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Examples of twice differentiable functions with continuous Laplacian and unbounded Hessian

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Abstract
We construct examples of twice differentiable functions in $\mathbb{R}^n$ with continuous Laplacian and unbounded Hessian. The same construction is also applicable to higher order differentiability.

1 Introduction

The standard Schauder theory states that if $\Delta u = f$ in $B_1(0) \subset \mathbb{R}^n$ and f is Hölder continuous ($C^{0,\alpha}$, $0 < \alpha < 1$), then u is $C^{2,\alpha}$. However, it fails when $\alpha = 0$, that is, if Δu is just continuous, then u may not be C^2, as shown by a standard example in $\mathbb{R}^2$ (see [3]):

$$w(x, y) = \begin{cases} (x^2 - y^2)\ln(-\ln(x^2 + y^2)) & 0 < x^2 + y^2 \leq \frac{1}{4}, \\ 0 & (x, y) = (0, 0). \end{cases}$$

This function has continuous Laplacian but is not C^2 because it is not twice differentiable at the origin. (Another such example can be obtained by replacing $x^2 - y^2$ with xy.)

The main goal of this paper is to construct (a family of) functions that are twice differentiable everywhere with continuous Laplacian and unbounded Hessian. These functions have only gained twice differentiablity at the origin over the above example; nevertheless, it appears that some effort is needed to achieve the gain.

Theorem 1.1 *Given any C^2 function $\varphi : (0, \infty) \to \mathbb{R}$ satisfying*

$$\lim_{s\to\infty} \varphi(s) = \infty, \quad \lim_{s\to\infty} \varphi'(s) = 0, \quad \lim_{s\to\infty} \varphi''(s) = 0,$$

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there is a function $u : \mathbb{R}^n \to \mathbb{R}$ *depending on* φ *with compact support, such that it is twice differentiable everywhere in* $\mathbb{R}^n$, *and it has continuous Laplacian and unbounded Hessian. In particular, u is not in* $C^2(\mathbb{R}^n)$.

Obviously there are many choices of such functions φ; for example, $\varphi(s) = s^\alpha$ with $0 < \alpha < 1$, $\varphi(s) = \ln(s)$, or $\varphi(s) = \ln\ln\cdots\ln s$ if $s > c$.

As a consequence, the following is a simple application to the Dirichlet problem.

Corollary 1.2 *There is a continuous function f such that the unique solution of the Dirichlet problem*

$$\begin{cases} \Delta u(x) = f(x) & \text{in } B_1(0), \\ u(x) = 0 & \text{on } \partial B_1(0) \end{cases}$$

is twice differentiable in $\overline{B_1(0)}$ *and has unbounded Hessian.*

For any positive integer k, the Schauder theory also asserts that if Δu is $C^{k,\alpha}$, then u is $C^{k+2,\alpha}$. Once again, it fails when $\alpha = 0$, that is, if Δu is just C^k, then u may not be C^{k+2}. As a result of our construction we have an extension of Theorem 1.1.

Theorem 1.3 *Given any* C^{k+2} *function* $\varphi : (0, \infty) \to \mathbb{R}$ *satisfying*

$$\lim_{s\to\infty} \varphi(s) = \infty, \quad \lim_{s\to\infty} \varphi'(s) = \cdots = \lim_{s\to\infty} \varphi^{(k+2)}(s) = 0,$$

there is a function $u : \mathbb{R}^n \to \mathbb{R}$ *depending on* φ *with compact support, such that u is* $(k+2)$*-times differentiable everywhere in* $\mathbb{R}^n$, Δu *is* C^k, *but* $D^{k+2}u$ *is unbounded. In particular, u is not in* $C^{k+2}(\mathbb{R}^n)$.

Corollary 1.4 *There is a* C^k *function f such that the unique solution of the Dirichlet problem*

$$\begin{cases} \Delta u(x) = f(x) & \text{in } B_1(0), \\ u(x) = 0 & \text{on } \partial B_1(0) \end{cases}$$

is $(k+2)$*-times differentiable in* $\overline{B_1(0)}$, *but* $D^{k+2}u$ *is unbounded.*

We would like to point out a dichotomy: although the Schauder theory fails when $\alpha = 0$ for each k, it is indeed true if $k = \infty$, since $\Delta u \in C^\infty$ does imply $u \in C^\infty$ by the elliptic theory.

According to Theorem 1.1, it would be rather natural to ask if unbounded Hessian is the only reason that hinders u from being C^2. Thus we propose the following problem.

Problem: If a function u is twice differentiable everywhere, Δu is continuous, and the Hessian of u is locally bounded, then is u always C^2?

We remark that the method of construction for Theorem 1.1 will not yield examples of twice differentiable function with continuous Laplacian and bounded Hessian without being C^2. On the other hand, if the function is not required to be twice differentiable everywhere, then there are simple examples of functions with continuous Laplacian and bounded Hessian, such as the following function ([4]) that is not twice differentiable at the origin.

$$\phi(x,y) = \begin{cases} (x^2-y^2)\sin(\ln(-\ln(x^2+y^2))) & 0 < x^2+y^2 \le \frac{1}{4}, \\ 0 & (x,y)=(0,0). \end{cases}$$

We also observe that if Δu is continuous, then the modulus of continuity of Du is of $o(L\log L)$. (If Δu is just bounded, then the modulus of continuity of Du is only of $O(L\log L)$. [5]) Precisely, the following is true.

Proposition 1.5 *Let u be a C^1 solution of $\Delta u = f$, where f is a continuous function on $B_1(0)$ in $\mathbb{R}^n$. Then for any $x, y \in B_{\frac{1}{2}}(0)$,*

$$\left|Du(x) - Du(y)\right| \le Cd\left(\sup_{B_1}|u| + \sup_{B_1}|f| + \int_d^1 \frac{\omega(r)}{r}dr\right),$$

where $d = |x-y|$, $\omega(r) = \sup_{|x-y|<r} |f(x) - f(y)|$, and C is a constant depending only on n.

Here we notice that

$$\lim_{d\to 0} \frac{d\int_d^1 \frac{\omega(r)}{r}dr}{d\ln d} = 0,$$

which can be easily proved by considering two cases: $\lim_{d\to 0}\int_d^1 \frac{\omega(r)}{r}dr < \infty$ or $\lim_{d\to 0}\int_d^1 \frac{\omega(r)}{r}dr = \infty$. It is this $o(L\log L)$ observation that motivated us to Theorems 1.1 and 1.3.

Theorems 1.1 and 1.3 will be proved in Sects. 3 and 4, respectively, after a thorough study of a building block function in Sect. 2. One of the ideas in the construction has its origin in [2] (and also [1]), where the inhomogeneous Cauchy-Riemann equation in the complex plane was considered. Since the proof of Proposition 1.5 is almost identical to that for Corollary 1 in [5], we include a detailed proof in the Appendix for the convenience of the reader.

2 A building block function

In this section we will look at a function that will become a building block and provide some crucial insights for the construction of examples for Theorem 1.1.

Recall that φ is a function satisfying

$$\lim_{s\to\infty}\varphi(s) = \infty, \quad \lim_{s\to\infty}\varphi'(s) = 0, \quad \lim_{s\to\infty}\varphi''(s) = 0. \tag{1}$$

Thus for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\lim_{|x|\to 0}\varphi(-\ln|x|^2) = \infty$. Nevertheless, the product of $\varphi(-\ln|x|^2)$ with positive powers of $|x|$ is well controlled, as shown by the following two simple lemmas that will be repeatedly used later in the construction.

Lemma 2.1 *For any* $\beta > 0$ *and* φ *satisfying* (1),

$$\lim_{|x|\to 0} |x|^\beta \varphi\big(-\ln |x|^2\big) = 0.$$

Proof Letting $s = \frac{1}{|x|}$, $\lim_{|x|\to 0} |x|^\beta \varphi\big(-\ln |x|^2\big) = \lim_{s\to\infty} \frac{\varphi(\ln s^2)}{s^\beta}$ is of $\frac{\infty}{\infty}$ type. By the L'Hopital's Rule,

$$\lim_{s\to\infty} \frac{\varphi(\ln s^2)}{s^\beta} = \lim_{s\to\infty} \frac{2\varphi'(\ln s^2)}{\beta s^\beta} = 0,$$

as long as $\beta > 0$. □

Lemma 2.2 *For any* $0 < \beta \le 1$ *and* φ *satisfying* (1), *there is a constant* C_φ *depending only on* φ, *such that*

$$\sup_{\substack{|x| \le \frac{2}{3} \\ 0<\beta\le 1}} \beta |x|^\beta \Big|\varphi\big(-\ln |x|^2\big)\Big| \le C_\varphi.$$

Proof By Lemma 2.1, we know

$$\lim_{|x|\to 0} \beta |x|^\beta \varphi\big(-\ln |x|^2\big) = 0.$$

When $|x| = \frac{2}{3}$,

$$\beta |x|^\beta \Big|\varphi\big(-\ln |x|^2\big)\Big| = \beta\Big(\frac{2}{3}\Big)^\beta \Big|\varphi\Big(\ln\Big(\frac{9}{4}\Big)\Big)\Big| \le \Big|\varphi\Big(\ln\Big(\frac{9}{4}\Big)\Big)\Big|,$$

which is a constant independent of t.

It remains to show that the local maximum of $\beta |x|^\beta \Big|\varphi\big(-\ln |x|^2\big)\Big|$ is also bounded by a constant depending only on φ. Denote $s = \frac{1}{|x|}$, then $s > 1$, and

$$\beta |x|^\beta \varphi\big(-\ln |x|^2\big)$$

is equivalent to

$$\lambda(s) = \beta \frac{\varphi(\ln s^2)}{s^\beta}.$$

We will find the local extremum of $\lambda(s)$. Because

$$\lambda'(s) = \beta \frac{2\varphi'(\ln s^2) - \beta\varphi(\ln s^2)}{s^{\beta+1}},$$

a critical point s_0 must satisfy

$$2\varphi'(\ln s_0^2) = \beta\varphi(\ln s_0^2).$$

At this point the local extremum of λ is

$$\lambda(s_0) = \beta \frac{\varphi(\ln s_0^2)}{s_0^\beta} = \beta \frac{\frac{2}{\beta}\varphi'(\ln s_0^2)}{s_0^\beta} = \frac{2\varphi'(\ln s_0^2)}{s_0^\beta},$$

and so

$$|\lambda(s_0)| = \left| \frac{2\varphi'(\ln s_0^2)}{s_0^\beta} \right| < 2\left|\varphi'(\ln s_0^2)\right|.$$

From (1) we know that $|\varphi'(\ln s_0^2)|$ is bounded by a constant depending only on φ. Thus the local maximum of $|\lambda(s)| = \beta|x|^\beta\left|\varphi\left(-\ln|x|^2\right)\right|$ is also bounded by a constant depending only on φ. This completes the proof. □

It is worth noting that Lemma 2.2 would not be true without the coefficient of β in the function. For example, let $\{x_k\}$ be a sequence of points in $\mathbb{R}^n$ with $|x_k| = e^{-2^k}$, and let $\beta_k = \frac{1}{2^k}$, then $|x_k| \to 0$ and $\beta_k \to 0$ as $k \to \infty$, but

$$|x_k|^{\beta_k}\varphi\left(-\ln|x_k|^2\right) = e^{-1}\varphi\left(2^{k+1}\right) \to \infty.$$

This difference will be crucial to our construction of functions with continuous Laplacian and unbounded Hessian.

For any $|x| \le \frac{1}{2}$, define a function $v(x)$ by

$$v(x) = \begin{cases} x_1x_2\varphi\left(-\ln|x|^2\right) & 0 < |x| \le \frac{1}{2}, \\ 0 & x = 0. \end{cases} \tag{2}$$

This function will be a building block for our construction, it generalizes the function w at the beginning of Sect. 1 from $\mathbb{R}^2$ to $\mathbb{R}^n$. It satisfies almost all the conditions in Theorem 1.1, except one that it is not twice differentiable at the origin.

Lemma 2.3 *The function v defined by* (2) *has continuous Laplacian and unbounded Hessian, but it is not twice differentiable at* 0.

Proof By definition, $v(x)$ is C^2 for all $x \ne 0$, and its derivatives are the following. (In the case $n \ge 3$, we use i and j to denote indices that are greater than or equal to 3.)

$$
\begin{aligned}
\frac{\partial v}{\partial x_1}(x) &= x_2\varphi\big(-\ln|x|^2\big) - \frac{2x_1^2x_2}{|x|^2}\varphi'\big(-\ln|x|^2\big),\\
\frac{\partial v}{\partial x_2}(x) &= x_1\varphi\big(-\ln|x|^2\big) - \frac{2x_1x_2^2}{|x|^2}\varphi'\big(-\ln|x|^2\big),\\
\frac{\partial v}{\partial x_i}(x) &= -\frac{2x_1x_2x_i}{|x|^2}\varphi'\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_1^2}(x) &= -\frac{6x_1x_2}{|x|^2}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1^3x_2}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1^3x_2}{|x|^4}\varphi''\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_2^2}(x) &= -\frac{6x_1x_2}{|x|^2}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2^3}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2^3}{|x|^4}\varphi''\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_i^2}(x) &= -\frac{2x_1x_2}{|x|^2}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2x_i^2}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2x_i^2}{|x|^4}\varphi''\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_1\partial x_2}(x) &= \varphi\big(-\ln|x|^2\big) - \frac{2(x_1^2+x_2^2)}{|x|^2}\varphi'\big(-\ln|x|^2\big)\\
&\quad + \frac{4x_1^2x_2^2}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1^2x_2^2}{|x|^4}\varphi''\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_1\partial x_i}(x) &= \frac{-2x_2x_i}{|x|^2}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1^2x_2x_i}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1^2x_2x_i}{|x|^4}\varphi''\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_2\partial x_i}(x) &= \frac{-2x_1x_i}{|x|^2}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2^2x_i}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2^2x_i}{|x|^4}\varphi''\big(-\ln|x|^2\big),\\
\frac{\partial^2 v}{\partial x_j\partial x_i}(x) &= -\frac{2x_1x_2}{|x|^2}\delta_{ij}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2x_ix_j}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1x_2x_ix_j}{|x|^4}\varphi''\big(-\ln|x|^2\big).
\end{aligned}
$$

We observe that each term in these derivatives is of the form $p(x)\varphi\big(-\ln|x|^2\big)$, $p(x)\varphi'\big(-\ln|x|^2\big)$, or $p(x)\varphi''\big(-\ln|x|^2\big)$, where $p(x)$ is homogeneous in x. For first derivatives, the degree of homogeneity is 1, and for second derivatives, the degree of homogeneity is 0. Because of this, by Lemma 2.1 and the choice of φ,

$$
\begin{aligned}
&\lim_{|x|\to 0}\frac{\partial v}{\partial x_1}(x) = \lim_{|x|\to 0}\frac{\partial v}{\partial x_2}(x) = \lim_{|x|\to 0}\frac{\partial v}{\partial x_i}(x) = 0,\\
&\lim_{|x|\to 0}\frac{\partial^2 v}{\partial x_1^2}(x) = \lim_{|x|\to 0}\frac{\partial^2 v}{\partial x_2^2}(x) = \lim_{|x|\to 0}\frac{\partial^2 v}{\partial x_i^2}(x)\\
&\qquad = \lim_{|x|\to 0}\frac{\partial^2 v}{\partial x_1 x_i}(x) = \lim_{|x|\to 0}\frac{\partial^2 v}{\partial x_2 x_i}(x) = \lim_{|x|\to 0}\frac{\partial^2 v}{\partial x_i x_j}(x) = 0.
\end{aligned}
$$

Thus all the first and second derivatives of v approach 0 as $|x|\to 0$, except for

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x_1\partial x_2}(x) &= \varphi\big(-\ln|x|^2\big) - \frac{2(x_1^2+x_2^2)}{|x|^2}\varphi'\big(-\ln|x|^2\big)\\
&\quad + \frac{4x_1^2x_2^2}{|x|^4}\varphi'\big(-\ln|x|^2\big) + \frac{4x_1^2x_2^2}{|x|^4}\varphi''\big(-\ln|x|^2\big).
\end{aligned}
$$

As $|x| \to 0$, its first term goes to ∞ and all the other terms go to 0, thus $\frac{\partial^2 v}{\partial x_1 \partial x_2}(x)$ is unbounded near the origin, which causes the Hessian of v to be unbounded.

On the other hand, because each diagonal entry of the Hessian has a removable discontinuity at the origin,

$$\begin{aligned}\Delta v(x) &= \frac{\partial^2 v}{\partial x_1^2}(x) + \frac{\partial^2 v}{\partial x_2^2}(x) + \sum_{i=3}^{n} \frac{\partial^2 v}{\partial x_i^2}(x) \\ &= -\frac{(2n+8)x_1x_2}{|x|^2}\varphi'\left(-\ln|x|^2\right) + \frac{4x_1x_2}{|x|^2}\Big(\varphi'\left(-\ln|x|^2\right) + \varphi''\left(-\ln|x|^2\right)\Big) \\ &\to 0 \quad \text{as } |x| \to 0.\end{aligned}$$

Lastly, we check the differentiablity of v at 0. It is differentiable because by Lemma 2.1,

$$\frac{|v(x) - v(0)|}{|x|} = \frac{\left|x_1x_2\varphi\left(-\ln|x|^2\right)\right|}{|x|} \le |x|\left|\varphi\left(-\ln|x|^2\right)\right| \to 0 \quad \text{as } |x| \to 0.$$

Therefore, all first derivatives of v equal 0 at the origin, and v is C^1 throughout $\mathbb{R}^n$. Computing the partial derivative $\frac{\partial^2 v}{\partial x_1^2}(0)$ by definition, we have

$$\frac{\partial^2 v}{\partial x_1^2}(0) = \lim_{h\to 0} \frac{\frac{\partial v}{\partial x_1}((h,0,\dots,0)) - \frac{\partial v}{\partial x_1}((0,\dots,0))}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.$$

Similarly, we also have

$$\frac{\partial^2 v}{\partial x_2^2}(0) = \cdots = \frac{\partial^2 v}{\partial x_n^2}(0) = 0.$$

Therefore, $\Delta v(0) = 0$. Consequently, Δv is continuous at 0. However, v is not twice differentiable at 0. To see that, we check the differentiability of $\frac{\partial v}{\partial x_1}$ at 0:

$$\frac{\left|\frac{\partial v}{\partial x_1}(x) - \frac{\partial v}{\partial x_1}(0)\right|}{|x|} = \frac{\left|x_2\varphi\left(-\ln|x|^2\right) - \frac{2x_1^2x_2}{|x|^2}\varphi'\left(-\ln|x|^2\right)\right|}{|x|}.$$

Let $x_1 = x_3 = \cdots = x_n = 0$, then $|x_2| = |x|$ and

$$\frac{\left|\frac{\partial v}{\partial x_1}(x) - \frac{\partial v}{\partial x_1}(0)\right|}{|x|} = \frac{\left|x_2\varphi\left(-\ln|x_2|^2\right)\right|}{|x_2|} = \left|\varphi\left(-\ln|x_2|^2\right)\right| \to \infty \quad \text{as } x_2 \to 0,$$

hence $\frac{\partial v}{\partial x_1}$ is not differentiable at 0. Similarly, $\frac{\partial v}{\partial x_2}$ is not differentiable at 0 either.

Interestingly,

$$\frac{\left|\frac{\partial v}{\partial x_i}(x) - \frac{\partial v}{\partial x_i}(0)\right|}{|x|} = \frac{\left|-\frac{2x_1x_2x_i}{|x|^2}\varphi'\left(-\ln|x|^2\right)\right|}{|x|} \le 2\left|\varphi'\left(-\ln|x|^2\right)\right| \to 0 \quad \text{as } x \to 0,$$

thus $\frac{\partial v}{\partial x_i}$ for all $i \ge 3$ are differentiable at 0.

Therefore, v fails to be twice differentiable at 0 because $\frac{\partial v}{\partial x_1}$ and $\frac{\partial v}{\partial x_2}$ are not differentiable at 0. This completes the proof of Lemma 2.3. □

3 Construction for Theorem 1.1

In this section, we will first “smooth out" v into a function that is C^2 at the origin, then we will combine a sequence of such functions through scaling and translation to create a desired function that is twice differentiable everywhere with continuous Laplacian and unbounded Hessian, thus proving Theorem 1.1.

Definition 3.1 Let $\eta : [0,\infty) \to [0,1]$ be a fixed, non-increasing C^∞ function such that

$$\eta(s) \equiv 1 \text{ for } 0 \le s \le \frac{1}{2} \quad \text{and} \quad \eta(s) \equiv 0 \text{ for } s \ge \frac{2}{3}. \tag{3}$$

For any $0 < t \le \frac{1}{2}$, define a function $u_t : \mathbb{R}^n \to \mathbb{R}$ $(n \ge 2)$ by

$$u_t(x) = \begin{cases} 0 & x = 0, \\ \eta(|x|)x_1x_2|x|^{2t}\varphi(-\ln|x|^2) & 0 < |x| < 1, \\ 0 & |x| \ge 1. \end{cases} \tag{4}$$

It follows immediately from Lemma 2.1 and (3) that u_t is continuous everywhere. Actually, it can be shown that $u_t \in C^2(\mathbb{R}^n)$, but we will not verify it here because it is not to be used in our construction.

What will be essential to our construction is the fact that all the first derivatives of u_t and second derivatives of the form $\frac{\partial^2 u_t}{\partial x_j^2}$ are uniformly bounded by constants independent of t. However, that is not the case for $\frac{\partial^2 u_t}{\partial x_1 \partial x_2}$, as will be shown later in this section.

Lemma 3.2 *There is a constant $C_{\eta,\varphi}$ depending only on η and φ, such that*

$$\sup_{x\in\mathbb{R}^n} \left|u_t(x)\right| \le C_{\eta,\varphi}. \tag{5}$$

$$\sup_{x\in\mathbb{R}^n} \left|\frac{\partial u_t}{\partial x_j}(x)\right| \le C_{\eta,\varphi} \quad \text{for} \quad j = 1,\dots,n. \tag{6}$$

$$\sup_{x\in\mathbb{R}^n} \left|\frac{\partial^2 u_t}{\partial x_j^2}(x)\right| \le C_{\eta,\varphi} \quad \text{for} \quad j = 1,\dots,n. \tag{7}$$

Proof By the definition of u_t,

$$\sup_{x\in\mathbb{R}^n} \left|u_t(x)\right| = \sup_{|x|\le\frac{2}{3}} \left|u_t(x)\right|, \quad \sup_{x\in\mathbb{R}^n} \left|\frac{\partial u_t}{\partial x_j}(x)\right| = \sup_{|x|\le\frac{2}{3}} \left|\frac{\partial u_t}{\partial x_j}(x)\right|,$$

and

$$\sup_{x\in\mathbb{R}^n}\left|\frac{\partial^2 u_t}{\partial x_j^2}(x)\right| = \sup_{|x|\le\frac{2}{3}}\left|\frac{\partial^2 u_t}{\partial x_j^2}(x)\right|,$$

so we assume $|x| \le \frac{2}{3}$. Furthermore, since $0 < t \le \frac{1}{2}$, when $\frac{1}{2} \le |x| \le \frac{2}{3}$, we know $|u_t(x)|$, $\left|\frac{\partial u_t}{\partial x_j}(x)\right|$, and $\left|\frac{\partial^2 u_t}{\partial x_j^2}(x)\right|$ are all bounded by a constant depending on η and φ and independent of t. Therefore it remains to show that when $|x| < \frac{1}{2}$, $|u_t(x)|$, $\left|\frac{\partial u_t}{\partial x_j}(x)\right|$, and $\left|\frac{\partial^2 u_t}{\partial x_j^2}(x)\right|$ are all bounded by a constant independent of t.

First, when $|x| < \frac{1}{2}$, we have $|u_t(x)| \le |x|^2\varphi(-\ln|x|^2)$. Then since $\lim_{|x|\to 0}|x|^2\varphi\big(-\ln|x|^2\big) = 0$ by Lemma 2.1, we know $\sup_{|x|<\frac{1}{2}}|u_t(x)|$ is bounded by a constant independent of t. Thus (5) is true.

Since $\eta(|x|) \equiv 1$ for $|x| \le \frac{1}{2}$, the derivatives of u_t when $0 < |x| < \frac{1}{2}$ are the following.

$$\begin{aligned}\frac{\partial u_t}{\partial x_1}(x) =& x_2|x|^{2t}\varphi(-\ln|x|^2) + 2tx_1^2x_2|x|^{2t-2}\varphi(-\ln|x|^2)\\ &- 2x_1^2x_2|x|^{2t-2}\varphi'(-\ln|x|^2).\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial u_t}{\partial x_2}(x) =& x_1|x|^{2t}\varphi(-\ln|x|^2) + 2tx_1x_2^2|x|^{2t-2}\varphi(-\ln|x|^2)\\ &- 2x_1x_2^2|x|^{2t-2}\varphi'(-\ln|x|^2).\end{aligned} \tag{9}$$

In the case $n \ge 3$, for any $i \ge 3$,

$$\frac{\partial u_t}{\partial x_i}(x) = 2tx_1x_2x_i|x|^{2t-2}\varphi(-\ln|x|^2) - 2x_1x_2x_i|x|^{2t-2}\varphi'(-\ln|x|^2). \tag{10}$$

The first and second terms in (8) are bounded by

$$|x|^{2t+1}\left|\varphi(-\ln|x|^2)\right|.$$

Because $|x| < \frac{1}{2}$, we have $|x|^{2t+1} \le |x|$, so

$$|x|^{2t+1}\left|\varphi(-\ln|x|^2)\right| \le |x|\left|\varphi(-\ln|x|^2)\right|.$$

By Lemma 2.1, $|x|\varphi(-\ln|x|^2)$ has a removable discontinuity at 0, therefore on the set $|x| < \frac{1}{2}$ it is bounded by a constant depending only on φ. Thus the first and second terms in (8) are bounded by a constant depending only on φ. The last term in (8),

$$2x_1^2x_2|x|^{2t-2}\varphi'(-\ln|x|^2),$$

is bounded by

$$2|x|^{2t+1}\left|\varphi'(-\ln|x|^2)\right|.$$

It is further bounded by

$$2|x|\left|\varphi'(-\ln|x|^2)\right|$$

since $|x| < \frac{1}{2}$. Because of (1), we know $|x|\varphi'(-\ln|x|^2)$ has a removable discontinuity at 0, therefore on the set $|x| < \frac{1}{2}$ it is bounded by a constant depending only on φ. Thus the last term in (8) is also bounded by a constant depending only on φ. Therefore, $\frac{\partial u_t}{\partial x_1}$ is bounded by a constant depending on φ only. In the same way, we can prove that $\frac{\partial u_t}{\partial x_2}$ and $\frac{\partial u_t}{\partial x_i}(i \geq 3)$ are also bounded by a constant depending on φ only. This proves (6).

Lastly, we prove (7). When $0 < |x| < \frac{1}{2}$,

$$\begin{aligned}\frac{\partial^2 u_t}{\partial x_1^2}(x) = {} & 6tx_1x_2|x|^{2t-2}\varphi(-\ln|x|^2) + 2t(2t-2)x_1^3x_2|x|^{2t-4}\varphi(-\ln|x|^2)\\ & - 2x_1x_2|x|^{2t-2}\varphi'(-\ln|x|^2) - 4tx_1^3x_2|x|^{2t-4}\varphi'(-\ln|x|^2)\\ & - 6x_1x_2|x|^{2t-2}\varphi'(-\ln|x|^2) - 2(2t-2)x_1^3x_2|x|^{2t-4}\varphi'(-\ln|x|^2)\\ & + 4x_1^3x_2|x|^{2t-4}\varphi''(-\ln|x|^2).\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial^2 u_t}{\partial x_2^2}(x) = {} & 6tx_1x_2|x|^{2t-2}\varphi(-\ln|x|^2) + 2t(2t-2)x_1x_2^3|x|^{2t-4}\varphi(-\ln|x|^2)\\ & - 2x_1x_2|x|^{2t-2}\varphi'(-\ln|x|^2) - 4tx_1x_2^3|x|^{2t-4}\varphi'(-\ln|x|^2)\\ & - 6x_1x_2|x|^{2t-2}\varphi'(-\ln|x|^2) - 2(2t-2)x_1x_2^3|x|^{2t-4}\varphi'(-\ln|x|^2)\\ & + 4x_1x_2^3|x|^{2t-4}\varphi''(-\ln|x|^2).\end{aligned} \tag{12}$$

In the case $n \geq 3$, for any $i \geq 3$,

$$\begin{aligned}\frac{\partial^2 u_t}{\partial x_i^2}(x) = {} & 2tx_1x_2|x|^{2t-2}\varphi(-\ln|x|^2) + 2t(2t-2)x_1x_2x_i^2|x|^{2t-4}\varphi(-\ln|x|^2)\\ & + (4-8t)x_1x_2x_i^2|x|^{2t-4}\varphi'(-\ln|x|^2) - 2x_1x_2x_i^2|x|^{2t-3}\varphi'(-\ln|x|^2)\\ & - 2x_1x_2|x|^{2t-2}\varphi'(-\ln|x|^2) + 4x_1x_2x_i^2|x|^{2t-4}\varphi''(-\ln|x|^2).\end{aligned} \tag{13}$$

To prove (7) we need to estimate each term of (11), (12), and (13).

We start with (11). Note that since $|x| < \frac{1}{2}$, the 3rd through 6th terms are bounded by

$$C|x|^{2t}\left|\varphi'(-\ln|x|^2)\right|,$$

which is further bounded by

$$C\left|\varphi'(-\ln|x|^2)\right|.$$

By (1), $\varphi'(-\ln|x|^2)$ has a removable discontinuity at 0, therefore on the set $|x| < \frac{1}{2}$ it is bounded by a constant depending only on φ. Similarly the 7th term is also bounded by a constant depending only on φ.

The first term,

$$6tx_1x_2|x|^{2t-2}\varphi(-\ln|x|^2),$$

and the second term,

$$2t(2t-2)x_1^3x_2|x|^{2t-4}\varphi(-\ln|x|^2),$$

are bounded by

$$Ct|x|^{2t}\left|\varphi(-\ln|x|^2)\right|.$$

By Lemma 2.2,

$$2t|x|^{2t}\left|\varphi(-\ln|x|^2)\right| \le C_\varphi,$$

where C_φ depends only on φ. Hence the first and second terms are bounded by a constant depending on φ. Therefore, we have proved that all the terms in (11) are uniformly bounded by a constant independent of t.

All the terms in (12) and (13) can be estimated in the same way, so this completes the proof of (7). □

Now we are ready to construct the main function, u, by "piecing together" a sequence of functions u_{t_k} as follows.

Choose two decreasing sequences of numbers $R_k \to 0$ and $r_k \to 0$, such that

$$R_k > r_k,$$

and for geometric reasons that will be explained later we also require

$$R_k - r_k > R_{k+1} + r_{k+1}; \tag{14}$$

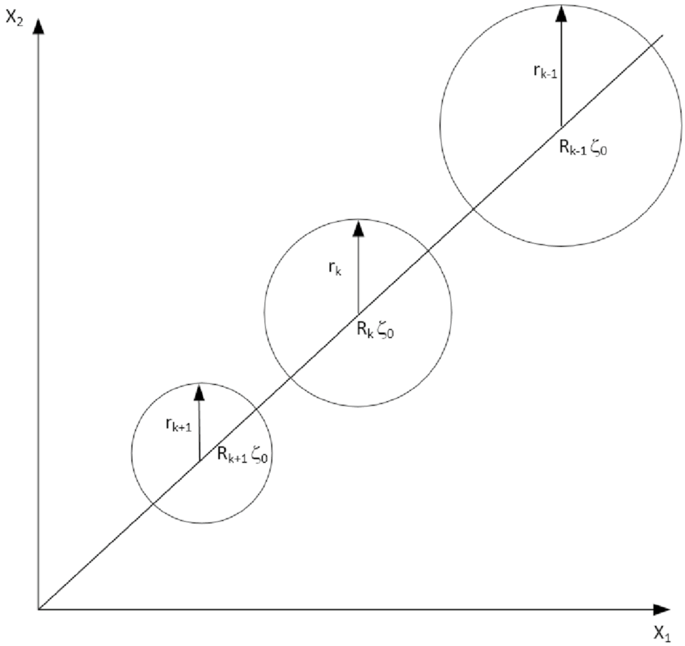
for example, we may choose $R_k = 10^{-k}$ and $r_k = 10^{-(k+1)}$.

We use ζ_0 to denote the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^n$ and choose a sequence $\{t_k\}$ such that $0 < t_k < \frac{1}{4}$ and $\lim_{k\to\infty} t_k = 0$. Define the function $u(x)$ by

$$u(x) = \sum_{k=1}^{\infty} \epsilon_k r_k^2 u_{t_k}\left(\frac{x - R_k\zeta_0}{r_k}\right), \tag{15}$$

where the only conditions on ϵ_k for now are $\epsilon_k > 0$ and $\lim_{k\to\infty} \epsilon_k = 0$, we do not need to assign specific values to ϵ_k until near the end of this section.

Condition (14) ensures that the balls centered at the points $R_k\zeta_0$ with radii r_k are mutually disjoint.



For each $k \in \mathbb{N}$, let B_k be the ball centered at the point $R_k\zeta_0$ with radius $\frac{2}{3}r_k$, then these B_k are also mutually disjoint. By (3) and (4), the support of each function $u_{t_k}\left(\frac{x-R_k\zeta_0}{r_k}\right)$ is the ball $\{x \in \mathbb{R}^n : |x - R_k\zeta_0| \le \frac{2}{3}r_k\}$, which is B_k. Therefore, although the definition of $u(x)$ appears to be an infinite sum, it actually is only a single term. For any given $x \in \mathbb{R}^n$, if x is not in any of the B_k, then

$$u(x) = 0,$$

otherwise

$$u(x) = \epsilon_k r_k^2 u_{t_k}\left(\frac{x - R_k\zeta_0}{r_k}\right) \quad \text{for some } k.$$

As $k \to \infty$, the radius of B_k goes down to 0 and its center moves toward the origin, but none of the balls B_k contains the origin. In fact, for any $j = 1, \dots, n$, the x_j-th coordinate hyperplane does not intersect any of the ball B_k. To see this, let

$$(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)$$

be an arbitrary point on the x_j-th coordinate hyperplane. The distance from this point to the center of the ball, $R_k\zeta_0 = \left(\frac{R_k}{\sqrt{2}}, \dots, \frac{R_k}{\sqrt{2}}\right)$, is

$$\sqrt{\left(\frac{R_k}{\sqrt{2}} - x_1\right)^2 + \cdots + \left(\frac{R_k}{\sqrt{2}} - x_{j-1}\right)^2 + \left(\frac{R_k}{\sqrt{2}}\right)^2 + \left(\frac{R_k}{\sqrt{2}} - x_{j+1}\right)^2 + \cdots + \left(\frac{R_k}{\sqrt{2}} - x_n\right)^2}$$
$$\geq \frac{R_k}{\sqrt{2}} > \frac{r_k}{\sqrt{2}} > \frac{2}{3} r_k,$$

since $\frac{1}{\sqrt{2}} \approx 0.71$ and $\frac{2}{3} \approx 0.67$. Thus $u = 0$ on all of the n coordinate hyperplanes, and consequently $u(0) = 0$. By (5) u_{t_k} is uniformly bounded by a constant independent of t_k, hence by definition $\lim_{|x| \to 0} u(x) = 0$. Therefore u is continuous at the origin, and thus continuous everywhere in $\mathbb{R}^n$.

Lemma 3.3 *The function $u(x)$ as defined in (15) is twice differentiable everywhere in $\mathbb{R}^n$, and all its first and second order partial derivatives at the origin are equal to 0.*

Proof By definition $u(x)$ is C^2 for all $x \neq 0$, so we only need to show it is twice differentiable at the origin. Because $u = 0$ on all the coordinate hyperplanes, for any $i, j = 1, \dots, n$,

$$\frac{\partial u}{\partial x_j}(0) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i \partial x_j}(0) = 0.$$

Thus

$$\lim_{|x| \to 0} \frac{u(x) - u(0) - \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0) x_i - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(0) x_i x_j}{|x|^2} = \lim_{|x| \to 0} \frac{u(x)}{|x|^2}$$

Recall that the balls B_k are mutually disjoint, so for any given $x \in \mathbb{R}^n$, either

$$u(x) = 0,$$

or

$$u(x) = \epsilon_k r_k^2 u_{t_k}\left(\frac{x - R_k \zeta_0}{r_k}\right) \quad \text{for some } k.$$

By (5) we have $|u_{t_k}| \leq C_{\eta,\varphi}$ which only depends on η and φ. Thus

$$\frac{|u(x)|}{|x|^2} \leq \frac{\epsilon_k r_k^2 C_{\eta,\varphi}}{\left(R_k - \frac{2}{3} r_k\right)^2} = \frac{\epsilon_k C_{\eta,\varphi}}{\left(\frac{R_k}{r_k} - \frac{2}{3}\right)^2} < \frac{\epsilon_k C_{\eta,\varphi}}{\left(1 - \frac{2}{3}\right)^2} \to 0 \quad \text{as } k \to \infty.$$

Hence we know that

$$\lim_{|x| \to 0} \frac{u(x) - u(0) - \sum_{i=1}^n \frac{\partial u}{\partial x_i}(0) x_i - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(0) x_i x_j}{|x|^2} = 0,$$

which means u is twice differentiable at the origin. This completes the proof. □

Lastly, we will show that u has continuous Laplacian but unbounded Hessian.

Lemma 3.4 *The function $u(x)$ as defined in (15) has continuous Laplacian everywhere in $\mathbb{R}^n$, and the partial derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ is unbounded near the origin.*

Proof Because $u = 0$ on all the coordinate hyperplanes,

$$\frac{\partial^2 u}{\partial x_j^2}(0) = 0.$$

For any given $x \in \mathbb{R}^n$, either

$$\frac{\partial^2 u}{\partial x_j^2}(x) = 0,$$

or

$$\frac{\partial^2 u}{\partial x_j^2}(x) = \epsilon_k \frac{\partial^2 u_{t_k}}{\partial x_j^2}\left(\frac{x - R_k\zeta_0}{r_k}\right) \quad \text{for some } k.$$

By (7) in Lemma 3.2,

$$\epsilon_k \left| \frac{\partial^2 u_{t_k}}{\partial x_j^2}\left(\frac{x - R_k\zeta_0}{r_k}\right) \right| \le \epsilon_k C_{\eta,\varphi} \to 0 \quad \text{as } k \to \infty.$$

Thus $\lim_{|x|\to 0} \frac{\partial^2 u}{\partial x_j^2}(x) = 0$, which implies that $\frac{\partial^2 u}{\partial x_j^2}(x)$ is continuous at 0. Since it is also continuous for all $x \neq 0$, it is continuous everywhere. This proves that $\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}$ is continuous everywhere in $\mathbb{R}^n$.

Next we will show $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ is unbounded near the origin. For general t,

$$\begin{aligned}
\frac{\partial^2 u_t}{\partial x_1 \partial x_2}(x) = {} & \eta''(|x|)x_1^2x_2^2|x|^{2t-2}\varphi(-\ln|x|^2) + \eta'(|x|)\big(x_1^2 + x_2^2\big)|x|^{2t-1}\varphi(-\ln|x|^2) \\
& + (4t-1)\eta'(|x|)x_1^2x_2^2|x|^{2t-3}\varphi(-\ln|x|^2) - 4\eta'(|x|)x_1^2x_2^2|x|^{2t-3}\varphi'(-\ln|x|^2) \\
& + \eta(|x|)|x|^{2t}\varphi(-\ln|x|^2) + 2t\eta(|x|)\big(x_1^2 + x_2^2\big)|x|^{2t-2}\varphi(-\ln|x|^2) \\
& - 2\eta(|x|)\big(x_1^2 + x_2^2\big)|x|^{2t-2}\varphi'(-\ln|x|^2) + 2t(2t-2)\eta(|x|)x_1^2x_2^2|x|^{2t-4}\varphi(-\ln|x|^2) \\
& + (4-8t)\eta(|x|)x_1^2x_2^2|x|^{2t-4}\varphi'(-\ln|x|^2) + 4\eta(|x|)x_1^2x_2^2|x|^{2t-4}\varphi''(-\ln|x|^2).
\end{aligned} \tag{16}$$

For each k, choose $x^{(k)} \in \mathbb{R}^n$ such that

$$\frac{x^{(k)} - R_k\zeta_0}{r_k} = \left(e^{-\frac{1}{4t_k}}, 0, \ldots 0\right).$$

Then $x^{(k)}$ is in the ball B_k and

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}\big(x^{(k)}\big) = \epsilon_k \frac{\partial^2 u_{t_k}}{\partial x_1 \partial x_2}\left(\frac{x^{(k)} - R_k\zeta_0}{r_k}\right) = \epsilon_k \frac{\partial^2 u_{t_k}}{\partial x_1 \partial x_2}\left(\left(e^{-\frac{1}{4t_k}}, 0, \ldots 0\right)\right). \tag{17}$$

Because $t_k < \frac{1}{4}$, we know $e^{-\frac{1}{4t_k}} < e^{-1} < \frac{1}{2}$, so in a neighborhood of $e^{-\frac{1}{4t_k}}$, $\eta(|x|) \equiv 1$ and $\eta'(|x|) = \eta''(|x|) = 0$. Also note that $x_2 = 0$ at the point $\left(e^{-\frac{1}{4t_k}}, 0, \dots 0\right)$, then by (16) we have

$$\begin{aligned}
&\frac{\partial^2 u_{t_k}}{\partial x_1 \partial x_2}\left(\left(e^{-\frac{1}{4t_k}}, 0, \dots 0\right)\right) \\
&= \left(e^{-\frac{1}{4t_k}}\right)^{2t_k} \varphi\left(-\ln\left(e^{-\frac{1}{4t_k}}\right)^2\right) + 2t_k\left(e^{-\frac{1}{4t_k}}\right)^{2t_k} \varphi\left(-\ln\left(e^{-\frac{1}{4t_k}}\right)^2\right) \\
&\quad - 2\left(e^{-\frac{1}{4t_k}}\right)^{2t_k} \varphi'\left(-\ln\left(e^{-\frac{1}{4t_k}}\right)^2\right) \\
&= e^{-\frac{1}{2}} \varphi\left(\frac{1}{2t_k}\right) + 2t_k\left(e^{-\frac{1}{4t_k}}\right)^{2t_k} \varphi\left(-\ln\left(e^{-\frac{1}{4t_k}}\right)^2\right) - 2e^{-\frac{1}{2}} \varphi'\left(\frac{1}{2t_k}\right),
\end{aligned}$$

where we purposefully did not simplify the second term. Thus (17) becomes

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_1 \partial x_2}\left(x^{(k)}\right) &= \epsilon_k e^{-\frac{1}{2}} \varphi\left(\frac{1}{2t_k}\right) + \epsilon_k \cdot 2t_k\left(e^{-\frac{1}{4t_k}}\right)^{2t_k} \varphi\left(-\ln\left(e^{-\frac{1}{4t_k}}\right)^2\right) \\
&\quad - 2\epsilon_k e^{-\frac{1}{2}} \varphi'\left(\frac{1}{2t_k}\right).
\end{aligned} \tag{18}$$

The second term in (18) goes to 0 because

$$\epsilon_k \left| 2t_k \left(e^{-\frac{1}{4t_k}}\right)^{2t_k} \varphi\left(-\ln\left(e^{-\frac{1}{4t_k}}\right)^2\right)\right| \le \epsilon_k C_\varphi$$

by Lemma 2.2. The third term in (18) goes to 0 because $\frac{1}{2t_k} \to \infty$ and $\lim_{s\to\infty} \varphi'(s) = 0$.

Now choose

$$\epsilon_k = \frac{1}{\sqrt{\varphi\left(\frac{1}{2t_k}\right)}}.$$

The first term in (18) becomes

$$\epsilon_k e^{-\frac{1}{2}} \varphi\left(\frac{1}{2t_k}\right) = e^{-\frac{1}{2}} \sqrt{\varphi\left(\frac{1}{2t_k}\right)} \to \infty \quad \text{as } k \to \infty.$$

Therefore,

$$\lim_{k\to\infty} \frac{\partial^2 u}{\partial x_1 \partial x_2}\left(x^{(k)}\right) = \infty.$$

This shows that $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ is not bounded near the origin, and the lemma is proved. □

4 Construction for Theorem 1.3

The idea for constructing higher order examples for Theorem 1.3 is the same as that for Theorem 1.1, and we only need to replace x_1x_2 by the real or imaginary part of $(x_1 + ix_2)^{k+2}$, where $k \in \mathbb{N}$. For example, if $k = 1$, then

$$(x_1 + ix_2)^3 = \left(x_1^3 - 3x_1x_2^2\right) + i\left(3x_1^2x_2 - x_2^3\right),$$

so we may use either $x_1^3 - 3x_1x_2^2$ or $3x_1^2x_2 - x_2^3$ in the construction. For general k,

$$(x_1 + ix_2)^{k+2} = \sum_{l=0}^{k+2} \binom{k+2}{l} x_1^{k+2-l}\left(ix_2\right)^l.$$

Evidently, the expressions for its real and imaginary parts are inconvenient to compute. Thus to simplify the calculations we use complex variable for the first two components of x: for any $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$, denote

$$z = x_1 + ix_2 \quad \text{and} \quad \bar{z} = x_1 - ix_2.$$

Then

$$x_1^2 + x_2^2 = z\bar{z}, \qquad |x|^2 = z\bar{z} + \sum_{j=3}^{n} x_j^2, \qquad \text{and} \qquad \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = 4\frac{\partial^2}{\partial \bar{z}\partial z},$$

and consequently

$$\frac{\partial |x|}{\partial z} = \frac{\bar{z}}{2|x|}, \qquad \frac{\partial |x|}{\partial \bar{z}} = \frac{z}{2|x|}, \qquad \text{and} \qquad \frac{\partial |x|}{\partial x_j} = \frac{x_j}{|x|} \quad (\text{when } j \geq 3).$$

Our strategy is to create a complex-valued function such that it is $(k+2)$-times differentiable in $\mathbb{R}^n$, its Laplacian is C^k, but $D^{k+2}u$ is unbounded. The real and imaginary parts of u are two real-valued functions, and at least one of them would be a desired function that satisfies all the conditions in Theorem 1.3. The proof is similar to that for Theorem 1.1, so we will only present the key calculations and noticeable differences without repeating the entire proof.

Recall that in the higher order case $\varphi(s)$ needs to be $(k+2)$-times differentiable and satisfy

$$\lim_{s\to\infty} \varphi(s) = \infty, \qquad \lim_{s\to\infty} \varphi'(s) = \cdots = \lim_{s\to\infty} \varphi^{(k+2)}(s) = 0. \tag{19}$$

The building block function in this case needs to be modified into

$$v(x) = z^{k+2}\varphi(-\ln |x|^2).$$

The Laplacian of v is

$$\begin{aligned}\Delta v &= 4\frac{\partial^2 v}{\partial \bar{z}\partial z} + \sum_{j=3}^{n} \frac{\partial^2 v}{\partial x_j^2}\\ &= -(2n + 4k)z^{k+2}|x|^{-2}\varphi'(-\ln |x|^2) + 4z^{k+2}|x|^{-2}\varphi''(-\ln |x|^2).\end{aligned}$$

It can be verified that Δv is C^k, the partial derivative $\frac{\partial^{k+2} v}{\partial z^{k+2}}$ is unbounded, and v is not $(k+2)$-times differentiable. Because this fact is not to be used in our constructions, we will not verify it here.

As in Sect. 3, the next step is to smooth out the function v. Define $u_t : \mathbb{R}^n \to \mathbb{R}$ $(n \geq 2)$ by

$$u_t(x) = \begin{cases} 0 & x = 0, \\ \eta(|x|) z^{k+2} |x|^{2t} \varphi(-\ln |x|^2) & 0 < |x| < 1, \\ 0 & |x| \geq 1, \end{cases} \tag{20}$$

where η is the same as in (3) and φ satisfies (19). This u_t is a complex-valued C^{k+2} function. The following is a key fact that will be used later.

Lemma 4.1 *For u_t defined by (20), the k-th partial derivatives of Δu_t are all bounded by a constant independent of t.*

Proof As in the proof of Lemma 3.2, we only need to prove that when $|x| < \frac{1}{2}$, all k-th partial derivatives of Δu_t are bounded by a constant independent of t. Since $\eta(|x|) \equiv 1$ when $|x| \leq \frac{1}{2}$, the Laplacian of u_t is

$$\Delta u_t = 4 \frac{\partial^2 u_t}{\partial \bar{z} \partial z} + \sum_{j=3}^{n} \frac{\partial^2 u_t}{\partial x_j^2},$$

where

$$\begin{aligned} \frac{\partial^2 u_t}{\partial \bar{z} \partial z}(x) = {} & (k+3) t z^{k+2} |x|^{2t-2} \varphi(-\ln |x|^2) + t(t-1) z^{k+3} \bar{z} |x|^{2t-4} \varphi(-\ln |x|^2) \\ & - (k+2) z^{k+2} |x|^{2t-2} \varphi'(-\ln |x|^2) - (2t-1) z^{k+3} \bar{z} |x|^{2t-4} \varphi'(-\ln(|x|^2) \\ & + z^{k+3} \bar{z} |x|^{2t-4} \varphi''(-\ln |x|^2), \end{aligned} \tag{21}$$

and for $j = 3, \dots, n$,

$$\begin{aligned} \frac{\partial^2 u_t}{\partial x_j^2}(x) = {} & 2t z^{k+2} |x|^{2t-2} \varphi(-\ln |x|^2) + 2t(2t-2) z^{k+2} x_j^2 |x|^{2t-4} \varphi(-\ln |x|^2) \\ & + (4-8t) z^{k+2} x_j^2 |x|^{2t-4} \varphi'(-\ln |x|^2) - 2 z^{k+2} |x|^{2t-2} \varphi'(-\ln |x|^2) \\ & + 4 z^{k+2} x_j^2 |x|^{2t-4} \varphi''(-\ln |x|^2). \end{aligned} \tag{22}$$

We will show that all the k-th partial derivatives of each term in (21) and (22) are bounded by a constant independent of t. In the subsequent discussions in this section, we will use C to denote a constant that depends on φ, n, k and is independent of t.

We start with (21). The first term of (21) is bounded by

$$(k+3) t z^{k+2} |x|^{2t-2} \left| \varphi(-\ln |x|^2) \right| \leq C t |x|^{k+2t} \left| \varphi(-\ln |x|^2) \right|. \tag{23}$$

Its first derivatives may be taken with respect to z, $\bar{z}$, or x_j $(j \geq 3)$.

- If we take its derivative with respect to z, then

$$\frac{\partial}{\partial z}\left((k+3)tz^{k+2}|x|^{2t-2}\varphi(-\ln|x|^2)\right)$$
$$= (k+3)(k+2)tz^{k+1}|x|^{2t-2}\varphi(-\ln|x|^2) + (k+3)tz^{k+2}(2t-2)|x|^{2t-3}$$
$$\left(\frac{\bar{z}}{2|x|}\right)\varphi(-\ln|x|^2) + (k+3)tz^{k+2}|x|^{2t-2}\left(\varphi'(-\ln|x|^2)\frac{-\bar{z}}{|x|^2}\right),$$

where the first term and the second term are bounded by

$$Ct|x|^{k-1+2t}\left|\varphi(-\ln|x|^2)\right|,$$

and the third term is bounded by

$$Ct|x|^{k-1+2t}\left|\varphi'(-\ln|x|^2)\right|.$$

Therefore, this derivative is bounded by

$$Ct|x|^{k-1+2t}\left|\varphi(-\ln|x|^2)\right| + Ct|x|^{k-1+2t}\left|\varphi'(-\ln|x|^2)\right|.$$

- If we take its derivative with respect to $\bar{z}$, then

$$\frac{\partial}{\partial \bar{z}}\left((k+3)tz^{k+2}|x|^{2t-2}\varphi(-\ln|x|^2)\right)$$
$$= (k+3)tz^{k+2}\left((2t-2)|x|^{2t-3}\frac{z}{2|x|}\right)\varphi(-\ln|x|^2)$$
$$+ (k+3)tz^{k+2}|x|^{2t-2}\left(\varphi'(-\ln|x|^2)\frac{-z}{|x|^2}\right).$$

By similar argument we know that this derivative is also bounded by

$$Ct|x|^{k-1+2t}\left|\varphi(-\ln|x|^2)\right| + Ct|x|^{k-1+2t}\left|\varphi'(-\ln|x|^2)\right|.$$

- If we take its derivative with respect to x_j, then

$$\frac{\partial}{\partial x_j}\left((k+3)tz^{k+2}|x|^{2t-2}\varphi(-\ln|x|^2)\right)$$
$$= (k+3)tz^{k+2}\left((2t-2)|x|^{2t-3}\frac{x_j}{|x|}\right)\varphi(-\ln|x|^2)$$
$$+ (k+3)tz^{k+2}|x|^{2t-2}\left(\varphi'(-\ln|x|^2)\frac{-2x_j}{|x|^2}\right).$$

Again, this derivative is bounded by

$$Ct|x|^{k-1+2t}\left|\varphi(-\ln|x|^2)\right| + Ct|x|^{k-1+2t}\left|\varphi'(-\ln|x|^2)\right|.$$

In conclusion, regardless of which variable we differentiate with, the first partial derivative of $(k+3)tz^{k+2}|x|^{2t-2}\varphi(-\ln|x|^2)$ is bounded by

$$Ct|x|^{k-1+2t}\left|\varphi(-\ln|x|^2)\right| + Ct|x|^{k-1+2t}\left|\varphi'(-\ln|x|^2)\right|.$$

Comparing the first term of this bound, $Ct|x|^{k-1+2t}\left|\varphi(-\ln|x|^2)\right|$, to the right hand side of (23), we see that the power of $|x|$ decreased from $k+2t$ to $k-1+2t$. By the same type of calculations, the k-th partial derivatives of $(k+3)tz^{k+2}|x|^{2t-2}\varphi(-\ln|x|^2)$ will be bounded by

$$Ct|x|^{2t}\left|\varphi(-\ln|x|^2)\right| + \sum_{l=1}^{k} Ct|x|^{k-l+2t}\left|\varphi^{(l)}(-\ln|x|^2)\right|.$$

Because

$$\lim_{s\to\infty}\varphi'(s) = \cdots = \lim_{s\to\infty}\varphi^{(k+2)}(s) = 0,$$

$\sum_{l=1}^{k} Ct|x|^{k-l+2t}\left|\varphi^{(l)}(-\ln|x|^2)\right|$ is bounded by a constant independent of t. By Lemma 2.2, $Ct|x|^{2t}\left|\varphi(-\ln|x|^2)\right|$ is also bounded by a constant independent of t. Therefore, the k-th derivatives of the first term of (21) are bounded by a constant independent of t.

All the other terms in (21) and (22) can be estimated in the same way. This completes the proof of the lemma. □

Then we define u by

$$u(x) = \sum_{l=1}^{\infty} \epsilon_l r_l^{k+2} u_{t_l}\left(\frac{x - R_l\zeta_0}{r_l}\right),$$

where R_l, r_l, ζ_0, t_l, and ϵ_l are the same as in Sect. 3; namely, R_l and r_l are decreasing sequences and

$$R_l - r_l > R_{l+1} + r_{l+1}, \quad \zeta_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{2}}\right), \quad \lim_{l\to\infty} t_l = 0, \quad \epsilon_l = \frac{1}{\sqrt{\varphi\left(\frac{1}{2t_l}\right)}}.$$

Thus by the same argument as in Sect. 3 we know that this infinite sum actually only has a single term for any given x value.

We first show that Δu is C^k. Note that here the power of r_l is $k+2$ as opposed to 2 in Sect. 3, so

$$\Delta u(x) = \sum_{l=1}^{\infty} \epsilon_l r_l^{k} \Delta u_{t_l}\left(\frac{x - R_l\zeta_0}{r_l}\right).$$

By Lemma 4.1, the k-th derivatives of Δu_{t_l} are uniformly bounded by a constant independent of t_l. Then since $\lim_{l\to\infty}\epsilon_l = 0$, we conclude that the k-th derivatives of Δu all approach 0 as $|x| \to 0$. Recall that by construction $u = 0$ on all of the coordinate hyperplanes, so all partial derivatives of u of any order is 0 at the origin. In particular, all the k-th derivatives of Δu at the origin is 0. Therefore, all the k-th derivatives of Δu are continuous at the origin, and consequently Δu is C^k throughout $\mathbb{R}^n$.

Next, we show that some of the $(k+2)$-th derivatives of u is unbounded. Precisely, we will show that $\frac{\partial^{k+2}u}{\partial z^{k+2}}$ is unbounded. We start with a close look at the first and second partial derivatives of u_t with respect to z.

Note that the power of z in

$$u_t = \eta(|x|)z^{k+2}|x|^{2t}\varphi(-\ln|x|^2)$$

is $k+2$. After one differentiation with respect to z, one of the terms in its derivative is

$$(k+2)\eta(|x|)z^{k+1}|x|^{2t}\varphi(-\ln|x|^2),$$

which is the first term in the following formula for $\frac{\partial u_t}{\partial z}$:

$$\begin{aligned}\frac{\partial u_t}{\partial z}(x) = &(k+2)\eta(|x|)z^{k+1}|x|^{2t}\varphi(-\ln|x|^2) + \frac{1}{2}\eta'(|x|)z^{k+2}\bar{z}|x|^{2t-1}\varphi(-\ln|x|^2)\\ &+ t\eta(|x|)z^{k+2}\bar{z}|x|^{2t-2}\varphi(-\ln|x|^2) - \eta(|x|)z^{k+2}\bar{z}|x|^{2t-2}\varphi'(-\ln|x|^2).\end{aligned} \tag{24}$$

After another differentiation with respect to z, there will be one term,

$$(k+1)(k+2)\eta(|x|)z^{k}|x|^{2t}\varphi(-\ln|x|^2).$$

where the power of z is k. That is the first term in the following formula for $\frac{\partial^2 u_t}{\partial z^2}$:

$$\begin{aligned}\frac{\partial^2 u_t}{\partial z^2}(x) = &(k+1)(k+2)\eta(|x|)z^{k}|x|^{2t}\varphi(-\ln|x|^2) + \frac{1}{4}\eta''(|x|)z^{k+2}\bar{z}^2|x|^{2t-2}\varphi(-\ln|x|^2)\\ &+ (k+2)\eta'(|x|)z^{k+1}\bar{z}|x|^{2t-1}\varphi(-\ln|x|^2) + \frac{4t-1}{2}\eta'(|x|)z^{k+2}\bar{z}^2|x|^{2t-3}\varphi(-\ln|x|^2)\\ &+ 2t(k+2)\eta(|x|)z^{k+1}\bar{z}|x|^{2t-2}\varphi(-\ln|x|^2) + t(t-1)\eta(|x|)z^{k+2}\bar{z}^2|x|^{2t-4}\varphi(-\ln|x|^2)\\ &- \eta'(|x|)z^{k+2}\bar{z}^2|x|^{2t-3}\varphi'(-\ln|x|^2) - 2(k+2)\eta(|x|)z^{k+1}\bar{z}|x|^{2t-2}\varphi'(-\ln|x|^2)\\ &- (2t-1)\eta(|x|)z^{k+2}\bar{z}^2|x|^{2t-4}\varphi'(-\ln|x|^2) + \eta(|x|)z^{k+2}\bar{z}^2|x|^{2t-4}\varphi''(-\ln|x|^2),\end{aligned} \tag{25}$$

After $(k+2)$-times of differentiation with respect to z, one of the terms in $\frac{\partial^{k+2}u_t}{\partial z^{k+2}}$ is

$$(k+2)!\eta(|x|)|x|^{2t}\varphi(-\ln|x|^2).$$

As will be shown later, this term is crucial to proving that $\frac{\partial^{k+2}u}{\partial z^{k+2}}$ is unbounded.

For each l, as we did in Sect. 3, choose $x^{(l)} \in \mathbb{R}^n$ such that

$$\frac{x^{(l)} - R_l\zeta_0}{r_l} = \left(e^{-\frac{1}{4t_l}}, 0, \dots 0\right).$$

As discussed in Sect. 3, in a neighborhood of $\left(e^{-\frac{1}{4t_l}}, 0, \dots, 0\right)$, $\eta(|x|) \equiv 1$ and $\eta'(|x|) = \eta''(|x|) = 0$. Therefore, when we evaluate $\frac{\partial u_t}{\partial z}$ and $\frac{\partial^2 u_t}{\partial z^2}$ in a neighborhood of $\left(e^{-\frac{1}{4t_l}}, 0, \dots, 0\right)$, all the terms in (24) and (25) that have an η' or η'' factor will disappear. For that reason in the discussion that follows, we will only consider the terms that have an η factor.

Then (24) becomes

$$(k+2)z^{k+1}|x|^{2t_l}\varphi(-\ln|x|^2)+t_lz^{k+2}\bar{z}|x|^{2t_l-2}\varphi(-\ln|x|^2)-z^{k+2}\bar{z}|x|^{2t_l-2}\varphi'(-\ln|x|^2).$$

Except the first term, the other terms in (24) are bounded by

$$|x|^{k+1+2t_l}\left|\varphi(-\ln|x|^2)\right|+|x|^{k+1+2t_l}\left|\varphi'(-\ln|x|^2)\right|.$$

And (25) becomes

$$\begin{aligned}&(k+1)(k+2)z^k|x|^{2t_l}\varphi(-\ln|x|^2)+2t_l(k+2)z^{k+1}\bar{z}|x|^{2t_l-2}\varphi(-\ln|x|^2)\\&\quad+t_l(t_l-1)z^{k+2}\bar{z}^2|x|^{2t_l-4}\varphi(-\ln(|x|^2)-2(k+2)z^{k+1}\bar{z}|x|^{2t_l-2}\varphi'(-\ln|x|^2)\\&\quad-(2t_l-1)z^{k+2}\bar{z}^2|x|^{2t_l-4}\varphi'(-\ln|x|^2)+z^{k+2}\bar{z}^2|x|^{2t_l-4}\varphi''(-\ln|x|^2).\end{aligned}$$

Except the first term, all the other terms in (25) are bounded by

$$Ct_l|x|^{k+2t_l}\left|\varphi(-\ln|x|^2)\right|+C|x|^{k+2t_l}\left|\varphi'(-\ln|x|^2)\right|+C|x|^{k+2t_l}\left|\varphi''(-\ln|x|^2)\right|.$$

By the same process, in a neighborhood of $\left(e^{-\frac{1}{4t_l}},0,\dots 0\right)$, $\frac{\partial^{k+2}u_{t_l}}{\partial z^{k+2}}$ is equal to

$$(k+2)!|x|^{2t_l}\varphi(-\ln|x|^2)$$

plus some other terms that are bounded by

$$Ct_l|x|^{2t_l}\left|\varphi(-\ln|x|^2)\right|+C|x|^{2t_l}\left|\varphi'(-\ln|x|^2)\right|+\cdots C|x|^{2t_l}\left|\varphi^{(k+2)}(-\ln|x|^2)\right|. \tag{26}$$

By Lemma 2.2 and the fact that

$$\lim_{s\to\infty}\varphi'(s)=\cdots=\lim_{s\to\infty}\varphi^{(k+2)}(s)=0,$$

we know (26) is bounded by a constant independent of t_l.

Now we look at

$$\frac{\partial^{k+2}u}{\partial z^{k+2}}(x^{(l)})=\epsilon_l\frac{\partial^{k+2}u_{t_l}}{\partial z^{k+2}}\left((e^{-\frac{1}{4t_l}},0,\dots,0)\right).$$

If we evaluate (26) at the point $\left(e^{-\frac{1}{4t_l}},0,\dots,0\right)$ and multiply the value with ϵ_l, the result will go to 0 as $\epsilon_l\to 0$.

The first term of $\frac{\partial^{k+2}u}{\partial z^{k+2}}(x^{(l)})$ is equal to

$$\epsilon_l(k+2)!\left(e^{-\frac{1}{4t_l}}\right)^{2t_l}\varphi\left(-\ln\left(e^{-\frac{1}{4t_l}}\right)^2\right)=\frac{(k+2)!}{\sqrt{e}}\epsilon_l\varphi\left(\frac{1}{2t_l}\right).$$

Recall that

$$\epsilon_l=\frac{1}{\sqrt{\varphi\left(\frac{1}{2t_l}\right)}},$$

hence $\epsilon_l \varphi\left(\frac{1}{2t_l}\right) \to \infty$ as $l \to \infty$. Consequently, $\frac{\partial^{k+2} u}{\partial z^{k+2}}(x^{(l)}) \to \infty$ as $l \to \infty$, which implies that $\frac{\partial^{k+2} u}{\partial z^{k+2}}$ is unbounded near the origin. Since $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$, as a result we know that some of the partial derivatives of u with respect to the x_1 and x_2 variables are unbounded near the origin.

Finally, we need to show that u is $(k+2)$-times differentiable at the origin. Recall that because $u = 0$ on all the coordinate hyperplanes, all partial derivatives of any order of u at the origin is 0. Then

$$\lim_{|x|\to 0} \frac{u(x) - \sum_{|\gamma|\le k+2} \frac{D^\gamma u(0)}{\gamma!} x^\gamma}{|x|^{k+2}} = \lim_{|x|\to 0} \frac{u(x)}{|x|^{k+2}}.$$

Note that $|x| \ge R_l - \frac{2}{3} r_l$, and similar to (5) in Sect. 3 we can prove u_{t_l} is uniformly bounded by a constant $C_{\eta,\varphi}$ depending only on η and φ, therefore

$$\frac{|u(x)|}{|x|^{k+2}} \le \frac{\epsilon_l r_l^{k+2} \left| u_{t_l}\left(\frac{x - R_l \zeta_0}{r_l}\right)\right|}{\left(R_l - \frac{2}{3} r_l\right)^{k+2}} \le \frac{\epsilon_l C_{\eta,\varphi}}{\left(\frac{R_l}{r_l} - \frac{2}{3}\right)^{k+2}} < \frac{\epsilon_l C_{\eta,\varphi}}{\left(1 - \frac{2}{3}\right)^{k+2}}.$$

It follows that $\lim_{|x|\to 0} \frac{u(x)}{|x|^{k+2}} = 0$, which implies u is $(k+2)$-times differentiable at the origin.

Thus we can conclude that as a complex-valued function, u is $(k+2)$-times differentiable at 0, Δu is C^k throughout $\mathbb{R}^n$, but $D^{k+2} u$ is unbounded near 0. The real and imaginary parts of u are two real-valued functions that are $(k+2)$-times differentiable at 0, their Laplacian are C^k throughout $\mathbb{R}^n$, and at least one of them has some unbounded $(k+2)$-th partial derivatives. Therefore, we have found a function that satisfies all the conditions in Theorem 1.3.

Appendix: Proof of Proposition 1.5

The proof of Proposition 1.5 is based on a method that was introduced in [5] and elaborated in detail in [3]. Note that after a translation we can assume x or y is at the origin, so we only need to prove that for $|z| < \frac{1}{16}$, (here z is a point in $\mathbb{R}^n$, not a complex variable as was used in the previous section), we have

$$\left| Du(z) - Du(0) \right| \le C|z| \left(\sup_{B_1} |u| + \sup_{B_1} |f| + \int_{|z|}^{1} \frac{w(r)}{r} dr \right). \tag{27}$$

For $|z| \ge \frac{1}{16}$ the estimate is also true by a covering argument (see [3]).

First, we recall three elementary estimates (see [3]) that will be used frequently in this proof.

If a function v satisfies $\Delta v = 0$ in B_r, then for any positive integer k,

$$\|D^k v\|_{L^\infty(B_{\frac{r}{2}})} \le C r^{-k} \|v\|_{L^\infty(B_r)}, \tag{28}$$

where C only depends on n and k.

If a function v satisfies $\Delta v = \lambda$ in B_r, where λ is a constant and $r < 1$, then

$$\|Dv\|_{L^\infty(B_{\frac{r}{2}})} \le C\left(r^{-1}\|v\|_{L^\infty(B_r)} + r|\lambda|\right). \tag{29}$$

If a function v satisfies $\Delta v = f$ in B_r, where f is a given bounded function, then the scaled maximum principle states that

$$\|u\|_{L^\infty(B_r)} \le \|u\|_{L^\infty(\partial B_r)} + Cr^2\|f\|_{L^\infty(B_r)}. \tag{30}$$

Now we are ready to prove (27). For $k = 0, 1, 2, \ldots$, let u_k be the solution to

$$\begin{cases} \Delta u_k = f(0) & \text{in} \quad B_{2^{-k}}, \\ u_k = u & \text{on} \quad \partial B_{2^{-k}}. \end{cases}$$

Then $\Delta(u_k - u) = f(0) - f$ in $B_{2^{-k}}$ and $u_k - u = 0$ on $\partial B_{2^{-k}}$. By the scaled maximum principle it follows that

$$\begin{aligned} \|u_k - u\|_{L^\infty(B_{2^{-k}})} \le& C\left(2^{-2k}\right)\|f(0) - f\|_{L^\infty(B_{2^{-k}})} \\ \le& C\left(2^{-2k}\right)\omega(2^{-k}), \end{aligned} \tag{31}$$

and therefore

$$\begin{aligned} \|u_{k+1} - u_k\|_{L^\infty(B_{2^{-k-1}})} \le& \|u_{k+1} - u\|_{L^\infty(B_{2^{-k-1}})} + \|u_k - u\|_{L^\infty(B_{2^{-k}})} \\ \le& C\left(2^{-2(k+1)}\right)\omega(2^{-(k+1)}) + C\left(2^{-2k}\right)\omega(2^{-k}) \\ \le& C\left(2^{-2k}\right)\omega(2^{-k}). \end{aligned} \tag{32}$$

Then since $u_{k+1} - u_k$ is harmonic, by (28) we have

$$\begin{aligned} \|Du_{k+1} - Du_k\|_{L^\infty(B_{2^{-k-2}})} \le& C\left(2^{k+1}\right)\|u_{k+1} - u_k\|_{L^\infty(B_{2^{-k-1}})} \\ \le& C\left(2^{-k}\right)\omega(2^{-k}). \end{aligned} \tag{33}$$

For any $|z| \le \frac{1}{16}$, choose $k \in \mathbb{N}$ such that

$$2^{-k-4} \le |z| \le 2^{-k-3}.$$

We will estimate $|Du(z) - Du(0)|$ by

$$|Du(z) - Du(0)| \le \left|Du(0) - Du_k(0)\right| + \left|Du(z) - Du_k(z)\right| + \left|Du_k(z) - Du_k(0)\right|. \tag{34}$$

We are going to estimate these three terms separately. First, we claim that

$$\lim_{k\to\infty} Du_k(0) = Du(0).$$

To see this, let $\tilde{u}(x) = u(0) + x \cdot Du(0)$ be the linear approximation of u at 0. Then $Du(0) = D\tilde{u}(0)$ and $|\tilde{u}(x) - u(x)| = o(|x|)$. Thus

$$
\begin{aligned}
&\left|Du_k(0) - Du(0)\right| \\
&\le \|Du_k - D\tilde{u}\|_{L^\infty(B_{2^{-k-1}})} \\
&\le C\left(2^k\right)\|u_k - \tilde{u}\|_{L^\infty(B_{2^{-k}})} \quad \left(\text{by (28)}\right) \\
&= C\left(2^k\right)\left(\|u_k - \tilde{u}\|_{L^\infty(\partial B_{2^{-k}})} + 2^{-2k}|f(0)|\right) \quad \left(\text{apply} \quad (30) \text{ to } \Delta(u_k - \tilde{u}) = f(0)\right) \\
&= C\left(2^k\right)\|u - \tilde{u}\|_{L^\infty(\partial B_{2^{-k}})} + C\left(2^{-k}\right)|f(0)| \\
&\le C\left(2^k\right) \cdot o(2^{-k}) + C\left(2^{-k}\right)|f(0)| \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

Then we can write $Du_k(0) - Du(0) = \sum_{j=k}^{\infty}\left(Du_j(0) - Du_{j+1}(0)\right)$, and consequently

$$
\begin{aligned}
|Du_k(0) - Du(0)| &\le \sum_{j=k}^{\infty} |Du_j(0) - Du_{j+1}(0)| \\
&\le C\sum_{j=k}^{\infty}\left(2^{-j}\right)\omega(2^{-j}) \quad \left(\text{by (33)}\right) \\
&\le C\sum_{j=k}^{\infty}\left(2^{-j}\right)\omega(2^{-k}) \\
&= C(2^{-k})\omega(2^{-k}) \\
&\le C|z|\omega(2^{-k}).
\end{aligned} \tag{35}
$$

Next, we estimate the term $\left|Du(z) - Du_k(z)\right|$. Let v_j be the solution of

$$
\begin{cases}
\Delta v_j = f(z) & \text{in } B_{2^{-j}}(z), \\
v_j = u & \text{on} \quad \partial B_{2^{-j}}(z).
\end{cases}
$$

By the same argument as before we can show

$$
|Dv_k(z) - Du(z)| \le C|z|\omega(2^{-k}).
$$

Because $\Delta(u_k - v_k) = f(0) - f(z)$ in $B_{2^{-k}}(0) \cap B_{2^{-k}}(z)$ and $B_{2^{-k-1}}(z) \subset B_{2^{-k}}(0) \cap B_{2^{-k}}(z)$,

$$
\begin{aligned}
|Dv_k(z) - Du_k(z)| &\le \|D(v_k - u_k)\|_{L^\infty\left(B_{2^{-k-2}(z)}\right)} \\
&\le C\left(2^{k+1}\|v_k - u_k\|_{L^\infty\left(B_{2^{-k-1}}(z)\right)} + 2^{-k-1}|f(0) - f(z)|\right) \quad \left(\text{by (29)}\right) \\
&= C\left(2^{k+1}\right)\|v_k - u_k\|_{L^\infty\left(B_{2^{-k-1}}(z)\right)} + C\left(2^{-k-1}\right)\omega(2^{-k-3}).
\end{aligned}
$$

Then

$$
\begin{aligned}
\left|Du(z) - Du_k(z)\right| &\le \left|Du_k(z) - Dv_k(z)\right| + \left|Dv_k(z) - Du(z)\right| \\
&\le C\left(2^{k+1}\right)\|v_k - u_k\|_{L^\infty\left(B_{2^{-k-1}}(z)\right)} + C\left(2^{-k-1}\right)\omega(2^{-k-3}) + C|z|\omega(2^{-k}) \\
&\le C\left(2^{k+1}\right)\|v_k - u_k\|_{L^\infty\left(B_{2^{-k-1}}(z)\right)} + C|z|\omega(2^{-k}).
\end{aligned} \tag{36}
$$

By (31) we know

$$\|u_k - u\|_{L^\infty(B_{2^{-k-1}}(z))} \le \|u_k - u\|_{L^\infty(B_{2^{-k}}(0))} \le C\big(2^{-2k}\big)\omega(2^{-k}),$$

and similarly we can prove

$$\|v_k - u\|_{L^\infty(B_{2^{-k-1}}(z))} \le C\big(2^{-2k}\big)\omega(2^{-k}),$$

so

$$\begin{aligned}\|u_k - v_k\|_{L^\infty(B_{2^{-k-1}}(z))} \le& \|u_k - u\|_{L^\infty(B_{2^{-k-1}}(z))} + \|v_k - u\|_{L^\infty(B_{2^{-k-1}}(z))} \\ \le& C\big(2^{-2k}\big)\omega(2^{-k}).\end{aligned}$$

Using this in (36) we have

$$\begin{aligned}\big|Du(z) - Du_k(z)\big| &\le C\big(2^{k+1}\big)\big(2^{-2k}\big)\omega(2^{-k}) + C|z|\omega(2^{-k}) \\ &\le C|z|\omega(2^{-k}).\end{aligned} \tag{37}$$

Now we only need to estimate $|Du_k(z) - Du_k(0)|$. Let

$$h_j = u_j - u_{j-1} \quad \text{for} \quad j = 1, \dots, k.$$

h_j is harmonic, so by (28)

$$\|D^2 h_j\|_{L^\infty(B_{2^{-j-1}}(0))} \le C\big(2^{2j}\big)\|h_j\|_{L^\infty(B_{2^{-j}}(0))}.$$

Thus,

$$\begin{aligned}\frac{|Dh_j(z) - Dh_j(0)|}{|z|} &\le \|D^2 h_j\|_{L^\infty(B_{2^{-k-3}}(0))} \\ &\le x\|D^2 h_j\|_{L^\infty(B_{2^{-j-1}}(0))} \\ &\le C\big(2^{2j}\big)\|h_j\|_{L^\infty(B_{2^{-j}}(0))} \\ &= C\big(2^{2j}\big)\|u_j - u_{j-1}\|_{L^\infty(B_{2^{-j}}(0))} \\ &\le C\big(2^{2j}\big)\big(2^{-2(j-1)}\big)\omega(2^{-(j-1)}) \quad \text{by (32)} \\ &\le C\omega(2^{-(j-1)}).\end{aligned}$$

Consequently,

$$\begin{aligned}|Du_k(z) - Du_k(0)| &\le |Du_1(z) - Du_1(0)| + \sum_{j=2}^{k} |Dh_j(z) - Dh_j(0)| \\ &\le |Du_1(z) - Du_1(0)| + \sum_{j=2}^{k} C|z|\omega(2^{-j+1}) \\ &\le |z|\|D^2 u_1\|_{L^\infty\left(B_{\frac{1}{4}}\right)} + C|z|\sum_{j=2}^{k}\omega(2^{-j+1}).\end{aligned}$$

Now we need to estimate $\|D^2u_1\|_{L^\infty(B_{\frac{1}{4}})}$.

Define a function

$$\zeta(x) = u_1(x) - \frac{f(0)}{2n}|x|^2 + \frac{f(0)}{8n}.$$

Then ζ is harmonic because $\Delta\zeta = \Delta u_1 - f(0) = 0$, and $\zeta = u_1 = u$ on $\partial B_{\frac{1}{2}}(0)$.

Furthermore, $D_{ij}\zeta = D_{ij}u_1$ when $i \neq j$, and $D_{ii}\zeta = D_{ii}u_1 - \frac{f(0)}{n}$.

Therefore,

$$\begin{aligned}\|D^2u_1\|_{L^\infty(B_{\frac{1}{4}})} &\le \|D^2\zeta\|_{L^\infty(B_{\frac{1}{4}})} + |f(0)| \\ &\le C\|\zeta\|_{L^\infty(B_{\frac{1}{2}})} + |f(0)| \\ &= C\|\zeta\|_{L^\infty(\partial B_{\frac{1}{2}})} + |f(0)| \\ &= C\|u\|_{L^\infty(\partial B_{\frac{1}{2}})} + |f(0)| \\ &\le C\|u\|_{L^\infty(B_1)} + |f(0)|.\end{aligned}$$

It follows that

$$|Du_k(z) - Du_k(0)| \le C|z|\|u\|_{L^\infty(B_1)} + |z||f(0)| + C|z|\sum_{j=2}^{k}\omega(2^{-j+1}). \tag{38}$$

Combining (34), (35), (37), and (38), we have

$$\begin{aligned}|Du(z) - Du(0)| &\le C|z|\omega(2^{-k}) + C|z|\|u\|_{L^\infty(B_1)} + |z||f(0)| + C|z|\sum_{j=2}^{k}\omega(2^{-j+1}) \\ &= |z|\left(\|u\|_{L^\infty(B_1)} + |f(0)| + \sum_{j=2}^{k+1}\omega(2^{-j+1})\right).\end{aligned}$$

Finally, note that since $\omega(r)$ is increasing with r increasing,

$$\begin{aligned}\int_{|z|}^{1}\frac{\omega(r)}{r}dr &\ge \int_{\frac{1}{2^{k+3}}}^{1}\frac{\omega(r)}{r}dr \\ &\ge \int_{\frac{1}{2^{k+3}}}^{\frac{1}{2^{k+2}}}\frac{\omega\left(\frac{1}{2^{k+3}}\right)}{r}dr + \int_{\frac{1}{2^{k+2}}}^{\frac{1}{2^{k+1}}}\frac{\omega\left(\frac{1}{2^{k+2}}\right)}{r}dr + \cdots + \int_{\frac{1}{2}}^{1}\frac{\omega\left(\frac{1}{2}\right)}{r}dr \\ &= (\ln 2)\omega\left(\frac{1}{2^{k+3}}\right) + (\ln 2)\omega\left(\frac{1}{2^{k+2}}\right) + \cdots + (\ln 2)\omega\left(\frac{1}{2}\right).\end{aligned}$$

Thus

$$\sum_{j=2}^{k+1}\omega(2^{-j+1}) < \omega\left(\frac{1}{2}\right) + \cdots + \omega\left(\frac{1}{2^{k+3}}\right) \le C\int_{|z|}^{1}\frac{\omega(r)}{r}dr.$$

Therefore, we have proved that

$$|Du(z) - Du(0)| \leq |z| \left(\|u\|_{L^\infty(B_1)} + |f(0)| + \int_{|z|}^{1} \frac{\omega(r)}{r} dr \right),$$

and this implies (27).

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