 In the following we will denote by $(C^*, \alpha^*, T^*_{d_2})$ a point in R_{g,d_2} constructed as in Construction 4.4, with D_1 general and taking $C_2 = \mathbb{P}^1$, C_1 hyperelliptic and (g_1, d_1, d_2) belonging to S. In particular the genus of C^* is $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. We observe that the conditions $g_1 \geq 3$, $d_2 \geq 4$, $d_2(g_1 - 2) > d_1 \geq g_1 + 5$ guarantee that C^* does exist, by Remark 4.5, and the surjectivity of the aforementioned Gaussian maps for the special point (see Corollary 4.6). We require C_1 to be hyperelliptic and $d_1 > d_2$, because in the proof of Proposition 6.2 we will need $h^0(C^*, T^*_{d_2}) = 1$ (we will use Proposition 5.1).

Proposition 6.2 *Fix (g_1, d_1, d_2) in S (6.2), and set $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1$. Then the Gaussian maps*

$$\Phi_{C,\omega_C,\omega_C(-T_{d_2}+\alpha)} : \mathrm{R}(\omega_C, \omega_C(-T_{d_2} + \alpha)) \to H^0(C, \omega_C^{\otimes 3}(-T_{d_2} + \alpha))$$

and

$$\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)} : \mathrm{R}(\omega_C(-T_{d_2}), \omega_C(-T_{d_2} + \alpha)) \to H^0(C, \omega_C^{\otimes 3}(-2T_{d_2} + \alpha))$$

are surjective for the general (C, α, T_{d_2}) in R_{g,d_2}.

Proof We will prove the result for $\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)}$. An identical proof gives the surjectivity of $\Phi_{C,\omega_C,\omega_C(-T_{d_2}+\alpha)}$. For the rest of the proof we denote by Φ^0 and Φ, the multiplication map $\Phi^0_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)}$, and the Gaussian map $\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)}$ respectively. Let X be the product $C_1 \times \mathbb{P}^1$, with $g(C_1) = g_1 \geq 3$, and C_1 hyperelliptic. Let $(C^*, \alpha^*, T^*_{d_2})$, $C^* \subset X$, be a marked Prym curve constructed as in 6.1.

First, we show that $h^0(C, \omega_C(-T_{d_2}))$, $h^0(C, \omega_C(-T_{d_2}+\alpha))$ and $h^0(C, \omega_C^{\otimes 2}(-2T_{d_2}+\alpha))$, are locally constant in a neighborhood of $(C^*, \alpha^*, T^*_{d_2})$. For the latter, it follows immediately from Riemann-Roch. So let's focus on the other two. By Riemann-Roch it is equivalent to show that $h^0(C, T_{d_2} + \alpha)$ and $h^0(C, T_{d_2})$ are locally constant in a neighborhood of $(C^*, \alpha^*, T^*_{d_2})$. For the special point we have $h^0(C^*, T^*_{d_2}) = 1$ since $d_1 > d_2$ by construction and $\mathrm{gon}(C^*) > \min(d_1, 2d_2) > d_2$ by Proposition 5.1. Next we show that $h^0(C^*, T^*_{d_2} + \alpha^*) = 0$. Consider:

$$0 \to \mathcal{O}_X(p_1^* p_{1,1} + p_1^*\alpha'(-C^*)) \to \mathcal{O}_X(p_1^* p_{1,1} + p_1^*\alpha') \to \mathcal{O}_{C^*}(p_1^* p_{1,1} + p_1^*\alpha') \to 0.$$

By Künneth formula we have that $H^1(X, \mathcal{O}_X(p_1^* p_{1,1} + p_1^*\alpha'(-C^*)) \simeq$

$$H^0(C_1, \mathcal{O}_{C_1}(p_{1,1} + \alpha'(-D_1)) \otimes H^1(C_2, \mathcal{O}_{C_2}(-D_2)) \\ \oplus \\ H^1(C_1, \mathcal{O}_{C_1}(p_{1,1} + \alpha'(-D_1)) \otimes H^0(C_2, \mathcal{O}_{C_2}(-D_2)).$$

Notice that the h^0 terms are zero by the hypothesis on the degrees of D_i, $i = 1, 2$. Now observe that choosing $p_{1,1} \in \operatorname{supp}(D_1)$ general in the construction, we can assume that $h^0(X, \mathcal{O}_X(p_1^* p_{1,1} + p_1^*\alpha')) = h^0(C_1, p_{1,1} + \alpha') = 0$. Therefore $h^0(C^*, T^*_{d_2} + \alpha) = h^0(C^*, \mathcal{O}_{C^*}(p_1^* p_{1,1} + p_1^*\alpha')) = 0$.

Now observe that since $g = (g_1 - 1)d_2 + d_1(d_2 - 1) + 1 \geq d_2$, if (C, T_{d_2}, α) is a general point in R_{g,d_2}, $h^0(C, T_{d_2}) = 1$. Analogously, since $g - 1 \geq d_2$, $h^0(C, T_{d_2} + \alpha) = 0$. Hence we are done: the dimensions of the spaces of global sections of the line bundles we are considering are locally constant in a neighborhood of the special point. By Proposition 4.7, Φ^0 is surjective for the special point $(C^*, \alpha^*, T^*_{d_2})$. Then the kernel of the multiplication map on global sections, $\mathrm{R}(\omega_C(-T_{d_2}), \omega_C(-T_{d_2} + \alpha))$, has constant dimension in a neighborhood of the special point. By Riemann-Roch, also $h^0(C, \omega_C^{\otimes 3}(-2T_{d_2} + \alpha))$ is locally constant. Since by Corollary 4.6, Φ is surjective for the special point, by semi-continuity it is surjective in a neighborhood. For the Gaussian map $\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)} : \mathrm{R}(\omega_C, \omega_C(-T_{d_2} + \alpha)) \to H^0(C, \omega_C^{\otimes 3}(-2T_{d_2} + \alpha))$ the proof is very similar. □

We observe that the previous result requires $d_2 \geq 4$, and hence we don't still have a surjectivity result for a general Prym curve with 2 or 3 marked points. We overcome this problem in the final theorem (Theorem 1.6).

Proof of Theorem 1.6 Let's deal with the first map. Let (C, α, T_d) and (C, α, T_{d_2}) be general points in $R_{g,d}$ and R_{g,d_2} respectively, such that $T_d \subseteq T_{d_2}$. In particular, we can suppose that $\operatorname{gon}(C) = \lfloor \frac{g+3}{2} \rfloor$. An easy calculation shows that $\lfloor \frac{g+3}{2} \rfloor > 2(d_2 + 2)$, and so we have that $\omega_C(-T_{d_2} + \alpha)$ is very ample. In fact, if $\omega_C(-T_{d_2} + \alpha)$ is not very ample, by Lemma 5.4 there exists a $g^1_{2(d_2+2)}$. Observe that also $\omega_C(-T_{d_2})$ and $\omega_C(-T_d)$ are very ample since otherwise the curve would admit a $g^1_{d_2+2}$ and a g^1_{d+2} respectively. Moreover, observe that $h^1(\omega_C(-T_d)) = h^1(\omega_C(-T_{d_2}))$ since $h^0(T_d) = h^0(T_{d_2}) = 1$. Denote by T_n the divisor whose support consists of the $n-$distinct points such that $T_{d_2} = T_d + T_n$. Then we can apply Proposition 2.2 with $L = \omega_C(-T_d)$, $n = d_2 - d$ (then $L - T_n = \omega_C(-T_{d_2})$), and $M = \omega_C(-T_{d_2} + \alpha)$. We obtain a surjective map

$$\operatorname{coker}(\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)}) \to \operatorname{coker}(\Phi_{C,\omega_C(-T_d),\omega_C(-T_{d_2}+\alpha)}).$$

Since $\operatorname{coker}(\Phi_{C,\omega_C(-T_{d_2}),\omega_C(-T_{d_2}+\alpha)}) = 0$ by Proposition 6.2, we conclude that $\operatorname{coker}(\Phi_{C,\omega_C(-T_d),\omega_C(-T_{d_2}+\alpha)})$ is zero. Now we use again Proposition 2.2 with $L = \omega_C(-T_d + \alpha)$, $n = d_2 - d$, and $M = \omega_C(-T_d)$ (in particular $L - T_n = \omega_C(-T_{d_2} + \alpha)$). Notice that since (C, α, T_d) and (C, α, T_{d_2}) are general points in $R_{g,d}$ (and R_{g,d_2} respectively), and $g - 1 \geq d_2 \geq d$, we have that $h^0(T_d + \alpha) = h^0(T_{d_2} + \alpha) = 0$. This gives by Serre-duality $h^1(L) = h^1(L - T_n)$. We then obtain a surjective map:

$$\operatorname{coker}(\Phi_{C,\omega_C(-T_{d_2}+\alpha),\omega_C(-T_d)}) \to \operatorname{coker}(\Phi_{C,\omega_C(-T_d+\alpha),\omega_C(-T_d)}).$$

Hence we conclude that $\operatorname{coker}(\Phi_{\omega_C(-T_d),\omega_C(-T_d+\alpha)}) = 0$ for the general point. The proof for

$$\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)}$$

is analogous. □

Example 1 Observe that choosing $d_2 = 4, d_1 = g_1 + l + 5$ with $g_1 \geq 3$ and $0 \leq l + 14 \leq 3g_1$, all the conditions of Theorem 1.6 are satisfied, and in this case $g = 7g_1 + 3l + 12$. Choosing $(g_1, l) \in \{(7+k, 5), (8+k, 3), (9+k, 1), (7+k, 6), (8+k, 4), (9+k, 2), (7+k, 7), (8+k, 5), (9+k, 3), (10+k, 1), (8+k, 6), (9+k, 4),\ k \geq 0\}$, we get all the genera greater than or equal to 76. Then for all $g \geq 76$, the Gaussian maps with 2,3 or 4−marked points are surjective by Theorem 1.6.

Remark 6.3 We expect our results regarding the surjectivity of 1.2 and 1.3 to be not sharp. In this remark, we compute the expected numerical range of degrees d and genus g such that one can expect the surjectivity of the Gaussian maps for the general point (C, α, T_d). Denote by Φ^0, Φ ($\Phi^{0'}$, $\Phi^{'}$) respectively $\Phi^0_{C,\omega_C,\omega_C(-T_d+\alpha)}$ and $\Phi_{C,\omega_C,\omega_C(-T_d+\alpha)}$ ($\Phi^0_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)}$ and $\Phi_{C,\omega_C(-T_d),\omega_C(-T_d+\alpha)}$), and denote by $R(g, d)$ ($R'(g, d)$) the kernel of Φ^0 ($\Phi^{0'}$). We first observe that a necessary condition for the surjectivity is $d \geq g-3$. Indeed, let (C, α, T_d) be a general point in $R_{g,d}$. Observe that $h^0(C, \omega_C(-T_d + \alpha)) = \max\{g - 1 - d, 0\}$. Then, if $d \geq g - 1$, $R(g, d) = 0$. If $d = g - 2$, $h^0(C, \omega_C(-T_d + \alpha)) = 1$ and Φ^0 ($\Phi^{0'}$) is injective in both cases. Then suppose $d \leq g - 3$. An easy calculation shows that in order to have the surjectivity of 1.2, we need $d \leq g - 7 - \frac{6}{g-2} + \frac{\mathrm{cork}(\Phi^0)}{g-2}$. In particular one can expect to have the surjectivity of 1.2 for every $g \geq 9$ and $d \leq g - 8$.

Analogously, an easy calculation shows that in order to have the surjectivity of 1.3, we need $d \leq g - 3$, if $g = 4$ or $g = 5$, and $d \leq g - \frac{5}{2} - \sqrt{8g - 7} + \mathrm{cork}(\Phi^{0'})$ if $g \geq 6$.

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