Banach spaces which always produce octahedral spaces of operators

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Abstract

We characterise those Banach spaces X which satisfy that $L(Y, X)$ is octahedral for every non-zero Banach space Y. They are those satisfying that, for every finite dimensional subspace Z, ℓ_∞ can be finitely-representable in a part of X kind of ℓ_1-orthogonal to Z. We also prove that $L(Y, X)$ is octahedral for every Y if, and only if, $L(\ell_p^n, X)$ is octahedral for every $n \in \mathbb{N}$ and $1 < p < \infty$. Finally, we find examples of Banach spaces satisfying the above conditions like $\mathrm{Lip}_0(M)$ spaces with octahedral norms or L_1-preduals with the Daugavet property.

Keywords Spaces of operators · Universally octahedral · Finite representability

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1 Introduction

According to [10, Remark II.5.2], the norm of a Banach space X is *octahedral* if, for every finite dimensional subspace E of X and every $\varepsilon > 0$, there exists $y \in S_X$ such that

$$\|x + \lambda y\| \geq (1 - \varepsilon)(\|x\| + |\lambda|) \text{ for every } x \in E \text{ and every } \lambda \in \mathbb{R}.$$

Octahedral norms were studied at the end of the eighties in succesive papers [10, 11] because it turns out that such norms characterise the containment of ℓ_1. Indeed, in [10, Theorem II.4] it is proved that a Banach space X contains an isomorphic copy of ℓ_1 if, and only if, X can be equivalently renormed with an octahedral norm.

However, octahedral norms have received much more attention in the recent years because in [6, Theorem 2.1] it is proved that X is octahedral if, and only if, every convex combination

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of w^*-slices of B_{X^*} has diameter two. Since then, octahedral norms and variations of such norms have been studied in many different contexts (see. e.g. [4, 8, 12, 18, 21]).

One of the areas where octahedrality has been intensively studied is in spaces of operators, that is, it has been analysed when the space of bounded operators $L(X, Y)$ between two Banach spaces X and Y is octahedral. The motivation for this interest comes from [1, Question (b)], where the authors asked when the projective tensor product $X \widehat{\otimes}_\pi Y$ satisfies that all the convex combination of slices of its unit ball have diameter two. Thanks to the duality $(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ and the above mentioned [6, Theorem 2.1], the above question is equivalent to determining when the norm of the space of operators is octahedral.

In [7, Theorem 3.5] it is proved that if Y^* and X are octahedral then H is octahedral for any subspace $H \subseteq L(Y, X)$ containing finite-rank operators $F(Y, X)$. More examples of octahedral spaces of operators were given in [14]. It was shown, however, that the above is not the case if we remove octahedrality on Y^*, and in fact octahedrality of $L(Y, X)$ is connected with finite-representability of Y in X. Indeed, in [17, Lemma 3.7] it is proved that if some subspace H of $L(Y, X)$ is octahedral and Y is uniformly convex then Y is finitely representable in X.

The connection between finite-representability and octahedrality of spaces of operators have shown to be much deeper. Indeed, a kind of converse is established in [19, Theorem 3.2] where it is proved that if X is a Banach space which is finitely representable in ℓ_1 and with the metric approximation property, then $L(X, Y)$ is octahedral if Y is octahedral.

In this note we will focus on the following problem: which Banach spaces X satisfies that $L(Y, X)$ is octahedral for every Banach space Y? We will refer to these spaces as *universally octahedral* (see Definition 3.1). Observe that, in order to solve a problem about octahedrality of spaces of vector-valued Lipschitz functions, it is proved in [19, Theorem 3.1] that $\mathrm{Lip}_0(M)$, the space of Lipschitz functions over M, is universally octahedral whenever $\mathrm{Lip}_0(M)$ is octahedral.

Anyway, in view of [17, Lemma 3.7] one should think that if X is universally octahedral then it should be an octahedral space such that every uniformly convex Banach space is finitely representable in it. This intuition is confirmed in Lemma 3.2, where we observe that a necessary condition for universal octahedrality is that, roughly speaking, given any finite dimensional subspace Z of X and any $\varepsilon > 0$, then any finite dimensional uniformly convex Banach space can be $(1 + \varepsilon)$-embedded in the space of "ε-orthogonal vectors to Z". In a further step, making use of approximations in Banach-Mazur distance, we obtain in Theorem 3.4 that we can replace in the above statement uniformly convex Banach spaces with ℓ_∞^n for every n. More precisely, we prove that if X is universally octahedral then, given any finite dimensional subspace Z of X, any $n \in \mathbb{N}$ and any $\varepsilon > 0$, we can find a norm-one operator $\Phi : \ell_\infty^n \longrightarrow X$ such that

$$\|z + \Phi(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $z \in Z$ and every $y \in \ell_\infty$.

The converse, making use of the finite-representability of every Banach space in c_0 together with the celebrated characterisation of L_1-preduals due to J. Lindenstrauss [20, Theorem 6.1], is proved in Theorem 3.6. As a consequence we obtain, in Theorem 3.7, that a Banach space X is universally octahedral if, and only if, $L(\ell_p^n, X)$ is octahedral for every $1 \leq p \leq \infty$ and every $n \in \mathbb{N}$, which is in turn equivalent to the condition that, given any finite dimensional subspace Z of X, any $n \in \mathbb{N}$ and any $\varepsilon > 0$, we can find a norm-one operator $\Phi : \ell_\infty^n \longrightarrow X$ such that

$$\|z + \Phi(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $z \in Z$ and every $y \in \ell_\infty^n$.

Observe that the above condition is strictly stronger than the mere finite-representability of ℓ_∞ in X. Indeed, in Example 3.8 we construct an example of a octahedral Banach space which contains an isomorphic copy of ℓ_∞ but failing the universal octahedrality. This shows that, in order to obtain universal octahedrality, the requierement that the copies of ℓ_∞^n can be found in the orthogonal part of any finite dimensional subspace can not be relaxed.

Another relevant example is given in Example 3.9, where it is shown that a universally octahedral space does not have to contain c_0 isomorphically.

In Sect. 4 we aim to find new examples of Banach spaces X which are universally octahedral. We begin by observing that a sufficient condition for universal octahedrality of a X is the following: for every finite dimensional subspace Z of X and every $\varepsilon > 0$, there exists a subspace Y of X which is isometrically isomorphic to c_0 and such that

$$\|z + y\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $z \in Z$ and every $y \in Y$ (we define this property in Definition 4.1 as c_0-*octahedral*). The reason to introduce this definition is double. The first one is to recover the technique followed in [19, Theorem 3.1], where it is proved that $\mathrm{Lip}_0(M)$ is universally octahedral when it is octahedral, but whose proof is based on [19, Lemma 3.3], where it is precisely proved that $\mathrm{Lip}_0(M)$ is c_0-octahedral. On the other hand, in spite of the fact that Example 3.9 shows that universal octahedrality does not imply c_0-octahedrality, there is a strong connection through ultrapower spaces. In fact, in Proposition 4.4 it is proved that X is universally octahedral if, and only if, $X_{\mathcal{U}}$ is c_0-octahedral for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. We end the paper with Theorem 4.6, where we prove that every L_1-predual which is octahedral is indeed universally octahedral, using recent tools developed in [22].

2 Notation and preliminary results

We will consider real Banach spaces. Given a Banach space X, we will denote the closed unit ball and the unit sphere of X by B_X and S_X respectively. We will also denote by X^* the topological dual of X. Given two Banach spaces X and Y denote by $L(X, Y)$ (respectively $F(X, Y)$) the space of linear bounded operators (respectively the finite-rank operators) from X to Y.

According to [2, Definition 11.1.1], given two Banach spaces X and Y, we say that *X is finitely representable in Y* if, given any finite dimensional subspace E of X and any $\varepsilon > 0$, there exist a subspace F of Y and a linear continuous bijection $T : E \longrightarrow F$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$. This notion encodes the idea that Y contains all the finite dimensional structure of X. Observe that every Banach space is finitely-representable in c_0 [2, Example 11.1.2]. Moreover, as a consequence of the Principle of Local Reflexivity (c.f. e.g. [9, Lemma 9.15]), for every Banach space X it follows that X^{**} is finitely representable in X. We refer the interested reader to [2, Chapter 11] and references therein for background about finite representability of Banach spaces.

Let us include here, for easy reference, the following lemma, which is extracted from [2, Lemma 11.1.11].

Lemma 2.1 *Let E be a finite dimensional Banach space and let $\{x_j : 1 \leq j \leq N\} \subseteq S_X$ be an ε-net of S_E. Let $T : E \longrightarrow X$ be a linear mapping such that $(1 - \varepsilon) \leq \|T(x_j)\| \leq (1 + \varepsilon)$*

holds for every $1 \leq j \leq N$. *Then, for every* $e \in E$, *we have*

$$\left(\frac{1 - 3\varepsilon}{1 - \varepsilon}\right) \|e\| \leq \|T(e)\| \leq \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right) \|e\|.$$

Given a sequence of Banach spaces $\{X_n : n \in \mathbb{N}\}$ we denote

$$\ell_\infty(\mathbb{N}, X_n) := \left\{ f : \mathbb{N} \longrightarrow \prod_{n \in \mathbb{N}} X_n : f(n) \in X_n \ \forall n \text{ and } \sup_{n \in \mathbb{N}} \|f(n)\| < \infty \right\}.$$

Given a non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, consider $c_{0, \mathcal{U}}(\mathbb{N}, X_n) := \{f \in \ell_\infty(\mathbb{N}, X_n) : \lim_{\mathcal{U}} \|f(n)\| = 0\}$. The *ultrapower of* $\{X_n : n \in \mathbb{N}\}$ *with respect to* $\mathcal{U}$ is the Banach space

$$(X_n)_{\mathcal{U}} := \ell_\infty(\mathbb{N}, X_n) / c_{0, \mathcal{U}}(\mathbb{N}, X_n).$$

We will naturally identify a bounded function $f : \mathbb{N} \longrightarrow \prod_{n \in \mathbb{N}} X_n$ with the element $(f(n))_{n \in \mathbb{N}}$. In this way, we denote by $(x_n)_{\mathcal{U}}$ or simply by (x_n), if no confusion is possible, the coset in $(X_n)_{\mathcal{U}}$ given by $(x_n)_{n \in \mathbb{N}} + c_{0, \mathcal{U}}(\mathbb{N}, (X_n))$.

From the definition of the quotient norm, it is not difficult to prove that $\|(x_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|$ holds for every $(x_n) \in (X_n)_{\mathcal{U}}$.

When $X_n = X$ holds for every $n \in \mathbb{N}$, the definition of the norm on $X_{\mathcal{U}}$ yields a canonical inclusion $j : X \longrightarrow X_{\mathcal{U}}$ given by the equation

$$j(x) := (x)_{\mathcal{U}}.$$

This inclusion is an into linear isometry, so X can be isometrically embedded in $X_{\mathcal{U}}$. Moreover, $X_{\mathcal{U}}$ is finitely representable in X [2, Proposition 11.1.12].

Given a Banach space X we say that X is an L_1-*predual* if $X^* = L_1(\mu)$ isometrically for some measure μ. Let us include here for easy reference in the text the following result.

Theorem 2.2 [20, Theorem 6.1] *Let* X *be a Banach space. The following assertions are equivalent:*

(1) X *is an* L_1-*predual.*
(2) *Every compact operator* $T : Y \longrightarrow X$ *has, for every* $\varepsilon > 0$ *and every Banach space* Z *containing* Y, *an extension* $\hat{T} : Z \longrightarrow X$ *such that* $\|\hat{T}\| \leq (1 + \varepsilon)\|T\|$.

Strongly related to $L_1(\mu)$-spaces are the L-summands. A projection $P : X \longrightarrow X$ on a Banach space X is said to be an L-*projection* if $\|x\| = \|Px\| + \|x - Px\|$ for every $x \in X$. The range of an L-projection is called an L-*summand*. We refer the reader to [15] for a vast background about L-summands.

3 Characterisation of universally octahedral spaces

Let us start with the main definition of the paper.

Definition 3.1 Let X be a Banach space. We will say that X is *universally octahedral* if $L(Y, X)$ is octahedral for every non-zero Banach space Y.

The aim of this section is to provide a characterisation of universally octahedral Banach spaces. In order to do so, let us start with the following preliminary lemma, which is a

strengthening of [17, Lemma 3.7]. Recall that a Banach space X is said to be *uniformly convex* if, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{c} x, y \in B_X \\ \|x + y\| > 2 - \delta(\varepsilon) \end{array}\right\} \Rightarrow \|x - y\| < \varepsilon.$$

Examples of uniformly convex Banach spaces are $L_p(\mu)$ for $1 < p < \infty$ thanks to Clarkson inequality (see [9, Chapter 9] for background about uniform convexity).

Lemma 3.2 *Let X be a Banach space and Y be a finite dimensional uniformly convex Banach space. Assume that $L(Y, X)$ is octahedral. Then, for every $\varepsilon > 0$ and for every finite dimensional subspace Z of X, there exists an element $T \in B_{L(Y,X)}$ such that*

$$\|z + T(y)\| \geq (1 - \varepsilon)^2 (\|z\| + \|y\|)$$

holds for every $y \in Y$ and every $z \in Z$.

Observe that the mapping T is a $(1+\varepsilon)$-isometry (just take $z = 0$) and that X is octahedral.

Proof Since Y is uniformly convex there exists a mapping $\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ and with the property that, given $\eta > 0$, if $x, y \in B_Y$ satisfy $\|x + y\| > 2 - \delta(\eta)$ then $\|x - y\| < \eta$.

Take $\varepsilon > 0$ and $\eta > 0$ small enough such that $\delta(\eta) + 4\eta < \varepsilon$. Pick a finite dimensional subspace Z of X. Take $\{y_1, \ldots, y_n\}$ a η-net of S_Y and take $\{z_1, \ldots, z_p\}$ a η-net of S_Z. For every $i \in \{1, \ldots, n\}$ take $f_i \in S_{Y^*}$ such that $f_i(y_i) = 1$.

Define $T_{ij} := f_i \otimes z_j \in L(Y, X)$ by $T_{ij}(x) := f_i(x)z_j$, and note that T_{ij} is a norm-one element. By the assumption that $L(Y, X)$ is octahedral we can find an operator $T \in S_{L(Y,X)}$ such that

$$\|T_{ij} + T\| > 2 - \delta(\eta)$$

holds for every $1 \leq i \leq n$ and $1 \leq j \leq p$.

Fix $1 \leq i \leq n$ and $1 \leq j \leq p$. By the definition of the operator norm we can find $y_{ij} \in S_Y$ such that $2 - \delta(\eta) < \|T_{ij}(y_{ij}) + T(y_{ij})\|$. By the Hahn-Banach theorem we can find $x_{ij}^* \in S_{X^*}$ such that

$$2 - \delta(\eta) < x_{ij}^*(f_i(y_{ij})z_j + T(y_{ij})).$$

Up to a change of sign we can assume with no loss of generality that $f_i(y_{ij}) \geq 0$. Since all the elements in the above inequality are norm-one elements we get that $f_i(y_{ij}) > 1 - \delta(\eta)$. Moreover, since $f_i(y_i) = 1$ we get that $\|y_i + y_{ij}\| > 2 - \delta(\eta)$, and the uniform convexity implies that $\|y_i - y_{ij}\| < \eta$. This implies that

$$2 - \delta(\eta) < x_{ij}^*(f_i(y_{ij})z_j + T(y_{ij})) \leq x_{ij}^*(f_i(y_i)z_j + T(y_i)) + 2\|y_i - y_{ij}\|$$

$$\leq \|z_j + T(y_i)\| + 2\eta.$$

Since $\{y_1, \ldots, y_n\}$ is a η-net in S_Y and $\{z_1, \ldots, z_p\}$ is a η-net in S_Z we conclude that

$$\|z + T(y)\| > 2 - \delta(\eta) - 4\eta > 2 - \varepsilon$$

holds for every $z \in S_Z$ and every $y \in S_Y$.

Let us conclude from here the desired result. To this end, take arbitrary $z \in S_Z$ and $y \in S_Y$, and take $t_1, t_2 \in [0, 1]$ such that $t_1 + t_2 = 1$. Let us estimate $\|t_1 z + t_2 T(y)\|$. Assume with

no loss of generality that $t_1 \geq t_2$ (the other case runs similar). Then

$$\|t_1 z + t_2 T(y)\| = \|t_1(z + T(y)) + (t_2 - t_1)T(y)\| \geq t_1\|z + T(y)\| - |t_2 - t_1|\|T(y)\|$$
$$\geq t_1(2 - \varepsilon) + t_2 - t_1 = t_1 + t_2 - t_1\varepsilon \geq 1 - \varepsilon.$$

Observe that this proves in particular that $\|T(y)\| \geq 1 - \varepsilon$ holds for every $y \in S_Y$ and, in consequence, $\|T(y)\| \geq (1 - \varepsilon)\|y\|$ for every $y \neq 0$.

Now, given $z \in Z \setminus \{0\}$ and $y \in Y \setminus \{0\}$, we get that

$$\frac{\|z + T(y)\|}{\|z\| + \|T(y)\|} = \left\| \frac{\|z\|}{\|z\| + \|T(y)\|} \frac{z}{\|z\|} + \frac{\|T(y)\|}{\|z\| + \|T(y)\|} \frac{T(y)}{\|T(y)\|} \right\| > 1 - \varepsilon,$$

from where $\|z + T(y)\| > (1 - \varepsilon)(\|z\| + \|T(y)\|) > (1 - \varepsilon)(\|z\| + (1 - \varepsilon)\|y\|) > (1 - \varepsilon)^2(\|z\| + \|y\|)$, and the lemma is proved.   □

Remark 3.3 Observe that, from the last part of the above proof, the following holds true: Given two Banach spaces X and Y with Y finite dimensional, the following assertions are equivalent:

(1) For every finite dimensional subspace Z of X and every $\varepsilon > 0$ we can find a norm-one operator $T : Y \longrightarrow X$ such that

$$\|z + T(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $y \in Y$ and every $z \in Z$.
(2) For every finite subsets $\{z_1, \ldots, z_n\} \subseteq S_X$ and $\{y_1, \ldots, y_m\} \subseteq S_Y$ and every $\varepsilon > 0$ there exists a norm-one operator $T : Y \longrightarrow X$ such that

$$\|z_i + T(y_j)\| > 2 - \varepsilon$$

holds for every $1 \leq i \leq n$ and $1 \leq j \leq m$.

We will use this remark throughout the text.

Now we are ready to prove the following necessary condition for a Banach space being universally octahedral.

Theorem 3.4 *Let X be a universally octahedral Banach space. Then, for every $\varepsilon > 0$, for every finite dimensional subspace Z of X and for every $n \in \mathbb{N}$, there exists an operator $T : \ell_\infty^n \longrightarrow X$ such that $\|T\| \leq 1$ and such that*

$$\|z + T(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $y \in \ell_\infty^n$ and every $z \in Z$.

Proof Observe that, given $x \in \mathbb{R}^n$, we have that $\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}}\|x\|_\infty$, so ℓ_∞^n is, for every $\varepsilon > 0$, $(1 + \varepsilon)$-isometric to a uniformly convex Banach space.

The proof is simple from now. Take $\varepsilon > 0$ and $n \in \mathbb{N}$, and take $p \in \mathbb{N}$ such that a suitable scalling of the formal identity $\phi : \ell_\infty^n \longrightarrow \ell_p^n$ satisfies

$$\sqrt{1 - \varepsilon}\|x\| \leq \|\phi(x)\| \leq \|x\| \ \forall x \in X.$$

Now set $Z \subseteq X$ be a finite dimensional subspace. Applying Lemma 3.2 we can find a bounded operator $T : \ell_p^n \longrightarrow X$ with $\|T\| \leq 1$ and such that

$$\|z + T(y)\| \geq \sqrt{1 - \varepsilon}(\|z\| + \|y\|)$$

holds for every $z \in Z$ and every $y \in Y$. Now $T \circ \phi$ is the desired operator.   □

Remark 3.5 A couple of remarks are pertinent.

(1) In the above proof we have only used that $L(\ell_p^n, X)$ is octahedral for every $n \in \mathbb{N}$ and every $1 < p < \infty$.
(2) Observe that this in particular implies that c_0 is finitely representable in X [2, Lemma 11.1.6], which implies that every Banach space is finitely representable in X. In particular ℓ_∞ is finitely representable in X, which implies that X has a trivial cotype [2, Theorem 11.1.14].

Now it is time to prove that the converse holds true.

Theorem 3.6 *Let X be a Banach space. Assume that, for every $\varepsilon > 0$, for every finite dimensional subspace Z of X and for every $n \in \mathbb{N}$, there exists an operator $T : \ell_\infty^n \longrightarrow X$ such that $\|T\| \leq 1$ and such that*

$$\|z + T(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $y \in \ell_\infty^n$ and every $z \in Z$.

Then, for every Banach space Y and for every subspace H of $L(Y, X)$ containing the finite rank operators, the norm of H is octahedral.

Proof Let Y be a non-zero Banach space and $H \subseteq L(Y, X)$ as in the hypothesis. In order to prove that H is octahedral pick $T_1, \ldots, T_n \in S_H$ and $\varepsilon > 0$, and let us find an element $\Psi \in S_H$ such that $\|T_i + \Psi\| > 2 - \varepsilon$ holds for every $1 \leq i \leq n$. This is enough by [13, Proposition 2.1]. In order to do so find, for every $i \in \{1, \ldots, n\}$, an element $y_i \in S_Y$ such that $\|T_i(y_i)\| > 1 - \varepsilon$. Set $Z := \text{span}\{T(y_i) : 1 \leq i \leq n\}$.

Set also $V := \text{span}\{y_1, \ldots, y_n\} \subseteq Y$. Since every Banach space is finitely representable in c_0 we can find an operator $\phi : V \longrightarrow c_0$ with $\|\phi(y_i)\| > 1 - \varepsilon$ for every i and such that $\|\phi\| < 1$. This operator can be extended by Theorem 2.2 to an operator $Q : Y \longrightarrow c_0$ which satisfies that $\|Q(y_i)\| > 1 - \varepsilon$ for every i and still $\|Q\| < 1$. By the definition of the c_0 norm we can find n large enough such that, if we define $P : c_0 \longrightarrow \ell_\infty^n$ the natural projection, we get $\|P(Q(y_i))\| > 1 - \varepsilon$ for every i.

Now, by the hypothesis, we can find an operator $T : \ell_\infty^n \longrightarrow X$ such that $\|T\| \leq 1$ and such that

$$\|z + T(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $y \in \ell_\infty^n$ and every $z \in Z$.

Now the desired operator is $\Psi := T \circ P \circ Q : Y \longrightarrow X$, which belongs to $F(Y, X) \subseteq H$. Observe that $\|\Psi\| \leq 1$. Moreover, given $1 \leq i \leq n$, we get

$$\|T_i + \Psi\| \geq \|T_i(y_i) + T(Q(P(y_i)))\| \geq (1 - \varepsilon)(\|T_i(y_i)\| + \|Q(P(y_i))\|)$$
$$\geq (1 - \varepsilon)(1 - \varepsilon + 1 - \varepsilon) = 2(1 - \varepsilon)^2.$$

Since ε was arbitrary we conclude the result.   □

As a consequence we get the following result.

Theorem 3.7 *Let X be a Banach space. The following are equivalent:*

(1) *For every Banach space Y and every $H \subseteq L(Y, X)$ containing the finite-rank operators, the space H is octahedral.*
(2) *X is universally octahedral.*

(3) *For every finite dimensional Banach space Y, the space $L(Y, X)$ is octahedral.*
(4) *For every finite dimensional uniformly convex Banach space Y, the space $L(Y, X)$ is octahedral.*
(5) *For every $1 < p < \infty$ and every $n \in \mathbb{N}$ the space $L(\ell_p^n, X)$ is octahedral.*
(6) *For every $\varepsilon > 0$, for every finite dimensional subspace Z of X and for every $n \in \mathbb{N}$, there exists an element $T : \ell_\infty^n \longrightarrow X$ with $\|T\| \leq 1$ and such that*

$$\|z + T(y)\| \geq (1 - \varepsilon)(\|z\| + \|y\|)$$

holds for every $y \in \ell_\infty^n$ and every $z \in Z$.

Proof (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5) are immediate, and (5)$\Rightarrow$(6) follows by Remark 3.5. Finally, (6)$\Rightarrow$(1) is Theorem 3.6. $\square$

Observe that condition (6) requires not only that ℓ_∞ is finitely representable in X, but also that ℓ_∞ must be finitely representable in a part of X which kind of ℓ_1-orthogonal to Z for any finite-dimensional subspace Z of X. This fact will become more clear in the following example.

Example 3.8 Let $X := \ell_\infty \oplus_1 \ell_1$. X is octahedral [13, Proposition 3.10] and clearly contains ℓ_∞ isometrically. However, we claim that X is not universally octahedral.

Indeed, assume by contradiction that X is universally octahedral. Let $Z := \text{span}\{(e_1, 0)\} \subseteq X = \ell_\infty \oplus_1 \ell_1$. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. By the above characterisation we can find $\phi : \ell_\infty^n \longrightarrow X$ with $\|\phi\| \leq 1$ and such that

$$\|(e_1, 0) \pm \phi(y)\| > (1 - \varepsilon)(1 + \|y\|)$$

holds for every $y \in \ell_\infty^n$.

Take $Q : X = \ell_\infty \oplus_1 \ell_1 \longrightarrow \ell_1$ the natural projection. Let us prove that $Q \circ \phi : \ell_\infty^n \longrightarrow \ell_1$ satisfies that

$$(1 - 2\varepsilon)\|y\| \leq \|Q(\phi(y))\| \leq \|y\|,$$

from where the arbitrariness of n and ε will imply that ℓ_∞ is finitely-representable in ℓ_1, which is a contradiction because ℓ_1 has cotype 2 and the finite representability of ℓ_∞ in a Banach space Z implies that Z fails to have cotype q for every $q < \infty$ [2, Theorem 11.1.14].

So take $x \in S_{\ell_\infty^n}$, call $\phi(x) := (a, b) \in \ell_\infty \oplus_1 \ell_1$ and notice that $\|a\|_\infty + \|b\|_1 \leq 1$. Note that

$$2(1 - \varepsilon) < \|(e_1, 0) \pm \phi(x)\| = \|(e_1 \pm a)\|_\infty + \|b\|_1.$$

It is direct computation that either $\|e_1 + a\|_\infty \leq 1$ or $\|e_1 - a\|_\infty \leq 1$. Assume without loss of generality that $\|e_1 + a\|_\infty \leq 1$. Now

$$2(1 - \varepsilon) \leq \|e_1 + a\|_\infty + \|b\|_1 \leq 1 + \|b\|_1,$$

which implies $\|b\|_1 \geq 1 - 2\varepsilon$.

This implies that $\|Q(\phi(x))\| \geq 1 - 2\varepsilon$. The arbitrariness of $x \in S_{\ell_\infty^n}$ forces that $\|Q(\phi(x))\| \geq 1 - 2\varepsilon$ holds for every $x \in S_{\ell_\infty^n}$, and a homogeneity argument yields that, for every $x \in \ell_\infty^n$, we get

$$(1 - 2\varepsilon)\|x\| \leq \|Q(\phi(x))\| \leq \|Q\|\|\phi\|\|x\| \leq \|x\|,$$

as desired.

Another exotic example is the following.

Example 3.9 Let $X := (\oplus_{i=1}^{\infty} \ell_{\infty}^i)_1$. It is immediate by the main characterisation that X is universally octahedral. Indeed, given $n \in \mathbb{N}$, $\varepsilon > 0$ and $\{z_1, \ldots, z_k\} \subseteq S_X$, it is enough by Remark 3.3 to find an operator $T : \ell_{\infty}^n \longrightarrow X$ with

$$\|z_i + T(y)\| > (1 - \varepsilon)(1 + \|y\|)$$

for every $1 \leq i \leq k$ and every $y \in S_{\ell_{\infty}^n}$. Up to a density argument we can assume that $z_i \in (\oplus_{i=1}^{\infty} \ell_{\infty}^i)$ have finite support (say contained in $\oplus_{i=1}^{p} \ell_{\infty}^i$). Take $q > \max\{p, n\}$. Now take the canonical inclusion operator $\phi : \ell_{\infty}^n \hookrightarrow \ell_{\infty}^q$. Take the canonical inclusion operator $j : \ell_{\infty}^q \hookrightarrow \oplus_{i=1}^{\infty} \ell_{\infty}^i$ by $j(x)(q) = x$ if $i = q$ and 0 otherwise. Now $T = j \circ \phi : \ell_{\infty}^n \longrightarrow X$ satisfies the desired requirement. In fact, given any $y \in \ell_{\infty}^n$ observe that, since the support of z_i and $T(y)$ are disjoint by construction, we obtain that

$$\|z_i + T(y)\| = \|z_i\| + \|T(y)\| = 1 + \|y\|$$

since T is an isometry. This proves that X is universally octahedral.

However X does not contain c_0 isomorphically. Indeed, $X = (c_0(\mathbb{N}, \ell_1^n))^*$ is a dual space, so if X contained c_0 then it would indeed contain ℓ_{∞} by [9, Theorem 6.39], which is impossible since X is clearly separable.

Let us end with an observation concerning the existence of L-orthogonal elements.

Remark 3.10 Let X be a Banach space. Following the notation of [21], we say that an element $u \in X^{**}$ is an *L-orthogonal element* if it satisfies

$$\|x + u\| = \|x\| + \|u\|$$

holds for every $x \in X$.

The existence of non-zero L-orthogonal elements is strongly connected with octahedral norms. It is a consequence of the Principle of Local Reflexivity (and explicitly mentioned in [11, Lemma 9.1]) that if X has a non-zero L-orthogonal element then the norm of X is octahedral. Moreover, the converse is true if X is separable [11, Lemma 9.1]. The question whether octahedrality implies the existence of non-zero L-orthogonals has remained open until the recent work [21], where many examples of octahedral spaces without any non-zero L-orthogonal element is exhibited.

A natural question at this point is whether or not there exists a Banach space X satisfying that, for every non-zero Banach space Y and for every $H \subseteq L(Y, X)$ containing $F(Y, X)$, the space H has non-zero L-orthogonal elements.

The answer is no. Indeed, given any Banach space X, taking $Y = \ell_2(I)$ and H to be the space of compact operators from Y to X (denoted by $K(Y, X)$), if H has a non-zero L-orthogonal element, then Y is isometrically isomorphic to a subspace of X^{**} by [21, Lemma 3.1], so it remains to take I big enough so that there is no injective mapping $\phi : I \longrightarrow X^{**}$ to conclude that $K(\ell_2(I), X)$ does not have any non-zero L-orthogonal element.

4 Examples

In this section we will analyse examples of universally octahedral Banach spaces. Observe that the condition in Theorem 3.6 is very difficult to check in a particular example. So, in order to provide examples where universal octahedrality holds, we give a criterion which implies it. In order to save notation, let us make the following definition.

Definition 4.1 Let X be a Banach space. We say that X is c_0-*octahedral* if, for every $x_1, \ldots, x_n \in S_X$ and every $\varepsilon > 0$, there exists a subspace $Y \subseteq X$ such that

(1) $\|x_i + y\| > (1 - \varepsilon)(1 + \|y\|)$ holds for every $y \in Y$ and;
(2) Y is isometric to c_0.

Remark 4.2 It is quite easy to prove (up to applying Remark 3.3) that c_0-octahedrality implies universal octahedrality. Moreover, a homogeneity argument similar to that of the end of Lemma 3.2 allows to show that a Banach space X is c_0-octahedral if, and only if, for every finite dimensional subspace $Z \subseteq X$ and every $\varepsilon > 0$, there exists a subspace $Y \subseteq X$ such that

(1) $\|z + y\| > (1 - \varepsilon)(\|z\| + \|y\|)$ holds for every $y \in Y$ and $z \in Z$ and;
(2) Y is isometric to c_0.

Example 4.3 By [19, Proposition 3.3] we have that if $\mathrm{Lip}_0(M)$ is octahedral (observe that $\mathrm{Lip}_0(M)$ is octahedral if, and only if, the predual $\mathcal{F}(M)$ has the strong diameter two property [6, Corollary 2.2], which is in turn equivalent to the fact that M is length [3, Theorem 1.5], which is the assumption of the above mentioned [19, Proposition 3.3]) then $\mathrm{Lip}_0(M)$ is c_0-octahedral (and consequently universally octahedral).

In general, there are Banach spaces which are universally octahedral but not c_0-octahedral, and one example is given in Example 3.9. Consequently, if X is universally octahedral it does not imply that X is c_0-octahedral. However, we can guarantee that all its ultrapowers are c_0-octahedral.

Proposition 4.4 *Let X be a Banach space and let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Then X is universally octahedral if, and only if, $X_{\mathcal{U}}$ is c_0-octahedral.*

Proof Let us start assuming that X is universally octahedral and let us prove that $X_{\mathcal{U}}$ is c_0-octahedral. To prove c_0-octahedrality, pick $(z_n^1), \ldots, (z_n^p) \in S_{X_{\mathcal{U}}}$. We can assume, up to changing of representative, that $\|z_n^i\| = 1$ holds for every $1 \le i \le p$ and $n \in \mathbb{N}$. Since X is universally octahedral then, for every $n \in \mathbb{N}$ we can find an operator $T_n : \ell_\infty^n \longrightarrow X$ with $\|T_n\| \le 1$ such that

$$\|z_n^i + T_n(x)\| \ge \left(1 - \frac{1}{n}\right)(1 + \|x\|)$$

holds for every $x \in \ell_\infty^n$. Now define $T : (\ell_\infty^n)_{\mathcal{U}} \longrightarrow X_{\mathcal{U}}$ by the equation

$$T((x_n)) := (T_n(x_n)) \ \forall (x_n) \in (\ell_\infty^n)_{\mathcal{U}}.$$

It is clear that $\|T\| \le 1$. Moreover, given $1 \le i \le p$ and $(x_n) \in (\ell_\infty^n)_{\mathcal{U}}$, we get

$$\|(z_n^i) + T((x_n))\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|z_n^i + T_n(x_n)\| \ge \lim_{\mathcal{U}} \left(1 - \frac{1}{n}\right)(1 + \|x_n\|)$$
$$= 1 + \lim_{\mathcal{U}} \|x_n\| = 1 + \|(x_n)\|_{\mathcal{U}},$$

from where $\|(z_n^i) + T((x_n))\|_{\mathcal{U}} = 1 + \|(x_n)\|_{\mathcal{U}}$. In particular, T is an isometry so, in order to finish the proof, it remains to show that $(\ell_\infty^n)_{\mathcal{U}}$ contains an isometric copy of c_0. Take $k \in \mathbb{N}$ and define $(x_n^k) \in (\ell_\infty^n)_{\mathcal{U}}$ by

$$x_n^k := \begin{cases} e_k & \text{if } n \ge k, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to prove that, given $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, for $n \geq k$ we obtain that $\| \sum_{i=1}^{k} \lambda_i x_n^i \| = \max_{1 \leq i \leq k} |\lambda_i|$, so

$$\left\| \sum_{i=1}^{k} \lambda_i (x_n^i) \right\|_{\mathcal{U}} = \lim_{\mathcal{U}} \left\| \sum_{i=1}^{k} \lambda_i x_n^i \right\| = \max_{1 \leq i \leq k} |\lambda_i|,$$

which proves that (x_n^k) is isometric to the c_0-basis. It remains to take $Y := \overline{\text{span}}\{T(x_n^k) : k \in \mathbb{N}\}$ to get the desired subspace.

For the converse, let $z_1, \ldots, z_q \in S_X$, $\varepsilon > 0$ and $n \in \mathbb{N}$, and let us find $T : \ell_\infty^n \longrightarrow X$ with $\|T\| \leq 1 + \varepsilon$ and such that

$$\|z_i + T(y)\| \geq (1 - \varepsilon)(1 + \|y\|)$$

holds for every $1 \leq i \leq q$. Call $j : X \longrightarrow X_{\mathcal{U}}$ the natural embedding by $j(x) := (x)_{\mathcal{U}}$. Take $\nu \in \mathbb{R}^+$ small enough such that $(1 + \nu)(1 - 3\nu)^{-1} < 1 + \varepsilon$ and such that $\nu(1 + \varepsilon) < \varepsilon$. Since $X_{\mathcal{U}}$ is c_0-octahedral we can find $Y \subseteq X_{\mathcal{U}}$ such that Y is isometric to c_0 and such that

$$\|j(z_i) + x_k\|_{\mathcal{U}} > \left(1 - \frac{\nu}{2}\right)(1 + \|(x_k)\|_{\mathcal{U}}) = \left(1 - \frac{\nu}{2}\right)\left(1 + \lim_{\mathcal{U}} \|x_k\|\right)$$

holds for every $(x_n) \in Y$. Since Y is isometric to c_0 there exists an isometry $\phi : \ell_\infty^n \longrightarrow Y$. Take $F \subseteq B_{\ell_\infty^n}$ a ν-net and observe that

$$\lim_{\mathcal{U}} \|z_i + \phi(z)(k)\| \geq \left(1 - \frac{\nu}{2}\right)(1 + \|z\|)$$

holds for every $z \in F$. Since F is finite we can find $A \in \mathcal{U}$ such that

$$(1 - \nu)(1 + \|z\|) \leq \|z_i + \phi(z)(k)\| \leq (1 + \nu)(1 + \|z\|)$$

and

$$(1 - \nu)\|z\| \leq \|\phi(z)(k)\| \leq (1 + \nu)\|z\|$$

holds for every $z \in F$ and every $k \in A$.

Now select any $k \in A$ and define a linear operator $T : \ell_\infty^n \longrightarrow X$ by $T(x) := \phi(x)(k)$ for every $x \in \ell_\infty^n$.

Observe, on the one hand, that given $z \in F$, we get

$$(1 - \nu)\|z\| \leq \|T(z)\| \leq (1 + \nu)\|z\|$$

and, since F is a ν-net of $B_{\ell_\infty^n}$ and $(1 + \nu)(1 - 3\nu)^{-1} < 1 + \varepsilon$, we get from Lemma 2.1 that

$$(1 - \varepsilon)\|z\| \leq \|T(z)\| \leq (1 + \varepsilon)\|z\|$$

holds for every $z \in S_{\ell_\infty^n}$. Now, given $x \in B_{\ell_\infty^n}$, take $z \in F$ such that $\|x - z\| < \nu$.

$$
\begin{aligned}
\|z_i + T(x)\| \geq \|z_i + T(z)\| - \|T(x - z)\| &> (1 - \nu)(1 + \|z\|) - (1 + \varepsilon)\|z - x\| \\
&\geq (1 - \nu)(1 + \|x\| - \|x - z\|) - (1 + \varepsilon)\nu \\
&> (1 - \nu)(1 + \|x\| - \nu) - (1 + \varepsilon)\nu \\
&\geq (1 - \nu)(1 + \|x\| - (1 + \|x\|)\nu) - \varepsilon \\
&\geq (1 - \nu)^2(1 + \|x\|) - (1 + \|x\|)\varepsilon \\
&= ((1 - \nu)^2 - \varepsilon)(1 + \|x\|).
\end{aligned}
$$

Now given $x \in \ell_\infty^n$ with $\|x\| > 1$ we finally get

$$
\begin{aligned}
\|z_i + T(x)\| &= \left\| \|x\| \left(z_i + T\left(\frac{x}{\|x\|} \right) \right) - (\|x\| - 1) z_i \right\| \\
&\geq \|x\| \left\| z_i + T\left(\frac{x}{\|x\|} \right) \right\| - (\|x\| - 1) \|z_i\| \\
&> \|x\| 2((1 - \nu)^2 - \varepsilon) - \|x\| + 1 \\
&= \|x\| (2 - 4\nu + 2\nu^2 - 2\varepsilon) - \|x\| + 1 \\
&= \|x\| (1 - 4\nu + 2\nu^2 - 2\varepsilon) + 1 \\
&> (1 - 4\nu + 2\nu^2 - 2\varepsilon)(\|x\| + 1),
\end{aligned}
$$

and the proof is finished.   □

Remark 4.5 Observe that in the above proof we have obtained in $X_{\mathcal{U}}$ c_0-octahedrality with the constant $\varepsilon = 0$. This is not surprising because J. D. Hardtke proved [16, Proposition 2.10] that, given any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, a Banach space X is octahedral if, and only if, $X_{\mathcal{U}}$ satisfies that for every $(z_n^1), \ldots, (z_n^k) \in S_{X_{\mathcal{U}}}$ there exists $(z_n) \in S_{X_{\mathcal{U}}}$ such that $\|(z_n^k) + (z_n)\| = 2$.

We end the paper by proving that L_1-preduals with octahedral norms are actually universally octahedral, which enlarges our class of Banach spaces which are universally octahedral.

Theorem 4.6 *Let X be an L_1-predual. If X is octahedral then X is universally octahedral.*

For the proof we will need the following lemma which says that octahedral L_1-preduals are close to be c_0-octahedral when we look at the bidual space.

Lemma 4.7 *Let X be an L_1 predual. If X is octahedral then, for every $x_1, \ldots, x_k \in S_X$ and every $\varepsilon > 0$ there exists a sequence $\{x_n^{**}\} \subseteq S_{X^{**}}$ such that*

(1) *$\{x_n^{**}\}$ is isometric to the usual basis of c_0 and,*
(2) *$\|x_i + v\| > 1 - \varepsilon + \|v\|$ holds for every $1 \leq i \leq k$ and $v \in \overline{\text{span}}\{x_n^{**} : n \in \mathbb{N}\}$.*

Proof Since X is octahedral then clearly X does not have any point of Fréchet differentiability. Since X is an L_1-predual we get that X has the Daugavet property [5, Theorem 2.4].

By [22, Theorem 3.2] we conclude that, for every $\varepsilon > 0$, the set

$$
V_i^\pm := \{ e^* \in \text{ext}\,(B_{X^*}) : e^*(\pm x_i) > 1 - \varepsilon \}
$$

is infinite for every $i \in \{1, \ldots, k\}$, where $\text{ext}\,(B_{X^*})$ stands for the set of extreme points of B_{X^*}.

Consequently we can construct, by an inductive argument, sequences $\{e_{in}^{\pm *}\}_{n \in \mathbb{N}}$ of elements such that $e_{in}^{\pm *} \in V_i^\pm$ for every $n \in \mathbb{N}$ and $1 \leq i \leq k$ and such that $e_{in}^{\pm *}$ and $e_{jm}^{\pm *}$ are linearly independent if $(i, n) \neq (j, m)$ and, moreover, $e_{ij}^{+ *}$ and $e_{ij}^{- *}$ are linearly independent too for every i, j.

Since $X^* = L_1(\mu)$ for certain μ it follows that, for every $1 \leq i \leq k$ and every $n \in \mathbb{N}$, there exists an L-projection $P_{in}^\pm : X^* \longrightarrow X^*$ such that $X^* := \mathbb{R} e_{in}^{\pm *} \oplus_1 \ker(P_{in}^\pm)$. Now set, for every $n \in \mathbb{N}$, $P_n := \sum_{i=1}^k (P_{in}^+ + P_{in}^-)$. By [22, Lemma 2.2] we get that $P_n : X^* \longrightarrow X^*$ is an L-projection with $B_{P_n(X^*)} = \text{aconv}\{e_{1n}^{\pm *}, \ldots, e_{kn}^{\pm *}\}$ (consequently $P_n(X^*) = \text{span}\{e_{1n}^{\pm *}, \ldots, e_{kn}^{\pm *}\}$ is isometrically ℓ_1^{2k}) and such that $\ker(P_n) = \bigcap_{i=1}^k \ker(P_{in}^+) \cap \ker(P_{in}^-)$. Consequently $X^* := P_n(X^*) \oplus_1 \ker(P_n) = \text{span}\{e_{1n}^{\pm *}, \ldots, e_{kn}^{\pm *}\} \oplus_1 \ker(P_n)$.

Given $n \in \mathbb{N}$ define $x_n^{**} : X^* \longrightarrow \mathbb{R}$ by the equation

$$x_n^{**}\left(\sum_{i=1}^{k}(\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*}) + z\right) = \sum_{i=1}^{k}\lambda_i^+ + \lambda_i^-$$

for all $\sum_{i=1}^{k}(\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*}) + z \in \text{span}\{e_{1n}^{\pm*}, \ldots, e_{kn}^{\pm*}\} \oplus_1 \ker(P_n) = X^*$. We claim that $\|x_n^{**}\| \leq 1$. Indeed, given $\sum_{i=1}^{k}(\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*}) + z \in \text{span}\{e_{1n}^{\pm*}, \ldots, e_{kn}^{\pm*}\} \oplus_1 \ker(P_n) = X^*$, we have that

$$\left|x_n^{**}\left(\sum_{i=1}^{k}(\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*}) + z\right)\right| = \left|\sum_{i=1}^{k}\lambda_i^+ + \lambda_i^-\right| \leq \sum_{i=1}^{k}|\lambda_i^+| + |\lambda_i^-|$$

$$\leq \left\|\sum_{i=1}^{k}\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*}\right\|$$

$$\leq \left\|\sum_{i=1}^{k}\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*}\right\| + \|z\|$$

$$= \left\|\sum_{i=1}^{k}\lambda_i^+ e_{in}^{+*} + \lambda_i^- e_{in}^{-*} + z\right\|,$$

which proves that $\|x_n^{**}\| \leq 1$. Let us prove that $\{x_n^{**}\}$ has the desired properties.

Now it is time to prove that $\{x_n^{**}\}$ is isometric to the c_0 basis. To this end pick $\lambda_1, \ldots \lambda_n \in \mathbb{R}$ and let us estimate the norm of $\sum_{i=1}^{n}\lambda_i x_i^{**}$. On the one hand, given $1 \leq i \leq n$ such that $\max_{1 \leq j \leq n}|\lambda_j| = |\lambda_i|$, set $\sigma_i := \text{sign}(\lambda_i)$ (with the convention $\text{sign}(0) = 1$). Moreover, given $1 \leq i \leq n$, define $\tau_i := +$ if $\sigma_i = 1$ and $\tau_i := -$ if $\sigma(i) = -1$. Then

$$\left\|\sum_{i=1}^{n}\lambda_i x_i^{**}\right\| \geq \sum_{i=1}^{n}\lambda_i x_i^{**}(\sigma_i e_{in}^{\tau_i *}) = \lambda_i \sigma_i = |\lambda_i|,$$

because clearly $e_{in} \in \bigcap_{j \neq i}\ker(P_j)$.

For the inequality from above, we can apply again [22, Lemma 2.2] to derive that $P := \sum_{j=1}^{n}P_j = \sum_{j=1}^{n}\sum_{i=1}^{k}(P_{ij}^+ + P_{ij}^-)$ is an L-projection, that $B_{P(X^*)} = \text{aconv}\{e_{ij}^{\pm*} : 1 \leq i \leq k, 1 \leq j \leq n\}$ and such that $\ker(P) = \bigcap_{j=1}^{n}\bigcap_{i=1}^{k}P_{ij}^{\pm}$. Now

$$\left\|\sum_{j=1}^{n}\lambda_j x_j^{**}\right\| = \sup_{x^* \in B_{X^*}}\sum_{j=1}^{n}\lambda_j x_j^{**}(x^*).$$

The decomposition $X^* = P(X^*) \oplus_1 \ker(P)$ implies that $B_{X^*} = \overline{\text{co}}(B_{P(X^*)} \cup B_{\ker(P)}) = \{\lambda u + (1 - \lambda)v : u \in B_{P(X^*)}, v \in B_{\ker(P)}, \lambda \in [0, 1]\}$. Since by construction $\sum_{j=1}^{n}\lambda_j x_j^{**}$ vanishes on $\ker(P)$ we get that

$$\sup_{x^* \in B_{X^*}}\sum_{j=1}^{n}\lambda_j x_j^{**}(x^*) = \sup_{x^* \in B_{P(X^*)}}\sum_{j=1}^{n}\lambda_j x_j^{**}(x^*)$$

Moreover, since $B_{P(X^*)} = \text{aconv}(\{e_{ij}^{\pm*} : 1 \leq i \leq k, 1 \leq j \leq n\})$ we get that

$$\sup_{x^* \in B_{P(X^*)}}\sum_{j=1}^{n}\lambda_j x_j^{**}(x^*) = \max_{1 \leq i \leq k, 1 \leq p \leq n}\left|\sum_{j=1}^{n}\lambda_j x_j^{**}(e_{ip}^{\pm*})\right|,$$

and since $x_j^{**}(e_{ip}^{\pm *}) = \delta_{jp}$ we concluce that the above maximum coincides with $\max\limits_{1 \leq j \leq n} |\lambda_j|$, as desired.

In order to finish the proof, take $v \in \mathrm{span}\{x_i^{**}\}$, so $v = \sum_{i=1}^{n} \lambda_i x_i^{**}$ with $\|v\| = \max\limits_{1 \leq i \leq n} |\lambda_i| = |\lambda_j|$ for some j. Let us prove that $\|x_i + v\| > 1 - \varepsilon + |\lambda_j|$ and the proof will be finished. To this end, let $1 \leq p \leq k$. If $\lambda_j \geq 0$ then

$$\left\| x_p + \sum_{i=1}^{n} \lambda_i x_i^{**} \right\| \geq x_p(e_{pj}^{+*}) + \sum_{i=1}^{n} \lambda_i x_i^{**}(e_{pj}^{+*}) > 1 - \varepsilon + \lambda_j = 1 - \varepsilon + |\lambda_j|.$$

On the other hand if $\lambda_j < 0$ we get

$$x_p(e_{pj}^{-*}) + \sum_{i=1}^{n} \lambda_i x_i^{**}(e_{pj}^{-*}) < -1 + \varepsilon + \lambda_j = -1 + \varepsilon - |\lambda_j| = -(1 - \varepsilon + |\lambda_j|),$$

from where $\|x_p + \sum_{i=1}^{n} \lambda_i x_i^{**}\| \geq 1 - \varepsilon + |\lambda_j|$ too, and the proof is finished. □

Proof of Theorem 4.6 Let $x_1, \ldots, x_k \in S_X$ and $\varepsilon > 0$. By Lemma 4.7 there exists a sequence $\{x_n^{**}\} \subseteq S_{X^{**}}$ such that $\|x_i + v\| > 1 - \varepsilon + \|v\|$ holds for every $1 \leq i \leq k$ and any $v \in \mathrm{span}\{x_n^{**} : n \in \mathbb{N}\}$, and such that $\{x_n^{**}\}$ is isometric to the c_0 basis.

Now let $n \in \mathbb{N}$ and take $\phi : \ell_\infty^n \longrightarrow \mathrm{span}\{x_j^{**} : j \in \mathbb{N}\}$ the canonical inclusion $\phi(e_j) := x_j^{**}$ for $1 \leq j \leq n$.

Now set $E := \mathrm{span}\{x_1, \ldots, x_k, x_1^{**}, \ldots, x_n^{**}\} \subseteq X^{**}$ and $\varepsilon > 0$. By the Principle of Local Reflexivity we can find an operator $P : E \longrightarrow X$ such that $P(e) = e$ for $e \in E \cap X$ and

$$(1 - \varepsilon)\|x\| \leq \|P(x)\| \leq \|x\|$$

holds for every $x \in E$. Now set $T := P \circ \phi : \ell_\infty^n \longrightarrow X$. Given $y \in \ell_\infty^n$ we have

$$
\begin{aligned}
\|x_i + T(y)\| = \|P(x_i) + P(\phi(y))\| = \|P(x_i + \phi(y))\| &\geq (1 - \varepsilon)(\|x_i + \phi(y)\|) \\
&\geq (1 - \varepsilon)(1 - \varepsilon + \|\phi(y)\|) \\
&= (1 - \varepsilon)^2(1 + \|y\|)
\end{aligned}
$$

The arbitrariness of ε and y yields the desired conclusion. □

Observe that we have proved that, given an L_1-predual X, we have that X is universally octahedral if, and only if, X is octahedral and if, and only if, X has the Daugavet property. See [22] for examples of L_1-preduals with the Daugavet property.

References

1. Abrahansen, T.A., Lima, V., Nygaard, O.: Remarks on diameter two properties. J. Convex Anal. **20**(2), 439–452 (2013)
2. Albiac, F., Kalton, N.J.: Topics in Banach Space Theory. Springer, Berlin (2006)
3. Avilés, A., Martínez-Cervantes, G.: Complete metric spaces with property (Z) are length spaces. J. Math. Anal. Appl. **473**(1), 334–344 (2019)
4. Avilés, A., Martínez-Cervantes, G., Rueda Zoca, A.: L-orthogonal elements and L-orthogonal sequences, to appear in Int. Math. Res. Not
5. Becerra, J., Martín, M.: The Daugavet property for Lindenstrauss spaces. In: Jesus, M.F.C., William, B.J. (eds.) Methods in Banach Space Theory. London Mathematical Society Lecture Note Series, vol. 337, pp. 91–96. (2006)
6. Becerra Guerrero, J., López-Pérez, G., Rueda Zoca, A.: Octahedral norms and convex combination of slices in Banach spaces. J. Funct. Anal. **266**(4), 2424–2435 (2014)
7. Becerra Guerrero, J., López-Pérez, G., Rueda Zoca, A.: Octahedral norms in spaces of operators. J. Math. Anal. Appl. **427**(1), 171–184 (2015)
8. Ciaci, S., Langemets, J., Lissitsin, A.: A characterization of Banach spaces containing $\ell_1(\kappa)$ via ball-covering properties, to appear in Isr. J. Math
9. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Pelant, J., Zizler, V.: Functional Analysis and Infinite dimensional Geometry. CMS Books in Mathematics. Springer, New York (2001)
10. Godefroy, G.: Metric characterization of first Baire class linear forms and octahedral norms. Studia Math. **95**(1), 1–15 (1989)
11. Godefroy, G., Kalton, N.J.: The ball topology and its applications. Contemp. Math. **85**, 195–238 (1989)
12. Haller, R., Langemets, J., Nadel, R.: Stability of average roughness, octahedrality, and strong diameter 2 properties of Banach spaces with respect to absolute sums. Banach J. Math. Anal. **12**(1), 222–239 (2018)
13. Haller, R., Langemets, J., Põldvere, M.: On duality of diameter 2 properties. J. Conv. Anal. **22**(2), 465–483 (2015)
14. Haller, R., Langemets, J., Põldvere, M.: Rough norms in spaces of operators. Math. Nachr. **290**(11–12), 1678–1688 (2017)
15. Harmand, P., Werner, D., Werner, W.: M-Ideals in Banach Spaces and Banach algebras. Lecture Notes in Mathematics, vol. 1547. Springer, Berlin (1993)
16. Hardtke, J.D.: Summands in locally almost square and locally octahedral spaces. Acta Comment. Univ. Tartu. Math. **22**(1), 149–162 (2018)
17. Langemets, J., Lima, V., Rueda Zoca, A.: Octahedral norms in tensor products of Banach spaces. Q. J. Math. **68**(4), 1247–1260 (2017)
18. Langemets, J., López-Pérez, G.: Bidual octahedral renormings and strong regularity in Banach spaces. J. Inst. Math. Jussieu **20**(2), 569–585 (2021)
19. Langemets, J., Rueda Zoca, A.: Octahedral norms in duals and biduals of Lipschitz-free spaces. J. Funct. Anal. **279**, 108557 (2020)
20. Lindenstrauss, J.: Extension of Compact Operators, vol. 48. Memoirs of the American Mathematical Society, Providence (1964)
21. López-Pérez, G., Rueda Zoca, A.: L-orthogonality, octahedrality and Daugavet property in Banach spaces. Adv. Math. **383**, 107719 (2021)
22. Martín, M., Rueda Zoca, A.: Daugavet property in projective symmetric tensor products of Banach spaces. Banach J. Math. Anal. **16**, 35 (2022)