



The enumerative geometry of cubic hypersurfaces: point and line conditions

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Received: 10 August 2022 / Accepted: 6 April 2023 / Published online: 3 May 2023
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Abstract


The set of smooth cubic hypersurfaces in \mathbb{P}^n is an open subset of a projective space. A compactification of the latter which allows to count the number of smooth cubic hypersurfaces tangent to a prescribed number of lines and passing through a given number of points is termed a *1-complete variety of cubic hypersurfaces*, in analogy with the space of complete quadrics. Imitating the work of Aluffi for plane cubic curves, we construct such a space in arbitrary dimensions by a sequence of five blow-ups. The counting problem is then reduced to the computation of five total Chern classes. In the end, we derive the desired numbers in the case of cubic surfaces.

Introduction

A famous moduli space in enumerative geometry is the *space of complete quadrics*. This is a compactification of the set of smooth quadric hypersurfaces in $\mathbb{P}(W) = \mathbb{P}^n$, where W is an $(n + 1)$ -dimensional vector space over an algebraically closed field \mathbb{k} . To construct this space, one starts with $V_0 = \mathbb{P}(\text{Sym}^2(W^*))$ and considers the sequence of n blow-ups obtained by iteratively blowing up the proper transforms of the loci of symmetric matrices with rank at most i . For details, we refer to [19] and the references therein. This variety has been used to answer the degree 2 case of questions like:

How many smooth degree d hypersurfaces in \mathbb{P}^n are tangent to $\binom{n+d}{d} - 1$ general linear spaces of various dimensions?

The solutions to these problems are classically called *characteristic numbers* (of the family of degree d hypersurfaces). In the case of quadrics, this question was first answered by Schubert [21], back in 1879. Later this was translated into a problem about the Chow ring of the space

MB has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - The Berlin Mathematics Research Center MATH⁺ (EXC-2046/1, project ID 390685689). AD is supported by Jan Draisma's Vici grant 639.033.514 from the NWO, Netherlands Organisation for scientific research, . AK is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 314838170, GRK 2297 MathCoRe.

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of complete quadrics, and beautiful results have been achieved [7, 23, 28]. More recently, the space of complete quadrics has proved useful to study some classical problems in algebraic statistics related to maximum likelihood estimation [19, 20]. Therefore, for quadrics, the problem of computing characteristic numbers is essentially solved. An explicit construction of a space where the above question translates into a cohomological problem is known.

Much less is understood when it comes to higher degree hypersurfaces. To our knowledge, the only cases where all characteristic numbers are known are plane cubics and plane quartics. For the latter, partial results were achieved in [3, 30], and later a full description was given in [29]. When it comes to numbers for cubic plane curves, we have to go back to around 150 years ago, when in the early 1870s Maillard [18] and Zeuthen [31] claimed to have computed them. Unfortunately, their methods were relying on assumptions that were not rigorously justified. It took more than a century to prove these numbers by using rigorous theoretical foundations provided by Fulton–MacPherson intersection theory, as in the works of Kleiman and Speiser [16], and Aluffi [1, 2]. A particularly interesting feature of [1] is that in order to compute the characteristic numbers the author constructs a *space of complete plane cubics*, which turns out to be the right compactification of the set of reduced plane cubics where to solve the characteristic numbers problem. In a similar fashion as for complete quadrics, the space of complete plane cubics is constructed through a sequence of blow-ups of the projective space parametrizing all cubic forms.

As far as we know, the case of higher-dimensional cubic hypersurfaces has been unexplored. Our aim in this paper is to generalize the space of complete plane cubics in [1] and to construct what we call a variety of *1-complete cubic hypersurfaces*, which is the right space where to answer the following question:

What is the number of smooth cubic hypersurfaces in \mathbb{P}^n passing through n_p general points and tangent to $\binom{n+3}{3} - n_p - 1$ general lines?

The paper is based on [1], whose construction we find out to be generalizable to higher dimensions in a direct way. We now give an outline of this construction and explain how a 1-complete variety of cubic hypersurfaces can be used to answer the above question.

The set of all cubic hypersurfaces in $\mathbb{P}(W)$ is naturally parametrized by the projective space $\mathbb{P}(\text{Sym}^3(W^*))$ of dimension $\binom{n+3}{3} - 1$. The classical theory of discriminants of univariate polynomials shows that the subset of $\mathbb{P}(\text{Sym}^3(W^*))$ of cubics tangent to a given line in $\mathbb{P}(W)$ is a hypersurface which we call a *line condition*, see also Remark 2.2. Similarly, a *point condition* is the hyperplane in $\mathbb{P}(\text{Sym}^3(W^*))$ of all cubics containing a given point in $\mathbb{P}(W)$. We want to count the finite number of *smooth* cubics in the intersection of n_p general point conditions and $\binom{n+3}{3} - n_p - 1$ general line conditions. However, the intersection of such hypersurfaces is not always generically transverse. A central role is indeed played by the locus where all line conditions intersect, which turns out to be the set of non-reduced cubics, i.e., those of the form $\lambda\mu^2$ for linear forms λ and μ . This simple description of the *base locus* is indeed the reason for our focus on *lines* rather than higher-dimensional linear spaces. The goal of our construction is to obtain a variety birational to $\mathbb{P}(\text{Sym}^3(W^*))$ but such that in this new space, the proper transforms of the line conditions will no longer intersect. This is what we called earlier a 1-complete variety of cubic hypersurfaces. It turns out that, as in [1], it is enough to blow-up five times along irreducible components of the loci where the proper transforms of the line conditions intersect. The ultimate goal of computing the characteristic numbers is then achieved by subtracting from the bound provided by Bézout's theorem a certain correction term which can be expressed via certain Chern classes of vector bundles arising in the blow-up process. Let us note that one crucial difference to [1], apart from the natural difficulties when dealing with higher dimensions, is that the center B_4 of

the fifth blow-up in our case is the projectivization of a vector bundle \mathcal{E} of rank > 1 which is not a priori known explicitly. Its identification and the computation of its Chern classes are difficult tasks and have no analog in [1]. In particular, Proposition 2.25 provides a new proof of the important Lemma 4.2 of [1].

The paper is organized as follows. In Sect. 1 we give the definition of a 1-complete variety of cubic hypersurfaces \tilde{V} and Theorem 1.4 proves that the intersection numbers we will compute in this variety coincide with the characteristic numbers we were aiming for.

Section 2 concerns the construction of the 1-complete variety \tilde{V} achieved by performing five blow-ups. In each subsection we spell out the details of each blow-up by expressing its equations, the support of the intersection of the proper transforms of the line conditions, and the equations for this intersection. Roughly, this intersection is then taken to be the center of the next blow-up. The construction ends with Corollary 2.23 where we show that the proper transforms of the line conditions no longer intersect.

Section 3 is devoted to the Chow rings of the five centers defined in the previous section and to the computation of the intersection classes needed for the correction term.

In the final Sect. 4, we gather the data computed so far and provide the characteristic numbers for cubic surfaces in projective 3-space. The proof of Theorem 3.14(ii) and Remark 3.15 explain what is missing to determine the characteristic numbers in higher dimensions.

The code used in this work together with computational results is available at

<https://mathrepo.mis.mpg.de/CountingCubicHypersurfaces>.

1 First associated hypersurfaces and the Hurwitz map

We fix an integer $d \geq 2$, an algebraically closed field \mathbb{k} of characteristic 0 or strictly greater than d , and a \mathbb{k} -vector space W of dimension $n + 1$ with $n \geq 2$. We refer to [13, Section 3.2.E] for the notion of higher associated hypersurfaces of a projective variety. Specifically, we are interested in the following case: Let $X := \mathcal{V}(f) \subseteq \mathbb{P}(W)$ be an integral projective hypersurface of degree $d \geq 2$, defined by an irreducible homogeneous polynomial f . If X is smooth, its *first associated hypersurface* $\mathcal{Z}_1(X) \subseteq \text{Gr}(2, W)$ consists of all lines $\ell \subseteq \mathbb{P}(W)$ such that ℓ is tangent to X at some point or, more precisely, $\dim(\ell \cap ET_x X) = 1$ for some point $x \in \ell \cap X$, where $ET_x X$ is the embedded tangent space of X at the point x . If instead X is singular, we first consider the lines ℓ for which there exists a smooth point satisfying the above conditions and then take the Zariski closure of this set in the Grassmannian $\text{Gr}(2, W)$.

In [13, Proposition 2.11] it is shown that $\mathcal{Z}_1(X)$ is an irreducible hypersurface in $\text{Gr}(2, W)$. Moreover, if X is smooth, [25, Theorem 1.1] shows that $\mathcal{Z}_1(X)$ is defined by an irreducible element Hu_f of degree $d(d - 1)$ in the projective coordinate ring of $\text{Gr}(2, W)$, called the *Hurwitz form*, written as a degree $d(d - 1)$ homogeneous polynomial in the Plücker coordinates, uniquely only up to the degree $d(d - 1)$ piece of the ideal generated by the Plücker relations. On the open set of $\mathbb{P}(\text{Sym}^d(W^*))$ parametrizing smooth degree d hypersurfaces, we can define a morphism sending X to the degree $d(d - 1)$ hypersurface $\mathcal{Z}_1(X)$ of $\text{Gr}(2, W)$. The set of hypersurfaces in $\text{Gr}(2, W)$ of this degree is parametrized by the projective space $|\mathcal{O}_{\text{Gr}(2, W)}(d(d - 1))|$.

Hence, we define the *Hurwitz map* to be the rational map

$$\text{Hu} : \mathbb{P}(\text{Sym}^d(W^*)) \dashrightarrow |\mathcal{O}_{\text{Gr}(2, W)}(d(d - 1))|, \quad [f] \longmapsto [\text{Hu}_f].$$

For instance, if $n = 2$, this map is simply the one taking a degree d plane curve into its dual curve. Following [25, Example 2.2], Hu_f can be computed as the resultant of the homogeneous polynomials of degree $d - 1$ in the variables s and t given by the two partial derivatives of $f(sv_0 + tw_0, \dots, sv_n + tw_n)$. As a polynomial in s and t , the coefficients of the latter are bihomogeneous of degree $(1, d)$ in the coefficients of f and the variables v_i, w_i , respectively. It follows that the polynomial Hu_f is bihomogeneous of degree $(2(d - 1), 2d(d - 1))$ with respect to the aforementioned variables. By [25, Example 2.2], Hu_f can even be expressed as a polynomial in the Plücker coordinates $p_{0,1}, p_{0,2}, \dots, p_{n-1,n}$ of the Grassmannian $Gr(2, W)$ given by the 2×2 minors $p_{ij} = v_i w_j - v_j w_i$. Hence, Hu_f is bihomogeneous of degree $(2(d - 1), d(d - 1))$ in the coefficients of f and the Plücker coordinates, respectively. Notice that the polynomial obtained in this way makes sense also for non-smooth, reducible and even non-reduced hypersurfaces $\mathcal{V}(f)$.

Remark 1.1 The rational map induced by the linear system generated by all line conditions in $H^0(\mathbb{P}(\text{Sym}^d(W^*)), \mathcal{O}(2(d - 1)))$ is closely related to Hu . Composing the former with a suitable linear embedding into $|\mathcal{O}_{Gr(2,W)}(d(d - 1))|$ gives the latter.

In the same line of [1], we define the *point condition* P^p and the *line condition* L^ℓ as the hypersurfaces in $\mathbb{P}(\text{Sym}^d(W^*))$ consisting of the degree d hypersurfaces, respectively, containing the point p and tangent to the line ℓ .

Lemma 1.2 *The indeterminacy locus of the Hurwitz map Hu is precisely the intersection of all line conditions, which in turn set-theoretically agrees with the subset $S_0 \subseteq \mathbb{P}(\text{Sym}^d(W^*))$ of the hypersurfaces defined by degree d homogeneous polynomials divisible by the square of some non-constant polynomial.*

Proof Fixing a line $\ell \in Gr(2, W)$, the polynomial $Hu_f(\ell)$ is a homogeneous degree $2(d - 1)$ polynomial in the coefficients of f . Its vanishing set agrees with $L^\ell \subseteq \mathbb{P}(\text{Sym}^d(W^*))$, hence for the first claim it is enough to see that $Hu_f(\ell)$, for fixed ℓ , is irreducible as a polynomial in the coefficients of f . This is clearly a property invariant under the action of PGL_n , so we can consider the line $\ell = \langle e_0, e_1 \rangle$. Then $Hu_f(\ell)$ is precisely the discriminant of the generic homogeneous degree d polynomial in two variables x_0, x_1 , and this is indeed known to be an irreducible polynomial of degree $2(d - 1)$ if $\text{char}(\mathbb{k}) \neq 2$.

The indeterminacy locus of Hu is the set of $[f]$ such that $Hu_f(\ell) = 0$ for every line ℓ . In this case, the singular locus of the closed subscheme $\mathcal{V}(f) \subseteq \mathbb{P}(W)$ must have codimension 0, otherwise the general line would intersect $\mathcal{V}(f)$ transversally in d distinct smooth points. But the singular locus of $\mathcal{V}(f)$ can only have codimension 0 if $\mathcal{V}(f)$ has a non-reduced component, so f is divisible by the square of some non-constant polynomial. \square

This allows us to present the following definition.

Definition 1.3 A *1-complete variety of degree d hypersurfaces* is a morphism $\pi: \tilde{V} \rightarrow \mathbb{P}(\text{Sym}^d(W^*))$ from a smooth projective variety \tilde{V} which is an isomorphism outside $\pi^{-1}(S_0)$ resolving Hu , i.e., such that the proper transforms of all line conditions L^ℓ in \tilde{V} do not intersect:

$$\begin{array}{ccc}
 \tilde{V} & & \\
 \pi \downarrow & \searrow \tilde{Hu} & \\
 \mathbb{P}(\text{Sym}^d(W^*)) & \dashrightarrow_{Hu} & |\mathcal{O}_{Gr(2,W)}(d(d - 1))|.
 \end{array}$$

An analogous construction for tangency with respect to s -dimensional planes instead of lines would lead to the definition of s -complete varieties of degree d hypersurfaces. For $s \geq 2$, however, the intersection of all s -plane conditions set-theoretically agrees with the subset of $\mathbb{P}(\text{Sym}^d(W^*))$ given by all degree d hypersurfaces with singular locus of dimension $\geq n - s$, which to our knowledge is not as easily parametrizable as the set S_0 of non-reduced hypersurfaces.

Theorem 1.4 *We write $V_0 := \mathbb{P}(\text{Sym}^d(W^*))$. Let \tilde{V} be a 1-complete variety of degree d hypersurfaces as above and let $F \subseteq V_0 \setminus S_0$ be a locally closed subvariety. Denote by $\tilde{F} \subseteq \tilde{V}$ the proper transform of the closure \bar{F} and by $\tilde{L}^\ell, \tilde{P}^p \subseteq \tilde{V}$ the line and point conditions of \tilde{V} , i.e., the proper transforms in \tilde{V} of the irreducible hypersurfaces $L^\ell, P^p \subseteq V_0$ corresponding to line and point conditions of V_0 , respectively, for the line $\ell \subseteq \mathbb{P}(W)$ and the point $p \in \mathbb{P}(W)$.*

- (i) *For any finite set of subvarieties $A_1, \dots, A_r \subseteq \tilde{V}$, there exist a point p and a line ℓ such that \tilde{P}^p and \tilde{L}^ℓ both intersect every A_i properly, i.e., in the expected dimension. In fact, this is the case for a general point and a general line.*
- (ii) *If $r = \dim(F)$, there exist r lines ℓ_1, \dots, ℓ_r such that the corresponding line conditions in \tilde{V} intersect \tilde{F} in finitely many points, mapping to F under π . Again, this is the case for general lines ℓ_1, \dots, ℓ_r .*
- (iii) *The number of elements of F , counted with multiplicity, passing through n_p general points and tangent to n_ℓ general lines such that $n_p + n_\ell = \dim(F)$ equals the degree of the 0-cycle $\tilde{P}^{n_p} \cdot \tilde{L}^{n_\ell} \cdot \tilde{F} \in \text{CH}_0(\tilde{V})$, where \tilde{P}, \tilde{L} denote the cycle classes of any point and line condition $\tilde{P}^p, \tilde{L}^\ell$ in \tilde{V} .*
- (iv) *Assume $\text{char}(\mathbb{k}) = 0$. Then for sufficiently general points and lines, the multiplicity of every element of F from (iii) is 1.*

Proof Both (ii) and (iii) follow from (i), the arguments being the same as in [1, Proposition 1]. For (i), we also mimic the strategy of [1, Proposition 1] and assume the conclusion is false, i.e., there are subvarieties A_1, \dots, A_r such that for every line ℓ , the line condition \tilde{L}^ℓ intersects non-properly at least one of the A_i . As \tilde{L}^ℓ is an irreducible hypersurface and all A_i are irreducible, this means that every line condition in \tilde{V} contains at least one of the A_i . Pick a point $p_i \in \tilde{\text{Hu}}(A_i)$ for each $i = 1, \dots, r$ and denote by $G^\ell \subseteq |\mathcal{O}_{\text{Gr}(2,W)}(d(d-1))|$ the hyperplane given by all degree $d(d-1)$ elements of the projective coordinate ring of $\text{Gr}(2, W)$ vanishing at ℓ . The coefficients of the linear equation defining G^ℓ are the monomials of degree $d(d-1)$ in the Plücker coordinates evaluated at the line ℓ . Now, by construction, $\text{Hu}(L^\ell \setminus S_0) \subseteq G^\ell$ and therefore $\tilde{\text{Hu}}(\tilde{L}^\ell) \subseteq G^\ell$. In particular, every hyperplane G^ℓ contains at least one of the finitely many points p_i . Dually, in $\check{\mathbb{P}}(H^0(\mathcal{O}_{\text{Gr}(2,W)}(d(d-1))))$ this means that all points corresponding to the hyperplanes G^ℓ are contained in the finite union of hyperplanes corresponding to the points p_i . However, the set of points corresponding to the G^ℓ is the image of the $d(d-1)$ -Veronese embedding $\text{Gr}(2, W) \rightarrow \check{\mathbb{P}}(H^0(\mathcal{O}_{\text{Gr}(2,W)}(d(d-1))))$ and thus irreducible. Hence, this image would have to be contained in a single hyperplane. In other words, switching back to the primal setting, there exists some p_i , say p_1 , that is contained in all hyperplanes G^ℓ . Then p_1 corresponds to a non-zero element in the degree $d(d-1)$ part of the projective coordinate ring of $\text{Gr}(2, W)$ that, as a polynomial in the Plücker coordinates, must vanish at all lines in $\mathbb{P}(W)$, hence on all of $\text{Gr}(2, W)$. This, of course, is impossible.

The statement of (iv) follows from [11, Theorem (d), p. 162]. □

2 A 1-complete variety of cubic hypersurfaces

This section is dedicated to our construction of a 1-complete variety of cubic hypersurfaces. We start from the projective space $V_0 := \mathbb{P}(\text{Sym}^3(W^*))$ parametrizing cubic hypersurfaces in $\mathbb{P}^n = \mathbb{P}(W)$ and blow-up five times along smooth centers. At each level, these are given by an irreducible component of the intersection of all proper transforms of the line conditions. We will also refer to cubic hypersurfaces as cubics.

We saw in the previous section that S_0 coincides with the set of cubic hypersurfaces divisible by the square of a non-constant polynomial. Hence, S_0 is the image of the morphism

$$\phi_0 : \mathbb{P}(W^*) \times \mathbb{P}(W^*) \longrightarrow \mathbb{P}(\text{Sym}^3(W^*)), \quad ([\lambda], [\mu]) \longmapsto [\lambda\mu^2]. \tag{1}$$

As ϕ_0 is injective, S_0 is a subvariety of $\mathbb{P}(\text{Sym}^3(W^*))$ of dimension $2n$. Let Δ denote the diagonal in $\mathbb{P}(W^*) \times \mathbb{P}(W^*)$. We write B_0 for the locus $\phi_0(\Delta)$ of triple hyperplanes.

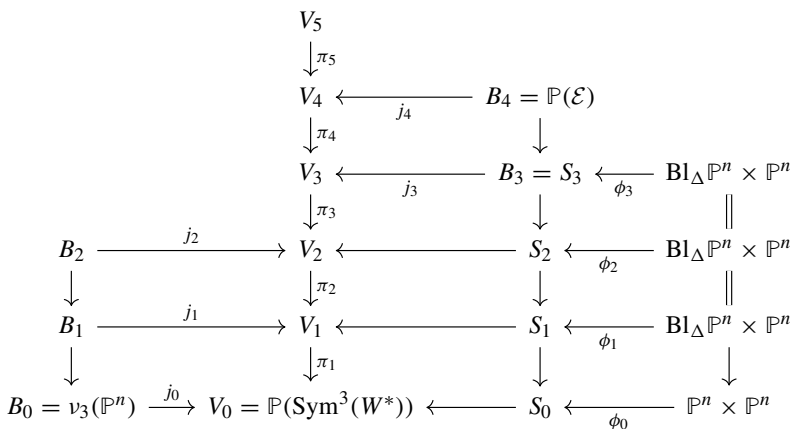
The following result is a direct generalization of [1, Lemma 0.1]. We will often use it without explicit reference. The statement does not depend on the choice of the line $\ell \subseteq \mathbb{P}^n$, hence can be verified on the equation of any line condition.

Lemma 2.1 *Let L^ℓ be the line condition in $\mathbb{P}(\text{Sym}^3(W^*))$ corresponding to $\ell \subseteq \mathbb{P}^n$. Then:*

- (i) *If $c \in L^\ell$, then L^ℓ is smooth at c if and only if c intersects ℓ with multiplicity exactly 2 at a point. In particular, the line conditions are generically smooth along the locus S_0 of non-reduced cubics.*
- (ii) *If c intersects ℓ with multiplicity 3 at a point, then L^ℓ has multiplicity 2 at c . In particular, the line conditions have multiplicity 2 along the locus B_0 of triple hyperplanes.*
- (iii) *The tangent hyperplane to L^ℓ at a smooth point c consists of the cubics containing the point of tangency of c to ℓ . The tangent cone in V_0 to L^ℓ at a cubic c intersecting ℓ in a triple point p is supported on the hyperplane in V_0 consisting of the cubics containing p .*

An immediate generalization of Lemma 0.2 in [1] implies that the map ϕ_0 is an isomorphism onto its image when restricted to $\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta$.

The next subsections explain in details the construction of the 1-complete variety of cubic hypersurfaces \tilde{V} . A schematic overview of this construction and the notation employed is englobed in the following diagram:



The center of each blow-up is denoted by B_i , while the blow-ups are called V_i . Hence V_{i+1} denotes the blow-up of V_i at the center B_i and $\pi_{i+1} : V_{i+1} \rightarrow V_i$ the corresponding blow-up map, for $i = 0, \dots, 4$. For $i \leq 3$, S_i indicates the proper transform of the locus S_0 in V_i .

The above diagram is analogous to the one in [1, p. 514]. Our construction of the 1-complete variety of cubic hypersurfaces V_5 is indeed a direct generalization to higher dimensions of the one performed by Aluffi for plane cubic curves. In particular, the same number of blow-ups is needed to empty the locus where the proper transforms of the line conditions intersect.

2.0 Space of cubic hypersurfaces

In what follows, let \mathbb{k} be an algebraically closed field of characteristic $\neq 2, 3$ and W a \mathbb{k} -vector space of dimension $n + 1$ with basis e_0, \dots, e_n . Let us introduce some notation in order to develop the first blow-up. Given a set of multi-indices $\mathcal{I} = \{I_1, \dots, I_n\}$, we denote (c_I) (resp. $[c_I]$) the vector of affine (resp. projective) coordinates $(c_{I_1}, \dots, c_{I_n})$ (resp. $[c_{I_1} : \dots : c_{I_n}]$). In particular, denote by $[a_I] = [a_{(0,0,0)} : \dots : a_{(n,n,n)}]$ the vector of $\binom{n+3}{3}$ projective coordinates for V_0 . More explicitly, each $a_{(i,j,k)}$ corresponds to the coefficient of the monomial $x_i x_j x_k$ in the equation for the associate cubic in \mathbb{P}^n , where we assume $i \leq j \leq k$. We denote by $[n]$ the set of natural numbers between 0 and n and by \mathcal{J} the set of multi-indices $(i, j, k) \in [n]^3$ with $i \leq j \leq k$ and $j \geq 1$. Then in the affine chart $D(a_{(0,0,0)})$, the ideal $\mathcal{I}(B_0)$ in V_0 determining the locus of triple hyperplanes is generated by the polynomials f_J , with $J \in \mathcal{J}$, given as follows:

$$\begin{aligned}
 f_{(0,i,i)} &:= 3a_{(0,i,i)} - a_{(0,0,i)}^2 && \text{for } i > 0, \\
 f_{(0,i,j)} &:= 3a_{(0,i,j)} - 2a_{(0,0,i)}a_{(0,0,j)} && \text{for } j > i > 0, \\
 f_{(i,i,i)} &:= 9a_{(i,i,i)} - a_{(0,0,i)}a_{(0,i,i)} && \text{for } i > 0, \\
 f_{(i,i,j)} &:= 3a_{(i,i,j)} - a_{(0,i,i)}a_{(0,0,j)} && \text{for } i, j > 0, i \neq j, \\
 f_{(i,j,k)} &:= 3a_{(i,j,k)} - a_{(0,i,j)}a_{(0,0,k)} && \text{for } k > j > i > 0. \tag{2}
 \end{aligned}$$

These polynomials will be needed to provide equations for the center of the first blow-up. Note that B_0 is a smooth complete intersection of codimension $\binom{n+3}{3} - 1 - n$ inside this open chart. In what follows, when we write the affine coordinates (a_I) we always assume the index $I = (0, 0, 0)$ to be excluded.

Remark 2.2 A line condition is a degree 4 hypersurface in V_0 . Indeed, fix the line $\ell = \mathcal{V}(x_2, \dots, x_n) \subseteq \mathbb{P}^n$. The tangency condition for a cubic to such line is given by the vanishing of the resultant of its derivatives with respect to x_0 and x_1 . Then the equation for the line condition L^ℓ in $D(a_{(0,0,0)}) \subseteq V_0$ is given by

$$a_{(0,0,1)}^2 a_{(0,1,1)}^2 + 18a_{(0,0,1)}a_{(0,1,1)}a_{(1,1,1)} - 4a_{(0,1,1)}^3 - 4a_{(0,0,1)}^3 a_{(1,1,1)} - 27a_{(1,1,1)}^2 = 0. \tag{3}$$

Using the action of PGL_n , we can recover the equation for L^ℓ for any line $\ell \subseteq \mathbb{P}^n$.

2.1 First Blow-up

Denote by V_1 the blow-up of the space V_0 along the center B_0 , and L_1 the proper transform in V_1 of a line condition L .

Coordinates I Let $([a_I], [b_J])$ denote the projective coordinates on $V_0 \times \mathbb{P}^{r-1}$, where $r = \binom{n+3}{3} - 1 - n$ is the codimension of B_0 as subvariety of V_0 and $J \in \mathcal{J}$. Then, by [9,

Exercise 17.14(b)] the blow-up V_1 is a closed subvariety of the affine chart $D(a_{(0,0,0)})$ given by the equations

$$f_{J_1}b_{J_2} - f_{J_2}b_{J_1} = 0,$$

where $J_1, J_2 \in \mathcal{J}$ and the f_J 's denote the equations in (2).

We restrict to the affine chart $D(b_{(0,1,1)})$, where V_1 can be described by the affine coordinates $(a_{(0,0,1)}, \dots, a_{(0,0,n)}, a', b_J)$, with $J \in \mathcal{J} \setminus \{(0, 1, 1)\}$, and the additional variable a' corresponds to the equation $f_{(0,1,1)} = 3a_{(0,1,1)} - a_{(0,0,1)}^2$. The equations for V_1 in this affine open set become

$$a' - f_{(0,1,1)} = 0, \quad f_J - b_J a' = 0, \quad \text{for all } J \in \mathcal{J} \setminus \{(0, 1, 1)\}.$$

The equation for the exceptional divisor E_1 inside V_1 is then $a' = 0$, and the (b_J) provide coordinates in the chosen affine chart for the fiber of E_1 over a point in B_0 . In what follows, we will always exclude the index $J = (0, 1, 1)$ when considering the affine coordinates (b_J) . Denote by $N_{\mathbb{P}(W^*)} \mathbb{P}(\text{Sym}^d(W^*))$ the normal bundle of the d -th Veronese embedding $v_d : \mathbb{P}(W^*) \hookrightarrow \mathbb{P}(\text{Sym}^d(W^*))$, where $\mathbb{P}(W^*)$ is embedded in $\mathbb{P}(\text{Sym}^d(W^*))$ via v_d .

Lemma 2.3 *Let $e \leq d$. Then there is a natural embedding of normal bundles*

$$\alpha_{e,d} : N_{\mathbb{P}(W^*)} \mathbb{P}(\text{Sym}^e(W^*)) \hookrightarrow N_{\mathbb{P}(W^*)} \mathbb{P}(\text{Sym}^d(W^*)),$$

given by “multiplication by λ^{d-e} ” in the fiber over $[\lambda] \in \mathbb{P}(W^*)$.

Proof We write $R := \mathbb{k}[x_0, \dots, x_n]$. The pullback of the Euler sequence on $\mathbb{P}(\text{Sym}^d(W^*))$ via v_d is

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(W^*)} \xrightarrow{v_d^*(\varepsilon)} \text{Sym}^d(W^*) \otimes \mathcal{O}_{\mathbb{P}(W^*)}(d) \rightarrow T\mathbb{P}(\text{Sym}^d(W^*))|_{\mathbb{P}(W^*)} \rightarrow 0,$$

where $v_d^*(\varepsilon)$ is induced by the graded R -module homomorphism

$$R \rightarrow \text{Sym}^d(W^*) \otimes_R R(d), \quad f \mapsto \sum_{|I|=d} \binom{d}{I} e_I \otimes (x^I f) = f \cdot (e_0 \otimes x_0 + \dots + e_n \otimes x_n)^d.$$

The fiber of $v_d^*(\varepsilon)$ over λ is therefore just multiplication by $\lambda^d = (\lambda_0 e_0 + \dots + \lambda_n e_n)^d$. More generally, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{O}_{\mathbb{P}(W^*)} & \longrightarrow & W^* \otimes \mathcal{O}_{\mathbb{P}(W^*)}(1) & \longrightarrow & T\mathbb{P}(W^*) \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha_{1,e} & & \downarrow \overline{\alpha_{1,e}} = dv_e \\ 0 & \rightarrow & \mathcal{O}_{\mathbb{P}(W^*)} & \rightarrow & \text{Sym}^e(W^*) \otimes \mathcal{O}_{\mathbb{P}(W^*)}(e) & \rightarrow & T\mathbb{P}(\text{Sym}^e(W^*))|_{\mathbb{P}(W^*)} \rightarrow 0 \\ & & \parallel & & \downarrow \alpha_{e,d} & & \downarrow \overline{\alpha_{e,d}} \\ 0 & \rightarrow & \mathcal{O}_{\mathbb{P}(W^*)} & \rightarrow & \text{Sym}^d(W^*) \otimes \mathcal{O}_{\mathbb{P}(W^*)}(d) & \rightarrow & T\mathbb{P}(\text{Sym}^d(W^*))|_{\mathbb{P}(W^*)} \rightarrow 0. \end{array}$$

In here, $\alpha_{e,d}$ is induced by the graded R -module homomorphism which is multiplication by $(e_0 \otimes x_0 + \dots + e_n \otimes x_n)^{d-e}$. It can be checked that $\overline{\alpha_{1,e}} = dv_e$ is the differential of the e -th Veronese embedding. Then $\alpha_{e,d}$ induces the embedding of normal bundles we are looking for. □

For us, $e = 2, d = 3$. The exceptional divisor is $E_1 \cong \mathbb{P}(N_{\mathbb{P}(W^*)} \mathbb{P}(\text{Sym}^3(W^*)))$ and we call B_1 the image of $\mathbb{P}(\alpha_{2,3})$ in E_1 . The proper transform of S_0 in V_1 will be denoted by S_1 .

Proposition 2.4 *The intersection of the proper transforms of all line conditions in V_1 is contained in the union $S_1 \cup B_1$.*

Proof It is enough to check that the intersection of the proper transforms of all line conditions and E_1 lies inside B_1 . The intersection of the proper transform L_1 of a line condition L with the fiber over $[\lambda^3] \in B_0$ is the image of the tangent cone of L at the point $[\lambda^3]$ in the projectivized normal bundle $\mathbb{P}(N_{B_0} V_0)$. By definition of $\alpha_{2,3}$ in Lemma 2.3, the fiber of B_1 over $[\lambda^3]$ consists of all cubics divisible by λ . Lemma 2.1(iii) implies that the intersection of all tangent cones at $[\lambda^3]$ of all line conditions is contained in the set of cubics containing the hyperplane λ . This shows the claim. \square

Lemma 2.5 *We have a commutative diagram*

$$\begin{array}{ccccc}
 \mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow[\cong]{\phi_1} & S_1 & \hookrightarrow & V_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{\phi_0} & S_0 & \hookrightarrow & V_0,
 \end{array}$$

where ϕ_1 is an isomorphism. In particular, S_1 is smooth.

Proof We write e for the exceptional divisor of $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$. The map ϕ_0 lifts to a map $\phi_1 : \mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \rightarrow S_1$ via the universal property of blowing up. Indeed, it can be checked that the pullback of the ideal sheaf $\mathcal{I}(B_0)$ via ϕ_0 is precisely the squared ideal sheaf $\mathcal{I}(\Delta)^2$ of the diagonal $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$, in particular the pullback of $\mathcal{I}(B_0)$ to $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ is an effective Cartier divisor, as needed. Clearly, ϕ_1 restricts to an isomorphism of $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \setminus e$ onto $S_1 \setminus E_1$. As $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ and S_1 are projective varieties, ϕ_1 is a closed map, so surjectivity follows. In order to prove the injectivity of ϕ_1 we observe that ϕ_0 is an injective morphism between varieties over an algebraically closed field, hence ϕ_0 is universally injective. Base-changing ϕ_0 along the blow-up map $\pi_1 : V_1 \rightarrow V_0$ hence gives an injection $(\mathbb{P}^n \times \mathbb{P}^n) \times_{V_0} V_1 \rightarrow V_1$. The blow-up closure lemma ensures that $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ is naturally a closed subscheme of $(\mathbb{P}^n \times \mathbb{P}^n) \times_{V_0} V_1$, and the composition $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \rightarrow V_1$ agrees with ϕ_1 , showing that ϕ_1 is injective. By [15, Corollary 14.10], it remains to show that $(d\phi_1)_p : T_p(\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n) \rightarrow T_{\phi_1(p)} V_1$ is injective for all p in the exceptional divisor e of $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$. This matter is local and invariant under the PGL_n -action, so we can assume p to lie in the fiber of $([1 : 0 : \dots : 0], [1 : 0 : \dots : 0]) \in \Delta$. Choose local coordinates

$$([1 : \lambda_1 : \dots : \lambda_n], [1 : \mu_1 : \dots : \mu_n]) \in \mathbb{P}^n \times \mathbb{P}^n.$$

The equations for Δ are $u_i := \lambda_i - \mu_i = 0$ for all $i \in \{1, \dots, n\}$. Thus, $\mathrm{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ is described by the points $(\mu_1, \dots, \mu_n, u_1, \dots, u_n, [s_1, \dots, s_n])$ such that $u_i s_j - u_j s_i = 0$ for all i, j . In the affine chart $D(s_1)$, the morphism ϕ_1 is given explicitly in the affine coordinates $(\mu_1, \dots, \mu_n, u_1, s_2, \dots, s_n)$ by

$$\begin{array}{ll}
 a_{(0,0,1)} = 3\mu_1 + u_1, & b_{(1,1,1)} = 2\mu_1, \\
 a_{(0,0,i)} = 3\mu_i + s_i u_1 & \text{for } i > 1, & b_{(0,1,i)} = 2s_i & \text{for } i > 1, \\
 a' = -u_1^2, & & b_{(0,i,j)} = 2s_i s_j & \text{for } j > i > 1, \\
 b_{(0,i,i)} = s_i^2 & \text{for } i > 1, & b_{(i,i,i)} = 2\mu_i s_i^2 & \text{for } i > 1, \\
 b_{(1,i,i)} = 2\mu_i s_i & \text{for } i > 1, & b_{(i,i,j)} = 2\mu_i s_i s_j & \text{for } i, j > 1, i \neq j, \\
 b_{(1,1,i)} = 2\mu_1 s_i & \text{for } i > 1, & b_{(i,j,k)} = 2s_k (\mu_i s_j + \mu_j s_i) & \text{for } k > j > i > 0.
 \end{array}$$

The exceptional divisor e now has equation $u_1 = 0$. This explicit description of ϕ_1 allows us to conclude the proof by checking the non-degeneracy of the Jacobian at every point. Indeed, the $2n$ row vectors in the Jacobian corresponding to $a_{(0,0,i)}$ for $1 \leq i \leq n$, to $b_{(0,1,i)}$ for $2 \leq i \leq n$ and to $b_{(1,1,1)}$ are linearly independent. \square

Lemma 2.6 *The set-theoretic intersection of B_1 and S_1 is $\phi_1(e)$. Moreover, the proper transforms of the line conditions are generically smooth and tangent to E_1 along B_1 .*

Proof Since ϕ_1 is an isomorphism, we have $\phi_1(e) = S_1 \cap E_1$ and it suffices to show $\phi_1(e) \subseteq B_1$. By invariance under projective transformations, it suffices to prove the inclusion for the fiber in E_1 over $[x_0^3] \in B_0$. Using the coordinates described above, the intersection of this fiber with B_1 in V_1 is described by the equations $a_{(0,0,i)} = 0$ for all $1 \leq i \leq n$, $a' = 0$ and $b_J = 0$ for the multi-indices $J \in \mathcal{J}$ with first entry being non-zero. The explicit description of ϕ_1 shows that the image of the fiber of $([1 : 0 : \dots : 0], [1 : 0 : \dots : 0]) \in \mathbb{P}^n \times \mathbb{P}^n$ satisfies all these equations, proving the claim.

For the second point, the invariance under the natural action of PGL_n on V_1 , allows us to verify the claim for the line condition corresponding to the line $\ell = \mathcal{V}(x_2, \dots, x_n)$. We can restrict to the affine open $D(a_{(0,0,0)}) \cap D(b_{(0,1,1)})$ where we have local coordinates (see Remark I). The equation for the line condition L^ℓ in $D(a_{(0,0,0)}) \subseteq V_0$ is given in (3). Plugging in $3a_{(0,1,1)} = a' + a_{(0,0,1)}^2$ and $27a_{(1,1,1)} = 3b_{(1,1,1)}a' + a_{(0,0,1)}(a' + a_{(0,0,1)}^2)$, we get the equation

$$(a')^2 \left(12b_{(1,1,1)}a_{(0,0,1)} - 4a_{(0,0,1)}^2 - 4a' - 9b_{(1,1,1)}^2 \right) = 0,$$

which outside of E_1 describes the proper transform L_1^ℓ of the line condition in the chosen affine chart, whose equation is therefore $-4a' - (3b_{(1,1,1)} - 2a_{(0,0,1)})^2 = 0$. Since the equation of E_1 in the local coordinates is $a' = 0$, every point of E_1 belonging to the proper transform is indeed a tangency point. Moreover, the equation shows that the proper transform is smooth in this entire affine open. \square

Lemma 2.7 *The ideal of $B_1 \subseteq V_1$ in the open $D(a_{(0,0,0)})$ is generated by the equations*

$$\begin{aligned} f_J &= 0 && \text{for all } J, \\ f'_{(i,i,i)} &:= 3b_{(i,i,i)} - 2a_{(0,0,i)}b_{(0,i,i)} = 0 && \text{for all } i > 0, \\ f'_{(i,i,j)} &:= 3b_{(i,i,j)} - a_{(0,0,i)}b_{(0,i,j)} = 0 && \text{for all } i, j > 0, i \neq j, \\ f'_{(i,j,k)} &:= 3b_{(i,j,k)} - a_{(0,0,i)}b_{(0,j,k)} - a_{(0,0,j)}b_{(0,i,k)} = 0 && \text{for all } k > j > i > 0. \end{aligned}$$

These equations clearly form a regular sequence, so B_1 is a complete intersection in the open chart. In the affine chart $D(b_{(0,1,1)})$, we can moreover replace the first set of conditions by $a' = 0$, as above.

Proof From the commutative diagram in the proof of Lemma 2.3, the fiber over $[\lambda^3] \in B_0$ of the normal bundle can be naturally identified with the vector space $\text{Sym}^3(W^*)/(\lambda^2x_0, \dots, \lambda^2x_n)$. We want to understand how an element in the latter corresponds to an element of the fiber $E_1|_{[\lambda^3]}$ if explicitly written in the coordinates from the description of V_1 in (I). The answer is provided by the conormal sequence

$$0 \rightarrow I/I^2 \rightarrow \Omega_{\mathbb{k}[a_I]} \otimes_{\mathbb{k}[a_I]} \mathbb{k}[a_I]/I \rightarrow \Omega_{\mathbb{k}[a_I]/I} \rightarrow 0.$$

Any point $k \in \text{Sym}^3(W^*)/\langle \lambda^2 x_0, \dots, \lambda^2 x_n \rangle$ can be uniquely represented as a cubic not containing the monomials $x_0^3, x_0^2 x_1, \dots, x_0^2 x_n$. If we write $k = k_{(0,1,1)} x_0 x_1^2 + \dots + k_{(n,n,n)} x_n^3$, then k corresponds to the element $\sum_J y_J \overline{f_J}$ in $(I/I^2 \otimes \mathbb{k}[a_J]/\mathfrak{m}_{[\lambda^3]})^\vee$ via

$$\begin{aligned} b_{(i,j,k)} &= 3k_{(i,j,k)} - a_{(0,0,k)} k_{(0,i,j)} && \text{for all } k > j > i > 0, \\ b_{(i,i,j)} &= 3k_{(i,i,j)} - a_{(0,0,j)} k_{(0,i,i)} && \text{for all } i, j > 0, i \neq j, \\ b_{(i,i,i)} &= 9k_{(i,i,i)} - a_{(0,0,i)} k_{(0,i,i)} && \text{for all } i > 0, \\ b_{(0,i,j)} &= 3k_{(0,i,j)} && \text{for all } j \geq i > 0. \end{aligned}$$

in the fiber over $[\lambda^3] \in B_0$ we have $a_{(0,0,i)} = 3\lambda_i$ and it is easy to see that the cubic k is divisible by $\lambda = x_0 + \lambda_1 x_1 + \dots + \lambda_n x_n$ if and only if k satisfies the equations

$$\begin{aligned} 3k_{(i,j,k)} &= a_{(0,0,i)} k_{(0,j,k)} + a_{(0,0,j)} k_{(0,i,k)} + a_{(0,0,k)} k_{(0,i,j)} && \text{for all } k > j > i > 0, \\ 3k_{(i,i,j)} &= a_{(0,0,j)} k_{(0,i,i)} + a_{(0,0,i)} k_{(0,i,j)} && \text{for all } i, j > 0, i \neq j, \\ 3k_{(i,i,i)} &= a_{(0,0,i)} k_{(0,i,i)} && \text{for all } i > 0. \end{aligned}$$

The claim can be deduced directly from this. □

2.2 Second Blow-up

Let $V_2 := \text{Bl}_{B_1} V_1$. This is smooth because so is B_1 . We denote $\pi_2 : V_2 \rightarrow V_1$ the blow-up map, and respectively $\tilde{E}_1, S_2, P_2, L_2$ the proper transforms of E_1, S_1, P_1, L_1 . Moreover, we define $B_2 := \tilde{E}_1 \cap E_2 = \mathbb{P}(N_{B_1} E_1)$, where E_2 denotes the exceptional divisor in V_2 .

Coordinates II Let $(a_{(0,0,1)}, \dots, a_{(0,0,n)}, a', b_J, [c_a, c_H])$ denote coordinates for the product space $(D(a_{(0,0,0)}) \cap D(b_{(0,1,1)})) \times \mathbb{P}^{r-1}$, where $r = \binom{n+3}{3} - \binom{n+2}{2} + 1$ is the codimension of B_1 as subvariety of V_1 and $J \in \mathcal{J}$. More precisely, c_a denotes a single variable and $[c_H]$ stands for the projective variables with multi-indices varying in the set $\mathcal{H} = \{(i, j, k) \in [n]^3, \text{ with } 1 \leq i \leq j \leq k\}$.

Thanks to Lemma 2.7, the blow-up V_2 in the open chart $D(a_{(0,0,0)}) \cap D(b_{(0,1,1)})$ is a closed subvariety given by the equations

$$c_a f'_H - a' c_H = 0, \quad c_{H_1} f'_{H_2} - c_{H_2} f'_{H_1} = 0,$$

for $H, H_1, H_2 \in \mathcal{H}$. We can choose the affine open of V_2 given by $D(c_{(1,1,1)})$, then these equations simplify to

$$c_a f'_{(1,1,1)} - a' = 0, \quad c_H f'_{(1,1,1)} - f'_H = 0,$$

where $H \in \mathcal{H} \setminus \{(1, 1, 1)\}$. Introducing the new variable $b' := f'_{(1,1,1)}$, essentially carrying the same information as $b_{(1,1,1)}$, this affine open of V_2 has affine coordinates $(a_{(0,0,i)}, b_{(0,j,k)}, b', c_a, c_H)$ with $H \neq (1, 1, 1)$ subject to no relations. In these coordinates, the equation for E_2 in V_2 becomes $b' = 0$ and the equation for the proper transform \tilde{E}_1 becomes $c_a = 0$. We will always exclude the index $H = (1, 1, 1)$ when considering the affine coordinates.

Lemma 2.8 Write $N_2 := N_{\mathbb{P}(W^*)} \mathbb{P}(\text{Sym}^2(W^*))$ and $N_3 := N_{\mathbb{P}(W^*)} \mathbb{P}(\text{Sym}^3(W^*))$ and let $p_1 : B_1 \rightarrow B_0$ be the restriction of the canonical map from the projective bundle $E_1 = \mathbb{P}(N_{B_0} V_0)$ to its base $B_0 \cong \mathbb{P}(W^*)$. Therefore, there is a natural isomorphism

$$N_{B_1} E_1 \cong p_1^*(N_3/N_2) \otimes_{\mathcal{O}_{B_1}} \mathcal{O}_{B_1}(1)$$

$$\begin{aligned} &\cong p_1^* \left(\frac{\text{Sym}^3(W^*) \otimes \mathcal{O}_{\mathbb{P}(W^*)}(3)}{\text{Sym}^2(W^*) \otimes \mathcal{O}_{\mathbb{P}(W^*)}(2)} \right) \otimes \mathcal{O}_{B_1}(1) \\ &\cong p_1^*(\text{Sym}^3(T\mathbb{P}(W^*))) \otimes \mathcal{O}_{B_1}(1). \end{aligned}$$

Hence, over a point $(\lambda, q) \in B_1$, the normal space $N_{B_1} E_1|_{(\lambda, q)}$ is naturally identified with $\text{Sym}^3(W^*)/(\lambda \cdot \text{Sym}^2(W^*))$. Points in B_2 can be thought of as triples consisting of a hyperplane λ together with a quadric q and a cubic c inside λ .

Proof The first isomorphism is given by [8, Proposition 9.13]. The Euler sequences for $T\mathbb{P}(W^*)$, $T\mathbb{P}(\text{Sym}^2(W^*))$, $T\mathbb{P}(\text{Sym}^3(W^*))$ then give the second and third equality. \square

Lemma 2.9 *The set-theoretical intersection of all proper transforms of the line conditions in V_2 is contained in the union of S_2 and the smooth variety $B_2 = \tilde{E}_1 \cap E_2$.*

Proof The variety S_2 is clearly a component of the intersection. By Lemma 2.6, the line conditions in V_1 are generically tangent to E_1 , and therefore the tangent space of each line condition is contained in the tangent space of E_1 . Hence, the intersection of the proper transforms of the line conditions with the exceptional divisor E_2 is contained in \tilde{E}_1 . \square

A similar reasoning as in Lemma 2.5 shows also the following.

Lemma 2.10 *The lift $\phi_2 : \text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \rightarrow V_2$ of ϕ_1 is explicitly given by*

$$\begin{aligned} a_{(0,0,1)} &= 3\mu_1 + u_1, & c_a &= u_1/2, \\ a_{(0,0,i)} &= 3\mu_i + s_i u_1 & \text{for } i > 1, & c_{(1,1,i)} &= s_i & \text{for } i > 1, \\ b_{(0,i,i)} &= s_i^2 & \text{for } i > 1, & c_{(1,i,i)} &= s_i^2 & \text{for } i > 1, \\ b_{(0,1,i)} &= 2s_i & \text{for } i > 1, & c_{(i,i,j)} &= s_i^2 s_j & \text{for } i, j > 1, \\ b_{(0,i,j)} &= 2s_i s_j & \text{for } j > i > 1, & c_{(i,i,i)} &= s_i^3 & \text{for } i \neq 0, 1, \\ b' &= -2u_1, & c_{(i,j,k)} &= 2s_i s_j s_k & \text{for } k > j > i > 0. \end{aligned}$$

Lemma 2.6 implies that the set-theoretic intersection of S_1 with B_1 is given by $\phi_1(e)$. It is not hard to see then that S_2 is isomorphic to S_1 , hence to $\text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$. Abusing notation, we will indicate with e the exceptional divisor of $\text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ as well as all its isomorphic images under the maps ϕ_i .

Lemma 2.11 *The following hold:*

- (i) B_2 intersects S_2 along e .
- (ii) The line conditions in V_2 are generically smooth along B_2 .

Proof First, recall that S_1 is tangent to E_1 along e . In fact, for any point $p \in e$ we have $T_{\phi_1(p)} S_1 = d\phi_1(T_p(\text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n))$. Working in the chosen affine chart for V_1 , since the entry relative to a' in the column vectors of the Jacobian is always zero, then $T_{\phi_1(p)} S_1$ is contained in the tangent space of E_1 . By invariance under projective transformations this is true everywhere. Thus, since B_1 intersects S_1 along $\phi_1(e)$ we have that $S_2 \cap E_2 \subseteq \tilde{E}_1 \cap E_2 = B_2$ because the tangent space of S_1 is contained in the tangent space of E_1 .

For the second claim, observe that the line conditions in V_1 are generically smooth along B_1 . The claim then follows from the blow-up closure lemma and the fact that the blow-up of a smooth variety is again smooth. \square

Remark 2.12 A cubic $k \in B_2|_{[\lambda, q]} \cong \mathbb{P}(\text{Sym}^3(W^*)/(\lambda \cdot \text{Sym}^2(W^*)))$, whose defining equation can be uniquely written (up to scaling) in the form $k = k_{(1,1,1)}x_1^3 + k_{(1,1,2)}x_1^2x_2 + \dots + k_{(n,n,n)}x_n^3$, not containing any monomial divisible by x_0 , is identified with the projective coordinates $[c_a, c_H]$ in Remark II via $c_a = 0$ and $k_H = 3c_H$ for all $H \in \mathcal{H}$ with at least two entries in $H = (i, j, k)$ being distinct, and $k_{(i,i,i)} = c_{(i,i,i)}$ for all $i \geq 1$. In particular, $S_2 \cap B_2$ consists of all triples $(\lambda, q, k) = (\lambda, g^2, g^3)$ for some hyperplane $g \in \mathbb{P}(W^*/\lambda)$ as follows from the explicit description of ϕ_2 in Lemma 2.10.

Proposition 2.13 Let $\bar{\lambda} := ([\lambda], [q], [k])$ be a point of B_2 , i.e. a hyperplane λ together with a quadric q and a cubic k . Consider the line condition L_2^ℓ in V_2 corresponding to a line $\ell \subseteq \mathbb{P}(W)$. Then:

- (i) ℓ intersects λ at the quadric q if and only if L_2^ℓ is tangent to E_2 at $\bar{\lambda}$
- (ii) ℓ intersects λ at the cubic k if and only if L_2^ℓ is tangent to \tilde{E}_1 at $\bar{\lambda}$

Proof We can assume the hyperplane λ to be $\mathcal{V}(x_0)$ and ℓ the line $\mathcal{V}(x_1, x_3, \dots, x_n)$. By plugging in the equations $c_{(1,2,2)}b' - 3b_{(2,2,2)} + 2a_{(0,0,2)}b_{(0,2,2)} = 0$ and $a' - c_a b' = 0$ in the equation of the proper transform of the line condition in V_1 , we get the equation for L^ℓ in local coordinates in V_3 , i.e.

$$4b_{(0,2,2)}^3 c_a + c_{(2,2,2)}^2 b'.$$

From Lemma 2.7, one has that the quadrics intersecting λ at its point of intersection with ℓ are given by the equation: $b_{(0,2,2)} = 0$. From Remark 2.12 the cubics intersecting λ in $\lambda \cap \ell$ are given by the equation $c_{(2,2,2)} = 0$. The statement on the tangency at E_2 and at \tilde{E}_1 follows from the direct computation with the equations. □

Remark 2.14 We can notice that if the line ℓ does not intersect the quadric q or the cubic k at the point $\bar{\lambda}$, then the line condition L^ℓ is smooth at $\bar{\lambda}$. This is clear from the proof of the previous lemma when $\lambda = x_0$ and $\ell = \mathcal{V}(x_1, x_3, \dots, x_n)$. The claim follows by invariance under projective transformations.

2.3 Third Blow-up

Let $V_3 := \text{Bl}_{B_2} V_2$. This is smooth because B_2 is. We stick to the notation $\pi_3 : V_3 \rightarrow V_2$ for the blow-up map and E_3 for the exceptional divisor. We denote L_3 the proper transform in V_3 of the a line condition $L_2 \subseteq V_2$, and S_3 is the proper transform of S_2 .

Coordinates III In the chosen chart for V_2 described in Remark II the base locus B_2 is given by $\mathcal{V}(c_a, b')$. Consider $(D(a_{(0,0,0)}) \cap D(b_{(0,1,1)}) \cap D(c_{(1,1,1)})) \times \mathbb{P}^1$ with coordinates $(a_{(0,0,i)}, b_{(0,j,k)}, b', c_a, c_H, [d_c, d_b])$. The blow-up of B_2 in the chosen chart of V_2 can be described as the subvariety determined by

$$b'd_c = d_b c_a.$$

In the affine chart $D(a_{(0,0,0)}) \cap D(b_{(0,1,1)}) \cap D(c_{(1,1,1)}) \cap D(d_c)$ of V_3 we can work with coordinates $(a_{(0,0,i)}, b_{(0,j,k)}, c_a, c_H, d_b)$. The exceptional divisor E_3 is cut out by $c_a = 0$ in this chart.

Remark 2.15 The line condition L_3^ℓ corresponding to $\ell := \mathcal{V}(x_1, x_3, \dots, x_n)$ has equation

$$4b_{(0,2,2)}^3 + c_{(2,2,2)}^2 d_b = 0.$$

Therefore, every other line condition obtained from this one by an induced action of the PGL_n -action preserving this chart will be of type

$$4f(b_J)^3 + g(c_H)^2 d_b = 0,$$

where f is a linear function in the (b_J) coordinates and g is a linear function in the (c_H) coordinates.

We now prove that the intersection of all line conditions coincides with S_3 .

Proposition 2.16 *The intersection of all line conditions in V_3 is supported on the smooth irreducible variety S_3 .*

Proof The base locus $B_2 = \mathbb{P}(N_{B_1}E_1)$ has codimension 2 in V_2 . The exceptional divisor $E_3 = \mathbb{P}(N_{B_2}V_2)$ is then a \mathbb{P}^1 -bundle over B_2 . Let $\bar{\lambda} := ([\lambda], [q], [k])$ be a fixed point in B_2 with $\pi_2 \circ \pi_1(\bar{\lambda}) = [\lambda^3] \in B_0$, i.e., a hyperplane λ together with a quadric q and a cubic k lying on λ . Thanks to Remark 2.14, a general line condition is smooth at $\bar{\lambda} \in B_2$, has codimension one, and contains B_2 . Its proper transform intersects the fiber of $\mathbb{P}(N_{B_2}V_2)$ over $\bar{\lambda}$ at most in one point. We need to check that line conditions in V_3 can only intersect in E_3 above $B_2 \cap S_2$.

The base locus $B_2 = E_2 \cap \tilde{E}_1$ is smooth of codimension 2 in V_2 . Therefore, the proper transforms of \tilde{E}_1 and E_2 in V_3 cut the fiber of E_3 over any $\bar{\lambda} \in B_2$ in different points r_1 and r_2 . From Proposition 2.13 it follows that if a line ℓ intersects q , then the line condition L_3^ℓ contains r_2 , while if ℓ intersects k , then the line condition L_3^ℓ contains the point r_1 .

We claim that in order for the line conditions to intersect over $\bar{\lambda}$ we must have $q = hg$ and $k = h^2g$ where h, g are linear forms on the hyperplane λ . In fact, suppose there is a point of q which is not in k . Then, we can take a line ℓ in \mathbb{P}^n passing through that point and not contained in λ . Thanks to Remark 2.14, the line condition L_2^ℓ is smooth at $\bar{\lambda}$ and L_3^ℓ intersects the fiber over $\bar{\lambda}$ in a unique point, necessarily in r_2 . Take now another line condition $L_2^{\ell'}$ in V_2 such that the line ℓ' does not intersect the cubic nor the quadric. The line condition $L_2^{\ell'}$ is a hypersurface which is smooth at $\bar{\lambda}$ and contains B_2 . If its proper transform intersects the fiber over $\bar{\lambda}$ in r_2 , then $L_2^{\ell'}$ is tangent to E_2 , and by Proposition 2.13 it must intersect the quadric.

Similarly, we can show that there is no point of q which is not in k . Hence, we proved that in order for the line conditions to intersect over $\bar{\lambda}$ we must have $q = k$ set-theoretically. But this is equivalent to $q = hg$ and $k = h^2g$ with h, g linear forms on the hyperplane λ .

By Remark 2.12, we just have to show that $g = h$. It is enough to show it for $\lambda = x_0$ because the locus $B_2 \cap S_2$ is invariant under the induced PGL_n -action on V_2 . Consider the point $\bar{x}_0 = ([x_0^3], [q], [k])$, where

$$q = (h_1x_1 + \dots + h_nx_n)(g_1x_1 + \dots + g_nx_n), \quad k = (h_1x_1 + \dots + h_nx_n)^2(g_1x_1 + \dots + g_nx_n)$$

are respectively a quadric and a cubic on the hyperplane $x_0 = 0$.

We claim there exists an index l such that \bar{x}_0 belongs to $D(b_{(0,l,l)})$. First, fix i and j such that \bar{x}_0 belongs to the affine chart $D(b_{(0,i,j)}) \cap D(c_{(i,i,j)})$. Then, we can work in this affine chart with its coordinates. For $t \in \mathbb{k}$, consider the line conditions $L_2^{x_i+tx_j}$ in V_2 corresponding to the line given by the vanishing of $x_i + tx_j = 0$ and of all coordinates except for x_0, x_i, x_j . Their equations in V_2 are

$$4(t^2b_{(0,i,i)} + b_{(0,j,j)} - t)^3c_a + (t^3c_{(i,i,i)} - 3t^2 - c_{(j,j,j)} + 3tc_{(i,j,j)})^2b' = 0$$

in the mentioned open chart, where $b' = f'_{(i,i,j)}$ and $a' = f_{(0,i,j)}$. Notice that thanks to Lemmas 2.7 and 2.12, we have a relation between the coordinates and the coefficients of q and k . Suppose $b_{(0,l,l)}(\bar{x}_0) = h_l g_l = 0$ for all indices. The line conditions $L_3^{x_i+t x_j}$ intersect the fiber over \bar{x}_0 in V_3 in the points

$$\left[-4t^3, (3t^2 - 3tc_{(i,j,j)}(\bar{x}_0))^2\right].$$

Recall that we are working in a chart such that $h_i^2 g_j = c_{(i,i,j)}(\bar{x}_0) \neq 0$, therefore $c_{(i,j,j)}(\bar{x}_0) = h_j^2 g_i = 0$ and the points become

$$[-4t^3, 9t^4] = [-4, 9t].$$

This is absurd because we are assuming these line conditions to intersect over \bar{x}_0 and we proved the claim. Without loss of generality, we put $l = 1$, and we work in the affine chart $D(a_{(0,0,0)}) \cap D(b_{(0,1,1)}) \cap D(c_{(1,1,1)})$ of V_2 , so we can assume $h_1 = g_1 = 1$.

We claim that h_i is zero if and only if g_i is zero for every index i . Suppose there exists an index i such that $h_i = 0$ and $g_i \neq 0$ and consider the line conditions $L_2^{x_i+t x_1}$ in V_2 with equations

$$4(t^2 b_{(0,i,i)} + 1 - t b_{(0,1,1)})^3 c_a + (t^3 c_{(i,i,i)} - 3t^2 c_{(i,i,1)} - 1 + 3t c_{(1,1,i)})^2 b' = 0$$

in the chosen affine chart. The line conditions $L_3^{x_i+t x_1}$ intersect the fiber over \bar{x}_0 in E_3 in the points

$$\begin{aligned} \left[4(1 - t b_{(0,1,i)}(\bar{x}_0))^3, (1 - 3t c_{(1,1,i)}(\bar{x}_0))^2\right] &= [4(1 - t h_i - t g_i)^3, (1 - t g_i)^2] \\ &= [4(1 - t g_i)^3, (1 - t g_i)^2]. \end{aligned}$$

Notice that for $g_i \neq 0$, these would give different points for different values of t which is absurd. The same reasoning holds for $g_i = 0$ and $h_i \neq 0$ proving then the claim. Finally, if we consider the line condition $L_2^{x_j}$ in V_2 corresponding to the line given by the vanishing of all coordinates except for x_0 and x_j then this has equation

$$4b_{(0,j,j)}^3 c_a + c_{(j,j,j)}^2 b' = 0$$

in the chosen open chart. If we assume $(b_{(0,j,j)}(\bar{x}_0), c_{(j,j,j)}(\bar{x}_0)) \neq (0, 0)$, we must have

$$\begin{aligned} &= [4(b_{(0,j,j)}(\bar{x}_0))^3, (c_{(j,j,j)}(\bar{x}_0))^2] \Leftrightarrow (b_{(0,j,j)}(\bar{x}_0))^3 = (c_{(j,j,j)}(\bar{x}_0))^2 \\ &\Leftrightarrow g_j^3 h_j^3 = g_j^4 h_j^2, \end{aligned}$$

and therefore $g_j = h_j$ when h_j is non-zero. □

Since S_3 will be the next center for the blow-up, we denote it with B_3 . From Lemma 2.11 follows that B_3 is isomorphic to S_2 . In particular, the isomorphism map $\phi_2 : \text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \rightarrow S_2$ defined in Lemma 2.10 lifts to the following map.

Lemma 2.17 *The lift $\phi_3 : \text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \rightarrow V_3$ of ϕ_2 on the chosen open charts is explicitly given by*

$$\begin{aligned}
 a_{(0,0,1)} &= 3\mu_1 + u_1, & c_{(1,1,i)} &= s_i & \text{for } i > 1, \\
 a_{(0,0,i)} &= 3\mu_i + s_i u_1 & & & \text{for } i > 1, \\
 b_{(0,i,i)} &= s_i^2 & & & \text{for } i > 1, \\
 b_{(0,1,i)} &= 2s_i & & & \text{for } i > 1, \\
 b_{(0,i,j)} &= 2s_i s_j & & & \text{for } j > i > 1, \\
 c_a &= u_1/2, & c_{(1,i,j)} &= s_i^2 & \text{for } i > 1, \\
 & & c_{(i,i,j)} &= s_i^2 s_j & \text{for } i, j > 1, \\
 & & c_{(i,j,k)} &= 2s_i s_j s_k & \text{for } k > j > i > 0, \\
 & & d_b &= -4.
 \end{aligned}$$

Remark 2.18 The equations

$$\begin{aligned}
 d_b + 4 &= 0, \\
 g_{(0,1,i)} &:= b_{(0,1,i)} - 2c_{(1,1,i)} = 0 & \text{for } i > 1, \\
 g_{(0,i,i)} &:= b_{(0,i,i)} - c_{(1,1,i)}^2 = 0 & \text{for } i > 1, \\
 g_{(0,i,j)} &:= b_{(0,i,j)} - 2c_{(1,1,i)}c_{(1,1,j)} = 0 & \text{for } j > i > 1, \\
 g_{(1,i,i)} &:= c_{(1,i,i)} - c_{(1,1,i)}^2 = 0 & \text{for } i > 1, \\
 g_{(i,i,j)} &:= c_{(i,i,j)} - c_{(1,1,i)}^2 c_{(1,1,j)} = 0 & \text{for } i, j > 1, \\
 g_{(i,j,k)} &:= c_{(i,j,k)} - 2c_{(1,1,i)}c_{(1,1,j)}c_{(1,1,k)} = 0 & \text{for } k > j > i > 0.
 \end{aligned}$$

cut out B_3 in the chosen affine open chart. Notice that the equations form a regular sequence and that B_3 is indeed a complete intersection of codimension $\binom{n+3}{3} - 1 - 2n$ in the chosen affine chart.

2.4 Fourth Blow-up

Recall that $B_3 = S_3$. Let $V_4 := \text{Bl}_{B_3} V_3$. We will write E_4 for the exceptional divisor and $\pi_4 : V_4 \rightarrow V_3$ for the blow-up map.

Coordinates IV In the chosen affine chart of V_3 the base locus B_3 is cut out by the equations in Remark 2.18. Consider $D(a_{(0,0,0)}) \cap D(b_{(0,1,1)}) \cap D(c_{(1,1,1)}) \cap D(d_c) \times \mathbb{P}^{\binom{n+3}{3}-2n-2}$ with coordinates $(a_{(0,0,i)}, b_{(0,j,k)}, c_a, c_H, d_b, [e_d, e_F])$, where e_d is a new coordinate and the multi-indices F vary in the set $\mathcal{F} = \{(i, j, k) \in [n^3], \text{ with } 0 \leq i \leq j \leq k, 1 < j\}$. The blow-up of V_3 along B_3 in the chosen affine chart can be described as the subvariety determined by

$$\begin{aligned}
 e_d g_{(i,j,k)} - (d_b + 4)e_{(i,j,k)} &= 0 & \text{for } (i, j, k) \in \mathcal{F}, \\
 e_{(i_1, j_1, k_1)} g_{(i_2, j_2, k_2)} - g_{(i_1, j_1, k_1)} e_{(i_2, j_2, k_2)} &= 0 & \text{for } (i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathcal{F}.
 \end{aligned}$$

In the affine chart $D(a_{(0,0,0)}) \cap D(y_{(0,1,1)}) \cap D(c_{(1,1,1)}) \cap D(d_c) \cap D(e_{(0,1,2)})$ of V_4 we can work with coordinates $(a_{(0,0,i)}, c_a, c_{(1,1,2)}, \dots, c_{(1,1,n)}, e_d, e_F, e')$ where $e' = g_{(0,1,2)}$ is used as a coordinate and \mathcal{F} is the same index set as above but we exclude $(0, 1, 2)$. The exceptional divisor E_4 is cut out by $e' = 0$ in this chart.

Proposition 2.19 *The intersection of all line conditions in V_4 is supported on a smooth subvariety B_4 of codimension $\binom{n+2}{3}$ inside E_4 . More precisely, $B_4 = \mathbb{P}(\mathcal{E})$ where \mathcal{E} is a vector subbundle of rank $\binom{n}{2}$ of the normal bundle $N_{B_3} V_3$.*

Proof We generalize the proof of [1, Proposition 4.1]. Let $R_\mu \subseteq V_0$ denote the subvariety of cubics containing the hyperplane μ . Clearly, $R_\mu \cong \mathbb{P}(\text{Sym}^2(W^*))$ is smooth. By Lemma 2.1, a line condition L^ℓ is smooth at $[\lambda\mu^2] \in S_0 \setminus B_0$ if the line ℓ intersects μ in a single point outside λ . Clearly, $T_{[\lambda\mu^2]}R_\mu \subseteq T_{[\lambda\mu^2]}L^\ell$ for every line ℓ , and Lemma 2.1 shows that

$$\bigcap_{\ell \subseteq \mathbb{P}(W) \text{ line}} T_{[\lambda\mu^2]}L^\ell = T_{[\lambda\mu^2]}R_\mu.$$

Clearly, finitely many lines suffice for the intersection of the tangent spaces to agree with $T_{[\lambda\mu^2]}R_\mu$ over every point $[\lambda\mu^2] \in B_3 \setminus e \cong S_0 \setminus B_0$. By Proposition 2.16, the intersection of the proper transforms L_3^ℓ in V_3 for all lines ℓ agrees set-theoretically with $S_3 = B_3$. The proper transforms L_4^ℓ in V_4 therefore only intersect in the exceptional divisor E_4 . We claim that their intersection is precisely the projectivization of a vector subbundle $\mathcal{E} \subseteq N_{B_3}V_3$. We construct \mathcal{E} as the intersection of the images of the tangent sheaves $\mathcal{T}L_3^\ell|_{B_3}$ in $N_{B_3}V_3$ corresponding to finitely many lines ℓ . The finiteness will ensure that the resulting subsheaf \mathcal{E} of $N_{B_3}V_3$ is coherent. First, we pick finitely many lines such that the intersection of the tangent spaces over every point $[\lambda\mu^2] \in B_3 \setminus e$ agrees with $T_{[\lambda\mu^2]}R_\mu$. The intersection of the images of the tangent sheaves in $N_{B_3}V_3$ of these line conditions defines a coherent subsheaf \mathcal{E}' which restricts to a vector subbundle over $B_3 \setminus e$. Then by Lemma 2.1 and a Zariski closure argument, every other line condition L_4^ℓ contains the projectivization $\mathbb{P}(\mathcal{E}'|_{B_3 \setminus e})$, and we have

$$\mathcal{E}'([\lambda\mu^2]) \cong T_{[\lambda\mu^2]}R_\mu / T_{[\lambda\mu^2]}S_0,$$

where $\mathcal{E}'(p)$ denotes the geometric fiber of \mathcal{E}' over the point p . The rank of \mathcal{E}' over $B_3 \setminus e$ is $r = \binom{n+2}{2} - 2n - 1 = \binom{n}{2}$. Next, we fix a point $p \in e = B_3 \cap E_3$ lying in our affine open chart. By Remark 2.15, in the chosen affine chart the equation for L_3^ℓ with ℓ any line passing through the point $[1 : 0 : \dots : 0]$ does not depend on the variable c_a , and the equation determining E_3 in V_3 is exactly $c_a = 0$. The transversality of such line conditions can therefore be checked outside of E_3 and hence in $S_0 \setminus B_0$. This shows at once that there are $\text{codim}(R_\mu, V_0) = \binom{n+2}{3}$ lines $\ell_i \subseteq \mathbb{P}(W)$ such that the line conditions $L_3^{\ell_i}$ are all smooth and intersect transversally at p . Moreover, employing the PGL_n -action and using that it acts transitively on e by Lemma 2.24, we obtain finitely many more lines such that the intersection of their tangent spaces at every point of e has dimension at most r . Let \mathcal{E} be the intersection of \mathcal{E}' with the images of the tangent sheaves in $N_{B_3}V_3$ of these new line conditions. Then \mathcal{E} is a coherent subsheaf of $N_{B_3}V_3$ which still restricts over $B_3 \setminus e$ to a vector subbundle of rank r and has rank $\leq r$ over every point of e . By upper semi-continuity of the rank, since \mathcal{E} is coherent, \mathcal{E} is a vector subbundle of $N_{B_3}V_3$ of rank r everywhere. As $\mathbb{P}(\mathcal{E})$ is an irreducible closed subset of V_4 , a Zariski closure argument then shows that it is contained in L_4^ℓ for every ℓ , so it is contained in the intersection of all line conditions in V_4 . Nevertheless, by construction $\mathbb{P}(\mathcal{E})$ also contains the intersection of some (and hence of all) line conditions in V_4 , proving equality. \square

2.5 Fifth Blow-up

Let $V_5 := \text{Bl}_{B_4}V_4$. Denote with E_5 the exceptional divisor and let $\pi_5 : V_5 \rightarrow V_4$ be the blow-up map. Let \tilde{E}_4 be the strict transform of E_4 .

Lemma 2.20 *We have the isomorphism*

$$N_{B_4}E_4 \cong (\pi_4|_{B_4})^*(N_{B_3}V_3/\mathcal{E}) \otimes \mathcal{O}_{B_4}(1).$$

Moreover, over $U := B_4 \setminus (\pi_4|_{B_4})^{-1}(e)$ the normal bundle $N_{B_4}E_4$ restricts to

$$N_U E_4 \cong (\pi_4|_U)^* \left(\frac{\text{Sym}^3(W^*) \otimes \mathcal{O}(1, 2)}{\text{Sym}^2(W^*) \otimes \mathcal{O}(1, 1)} \right) \otimes \mathcal{O}_U(1),$$

where $\mathcal{O}(a, b)$ denotes the pullback to $\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta$. In particular, the fiber of $N_{B_4}E_4$ over some point of $B_4 \setminus \pi_4^{-1}(e)$ mapping to $[\lambda\mu^2] \in B_3 \setminus e$ is given by $\text{Sym}^3(W^*)/(\mu \cdot \text{Sym}^2(W^*))$.

The proof is similar to that of Lemma 2.8. We can now start to understand the intersection of all line conditions inside V_4 .

Lemma 2.21 *Fix a line ℓ of \mathbb{P}^n and a cubic $\lambda\mu^2$ such that ℓ does not intersect $\lambda \cap \mu$. The strict transform L_5^ℓ in V_5 contains a point p in $E_5 \cap \tilde{E}_4$ with $(\pi_4 \circ \pi_5)(p) = [\lambda\mu^2]$ if and only if the line ℓ intersects the cubic on μ associated with p , i.e. the element of $\text{Sym}^3(W^*)/(\mu \cdot \text{Sym}^2(W^*))$.*

Proof By assumption, L_3^ℓ and its proper transforms are smooth at every point over $[\lambda\mu^2] \in B_3$. We have $(L_5^\ell \cap \tilde{E}_4 \cap E_5)|_{\pi_5(p)} = \mathbb{P}(N_{B_4}(L_4 \cap E_4)|_{\pi_5(p)})$. Since $L_4 \cap E_4|_U = \mathbb{P}(N_{B_3}L_3|_U)$ on the smooth locus U of L_3^ℓ inside B_3 , we have the canonical isomorphisms

$$N_{B_4}(L_4 \cap E_4)|_{\pi_5(p)} \cong ((\pi_4|_{B_4})^*(N_{B_3}L_3^\ell/\mathcal{E}) \otimes \mathcal{O}_{B_4}(1))|_{\pi_5(p)} \cong (N_{B_3}L_3^\ell/\mathcal{E})|_{[\lambda\mu^2]}.$$

Knowing that $T_{[\lambda\mu^2]}L_3^\ell$ is given by those cubics containing $\ell \cap \mu$ by Lemma 2.1(iii) and that the fiber of \mathcal{E} at $[\lambda\mu^2]$ is the quotient of the cubics containing μ by the tangent space of B_3 at $[\lambda\mu^2]$, we conclude that the projective fiber of this bundle over the point $\pi_5(p)$ of B_4 is exactly given by those cubics on μ touching ℓ . □

Lemma 2.22 *There exists a point $[\lambda\mu^2]$ with $\lambda \neq \mu$ in B_3 such that for every point $\overline{\lambda\mu^2}$ in B_4 with $\pi_3(\overline{\lambda\mu^2}) = [\lambda\mu^2]$ the intersection of the line conditions in the fiber $(E_5)|_{\overline{\lambda\mu^2}}$ is contained in the proper transform \tilde{E}_4 of E_4 in V_5 .*

Proof Consider the chart in V_3 given by $D(a_{(0,0,0)}) \cap D(b_{(0,1,1)}) \cap D(c_{(1,1,1)}) \cap D(d_a)$. We can now choose any point in $B_3 \setminus e$; we will choose our favourite one $P := [(x_2+x_0)^2(x_2+x_1+x_0)]$. Notice that this is indeed contained in the chart. We will denote with (P, Q) a point in the fiber of B_4 over P , where $Q \in \mathbb{P}(R_{(x_2+x_0)}/(TB_3)_P)$ and where $R_{(x_2+x_0)}$ is the space of cubics which are divisible by (x_2+x_0) . Points $P_{\epsilon,q}$ in $R_{(x_2+x_0)}$ can be uniquely written up to constants as $P_{\epsilon,q} := (x_2+x_0)q$ in the projective coordinates $[q_{ij}]_{i,j \in [n]}$ of the quadric q in $(n+1)$ variables. In this coordinates, the tangent space $(TB_3)_P$ is given by

$$\begin{cases} q_{00} + q_{22} = q_{02} \\ q_{0j} - q_{2j} = 0 & \text{for all } j \neq 0, 1, 2, \\ q_{ij} = 0 & \text{for all } i, j \neq 0, 1, 2. \end{cases}$$

Denoting $\pi_P : R_{(x_2+x_0)} \setminus (TB_3)_P \rightarrow (B_4)_P$ the quotient map followed by the projectivization, every point (P, Q) can be represented in a non-unique way as $\pi_P([q_{ij}]_{i,j \in [n]})$. We now want to show that for every point $(P, Q) = \pi_P([q_{ij}]_{i,j \in [n]})$ there exists a sequence of line conditions L_4^m in V_4 which are smooth at (P, Q) and such that the hyperplanes $(TL_4^m)_{(P,Q)}$ tend to $(TE_4)_{(P,Q)}$ as vector subspaces of $(TV_4)_{(P,Q)}$. This proves the lemma, as the intersection of all line conditions in $(E_5)_{(P,Q)}$ will be the same as the intersection of all line conditions and $(\tilde{E}_4)_{(P,Q)}$.

Before choosing appropriate line conditions, let us compute the projective coordinates $[e_d, e_{(0,1,2)}, \dots, e_{(0,n,n)}, e_{(1,1,2)}, \dots, e_{(n,n,n)}]$ for $(P, Q) = \pi_P([q_{ij}]_{i,j \in [n]})$ as functions of $[q_{ij}]$. We get the following coordinates for the point (P, Q) :

$$\begin{aligned}
 e_d &= 0, \\
 e_{(0,1,j)} &= 0 && \text{for } j \neq 0, 1, && e_{(1,j,j)} &= -\frac{3}{2}q_{jj} && \text{for } j \neq 0, 1, 2, \\
 e_{(0,2,2)} &= 3(q_{02} - q_{00} - q_{22}), && && e_{(1,2,j)} &= \frac{3}{2}(q_{0j} - q_{2j}) && \text{for } j \neq 0, 1, 2, \\
 e_{(0,j,j)} &= -3q_{jj} && \text{for } j \neq 0, 1, 2, && e_{(1,i,j)} &= -\frac{3}{2}q_{ij} && \text{for } i, j \neq 0, 1, 2, \\
 e_{(0,2,j)} &= 3(q_{0j} - q_{2j}) && \text{for } j \neq 0, 1, 2, && e_{(1,i,i)} &= \frac{3}{2}q_{ii} && \text{for } i, j \neq 0, 1, 2, \\
 e_{(0,i,j)} &= -3q_{ij} && \text{for } i, j \neq 0, 1, 2, && e_{(i,i,j)} &= 0 && \text{for } i, j \neq 0, 1, \\
 e_{(1,2,2)} &= \frac{3}{2}(q_{02} - q_{00} - q_{22}), && && e_{(j,j,j)} &= 0 && \text{for } j \neq 0, 1.
 \end{aligned}$$

Notice that this makes sense as long as $[q_{ij}]_{i,j \in [n]} \notin (TB_3)_P$, which is the case we are interested in.

We will use the notation $L^{j,t}$ for line conditions associated to the lines

$$\mathcal{V}(x_1 + tx_j, x_2, \dots, \hat{x}_j, \dots, x_n).$$

The proper transform $L_3^{j,t}$ of these line conditions in the chosen affine chart for V_3 are given by

$$4(t^2 + b_{(0,j,j)} - tb_{(0,1,j)})^3 + (t^3 - 3t^2c_{(1,1,j)} - c_{(j,j,j)} + 3tc_{(1,j,j)})^2 db = 0.$$

Notice that the line condition $L_3^{j,0}$ is singular at P for any $j \neq 0, 1$, but the line conditions $L_3^{j,t}$ for $t \neq 0$ are not, and therefore the proper transforms $L_4^{j,t}$ in V_4 are smooth at every point $(P, Q) \in B_4$. Now consider the proper transform of such line condition in a chart of V_4 different from $D(e_d)$ with coordinate e' . Notice that we can do that because $e_d(P, Q) = 0$ for every choice of Q . Since we are interested in the gradient of the equation evaluated on points in $B_4 \subseteq E_4 = \{e' = 0\}$, we can just look at the gradient of the following equation:

$$\begin{aligned}
 &\left(12(t - c_{(1,1,j)})^4(e_{(0,j,j)} - te_{(0,1,j)}) + e_d(t - c_{(1,1,j)})^6 - 8(t - c_{(1,1,j)})^3(3te_{(1,j,j)} - e_{(j,j,j)}) + \right. \\
 &\quad \left. e' \left(12(t - c_{(1,1,j)})^2(e_{(0,j,j)} - te_{(0,1,j)})^2 + 2e_d(t - c_{(1,1,j)})^3(3te_{(1,j,j)} - e_{(j,j,j)}) - \right. \right. \\
 &\quad \left. \left. 4(3te_{(1,j,j)} - e_{(j,j,j)})^2 \right) \right) = 0.
 \end{aligned}$$

If we look at partial derivatives ∂_y with respect to variables $y \neq e'$ evaluated at the point (P, Q) , we have that $\frac{\partial_y}{\partial a} = 0$ for $a \in \{0, 1, 2\}$, and this follows from $c_{(1,1,j)}(P) = 0$. If we look at partial derivatives $\partial_{e'}$ evaluated at the point $(P, Q) = \pi_P([q_{ij}]_{i,j \in [n]})$, this is given by

$$12t^2(e_{(0,j,j)})^2 - 4(3te_{(1,j,j)})^2.$$

We see that the partial derivative $\frac{\partial_{e'}}{\partial t}$ is given by

$$12(e_{(0,j,j)}^2 - 3e_{(1,j,j)}^2).$$

For $j = 2$ then this becomes

$$27 (4 (q_{02} - q_{00} - q_{22})^2 - 3 (q_{02} - q_{00} - q_{22})^2) = 27 (q_{02} - q_{00} - q_{22})^2,$$

and the claim follows from this quantity being non-zero at the point (P, Q) .

Suppose instead $q_{02} - q_{00} - q_{22} = 0$, then we can look at different line conditions assuming $e_{(0,2,2)}(P, Q) = e_{(1,2,2)}(P, Q) = 0$. Take $L_4^{j,t}$ where $j \neq 2$. If we repeat the same reasoning everything remains the same but in the end we get that $\frac{\partial_{e'}}{t^2}$ is given by

$$27 (4q_{jj}^2 - 3q_{jj}^2) = 27q_{jj}^2.$$

Once again, we obtain the claim if this quantity is different from zero for our (P, Q) . If instead $c_{jj} = 0$ for every $j \neq 2$, then we can look at different line conditions assuming $e_{(0,j,j)}(P, Q) = e_{(1,j,j)}(P, Q) = 0$ for every j . Let us denote with $L^{i,j,t}$ the line conditions associated to the lines $\mathcal{V}(x_1 + tx_j, x_2, \dots, \hat{x}_j, \dots, \hat{x}_i, x_i + x_j, \dots, x_n)$ for $i, j \neq 0, 1$. The proper transform $L_3^{i,j,t}$ of these line conditions in the chosen affine chart for V_3 are given by

$$4F_{i,j,t}^3 + G_{i,j,t}^2 db = 0,$$

where

$$F_{i,j,t} = t^2 + b_{(0,j,j)} + b_{(0,i,i)} - b_{(0,i,j)} - tb_{(0,1,j)} + tb_{(0,1,i)},$$

and

$$G_{i,j,t} = t^3 + c_{(i,i,i)} + 3tc_{(1,i,i)} + 3t^2c_{(1,1,i)} + 3c_{(i,j,j)} - 3c_{(i,i,j)} + 3tc_{(1,j,j)} - 3t^2c_{(1,1,j)} - 3tc_{(1,i,j)} - c_{(j,j,j)}.$$

If we now consider the proper transform of this line condition in a chart of V_4 different from $e_d \neq 0$, repeating a similar reasoning to before we can see that for partial derivatives ∂_y with respect to variables $y \neq e'$ evaluated at the point (P, Q) , we have $\frac{\partial_y}{t^a} = 0$ for $a \in \{0, 1, 2\}$, and this follows again from the fact that $c_{(1,1,i)}(P) = c_{(1,1,j)}(P) = 0$ for our point P . If we look at partial derivatives $\partial_{e'}$ for the variable e' evaluated at the point $(P, Q) = \pi_P([q_{ij}]_{i,j \in [n]})$, this is given by

$$12t^2 (e_{(0,i,j)})^2 - 4 (3te_{(1,i,j)})^2.$$

But then we see that $\frac{\partial_{e'}}{t^2}$ is given by

$$12 (e_{(0,i,j)}^2 - 3e_{(1,i,j)}^2).$$

If $j = 2$ then this becomes

$$27 (4(q_{0i} - q_{2i})^2 - 3 (q_{0i} - q_{2i})^2) = 27(q_{0i} - q_{2i})^2$$

and we obtain the claim if this quantity is different from zero for our (P, Q) . If instead we also have $q_{0i} - q_{2i} = 0$ for every i , then we can look at different line conditions. Take $L^{i,j,t}$ where $j \neq i \neq \{0, 1, 2\}$. If we repeat the same reasoning everything remains the same but in the end we get that $\frac{\partial_{e'}}{t^2}$ is given by

$$27 (4q_{ij}^2 - 3q_{ij}^2) = 27q_{ij}^2.$$

Once again, we obtain the claim if this quantity is different from zero for our (P, Q) . Finally, if $q_{ij} = 0$ for every ij as before, then this implies $[q_{ij}]_{i,j \in [n]} \in (TB_3)_P$, but this is not possible. This concludes the proof. \square

Corollary 2.23 *The intersection of all line conditions in V_5 is empty.*

Proof We need to show that the line conditions do not intersect in E_5 . Thanks to Remark 2.15 and the fact that the equations in Remark 2.18 do not involve the variable c_a , we can show it over fibers corresponding to points in $B_4 \setminus (\pi_4|_{B_4})^{-1}(e)$. By the PGL_n -action we can just look at one single fiber on a point $\overline{\lambda\mu^2}$ of B_4 , with $\pi_4(\overline{\lambda\mu^2}) = [\lambda\mu^2]$. The claim then follows from Lemma 2.22. \square

The previous lemma proves that line conditions separate in V_5 and that this space is a 1-complete variety of cubic hypersurfaces.

2.6 Identifying the vector bundle \mathcal{E} on e

We now give a more explicit description of the bundle $\mathcal{E}|_e$, which will be useful for understanding the total Chern class $c(\mathcal{E})$.

Lemma 2.24 *The natural action of PGL_n on the exceptional divisor $e \subseteq \text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ is transitive.*

Proof We have $e = \mathbb{P}(N_\Delta \mathbb{P}^n \times \mathbb{P}^n) = \mathbb{P}(T\Delta)$ where the isomorphism $N_\Delta \mathbb{P}^n \times \mathbb{P}^n \cong T\Delta$ is provided by any multiple of the difference of the differentials of the projections, e.g. $dpr_1 - dpr_2$. Fix now two points $[\lambda], [\mu] \in \Delta$ and two non-zero normal vectors $(v_1, v_2) \in N_\Delta \mathbb{P}^n \times \mathbb{P}^n|_{[\lambda]}$ and $(w_1, w_2) \in N_\Delta \mathbb{P}^n \times \mathbb{P}^n|_{[\mu]}$. These two normal vectors are represented by two curves $\mathbb{A}^1 \rightarrow \mathbb{P}^n \times \mathbb{P}^n, t \mapsto ([\lambda + tv_1], [\lambda + tv_2])$ and $t \mapsto ([\mu + tw_1], [\mu + tw_2])$, respectively. We then only need to find $A \in \text{PGL}_n = \text{GL}_{n+1}/\sim$ with $A\lambda = \mu$ and $A(v_1 - v_2) = w_1 - w_2$. Such a A exists if $v_1 - v_2$ is not a multiple of λ and $w_1 - w_2$ is not a multiple of μ . Both conditions are satisfied by the requirement that (v_1, v_2) and (w_1, w_2) are both non-zero normal vectors. \square

Proposition 2.25 *We have $\mathcal{E}|_e \cong \text{Sym}^2(T_{e/\Delta})$.*

Remark 2.26 The geometric intuition behind this proposition is as follows. The fiber of $\text{Sym}^2(T_{e/\Delta})$ over a point $([\lambda], [g]) \in e$ is $\frac{\text{Sym}^2(W^*/\lambda)}{g \cdot (W^*/\lambda)}$, the quadrics on g . This makes much sense, given that over a point $[\lambda\mu^2] \in B_3 \setminus e$, the fiber of \mathcal{E} is naturally identified with $\frac{\text{Sym}^2(W^*)}{(\lambda \cdot W^* + \mu \cdot W^*)}$, the quadrics on $\lambda \cap \mu$. Fixing λ , as μ approaches λ along some curve, $\lambda \cap \mu$ can be seen as a sequence of hyperplanes inside λ with some limiting hyperplane g inside λ . Along this sequence, the quadrics on $\lambda \cap \mu$ should indeed approach the quadrics on g .

Remark 2.27 It follows from the relative Euler sequence of the projective bundle e over Δ that

$$\text{Sym}^2(T_{e/\Delta}) \cong \frac{\pi_e^*(\text{Sym}^2(T\Delta)) \otimes \mathcal{O}_e(2)}{\pi_e^*(T\Delta) \otimes \mathcal{O}_e(1)},$$

the total Chern class of which can be computed using the Chern classes of $T\Delta$.

Proof of Proposition 2.25 From the relative Euler sequence for the projective bundles $e = \mathbb{P}(T\Delta)$ and $B_1 = \mathbb{P}(\text{Sym}^2(T\Delta))$ we obtain

$$N_e B_1 \cong \frac{T_{B_1/B_0}|_e}{T_{e/\Delta}} \cong \text{Sym}^2(T_{e/\Delta}).$$

We now first give an embedding of $N_e B_1$ into $N_{B_3} V_3|_e$. In a second step we show that the image agrees with $\mathcal{E}|_e$. For the first step, we observe the chain of natural inclusions of geometric vector bundles

$$T_{B_2/B_0}|_e \subseteq T B_2|_e \cong T\mathbb{P}(N_{B_2} E_2)|_e \subseteq T\mathbb{P}(N_{B_2} V_2)|_e = T E_3|_e \subseteq T V_3|_e, \tag{4}$$

using in the first step that $B_2 = E_2 \cap \tilde{E}_1$, so $N_{B_2} E_2$ is a line bundle and therefore the restriction of π_3 is an isomorphism $\mathbb{P}(N_{B_2} E_2) \cong B_2$. In order to embed $T_{B_1/B_0}|_e$ into $T_{B_2/B_0}|_e$, note that B_2 is actually a fiber product over B_0 . To be precise, it follows from Lemma 2.8 that

$$B_2 = \mathbb{P}(N_{B_1} E_1) \cong \mathbb{P}(p_1^*(\text{Sym}^3(T\Delta))) = \mathbb{P}(\text{Sym}^3(T\Delta)) \times_{B_0} B_1.$$

The restriction $p_2 : B_2 \rightarrow B_1$ of π_2 agrees under this identification with the projection to the second factor. Under the natural identifications $B_0 = \Delta$ and $B_1 = \mathbb{P}(\text{Sym}^2(T\Delta))$, the inclusion $e \subseteq B_2$ corresponds to the map

$$e = \mathbb{P}(T\Delta) \xrightarrow{(v_3, v_2)} \mathbb{P}(\text{Sym}^3(T\Delta)) \times_{\Delta} \mathbb{P}(\text{Sym}^2(T\Delta)),$$

where v_2, v_3 denote the relative second and third Veronese embeddings. On the fiber over $[\lambda] \in \Delta = \mathbb{P}(W^*)$, these map a linear form $[g] \in \mathbb{P}(W^*/\lambda) = e|_{[\lambda]}$ to its second respectively third power. Consider now the following diagram (where we omit the pullback signs and identify $B_0 = \Delta$):

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 T_{B_2/B_0} & \overset{\text{---}}{\dashrightarrow} & & & T_{B_1/B_0} & & \\
 & \searrow & & & \downarrow & & \\
 & & T B_2 & \longrightarrow & T B_1 & & \\
 & & \downarrow & \square & \downarrow & & \\
 0 & \longrightarrow & T_{\mathbb{P}(\text{Sym}^3(T\Delta))/\Delta} & \longrightarrow & T\Delta = T B_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

The induced dashed maps provide an isomorphism $T_{B_2/B_0} \cong T_{\mathbb{P}(\text{Sym}^3(T\Delta))/\Delta} \oplus T_{B_1/B_0}$. We define the embedding

$$s : T_{B_1/B_0}|_e \hookrightarrow T_{\mathbb{P}(\text{Sym}^3(T\Delta))/\Delta}|_e \oplus T_{B_1/B_0}|_e \cong T_{B_2/B_0}|_e$$

by prescribing it to be the identity on the second factor. On the first factor we define it via

$$\begin{array}{ccc}
 T_{B_1/B_0}|_e & \hookrightarrow & T_{\mathbb{P}(\text{Sym}^3(T\Delta))/\Delta}|_e \\
 \cong \downarrow & & \downarrow \cong \\
 \frac{\pi_e^*(\text{Sym}^2(T\Delta)) \otimes \mathcal{O}_e(2)}{\mathcal{O}_e} & \hookrightarrow & \frac{\pi_e^*(\text{Sym}^3(T\Delta)) \otimes \mathcal{O}_e(3)}{\mathcal{O}_e},
 \end{array}$$

given on the fiber over $([\lambda], [g]) \in e$ (i.e. $[\lambda] \in \Delta = \mathbb{P}(W^*)$ and $g \in W^*/\lambda$) by sending a quadric $q \in \text{Sym}^2(W^*/\lambda)/(g^2)$ to $\text{cst} \cdot g \cdot q \in \text{Sym}^3(W^*/\lambda)/(g^3)$, where cst is some non-zero constant still to be specified. (Up to multiplication by cst , this is a relative version of the map $\alpha_{2,3}$ from Lemma 2.3.) Denoting by $e_1 \subseteq B_1$ and $e_2 \subseteq B_2$ the images of $\phi_1(e)$ and $\phi_2(e)$, respectively, we want $s(T_{e_1/\Delta}) = T_{e_2/\Delta} \subseteq T_{B_2/B_0}|_{e_2}$. This is achieved precisely for the choice $\text{cst} = \frac{3}{2}$. Composing with (4), the embedding $s : T_{B_1/B_0}|_e \hookrightarrow T_{B_2/B_0}|_e$ now provides an embedding of geometric vector bundles $T_{B_1/B_0}|_e \hookrightarrow T_{V_3}|_e$. Composing further with the quotient map $T_{V_3}|_e \rightarrow N_{B_3}V_3|_e$, the kernel is then precisely $T_{e_1/B_0} \subseteq T_{B_1/B_0}|_{e_1}$. Hence, we obtain an embedding of geometric vector bundles

$$N_e B_1 \hookrightarrow N_e E_3 \subseteq N_{B_3} V_3|_e.$$

We denote by $\mathcal{F} \subseteq N_e E_3 \subseteq N_{B_3} V_3|_e$ its image. It is enough to show $\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{E}|_e) = B_4 \cap \pi_4^{-1}(e)$. As the embedding $N_e B_1 \hookrightarrow N_{B_3} V_3|_e$ is PGL_n -equivariant, it is enough to show the equality $\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{E}|_e)$ for the fiber over a single point of e , using that PGL_n acts transitively on e by Lemma 2.24. We pick the point $([\lambda], [g]) = ([x_0], [x_1]) \in e$. In the explicit coordinates of Subsection 2.4, the fiber of B_4 over this point is defined (in the affine chart where $e_{(0,1,2)} = 1$) by the equations

$$\begin{aligned} 0 &= e', \\ 0 &= e_d, \\ 0 &= e_{(0,i,j)} - 2e_{(1,i,j)} && \text{for all } i, j > 1, \\ 0 &= e_F && \text{for all } F \neq (0, i, j), (1, i, j). \end{aligned}$$

This follows from the fact that the same equations hold for the fiber of B_4 over the point $[(x_0 + x_1 + x_2)(x_0 + x_2)^2] \in B_3 \setminus e$, see the proof of Lemma 2.22. This point has the same $b_{(0,i,j)}$ and $c_{(i,j,k)}$ coordinates as $([x_0], [x_1]) \in e$. By Remark 2.18, the equations for B_3 inside V_3 only depend on those, and the same is true for the equations of the line conditions from Remark 2.15. Therefore, the fiber of B_4 over $([x_0], [x_1]) \in e$ is indeed defined by the same equations in the (e_F, e_d, e') -coordinates as the fiber over $[(x_0 + x_1 + x_2)(x_0 + x_2)^2] \in B_3 \setminus e$. Finally, the explicit description of the embedding $s : T_{B_1/B_0}|_e \hookrightarrow T_{B_2/B_0}|_e$ above provides a way to check that all points in the chart satisfying the above equations lie inside $\mathbb{P}(\mathcal{F}|_{([x_0],[x_1])})$. Namely, if we start with a tangent vector associated to a quadric $q = \sum_{2 \leq i \leq j} q_{(i,j)} x_i x_j \in T_{B_1/B_0}|_{([x_0],[x_1])}$, it is represented by a curve in our usual affine open chart of B_2 given by sending $t \in \mathbb{A}^1$ to $b_{(0,i,j)} = q_{(i,j)} \cdot t$, $c_{(1,i,j)} = \frac{\text{cst}}{3} \cdot q_{(i,j)} \cdot t$ for all $i, j > 1$ and all other coordinates equal to 0. Tracing the proper transform of this curve in V_3 , we obtain that this tangent vector corresponds to the point in E_4 with coordinates $e' = e_d = 0$, $e_{(0,i,j)} = q_{(i,j)}$, $e_{(1,i,j)} = \frac{\text{cst}}{3} \cdot q_{(i,j)}$ and all other $e_F = 0$. This satisfies the above equations exactly for the choice $\text{cst} = \frac{3}{2}$. We get that $\mathbb{P}(\mathcal{F}|_{([x_0],[x_1])})$ contains a dense open subset of $\mathbb{P}(\mathcal{E}|_{([x_0],[x_1])})$ and hence the entire fiber. As their dimensions agree, we obtain equality, and with this we conclude that $\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{E}|_e)$. \square

3 Intersection rings and Chern classes

In the following subsections we collect details about the Chow rings of the centers of the blow-ups. This is the last step needed to compute the characteristic numbers. In particular, we find generators, describe the degree of the product of those generators, and find the Chern classes $c(N_{B_i} V_i)$ of the normal bundle of B_i inside V_i . Recall from the beginning of [1, Section 2]

that for a nonsingular variety V of dimension n , and a nonsingular closed subvariety B in V of codimension d , the *full intersection class* of a pure-dimensional subscheme X in V is defined as:

$$B \circ X := c(N_B V) \cap s(B \cap X, X)$$

in the Chow group $A_*(B \cap X)$ of $B \cap X$, with $s(B \cap X, X)$ denoting the Segre class. In the end of this section, we compute the full intersection classes $B_i \circ P_i$ and $B_i \circ L_i$.

Remark 3.1 For $X \subseteq V_i$ being a divisor and $j_i : B_i \hookrightarrow V_i$, we have

$$B_i \circ X = e_{B_i} X[B_i] + j_i^*[X] \tag{5}$$

where $e_{B_i} X$ denotes the multiplicity of X along B_i .

3.0 Chow ring of B_0

The following results directly generalize from [1]:

Lemma 3.2 *The intersection ring of $B_0 \simeq \mathbb{P}^n$ is generated by the hyperplane class h . Moreover, $c(N_{B_0} V_0) = (1 + 3h)^{\binom{n+3}{3}} / (1 + h)^{n+1}$.*

Lemma 3.3 *The full intersection classes of point and line conditions in V_0 with respect to B_0 are*

$$B_0 \circ P = 3h \quad B_0 \circ L = 2 + 12h.$$

3.1 Chow ring of B_1

The center of the second blow-up is B_1 , this was described in Subsection 2.1.

Lemma 3.4 *The variety B_1 has dimension $\binom{n+2}{2} - 2$.*

(i) *The intersection ring of B_1 is generated by the pullback h of h via the map $\pi_1 : B_1 \rightarrow B_0$ and the pullback ϵ of $[E_1]$ via the inclusion map $j_1 : B_1 \hookrightarrow V_1$. Consider the sequence $\{a_s\}$ obtained with the following recursion*

$$\begin{cases} a_0 = 1 \\ a_s = \binom{n+1}{s} - \sum_{i=1}^s 2^i \binom{\dim B_1 + 2}{i} a_{s-i}. \end{cases}$$

We get

$$\begin{aligned} \int_{B_1} h^s \epsilon^{\dim B_1 - s} &= 0 && \text{for } s \in [n + 1, \dim B_1], \\ \int_{B_1} h^s \epsilon^{\dim B_1 - s} &= (-1)^{\dim B_1 - s} a_{n-s} && \text{for } s \in [0, n]. \end{aligned}$$

(ii) $c(N_{B_1} V_1) = (1 + \epsilon)(1 + 3h - \epsilon)^{\binom{n+3}{3}} / (1 + 2h - \epsilon)^{\binom{n+2}{2}}$.

Proof The proof is exactly as in [1]. Notice that $\mathcal{O}_{E_1}(-1) = \mathcal{O}_{V_1}(E_1)|_{E_1}$, so that $c_1(\mathcal{O}_{E_1}(-1)) = c_1(\mathcal{O}_{V_1}(E_1)|_{E_1}) = j_1^*([E_1])$. The coefficients in

$$s(N_{V_2(\mathbb{P}^n)} \mathbb{P}^{\binom{n+2}{2}-1}) = \frac{(1 + h)^{n+1}}{(1 + 2h)^{\binom{n+2}{2}}} = a_0 + a_1 h + \dots + a_n h^n$$

Table 1 The intersection number of $\int_{B_1} h^j \epsilon^{8-j}$ is given in column h^j

	1	h	h^2	h^3
h^j	-996	-146	-16	-1

are computed by equating the coefficients of the powers of h . From the above expression one gets the relation

$$\binom{n+1}{s} = \sum_{i=0}^s 2^{s-i} \binom{n+2}{s-i} a_i$$

from which we attain the recursive formula for the a_i 's stated above. □

Remark 3.5 ($n = 3$) In this case we have:

$$s(N_{v_2(\mathbb{P}^3)} \mathbb{P}^9) = 1 - 16h + 146h^2 - 996h^3.$$

The center B_1 has dimension 8 and $\int_{B_1} h^j \epsilon^{8-j} = 0$ when $j \geq 4$, while the other intersection numbers are summarized in Table 1,

Lemma 3.6 We have $\pi_1^*(P) = P_1$ and $\pi_1^*(L) = L_1 + 2E_1$. The full intersection classes of point and line conditions with respect to B_1 are:

$$B_1 \circ P_1 = 3h, \quad B_1 \circ L_1 = 1 + 12h - 2\epsilon.$$

3.2 Chow ring of B_2

The third center of blow-up is B_2 , which was described in the Subsection 2.2.

Lemma 3.7 B_2 is a $\mathbb{P}^{\binom{n+2}{3}-1}$ -bundle over B_1 and it has dimension $\binom{n+3}{3} - 3$.

(i) Consider $\pi_{2|B_2} : B_2 \rightarrow B_1$ and the inclusion $j_2 : B_2 \hookrightarrow V_2$. Then the intersection ring of B_2 is generated by the pullback of the classes h and ϵ along the projection π_2 and by ϕ the pullback of $[E_2]$ along the inclusion j_2 . Let $\{c_{j,k}\}_{j \in [n], k \in [\dim B_1]}$ be the sequence obtained recursively for $j+k \leq \dim B_1$, following the lexicographic order:

$$\begin{cases} c_{0,0} = 1 \\ c_{j,k} = (-1)^k 2^j \binom{\dim B_1 + 2}{j,k} - \sum_{\substack{a \leq j, b \leq k \\ (a,b) \neq (j,k)}} c_{a,b} \binom{\dim B_2 + 3}{j-a, k-b} 3^{j-a} (-1)^{k-b}, \end{cases}$$

where $\binom{a}{b,c} = \frac{a!}{(a-b-c)!b!c!}$. For $j \in [n], k \in [\dim B_1]$, we get

$$\int_{B_2} h^j \epsilon^k \phi^{\dim B_2 - j - k} = 0$$

for $j+k \in [\dim B_1 + 1, \dim B_2]$ and

$$\int_{B_2} h^j \epsilon^k \phi^{\dim B_2 - j - k} = (-1)^{\dim B_2 - j - k} \sum_{\substack{a+b=\dim B_1 - j - k \\ a \in [n-j], b \in [\dim B_1]}} c_{a,b} (-1)^{\dim B_1 - a - j} a_{n-a-j}$$

Table 2 The intersection number of $\int_{B_2} h^j \epsilon^k \phi^{17-j-k}$ is given in row h^j and column ϵ^k

	1	ϵ	ϵ^2	ϵ^3	ϵ^4	ϵ^5	ϵ^6	ϵ^7	ϵ^8
1	-1370200	-641680	251160	24388	-49400	12900	4460	-4120	996
h	-345280	-3640	31668	-10790	-320	1860	-820	146	0
h^2	-40040	8008	0	-880	440	-120	16	0	0
h^3	-2002	715	-220	55	-10	1	0	0	0

for $j + k \in [0, \dim B_1]$, where the a_i in the sum are the numbers obtained by recursion in Lemma 3.4.

(ii) Moreover $c(N_{B_2} V_2) = (1 + \phi)(1 + \epsilon - \phi)$.

Proof Since $B_2 = \mathbb{P}(N_{B_1} V_1)$, the first point follows from [12, Example 8.3.4]. Point (ii) in Lemma 3.4 implies

$$c(N_{B_1} E_1) = \frac{(1 + 3h - \epsilon)^{\binom{n+3}{3}}}{(1 + 2h - \epsilon)^{\binom{n+2}{2}}}.$$

Hence recalling that $h^{n+1} = 0$ and $\epsilon^{\binom{n+2}{2}} = 0$, we have

$$s(N_{B_1} E_1) = \frac{(1 + 2h - \epsilon)^{\binom{n+2}{2}}}{(1 + 3h - \epsilon)^{\binom{n+3}{3}}} = \sum_{j \in [n], k \in [\binom{n+2}{2}]} c_{j,k} h^j \epsilon^k.$$

Note that this relation allows us to give a recursive formula for the coefficients $c_{j,k}$ (following the lexicographic order on (j, k)). For the second point we have that $B_2 = \tilde{E}_1 \cap E_2$ hence $c(N_{B_2} V_2) = c(N_{E_2} V_2)c(N_{\tilde{E}_1} V_2)$. Note that $N_{E_2} V_2$ is a line bundle on E_2 and $N_{\tilde{E}_1} V_2$ is a line bundle on E_1 . Since $N_{E_2} V_2 = \mathcal{O}_{E_2}(-1)$, we get $c(N_{E_2} V_2) = 1 + \phi$. The isomorphism $N_{\tilde{E}_1} V_2 \simeq \pi_1^*(N_{E_1} V_1) \otimes \mathcal{O}_{E_2}(-E_2)$ implies $c(N_{\tilde{E}_1} V_2) = 1 + \epsilon - \phi$. Hence, one concludes that

$$c(N_{B_2} V_2) = c(N_{E_2} V_2)c(N_{\tilde{E}_1} V_2) = (1 + \phi)(1 + \epsilon - \phi).$$

□

Remark 3.8 ($n = 3$) In the case of cubic surfaces, B_2 has dimension 17 and $\int_{B_2} h^j \epsilon^k \phi^{17-j-k} = 0$ with $j + k > 8$ or $j > 3$, while the remaining integrals are summarized in Table 2.

Lemma 3.9 We have $\pi_2^*(P_1) = P_2$ and $\pi_2^*(L_1) = L_2 + E_2$. The full intersection classes of point and line conditions with respect to B_2 are:

$$B_2 \circ P_2 = 3h, \quad B_2 \circ L_2 = 1 + 12h - 2\epsilon - \phi.$$

Proof As divisor classes we have $\pi_2^*(P_1) = P_2$ and $\pi_2^*(L_1) = L_2 + E_2$, because L_1 is generically smooth along B_1 . In the intersection ring of B_2 we get $j_2^*(P_2) = 3h$ and $j_2^*(L_2) = 12h - 2\epsilon - \phi$. The claim follows by Remark 3.1 observing that in our case the divisor P_2 does not contain B_2 and L_2 is smooth along B_2 . □

3.3 Chow ring of B_3

Recall from Sect. 2.3 that B_3 denotes the fourth center in our sequence of blow-ups, and it is defined as the proper transform in V_3 of S_0 . Moreover, in Lemma 2.17, we described the isomorphism ϕ_3 of S_3 with $\text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$.

Theorem 3.10 *We identify B_3 with $\text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$.*

- (i) *We have two natural projections of $\text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n$ onto \mathbb{P}^n . Let l, m be the pullbacks of the hyperplane classes in \mathbb{P}^n through these projections, and let e denote the exceptional divisor. Consider the sequence $\{d_s\}$ obtained recursively:*

$$\begin{cases} d_0 = 1 \\ d_s = -\sum_{i=0}^{s-1} \binom{n+1}{s-i} d_i. \end{cases}$$

The intersection ring of B_3 is generated by l, m, e subject to $em^s = el^s$ for every s , $l^{n+1} = m^{n+1} = 0$ and

$$\int_{B_3} m^n l^n = 1, \quad \int_{B_3} l^s e^{2n-s} = (-1)^{2n-s-1} d_{n-s}$$

with $s \in [n]$. All the remaining intersection numbers vanish.

- (ii) *($n = 3$) In the case of cubic surfaces, we have*

$$\begin{aligned} c(N_{B_3} V_3) = & 1672560l^3 m^3 - 66343820m^3 e^3 + 36537350m^2 e^4 - 10851224me^5 + 1356403e^6 + \\ & 209440l^3 m^2 + 474320l^2 m^3 + 8045100m^3 e^2 - 5907690m^2 e^3 + 2193180me^4 - \\ & 328977e^5 + 15960l^3 m + 53560l^2 m^2 + 81680lm^3 - 582940m^3 e + 642110m^2 e^2 - \\ & 317840me^3 + 59595e^4 + 560l^3 + 3720l^2 m + 8400lm^2 + 6460m^3 - 42166m^2 e + \\ & 31308me^2 - 7827e^3 + 120l^2 + 536lm + 610m^2 - 1880me + 705e^2 + 16l + 36m - \\ & 39e + 1. \end{aligned}$$

Proof We follow the proof of [1, Theorem III (4)]. Consider the diagram:

$$\begin{array}{ccc} e & \xrightarrow{j} & \text{Bl}_\Delta \mathbb{P}^n \times \mathbb{P}^n \\ \downarrow g & & \downarrow f \\ \Delta & \xrightarrow{i} & \mathbb{P}^n \times \mathbb{P}^n. \end{array}$$

By definition we have $s(\Delta, \mathbb{P}^n \times \mathbb{P}^n) = s(N_\Delta(\mathbb{P}^n \times \mathbb{P}^n))$ and

$$s_i(N_\Delta(\mathbb{P}^n \times \mathbb{P}^n)) = g_*((- \zeta)^{n-1+i})$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}(e)}(-1)) = j^*(e)$. Hence we get

$$s(\Delta, \mathbb{P}^n \times \mathbb{P}^n) = g_* \left(\sum_{i=0}^n (-1)^{n-1+i} (j^*e)^{n-1+i} \right).$$

However, we can also compute $s(\Delta, \mathbb{P}^n \times \mathbb{P}^n)$ as the inverse of the Chern class of $N_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$. This gives us

$$s(\Delta, \mathbb{P}^n \times \mathbb{P}^n) = \frac{1}{(1+k)^{n+1}} = \left(\sum_{i=0}^n (-1)^i k^i \right)^{n+1}.$$

Table 3 The intersection number of $\int_{B_3} l^j e^{6-j}$ is given in column l^j

1	l	l^2	l^3
20	10	4	1

Applying the projection formula and the above observations we get

$$\begin{aligned} \int_{B_3} f^*(l)^i e^{2n-i} &= \int_{B_3} j_* \left(j^*(f^*l^i) j^*(e^{2n-i-1}) \right) = \int_{\Delta} k^i g_* \left(j^*(e)^{2n-i-1} \right) = \\ &= \int_{\Delta} k^i (-1)^{2n-i-1} s(\Delta, \mathbb{P}^n \times \mathbb{P}^n) = (-1)^{2n-i-1} d_{n-i}. \end{aligned}$$

For proving point (ii), we can compute $\phi_3^*(c(TV_3))$ applying [12, Theorem 15.4] multiple times. Notice that to do this, computing $c(TB_1)$ and $c(TB_2)$ is also needed. Hence we observe that

$$c(TB_1) = \frac{(i_1 \circ j_1)^* c(TV_1)}{c(N_{B_1} V_1)}$$

and then we can again use [12, Theorem 15.4] to compute the numerator. Finally, $c(TB_3)$ is computed applying [12, Theorem 15.4] considering $B_3 = \text{Bl}_{\Delta} \mathbb{P}^n \times \mathbb{P}^n$. \square

Remark 3.11 Notice that in principle it is possible to compute $c(N_{B_3} V_3)$ for any n , using the strategy of the proof. However, this can be computationally difficult. For instance, here is the result for $n = 4$:

$$\begin{aligned} c(N_{B_3} V_3) &= 8604607900l^4m^4 + 1511859296400m^4e^4 - 956335227000m^3e^5 + 379626653775m^2e^6 - \\ &86448428700me^7 + 8644842870e^8 + 699244875l^4m^3 + 1520696100l^3m^4 - \\ &107772730500m^4e^3 + 85215404025m^3e^4 - 40592536260m^2e^5 + 10784338950me^6 - \\ &1232495880e^7 + 40828725l^4m^2 + 117863200l^3m^3 + 192910550l^2m^4 + 5484228225m^4e^2 - \\ &5781808210m^3e^3 + 3442721815m^2e^4 - 1097565900me^5 + 146342120e^6 + 1525545l^4m + \\ &6578880l^3m^2 + 14291235l^2m^3 + 15643810lm^4 - 177497950m^4e + 280693735m^3e^2 - \\ &222848500m^2e^3 + 88807250me^4 - 14209160e^5 + 27405l^4 + 235480l^3m + 764065l^2m^2 + \\ &1109920lm^3 + 609280m^4 - 8685470m^3e + 10343355m^2e^2 - 5495900me^3 + 1099180e^4 + \\ &4060l^3 + 26245l^2m + 56940lm^2 + 41475m^3 - 306580m^2e + 244350me^2 - 65160e^3 + \\ &435l^2 + 1880lm + 2045m^2 - 6950me + 2780e^2 + 30l + 65m - 76e + 1. \end{aligned}$$

Remark 3.12 ($n = 3$) In this case the dimension of B_3 is 6 and $\int_{B_3} l^j m^k e^{6-j-k} = 0$ with $j > 3$ or $k > 3$, $\int_{B_3} l^3 m^3 = 1$ while the other integrals are summarised in Table 3, remembering that $\int_{B_3} l^j m^k e^{6-j-k} = \int_{B_3} l^{j+k} e^{6-j-k}$ for $j + k < 6$.

Lemma 3.13 We have $\pi_3^*(P_2) = P_3$ and $\pi_3^*(L_2) = L_3 + E_3$. The full intersection classes of point and line conditions with respect to B_3 are:

$$B_3 \circ P_3 = l + 2m, \quad B_3 \circ L_3 = 1 + 4l + 8m - 6e.$$

Proof In the intersection ring of B_3 we get $j_3^*(P_3) = l + 2m$ and $j_3^*(L_3) = 4(l + 2m) - 2(2e) - e - e = 4l + 8m - 6e$. The assertion is then proved by noticing that P_3 does not contain B_3 and L_3 is smooth along B_3 . \square

Table 4 The intersection number of $\int_{B_4} m^j e^k z^{8-j-k}$ is given in row m^j and column e^k , while $\int_{B_4} m^j l^k z^{8-j-k}$ is given in row l^k and column m^j

	e	e^2	e^3	e^4	e^5	e^6
1	-1820	-580	340	12	-60	20
m	-890	190	54	-42	10	0
m^2	0	68	-24	4	0	0
m^3	51	-9	1	0	0	0
	1	m	m^2	m^3		
1	13720	1610	-600	-175		
l	1610	-230	-35	21		
l^2	-600	-35	46	6		
l^3	-175	21	6	1		

3.4 Chow ring of B_4

The fifth center we blow-up is B_4 , which was described in Subsection 2.4. In particular, recall that $B_4 = \mathbb{P}(\mathcal{E})$ and that we have an isomorphism of $\mathcal{E}|_e$ given in Proposition 2.25.

Theorem 3.14 *We identify B_4 with $\mathbb{P}(\mathcal{E})$.*

- (i) *Let l, m, e be the pullbacks of the generators of the Chow ring of B_3 through the projection $\pi_4|_{B_4} : B_4 \rightarrow B_3$. The Chow ring of B_4 is generated by l, m, e and z where z is the first Chern class of $\mathcal{O}_{B_4}(-1)$. The intesection numbers in the case $n = 3$ are collected in Table 4.*
- (ii) *($n = 3$) In the case of cubic surfaces, we have*

$$\begin{aligned}
 c(N_{B_4} V_4) = & -8540e^6z^2 - 45500l^2m^2z^4 - 109900lm^3z^4 + 280350m^2e^2z^4 - 325500me^3z^4 \\
 & + 106575e^4z^4 + 13440l^2mz^5 + 44800lm^2z^5 + 47320m^3z^5 - 235200m^2e^2z^5 + 174720me^2z^5 \\
 & - 43680e^3z^5 - 1260l^2z^6 - 7560lmz^6 - 11620m^2z^6 + 30240mez^6 - 11340e^2z^6 + 480lz^7 \\
 & + 1440mz^7 - 1440ez^7 - 75z^8 + 12810e^6z + 251300l^2m^3z^2 + 195650me^4z^2 - 108220e^5z^2 \\
 & - 45500l^2m^2z^3 - 109900lm^3z^3 + 280350m^2e^2z^3 - 325500me^3z^3 + 106575e^4z^3 + 630l^2z^5 \\
 & + 3780lmz^5 + 5810m^2z^5 - 15120mez^5 + 5670e^2z^5 - 420lz^6 - 1260mz^6 + 1260ez^6 + 90z^7 \\
 & - 4270e^6 - 201040l^2m^3z - 156520me^4z + 86576e^5z + 81900l^2m^2z^2 + 197820lm^3z^2 \\
 & - 504630m^2e^2z^2 + 585900me^3z^2 - 191835e^4z^2 - 13440l^2mz^3 - 44800lm^2z^3 - 47320m^3z^3 \\
 & + 235200m^2e^2z^3 - 174720me^2z^3 + 43680e^3z^3 + 630l^2z^4 + 3780lmz^4 + 5810m^2z^4 \\
 & - 15120mez^4 + 5670e^2z^4 - 42z^6 + 50260l^2m^3 + 39130me^4 - 21644e^5 - 45500l^2m^2z \\
 & - 109900lm^3z + 280350m^2e^2z - 325500me^3z + 106575e^4z + 13440l^2mz^2 + 44800lm^2z^2 \\
 & + 47320m^3z^2 - 235200m^2e^2z^2 + 174720me^2z^2 - 43680e^3z^2 - 1260l^2z^3 - 7560lmz^3 \\
 & - 11620m^2z^3 + 30240mez^3 - 11340e^2z^3 + 420lz^4 + 1260mz^4 - 1260ez^4 - 42z^5 + 9100l^2m^2 \\
 & + 21980lm^3 - 56070m^2e^2 + 65100me^3 - 21315e^4 - 5760l^2mz - 19200lm^2z - 20280m^3z \\
 & + 100800m^2ez - 74880me^2z + 18720e^3z + 900l^2z^2 + 5400lmz^2 + 8300m^2z^2 - 21600mez^2 \\
 & + 8100e^2z^2 - 480lz^3 - 1440mz^3 + 1440ez^3 + 90z^4 + 960l^2m + 3200lm^2 + 3380m^3 - 16800m^2e \\
 & + 12480me^2 - 3120e^3 - 315l^2z - 1890lmz - 2905m^2z + 7560mez - 2835e^2z + 270lz^2 + 810mz^2 \\
 & - 810ez^2 - 75z^3 + 45l^2 + 270lm + 415m^2 - 1080me + 405e^2 - 80lz - 240mz + 240ez + 35z^2 \\
 & + 10l + 30m - 30e - 9z + 1.
 \end{aligned}$$

Proof To compute the intersection numbers, we follow the usual strategy adopted in the previous proofs, using

$$s(\mathcal{E}) = (\pi_4|_{B_4})_* \left(\sum_{i=0}^{\dim(B_4)} (-1)^i z^i \right).$$

The Segre class $s(\mathcal{E})$ is computed explicitly in the case $n = 3$ thanks to Remark 2.27. In fact, we have the Chern Classes $c(\mathcal{E}|_{B_3 \setminus e})$ and $c(\mathcal{E}|_e)$. The first one is in the Chow ring of $B_3 \setminus e$, which thanks to the excision theorem is described by

$$\frac{\mathbb{Z}[l, m]}{(l^{n+1}, m^{n+1}, [\Delta])}$$

where $[\Delta] = \sum_{i=0}^n l^{n-i} m^i$. Therefore $c(\mathcal{E}|_{B_3 \setminus e}) = \sum_{d=0}^{2n} \sum_{j=0}^d o_{j,d-j} m^j n^{d-j}$.

The second one is an element of the Chow ring of e described by

$$\frac{\mathbb{Z}[k, \zeta]}{(k^{n+1}, \zeta^{2n}, (n+1)k^n + \sum_{i=1}^n \zeta^i k^{n-i} c_{n-i}(T\Delta))}$$

where ζ is just the pullback of the class e through the inclusion of e in B_3 , and $c_i(T\Delta)$ is the i -th Chern class of the tangent space of the diagonal. Therefore $c(\mathcal{E}|_{B_3 \setminus e}) = \sum_{d=0}^{2n} \sum_{j=0}^d u_{j,d-j} k^j \zeta^{d-j}$. In general, knowing $c(\mathcal{E}|_{B_3 \setminus e})$ and $c(\mathcal{E}|_e)$ allows us to reconstruct $c(\mathcal{E})$ up to degree n . In fact, for $d < n$ we have

$$c_d(\mathcal{E}) = \sum_{j=0}^d o_{j,d-j} m^j l^{d-j} + \sum_{j=0}^{d-1} u_{j,d-j} m^j e^{d-j}.$$

For $d = n$, call then $\sum_{j=0}^n o_{j,n-j} = u_{n,0} + (n+1)w$ for some $w \in \mathbb{Z}$ and

$$c_n(\mathcal{E}) = \sum_{j=0}^n o_{j,n-j} m^j n^{n-j} - w[\Delta] + \sum_{j=0}^{n-1} u_{j,n-j} k^j \zeta^{n-j}.$$

When $n = 3$, since $\text{rk}(\mathcal{E}) = 3$, we get the entire total Chern class.

For proving (ii), we can follow the strategy of [1, Theorem III(ii)] and Lemma 2.20. \square

Remark 3.15 We are not able to recover the Chern class $c(\mathcal{E})$ in the case $n > 3$. Indeed, knowing $c(\mathcal{E}|_e)$ and $c(\mathcal{E}|_{B_3 \setminus e})$ we can recover $c(\mathcal{E})$ only up to integer multiples of $m^k[\Delta] = m^k(\sum_{i=1}^n e^i m^{n-i} c_{n-i}(T\Delta))$ for $k \geq 1$.

Lemma 3.16 We have $\pi_4^*(P_3) = P_4$ and $\pi_4^*(L_3) = L_4 + E_4$. The full intersection classes of point and line conditions with respect to B_3 are:

$$B_4 \circ P_4 = l + 2m, \quad B_4 \circ L_4 = 1 + 4l + 8m - 6e - z.$$

4 Characteristic numbers for cubic surfaces, and something more

In this section we gather all information from Sect. 3 in order to compute the characteristic numbers with respect to line conditions for smooth cubic surfaces. Recall that for $n = 3$ the moduli space of cubic surfaces is $V_0 = \mathbb{P}^{19}$ and that in Sect. 2 we constructed a 1-complete space of cubic surfaces denoted V_5 . Moreover, V_{i+1} is the blow-up of V_i with center B_i , and

P_i and L_i are the proper transforms, in V_i , of a point and line condition from V_0 . Thanks to Theorem 1.4 and [1, Theorem II] we obtain the following.

Lemma 4.1 *The number $\mathcal{N}(n_p, n_\ell)$ of smooth cubic surfaces containing n_p given points and tangent to n_ℓ given lines in general position with $n_p + n_\ell = 19$ is*

$$\mathcal{N}(n_p, n_\ell) = 4^{n_\ell} - \sum_{i=0}^4 \int_{B_i} \frac{(B_i \circ P_i)^{n_p} (B_i \circ L_i)^{n_\ell}}{c(N_{B_i} V_i)}$$

Theorem 4.2 *We have*

$$\mathcal{N}(n_p, n_\ell) = \begin{cases} 4^{19-n_p}, & n_p \in \{7, 8, \dots, 19\}, \\ 67107584, & n_p = 6, \\ 268391296, & n_p = 5, \\ 1072926016, & n_p = 4, \\ 4266198896, & n_p = 3, \\ 16615227040, & n_p = 2, \\ 61810371328, & n_p = 1, \\ 213642327616, & n_p = 0. \end{cases}$$

Proof The proof is merely computational.

$$\int_{B_0} \frac{(3h)^{n_p} (2 + 12h)^{n_\ell} (1 + h)^4}{(1 + 3h)^{20}} = \begin{cases} 1769472, & n_p = 3 \\ 54263808, & n_p = 2 \\ 877658112, & n_p = 1 \\ 9948889088, & n_p = 0 \end{cases}$$

$$\int_{B_1} \frac{(3h)^{n_p} (1 + 12h - 2\epsilon)^{n_\ell} (1 + 2h - \epsilon)^{10}}{(1 + \epsilon)(1 + 3h - \epsilon)^{20}} = \begin{cases} 434889, & n_p = 3 \\ 13011156, & n_p = 2 \\ 203305944, & n_p = 1 \\ 2199770536, & n_p = 0 \end{cases}$$

$$\int_{B_2} \frac{(3h)^{n_p} (1 + 12h - 2\epsilon - \phi)^{n_\ell}}{(1 + \phi)(1 + \epsilon - \phi)} = \begin{cases} 17951031, & n_p = 3 \\ 443328300, & n_p = 2 \\ 5677810728, & n_p = 1 \\ 49885157976, & n_p = 0 \end{cases}$$

$$\int_{B_3} \frac{(l + 2m)^{n_p} (1 + 4l + 8m - 6e)^{n_\ell}}{c(N_{B_3} V_3)} = \begin{cases} 160, & n_p = 6 \\ 6240, & n_p = 5 \\ 130224, & n_p = 4 \\ 1426504, & n_p = 3 \\ 8284040, & n_p = 2 \\ 7701512, & n_p = 1 \\ -337368096, & n_p = 0 \end{cases}$$

$$\int_{B_4} \frac{(B_4 \circ P_4)^{n_p} (B_4 \circ L_4)^{n_\ell}}{c(N_{B_4} V_4)} = \begin{cases} 1120, & n_p = 6 \\ 37920, & n_p = 5 \\ 685584, & n_p = 4 \\ 7186504, & n_p = 3 \\ 45754840, & n_p = 2 \\ 142629112, & n_p = 1 \\ -460870176, & n_p = 0 \end{cases}$$

□

Remark 4.3 As explained in Remark 3.15, we cannot have a similar result for $n > 3$. Looking at the computations in [MathRepo](#), it seems that for $n_p \in [n, 2n]$ the last correction term is not affected by the ambiguity explained in Remark 3.15 of the Chern class $c(\mathcal{E})$. We therefore conjecture that, for every n , the numbers given by the code for $n_p \geq n$ are the characteristic numbers.

It is an interesting question whether the characteristic numbers above can be attained by numerical algebraic geometry methods. However, the numbers in Theorem 4.2 increase fast and it could be numerically challenging to compute them. One could try instead to compute the correction term that needs to be subtracted from 4^{19-n_p} by numerical software, e.g. `HomotopyContinuation.jl` [6]. We did not pursue this direction, it would be in any case an interesting problem for experts in numerical algebraic geometry.

4.0 Crumbs of hyperplanes tangency conditions

We stick to the conventions of Sect. 1. In the case of hyperplane tangency conditions for degree d hypersurfaces, the base locus is in general hard to parametrize if compared with the base locus of tangency with respect to lines, where one has obvious maps as in (1). In fact, hyperplane conditions in $\mathbb{P}(\text{Sym}^d(W^*))$ intersect in the locus B_0^H of hypersurfaces with positive-dimensional singular locus

$$B_0^H(d, n) = \{[h] \in \mathbb{P}(\text{Sym}^d(W^*)), \mid \dim \text{Sing}(\mathcal{V}(h)) \geq 1\},$$

as a Bertini-type argument shows. This set has been studied in [24, 27]. Building on [5], it is shown in [17, Lemma 2.4] that an integral cubic hypersurface in \mathbb{P}^n which is not normal, i.e., whose singular locus has dimension $n - 2$, is necessarily singular exactly along a linear subspace of \mathbb{P}^n of dimension $n - 2$. For $n = 3$, this means that the cubic form defining a cubic surface with singular locus of dimension ≥ 1 is either reducible or the singular locus contains a line in \mathbb{P}^3 (or both). This fact is also known classically [22, p. 144]. It is not hard to see that the first set has dimension 12. The dimension of the second set is 13 because it is birational to

$$\{(\ell, f) : (\text{grad} f)|_\ell \equiv 0\} \subseteq \text{Gr}(2, 4) \times \mathbb{P}(\text{Sym}^3(W^*)),$$

a \mathbb{P}^9 -bundle over $\text{Gr}(2, 4)$. We deduce that Slavov’s theorem [24, Theorem 1.1] is also true for cubic surfaces. The work [26] lists the finitely many PGL-orbits for cubic surfaces with positive-dimensional singular locus and studies containments among their closures.

We can derive some characteristic numbers for hyperplane conditions without any complicated construction.

For a hyperplane $H \subseteq \mathbb{P}(W)$, we define L^H to be the hypersurface in $\mathbb{P}(\text{Sym}^d(W^*))$ parametrizing all degree d hypersurfaces tangent to H .

Remark 4.4 The variety L_H of all degree d hypersurfaces in $\mathbb{P}(W)$ tangent to H has degree $n(d-1)^{n-1}$. Indeed, assume $H = \mathcal{V}(x_0)$ and $g \in \mathbb{P}(\text{Sym}^d(W^*))$. Asking for the hypersurface g to be tangent to H corresponds to the vanishing of the resultant of the polynomials $\partial_{x_i} g(0, x_1, \dots, x_n)$, for $i \in \{1, \dots, n\}$. This is the resultant of n homogeneous polynomials of degree $d-1$ in n variables, hence it has degree $n(d-1)^{n-1}$.

Knowing the degree of the variety L^H , it is immediate to compute some of the characteristic numbers for degree d hypersurfaces in $\mathbb{P}(W)$.

Lemma 4.5 *Let $d = 5$, $d \geq 7$ and n arbitrary or $(d, n) = (3, 3)$. If $n_H < n(d-2) + 3$, the number $\mathcal{N}^H(n_p, n_H)$ of smooth degree d hypersurfaces in $\mathbb{P}(W)$ tangent to n_H general hyperplanes and passing through $n_p = \binom{n+d}{d} - 1 - n_H$ general points equals $\mathcal{N}^H(n_p, n_H) = (n(d-1)^{n-1})^{n_H}$.*

Proof If we consider n_H hyperplane conditions with n_H strictly less than the codimension of $B_0^H(d, n)$, the claim follows from Bézout's theorem. The codimension of $B_0^H(d, n)$ is known for $d = 5$ or $d \geq 7$ and arbitrary n by [27, Theorem 1.6] and equals

$$\text{codim} B_0^H(d, n) = n(d-2) + 3.$$

Moreover, this codimension holds true also in the case of cubic surfaces by the above discussion. \square

Acknowledgements The authors wish to thank Mateusz Michałek for presenting the problem to us and for the precious help offered throughout the way. Special thanks go to Paolo Aluffi for his hints and the time he dedicated to answer our long emails about his thesis. We also thank Tim Seynnaeve and Fulvio Gesmundo for valuable discussions. Finally, we are grateful to the organizers of the online workshop REACT, which gave us the opportunity to meet and start this project.

Funding Open Access funding enabled and organized by Projekt DEAL.

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