Fuglede's theorem in generalized Orlicz–Sobolev spaces

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Abstract

In this paper, we show that Orlicz–Sobolev spaces $W^{1,\varphi}(\Omega)$ can be characterized with the ACL- and ACC-characterizations. ACL stands for absolutely continuous on lines and ACC for absolutely continuous on curves. Our results hold under the assumptions that $C^1(\Omega)$ functions are dense in $W^{1,\varphi}(\Omega)$, and $\varphi(x, \beta) \geq 1$ for some $\beta > 0$ and almost every $x \in \Omega$. The results are new even in the special cases of Orlicz and double phase growth.

Keywords Generalized Orlicz space · Orlicz–Sobolev space · ACC · ACL · Modulus

Mathematics Subject Classification 46E35

1 Introduction

In this paper, we study the ACL- and ACC-characterizations of Orlicz–Sobolev spaces $W^{1,\varphi}(\Omega)$, where φ has generalized Orlicz growth and $\Omega \subset \mathbb{R}^n$ is an open set. ACL stands for absolutely continuous on lines and ACC for absolutely continuous on curves. Special cases of Orlicz growth include the constant exponent case $\varphi(x, t) = t^p$, the Orlicz case $\varphi(x, t) = \varphi(t)$, the variable exponent case $\varphi(x, t) = t^{p(x)}$, and the double phase case $\varphi(x, t) = t^p + a(x)t^q$. Generalized Orlicz and Orlicz–Sobolev spaces on $\mathbb{R}^n$ have been recently studied for example in [4, 5, 13], and in a more general setting in [1, 12]. ACC-characterization has been used for example in [9] to study properties of capacities in the variable exponent case.

The ACL-characterization of the classical constant exponent Sobolev spaces was given by Nikodym [11]. It states that a function $u \in L^p(\Omega)$ belongs to $W^{1,p}(\Omega)$ if and only if it has representative $\tilde{u}$ that is absolutely continuous on almost every line segment parallel to the coordinate axes and the classical partial derivatives of $\tilde{u}$ belong to $L^p(\Omega)$. Moreover the classical partial derivatives are equal to the weak partial derivatives. Fuglede [6] gave a finer version of this characterization, namely, the ACC-characterization. The ACC-characterization states that a function $u \in L^p(\Omega)$ belongs to $W^{1,p}(\Omega)$ if and only if it has representative $\tilde{u}$ that is absolutely continuous on every rectifiable curve outside a family of zero p -modulus and the (classical) partial derivatives $\tilde{u}$ belong to $L^p(\Omega)$.

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In [8], it was shown that variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ also has the ACL- and ACC-characterizations, if the exponent satisfies suitable conditions and $C^1(\Omega)$ functions are dense. In Section 8 of [12], it was shown that the results hold in the space $W^{1,\varphi}(\mathbb{R}^n)$, if $C^1(\mathbb{R}^n)$ -functions are dense and φ satisfies certain conditions. In this paper, we generalize the results even further. We show that the results hold for the space $W^{1,\varphi}(\Omega)$, and we do so using fewer assumptions than in [8] or [12]. There are two assumptions we need to make: First that $C^1(\Omega)$ functions are dense in $W^{1,\varphi}(\Omega)$. And second, that $\varphi(x, \beta) \geq 1$ for some $\beta > 0$ and almost every $x \in \Omega$. To best of our knowledge, the results are new even in the special cases of Orlicz and double phase growth.

We base our approach on [8], but make some modifications to both make the results more general and simplify some of the results. One difference is that we use a slightly different definition for the modulus of a curve family. Our definition of is based on the norm, while the definition in [8] is based on the modular. The reason for defining the modulus differently has to do with the fact that modular convergence is a weaker concept than norm convergence. Another difference with [8] is that we do not use the theory of capacities. This has two advantages: First, the use of capacities would force us to make some extra assumptions on φ . Second, we can prove our results directly in $W^{1,\varphi}(\Omega)$, for any $\Omega \subset \mathbb{R}^n$, whereas in [8] the results are first proven in the case $\Omega = \mathbb{R}^n$, and this case is then used to prove the results for $\Omega \subset \mathbb{R}^n$.

The structure of this paper is as follows: Sect. 2 covers preliminaries about generalized Orlicz and Orlicz–Sobolev spaces. In Sect. 3 we define and discuss the modulus of a curve family. In Sect. 4 we prove two lemmas, which we will need in order to prove our main results. In Sect. 5 we prove our main results, the ACL- and ACC-characterizations of $W^{1,\varphi}(\Omega)$.

Let us say a few words about why one might be interested in studying ACL- or ACC-characterizations. One reason is that ACL-functions have classical partial derivatives almost everywhere, and ACC-functions are a subclass of ACL-functions under the assumptions we use. ACL- and ACC-functions also have some nice closure properties, for example the product and the maximum of two ACC-functions is an ACC-function, and the composition of an ACC-function with a Lipschitz function is an ACC-function, and similar results hold for ACL. Another reason for studying ACC-characterization in particular is that the theory can be applied in a more general setting. In a general metric space, the concept of direction does not really make sense, so the concept of an ACL-functions cannot be used. But the concept of an ACC-function can still be defined, and has been used in the study of Newtonian spaces on general metric spaces, see [3, 10] for example.

2 Preliminaries

Throughout this paper, we assume that $\Omega \subset \mathbb{R}^n$ is an open set. The following definitions are as in [7], which we use as a general reference to background theory in generalized Orlicz spaces.

Definition 2.1 We say that $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$ is a *weak Φ -function*, and write $\varphi \in \Phi_w(\Omega)$, if the following conditions hold

- For every measurable $f : \Omega \rightarrow [-\infty, \infty]$ the function $x \mapsto \varphi(x, |f|)$ is measurable, and for every $x \in \Omega$ the function $t \mapsto \varphi(x, t)$ is non-decreasing.
- $\varphi(x, 0) = \lim_{t \rightarrow 0+} \varphi(x, t) = 0$ and $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$ for every $x \in \Omega$.
- The function $t \mapsto \frac{\varphi(x,t)}{t}$ is L -almost increasing for $t > 0$ uniformly in Ω . “Uniformly” means that L is independent of x .

If $\varphi \in \Phi_w(\Omega)$ is additionally convex and left-continuous, then φ is a *convex Φ -function*, and we write $\varphi \in \Phi_c(\Omega)$.

Two functions φ and ψ are *equivalent*, $\varphi \simeq \psi$, if there exists $L \geq 1$ such that $\psi(x, \frac{t}{L}) \leq \varphi(x, t) \leq \psi(x, Lt)$ for every $x \in \Omega$ and every $t > 0$. Equivalent Φ -functions give rise to the same space with comparable norms.

We define the left-inverse of φ by setting

$$\varphi^{-1}(x, \tau) := \inf\{t \geq 0 : \varphi(x, t) \geq \tau\}.$$

2.1 Assumptions

We state some assumptions for later reference.

(A0) There exists $\beta \in (0, 1)$ such that $\varphi(x, \beta) \leq 1 \leq \varphi(x, 1/\beta)$ for almost every x .

(A1) There exists $\beta \in (0, 1)$ such that, for every ball B and a.e. $x, y \in B \cap \Omega$,

$$\beta\varphi^{-1}(x, t) \leq \varphi^{-1}(y, t) \quad \text{when } t \in \left[1, \frac{1}{|B|}\right].$$

(A2) For every $s > 0$ there exist $\beta \in (0, 1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\beta\varphi^{-1}(x, t) \leq \varphi^{-1}(y, t)$$

for almost every $x, y \in \Omega$ and every $t \in [h(x) + h(y), s]$.

(aInc) $_p$ There exist $L \geq 1$ such that $t \mapsto \frac{\varphi(x, t)}{t^p}$ is L -almost increasing in $(0, \infty)$.

(aDec) $_q$ There exist $L \geq 1$ such that $t \mapsto \frac{\varphi(x, t)}{t^q}$ is L -almost decreasing in $(0, \infty)$.

We say that φ satisfies (aInc), if it satisfies (aInc) $_p$ for some $p > 1$. Similarly, φ satisfies (aDec), if it satisfies (aDec) $_q$ for some $q > 1$. We write (Inc) if the ratio is increasing rather than just almost increasing, similarly for (Dec). See [7, Table 7.1] for an interpretation of the assumptions in some special cases.

2.2 Generalized Orlicz spaces

We recall some definitions. We denote by $L^0(\Omega)$ the set of measurable functions in Ω .

Definition 2.2 Let $\varphi \in \Phi_w(\Omega)$ and define the *modular* ϱ_φ for $f \in L^0(\Omega)$ by

$$\varrho_\varphi(f) := \int_\Omega \varphi(x, |f(x)|) dx.$$

The *generalized Orlicz space*, also called Musielak–Orlicz space, is defined as the set

$$L^\varphi(\Omega) := \left\{ f \in L^0(\Omega) : \lim_{\lambda \rightarrow 0^+} \varrho_\varphi(\lambda f) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^\varphi(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \leq 1 \right\}. \quad (2.1)$$

If the set is clear from the context we abbreviate $\|f\|_{L^\varphi(\Omega)}$ by $\|f\|_\varphi$.

The following lemma is a direct consequence of the proof of [7, Theorem 3.3.7].

Lemma 2.3 If (f_i) is a Cauchy sequence in $L^\varphi(\Omega)$ such that the pointwise limit $f(x) := \lim_{i \rightarrow \infty} f_i(x)$ ($\pm\infty$ allowed) exists for almost every $x \in \Omega$, then f is the limit of (f_i) in $L^\varphi(\Omega)$.

Definition 2.4 A function $u \in L^\varphi(\Omega)$ belongs to the Orlicz–Sobolev space $W^{1,\varphi}(\Omega)$ if its weak partial derivatives $\partial_1 u, \dots, \partial_n u$ exist and belong to the space $L^\varphi(\Omega)$. For $u \in W^{1,\varphi}(\Omega)$, we define the norm

$$\|u\|_{W^{1,\varphi}(\Omega)} := \|u\|_\varphi + \|\nabla u\|_\varphi.$$

Here $\|\nabla u\|_\varphi$ is short for $\|\nabla u\|_\varphi$. Again, if Ω is clear from the context, we abbreviate $\|u\|_{W^{1,\varphi}(\Omega)}$ by $\|u\|_{1,\varphi}$.

Many of our results need the assumption that $C^1(\Omega)$ -functions are dense in $W^{1,\varphi}(\Omega)$. A sufficient condition is given by [7, Theorem 6.4.7], which states that $C^\infty(\Omega)$ -functions are dense in $W^{1,\varphi}(\Omega)$, if φ satisfies (A0), (A1), (A2) and (aDec). By [7, Lemma 4.2.3], (A2) can be omitted, if Ω is bounded.

3 Modulus of a family of curves

By a curve, we mean any continuous function $\gamma : I \rightarrow \mathbb{R}^n$, where $I = [a, b]$ is a closed interval. If a curve γ is rectifiable, we may assume that $I = [0, \ell(\gamma)]$, where $\ell(\gamma)$ denotes the length of γ . We denote the image of γ by $\text{Im}(\gamma)$, and by $\Gamma_{\text{rect}}(\Omega)$ we denote the family of all rectifiable curves γ such that $\text{Im}(\gamma) \subset \Omega$. Let $\Gamma \subset \Gamma_{\text{rect}}(\Omega)$. We say that a Borel function $u : \Omega \rightarrow [0, \infty]$ is Γ -admissible, if

$$\int_\gamma u \, ds \geq 1$$

for all $\gamma \in \Gamma$, where ds denotes the integral with respect to curve length. We denote the set of all Γ -admissible functions by $F_{\text{adm}}(\Gamma)$.

Definition 3.1 Let $\Gamma \subset \Gamma_{\text{rect}}(\Omega)$. Let $\varphi \in \Phi_w(\Omega)$. We define the φ -modulus of Γ by

$$M_\varphi(\Gamma) := \inf_{u \in F_{\text{adm}}(\Gamma)} \|u\|_\varphi.$$

If $F_{\text{adm}}(\Gamma) = \emptyset$, we set $M_\varphi(\Gamma) := \infty$. A family of curves Γ is *exceptional*, if $M_\varphi(\Gamma) = 0$.

The definition above is as in [10]. The following lemma gives some useful properties of the modulus. Items (a) and (b) are items (a) and (c) of [10, Lemma 4.5], and item (c) is a consequence of [10, Proposition 4.8]. To use the lemma, we must check that $L^\varphi(\Omega)$ satisfies conditions (P0), (P1), (P2) and (RF) stated at the beginning of section 2 in [10]. The conditions (P0), (P1) and (P2) are easy to check. For (RF) to hold, there must exist $c \geq 1$ such that

$$\left\| \sum_{i=1}^{\infty} u_i \right\|_\varphi \leq \sum_{i=1}^{\infty} c^i \|u_i\|_\varphi$$

holds for non-negative $u_i \in L^\varphi(\Omega)$. This is an easy consequence of [7, Lemma 3.2.5], which states that there exists $c \geq 1$ such that

$$\left\| \sum_{i=1}^{\infty} u_i \right\|_\varphi \leq c \sum_{i=1}^{\infty} \|u_i\|_\varphi.$$

Lemma 3.2 *Let $\varphi \in \Phi_w(\Omega)$, then the φ -modulus has the following properties:*

- (a) *if $\Gamma_1 \subset \Gamma_2$, then $M_\varphi(\Gamma_1) \leq M_\varphi(\Gamma_2)$,*
- (b) *if $M_\varphi(\Gamma_i) = 0$ for every $i \in \mathbb{N}$, then $M_\varphi(\bigcup_{i=1}^\infty \Gamma_i) = 0$.*
- (c) *$M_\varphi(\Gamma) = 0$ if and only if there exists a non-negative Borel function $u \in L^\varphi(\Omega)$ such that $\int_\gamma u \, ds = \infty$ for every $\gamma \in \Gamma$.*

In [6], the L^p -modulus was originally defined by

$$M_p(\Gamma) := \inf_{u \in F_{adm}(\Gamma)} \int_\Omega u^p \, dx.$$

This differs from Definition 3.1 in that the infimum is taken over the modulars of admissible functions instead of their norms. A similar approach was taken in the variable exponent case in [8]. Following the original approach, we could have defined the modulus by

$$\tilde{M}_\varphi(\Gamma) := \inf_{u \in F_{adm}(\Gamma)} \int_\Omega \varphi(x, u(x)) \, dx.$$

In the case $\varphi(x, t) = t^p$, where $1 \leq p < \infty$, we have $\tilde{M}_\varphi(\Gamma) = M_\varphi(\Gamma)^p$. Thus in this special case $\tilde{M}_\varphi(\Gamma) = 0$ if and only if $M_\varphi(\Gamma) = 0$. Since we are only interested in whether a family of curves is exceptional or not, in this case it does not matter whether we use M_φ or $\tilde{M}_\varphi$.

In the general case, the situation is somewhat more complicated. Let $\varphi \in \Phi_w(\Omega)$. By [7, Corollary 3.2.8], if $\|u\|_\varphi < 1$, then $\varrho_\varphi(u) \lesssim \|u\|_\varphi$. Thus $M_\varphi(\Gamma) = 0$ implies $\tilde{M}_\varphi(\Gamma) = 0$. The converse implication does not necessarily hold, as the next example shows, which is the main reason for using norms instead of modulars in Definition 3.1.

Example 3.3 Define $\varphi \in \Phi_w(\mathbb{R}^2)$ by

$$\varphi(x, t) := \begin{cases} 0 & \text{if } t \leq 1, \\ t - 1 & \text{if } t > 1. \end{cases}$$

For $y \in [0, 1]$, let $\gamma_y : [0, 1] \rightarrow \mathbb{R}^2$, $z \mapsto (y, z)$, and let $\Gamma := \{\gamma_y : y \in [0, 1]\}$. Let $u = 1$ everywhere. Then

$$\int_\gamma u(s) \, ds = 1$$

for every $\gamma \in \Gamma$, and therefore $u \in F_{adm}(\Gamma)$. Since $\varphi(x, u(x)) = 0$ for every $x \in \mathbb{R}^2$, we have $\varrho_\varphi(u) = 0$, and thus $\tilde{M}_\varphi(\Gamma) = 0$.

To show that $M_\varphi(\Gamma) > 0$, suppose on the contrary, that $M_\varphi(\Gamma) = 0$. Then by Lemma 3.2(c) there exists some $v \in L^\varphi(\mathbb{R}^2)$ such that $\int_\gamma v \, ds = \infty$ for every $\gamma \in \Gamma$. Thus

$$\int_{[0,1]} v(y, z) \, dz = \int_{\gamma_y} v \, ds = \infty$$

for every $y \in [0, 1]$. Let $\lambda > 0$. Since $\varphi(x, t) \geq t - 1$ for every $x \in \mathbb{R}^2$ and every $t \geq 0$, Fubini's theorem implies that

$$\int_{\mathbb{R}^2} \varphi(x, \lambda v(x)) \, dx \geq \int_{[0,1]} \int_{[0,1]} \lambda v(y, z) - 1 \, dz \, dy = \infty - \int_{[0,1]} \int_{[0,1]} 1 \, dz \, dy = \infty$$

Since $\lambda > 0$ was arbitrary, it follows by (2.1) that $\|v\|_\varphi = \infty$. But this is impossible, since $v \in L^\varphi(\mathbb{R}^2)$. Thus the assumption that $M_\varphi(\Gamma) = 0$ must be wrong and $M_\varphi(\Gamma) > 0$.

Note that if $\varphi \in \Phi_w(\Omega)$ satisfies $(\text{aDec})_q$ for $1 \leq q < \infty$, then, by [7, Lemma 3.2.9] (since φ satisfies $(\text{aInc})_1$ by definition) we have

$$\|u\|_\varphi \lesssim \max\{\varrho_\varphi(u), \varrho_\varphi(u)^{\frac{1}{q}}\}.$$

Thus, if φ satisfies (aDec) , then $\tilde{M}_\varphi(\Gamma) = 0$ if and only if $M_\varphi(\Gamma) = 0$.

4 Fuglede's lemma

Lemma 4.1 (Fuglede's lemma) *Let $\varphi \in \Phi_w(\Omega)$, and let (u_i) be a sequence of non-negative Borel functions converging to zero in $L^\varphi(\Omega)$. Then there exists a subsequence (u_{i_k}) and an exceptional set $\Gamma \subset \Gamma_{\text{rect}}(\Omega)$ such that for all $\gamma \notin \Gamma$ we have*

$$\lim_{k \rightarrow \infty} \int_\gamma u_{i_k} ds = 0.$$

Proof Let $(v_k) := (u_{i_k})$ be a subsequence of (u_i) , such that

$$\|v_k\|_\varphi \leq 2^{-k}.$$

Let $\Gamma \subset \Gamma_{\text{rect}}(\Omega)$ be the family of curves γ , such that $\int_\gamma v_k ds \not\rightarrow 0$ as $k \rightarrow \infty$. For every $j \in \mathbb{N}$, let

$$w_j := \sum_{k=1}^j v_k.$$

Since every v_k is a non-negative Borel function, it follows that every w_j is also a non-negative Borel function. And since the sequence $(w_j(x))$ is increasing for every $x \in \Omega$, it follows that the limit $w(x) := \lim_{j \rightarrow \infty} w_j(x)$ (possibly ∞) exists. By [7, Corollary 3.2.5], if $j < m$, then

$$\|w_m - w_j\|_\varphi = \left\| \sum_{k=j+1}^m v_k \right\|_\varphi \leq \sum_{k=j+1}^m \|v_k\|_\varphi \leq \sum_{k=j+1}^m 2^{-k} < 2^{-j},$$

which implies that (w_j) is a Cauchy sequence in $L^\varphi(\Omega)$. By Lemma 2.3, w is the limit of (w_j) in $L^\varphi(\Omega)$, which implies that $w \in L^\varphi(\Omega)$, and therefore $\|w\|_\varphi < \infty$.

Suppose now that $\gamma \in \Gamma$. Then

$$\int_\gamma w ds = \sum_{k=1}^{\infty} \int_\gamma v_k ds = \infty,$$

because $\sum_{k=1}^{\infty} \int_\gamma v_k ds < \infty$ would imply that $\lim_{k \rightarrow \infty} \int_\gamma v_k ds = 0$. Thus w/m is Γ -admissible for every $m \in \mathbb{N}$. Since $\lim_{m \rightarrow \infty} \|w/m\|_\varphi = \lim_{m \rightarrow \infty} \|w\|_\varphi/m = 0$, we have $M_\varphi(\Gamma) = 0$. $\square$

Let $E \subset \Omega$. We denote by Γ_E the set of all curves $\gamma \in \Gamma_{\text{rect}}(\Omega)$, such that the $E \cap \text{Im}(\gamma)$ is nonempty.

The next lemma is, in a sense, a combination of [8, Lemma 3.1] and [2, Lemma 5.1]. The former of the aforementioned lemmas states that if $C^1(\mathbb{R}^n)$ functions are dense in the variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ < \infty$, then Γ_E is exceptional whenever $E \subset \mathbb{R}^n$ is of capacity zero. The latter states that if $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfies (aInc)

and (aDec), then for every Cauchy sequence in $C(\mathbb{R}^n) \cap W^{1,\varphi}(\mathbb{R}^n)$ there exists a subsequence which converges pointwise outside a set of zero capacity. The beginning of the proof of our lemma is similar to [2, Lemma 5.1], but then we use the ideas from [8, Lemma 3.1] and modify the proof to replace convergence outside a set of capacity zero by convergence outside a set E , such that Γ_E is exceptional. The reason that we do not simply prove a direct generalization of [8, Lemma 3.1] and then use [2, Lemma 5.1] is, that our proof avoids the use of capacities. This has two advantages: First, we can drop the assumptions (aInc) and (aDec). And second, our new result works in $W^{1,\varphi}(\Omega)$ for any $\Omega \subset \mathbb{R}^n$, while in [8, Lemma 3.1] and [2, Lemma 5.1] we have $\Omega = \mathbb{R}^n$.

Lemma 4.2 *Let $\varphi \in \Phi_w(\Omega)$ and let (u_i) be a Cauchy sequence of functions in $C^1(\Omega) \cap W^{1,\varphi}(\Omega)$. Then there exists a set E and a subsequence (u_{i_k}) such that $M_\varphi(\Gamma_E) = |E| = 0$ and (u_{i_k}) converges pointwise everywhere outside E .*

Proof By [7, Lemma 3.3.6] there exists a subsequence of (u_i) that converges pointwise almost everywhere. Thus we can choose a subsequence $(v_k) := (u_{i_k})$, such that (v_k) converges pointwise almost everywhere, and

$$\|v_{k+1} - v_k\|_{1,\varphi} < 4^{-k}$$

for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, let $f_k := 2^k(v_{k+1} - v_k) \in C^1(\Omega) \cap W^{1,\varphi}(\Omega)$. For every $j \in \mathbb{N}$, let

$$g_j := \sum_{k=1}^j |f_k| \quad \text{and} \quad h_j := \sum_{k=1}^j |\nabla f_k|.$$

Since the sequences $(g_j(x))$ and $(h_j(x))$ are increasing for every $x \in \Omega$, the limits $g(x) := \lim_{j \rightarrow \infty} g_j(x)$ and $h(x) := \lim_{j \rightarrow \infty} h_j(x)$ (possibly ∞) exist. Since the functions g_j are continuous, g is a Borel function. If $j < m$, then by [7, Corollary 3.2.5]

$$\|g_m - g_j\|_\varphi \lesssim \sum_{k=j+1}^m \|f_k\|_\varphi \leq \sum_{k=j+1}^\infty \|f_k\|_{1,\varphi} < \sum_{k=j+1}^\infty 2^{-k} = 2^{-j},$$

which implies that (g_j) is a Cauchy sequence in $L^\varphi(\Omega)$. By Lemma 2.3, g is the limit of (g_j) in $L^\varphi(\Omega)$. Similarly, since

$$\|h_m - h_j\|_\varphi \lesssim \sum_{k=j+1}^m \|\nabla f_k\|_\varphi \leq \sum_{k=j+1}^\infty \|f_k\|_{1,\varphi} < 2^{-j},$$

we find that h is the limit of h_j in $L^\varphi(\Omega)$.

Since $f_k \in C^1(\Omega)$, for any $k \in \mathbb{N}$ we have

$$||f_k(x)| - |f_k(y)|| \leq |f_k(x) - f_k(y)| \leq \int_\gamma |\nabla f_k| ds$$

for every $x, y \in \Omega$ and any $\gamma \in \Gamma_{rect}(\Omega)$ containing x and y . Thus for every $j \in \mathbb{N}$ we have

$$|g_j(x) - g_j(y)| \leq \sum_{k=1}^j ||f_k(x)| - |f_k(y)|| \leq \sum_{k=1}^j \int_\gamma |\nabla f_k| ds = \int_\gamma h_j ds, \quad (4.1)$$

for every $x, y \in \Omega$ and any $\gamma \in \Gamma_{rect}(\Omega)$ containing x and y .

Denote by E the set of points $x \in \Omega$ such that the sequence $(v_k(x))$ does not converge. Since (v_k) converges pointwise almost everywhere, we have $|E| = 0$. It is easy to see that if $x \in E$, then $x \in \{|f_k| > 1\}$ for infinitely many $k \in \mathbb{N}$, and therefore $g(x) = \infty$. Thus

$$E \subset E_\infty := \{x \in \Omega : g(x) = \infty\},$$

and $\Gamma_E \subset \Gamma_{E_\infty}$. Next we construct a set $\Gamma \subset \Gamma_{\text{rect}}(\Omega)$ such that $\Gamma_{E_\infty} \subset \Gamma$ and $M_\varphi(\Gamma) = 0$. It then follows by Lemma 3.2(a) that $M_\varphi(\Gamma_E) = M_\varphi(\Gamma_{E_\infty}) = 0$.

By Lemma 4.1, considering a subsequence if necessary, we find an exceptional set $\Gamma_1 \subset \Gamma_{\text{rect}}(\Omega)$ such that

$$\lim_{j \rightarrow \infty} \int_\gamma h - h_j \, ds = 0$$

for every $\gamma \in \Gamma_{\text{rect}}(\Omega) \setminus \Gamma_1$. Let

$$\Gamma_2 := \left\{ \gamma \in \Gamma_{\text{rect}}(\Omega) : \int_\gamma g \, ds = \infty \right\} \quad \text{and} \quad \Gamma_3 := \left\{ \gamma \in \Gamma_{\text{rect}}(\Omega) : \int_\gamma h \, ds = \infty \right\}.$$

For every $m \in \mathbb{N}$, the function g/m is Γ_2 admissible, hence $M_\varphi(\Gamma_2) \leq \|g\|_\varphi/m$. Thus it follows that $M_\varphi(\Gamma_2) = 0$. Similarly, we see that $M_\varphi(\Gamma_3) = 0$. Let $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. By Lemma 3.2(b) $M_\varphi(\Gamma) = 0$.

It remains to show that $\Gamma_{E_\infty} \subset \Gamma$. Suppose that $\gamma \in \Gamma_{\text{rect}}(\Omega) \setminus \Gamma$. Since $\gamma \notin \Gamma_2$, there must exist some $y \in \text{Im}(\gamma)$ with $g(y) < \infty$. By (4.1), for any $x \in \text{Im}(\gamma)$ and any $j \in \mathbb{N}$ we have

$$g_j(x) \leq g_j(y) + |g_j(x) - g_j(y)| \leq g_j(y) + \int_\gamma h_j \, ds.$$

Since $\gamma \notin \Gamma_1$, it follows that

$$\lim_{j \rightarrow \infty} \int_\gamma h_j \, ds = \int_\gamma h \, ds,$$

where the right-hand side is finite because $\gamma \notin \Gamma_3$. Thus we have

$$g(x) = \lim_{j \rightarrow \infty} g_j(x) \leq \lim_{j \rightarrow \infty} \left(g_j(y) + \int_\gamma h_j \, ds \right) = g(y) + \int_\gamma h \, ds < \infty.$$

Since $x \in \text{Im}(\gamma)$ was arbitrary, it follows that $\gamma \notin \Gamma_{E_\infty}$. And since $\gamma \notin \Gamma$ was arbitrary, it follows that $\Gamma_{E_\infty} \subset \Gamma$. $\square$

5 Fuglede's Theorem

We begin this section by defining some notations. Let $k \in \{1, 2, \dots, n\}$. If $z \in \mathbb{R}$ and $y = (y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ we define

$$(y, z)_k := (y_1, \dots, y_{k-1}, z, y_k, \dots, y_{n-1}) \in \mathbb{R}^n.$$

For every $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we write $\tilde{x}_k := (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. With these notations, we have $x = (\tilde{x}_k, x_k)_k$. We define $\tilde{\Omega}_k \subset \mathbb{R}^{n-1}$ by

$$\tilde{\Omega}_k := \{\tilde{x}_k : x \in \Omega\} = \{y \in \mathbb{R}^{n-1} : (y, z)_k \in \Omega \text{ for some } z \in \mathbb{R}\}.$$

The set $\tilde{\Omega}_k$ is, in a sense, the orthogonal projection of Ω into the space $\{x \in \mathbb{R}^n : x_k = 0\}$, but strictly speaking this is not true, since a projection is a function $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$, but

$\tilde{\Omega}_k \subset \mathbb{R}^{n-1}$. For every $y \in \tilde{\Omega}_k$, we let $Z_k(y) \subset \mathbb{R}$ be the set of points z , such that $(y, z)_k \in \Omega$. Note that $\Omega = \{(y, z)_k : y \in \tilde{\Omega}_k \text{ and } z \in Z_k(y)\}$.

Since we will be using Lebesgue measures with different dimensions simultaneously, we will use subscripts to differentiate them, i.e. m -dimensional measure will be denoted by $|\cdot|_m$.

Definition 5.1 We say that $u : \Omega \rightarrow \mathbb{R}$ is *absolutely continuous on lines*, $u \in ACL(\Omega)$, if it is absolutely continuous on almost every line segment in Ω parallel to the coordinate axes. More formally, let $k \in \{1, 2, \dots, n\}$ and let $E_k \subset \tilde{\Omega}_k$ be the set of points y such that the function

$$f_y : Z_k(y) \rightarrow [-\infty, \infty], \quad f_y(z) = u((y, z)_k)$$

is absolutely continuous on every compact interval $[a, b] \subset Z_k(y)$. Then $u \in ACL(\Omega)$ if and only if $|\tilde{\Omega}_k \setminus E_k|_{n-1} = 0$ for every k .

Let $u \in ACL(\Omega)$. Absolute continuity implies that the classical partial derivative $\partial_k u$ of $u \in ACL(\Omega)$ exist for every $x \in \Omega$ such that $\tilde{x}_k \in E_k$. Since $|\tilde{\Omega}_k \setminus E_k|_{n-1} = 0$, it follows by Fubini's theorem that $\partial_k u$ exists for almost every $x \in \Omega$. Another application of Fubini's theorem shows that the classical partial derivative is equal to the weak partial derivative, see [14, Theorem 2.1.4]. Since the partial derivatives exist almost everywhere, it follows that the gradient ∇u exists almost everywhere. A function $u \in ACL(\Omega)$ is said to belong to $ACL^\varphi(\Omega)$, if $|\nabla u| \in L^\varphi(\Omega)$.

The following lemma follows immediately from the definitions of $L^\varphi(\Omega)$, $ACL^\varphi(\Omega)$ and $W^{1,\varphi}(\Omega)$.

Lemma 5.2 If $\varphi \in \Phi_w(\Omega)$, then $ACL^\varphi(\Omega) \cap L^\varphi(\Omega) \subset W^{1,\varphi}(\Omega)$.

Definition 5.3 For any $u : \Omega \rightarrow \mathbb{R}$, we define $\Gamma_{NAC}(u) \subset \Gamma_{rect}(\Omega)$ as the family of curves $\gamma : [0, \ell(\gamma)] \rightarrow \Omega$ such that $u \circ \gamma$ is not absolutely continuous on $[0, \ell(\gamma)]$. If $M_\varphi(\Gamma_{NAC}(u)) = 0$, then we say that u is *absolutely continuous on curves*, $u \in ACC(\Omega)$.

In the next lemma, we show that $ACC(\Omega)$ is a subset of $ACL(\Omega)$, if φ satisfies a suitable condition.

Lemma 5.4 Let $\varphi \in \Phi_w(\Omega)$ and assume that φ satisfies the following condition:

$$\text{there exist } \beta > 0 \text{ such that } \varphi(x, \beta) \geq 1 \text{ for almost every } x \in \Omega. \quad (5.1)$$

Then

$$ACC(\Omega) \subset ACL(\Omega).$$

Remark 5.5 Note that (A0) implies (5.1), but not the other way around, since we do not assume that $\varphi(x, 1/\beta) \leq 1$. We also note (5.1) is equivalent to

$$\text{there exist } \beta > 0 \text{ and } \delta > 0 \text{ such that } \varphi(x, \beta) \geq \delta \text{ for almost every } x \in \Omega. \quad (5.2)$$

It is clear that (5.1) is just a special case of (5.2) with $\delta = 1$. It is also clear that (5.2) implies (5.1), if $\delta > 1$. Suppose then, that φ satisfies (5.2) with $0 < \delta < 1$. Then

$$\frac{\delta}{\beta} \leq \frac{\varphi(x, \beta)}{\beta} \quad (5.3)$$

for almost every $x \in \Omega$. By (aInc)₁ (which φ satisfies by definition of Φ_w), there exist a constant $a \geq 1$ such that

$$\frac{\varphi(x, \beta)}{\beta} \leq a \frac{\varphi(x, t)}{t} \quad (5.4)$$

for almost every $x \in \Omega$ and every $t \geq \beta$. Choosing $t := a\beta/\delta > \beta$, it follows from (5.3) and (5.4) that $\varphi(x, a\beta/\delta) \geq 1$ for almost every $x \in \Omega$, and therefore φ satisfies (5.1). Thus the choice $\delta = 1$ in (5.1) has no special meaning, except for making notations simpler by getting rid of δ .

Proof of Lemma 5.4 Let $u \in \text{ACC}(\Omega)$, and let $k \in \{1, \dots, n\}$ and let $E_k \subset \mathbb{R}^{n-1}$ be as in Definition 5.1. By Lemma 3.2, there exists a non-negative Borel function $v \in L^\varphi(\Omega)$ such that $\int_\gamma v \, ds = \infty$ for every $\gamma \in \Gamma_{\text{NAC}}(u)$. For every $y \in \tilde{\Omega}_k \setminus E_k$, let $I(y) \subset Z_k(y)$ be some compact interval such that v is not absolutely continuous on $I(y)$, and let $\gamma_y : [0, |I(y)|_1] \rightarrow \Omega$ be a parametrization of $I(y)$. Since $\gamma_y \in \Gamma_{\text{NAC}}(u)$, it follows that $\int_{I(y)} v((y, z)_k) \, dz = \int_{\gamma_y} v(s) \, ds = \infty$.

From (5.3) (with $\delta = 1$) and (5.4) we get

$$\varphi(x, t) \geq \frac{t}{a\beta}$$

for almost every $x \in \Omega$ and every $t \geq \beta$. Since $\varphi(x, t) \geq 0$, it follows that

$$\varphi(x, t) \geq \frac{t}{a\beta} - \frac{1}{a} \quad (5.5)$$

for almost every $x \in \Omega$ and every $t \geq 0$. Let $\lambda > \|v\|_\varphi$. By (2.1) and Fubini's theorem we have

$$\begin{aligned} 1 &\geq \int_\Omega \varphi\left(x, \frac{v(x)}{\lambda}\right) dx = \int_{\tilde{\Omega}_k} \int_{Z_k(y)} \varphi\left((y, z)_k, \frac{v((y, z)_k)}{\lambda}\right) dz dy \\ &\geq \int_{\tilde{\Omega}_k \setminus E_k} \int_{I(y)} \varphi\left((y, z)_k, \frac{v((y, z)_k)}{\lambda}\right) dz dy. \end{aligned} \quad (5.6)$$

By (5.5) we have

$$\int_{I(y)} \varphi\left((y, z)_k, \frac{v((y, z)_k)}{\lambda}\right) dz \geq \int_{I(y)} \frac{v((y, z)_k)}{a\beta\lambda} dz - \int_{I(y)} \frac{1}{a} dz.$$

Since $\int_{I(y)} v((y, z)_k) \, dz = \infty$, the first integral on the right-hand side is infinite, and since $I(y)$ is compact, the second integral is finite. Thus

$$\int_{I(y)} \varphi\left((y, z)_k, \frac{v((y, z)_k)}{\lambda}\right) dz = \infty.$$

Inserting this into (5.6), we get

$$\begin{aligned} 1 &\geq \int_{\tilde{\Omega}_k \setminus E_k} \int_{I(y)} \varphi\left((y, z)_k, \frac{v((y, z)_k)}{\lambda}\right) dz dy \\ &= \int_{\tilde{\Omega}_k \setminus E_k} \infty dy. \end{aligned}$$

This is possible only if $|\tilde{\Omega}_k \setminus E_k|_{n-1} = 0$. Thus $u \in \text{ACL}(\Omega)$. $\square$

The next example shows that the assumption (5.1) in the preceding lemma is not redundant.

Example 5.6 Let $\Omega = \mathbb{R}^2$. For $x = (y, z) \in \mathbb{R}^2$, let

$$\varphi(x, t) := \begin{cases} t & \text{if } y = 0, \\ 0 & \text{if } y \neq 0 \text{ and } t \leq |y|^{-1}, \\ t & \text{if } y \neq 0 \text{ and } t > |y|^{-1}. \end{cases}$$

It easily follows from [7, Theorem 2.5.4] that $\varphi \in \Phi_w(\mathbb{R}^2)$. Define $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$u(y, z) := \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y = 0, \\ 2 & \text{if } y > 0. \end{cases}$$

It is trivial that $u \notin ACL(\mathbb{R}^2)$. It is however the case that $u \in ACC(\mathbb{R}^2)$.

It is easy to see, that $\Gamma_{MAC}(u) = \Gamma_E$, where $E := \{(y, z) \in \mathbb{R}^2 : y = 0\}$. Define $v : \mathbb{R}^2 \rightarrow [0, \infty]$ by

$$v(y, z) := \begin{cases} \infty & \text{if } y = 0, \\ |y|^{-1} & \text{if } y \neq 0. \end{cases}$$

Since the set

$$\{(y, z) \in \mathbb{R}^2 : v(y, z) > r\} = \{(y, z) \in \mathbb{R}^2 : |y| < r^{-1}\}$$

is open for every $r \in \mathbb{R}$, it follows that v is a Borel function. Fix $\gamma \in \Gamma_E$. For every $a \in [0, \ell(\gamma)]$, we write $(y_a, z_a) := \gamma(a)$. Now, there exists some $b \in [0, \ell(\gamma)]$ with $y_b = 0$. Since γ is parametrized by arc-length, we have

$$|y_a| = |y_a - y_b| \leq |\gamma(a) - \gamma(b)| \leq |a - b|$$

for every $a \in [0, \ell(\gamma)]$. If $a \neq b$, then $v(\gamma(a)) \geq |a - b|^{-1}$, since if $y_a = 0$, then $v(\gamma(a)) = \infty$, and if $y_a \neq 0$, then $v(\gamma(a)) = |y_a|^{-1} \geq |a - b|^{-1}$. Thus

$$\int_{\gamma} v \, ds = \int_0^b v(\gamma(a)) \, da + \int_b^{\ell(\gamma)} v(\gamma(a)) \, da \geq \int_0^b \frac{1}{|a - b|} \, da + \int_b^{\ell(\gamma)} \frac{1}{|a - b|} \, da = \infty.$$

Since this holds for all $\gamma \in \Gamma_E$, by Lemma 3.2(c), to show that $M_{\varphi}(\Gamma_E) = 0$, it suffices to show that $v \in L^{\varphi}(\mathbb{R}^2)$. If $x = (y, z)$ and $y \neq 0$, then $\varphi(x, v(x)) = \varphi(x, |y|^{-1}) = 0$. Thus $\varphi(x, v(x)) = 0$ almost everywhere, and $\varrho_{\varphi}(v) = 0$. By (2.1), it follows that $\|v\|_{\varphi} \leq 1$, and therefore $v \in L^{\varphi}(\mathbb{R}^2)$.

We know that ∇u exists for every $u \in ACL(\Omega)$. Thus, if φ satisfies (5.1), then Lemma 5.4 implies that ∇u exists for every $u \in ACC(\Omega)$. We say that $u \in ACC^{\varphi}(\Omega)$, if $u \in ACC(\Omega)$ and $\nabla u \in L^{\varphi}(\Omega)$.

Theorem 5.7 (Fuglede's theorem) *Let $\varphi \in \Phi_w(\Omega)$ satisfy (5.1). If $C^1(\Omega)$ -functions are dense in $W^{1,\varphi}(\Omega)$, then $u \in W^{1,\varphi}(\Omega)$ if and only if $u \in L^{\varphi}(\Omega)$ and it has a representative that belongs to $ACC^{\varphi}(\Omega)$. In short*

$$ACC^{\varphi}(\Omega) \cap L^{\varphi}(\Omega) = W^{1,\varphi}(\Omega).$$

Proof By Lemmas 5.2 and 5.4, we have

$$ACC^{\varphi}(\Omega) \cap L^{\varphi}(\Omega) \subset ACL^{\varphi}(\Omega) \cap L^{\varphi}(\Omega) \subset W^{1,\varphi}(\Omega).$$

Thus it suffices to show that $W^{1,\varphi}(\Omega) \subset ACC^{\varphi}(\Omega)$. Since $|\nabla u| \in L^{\varphi}(\Omega)$ whenever $u \in W^{1,\varphi}(\Omega)$, we only have to show that $W^{1,\varphi}(\Omega) \subset ACC(\Omega)$.

Suppose that $u \in W^{1,\varphi}(\Omega)$. Let (u_i) be a sequence of functions in $C^1(\Omega) \cap W^{1,\varphi}(\Omega)$ converging to u in $W^{1,\varphi}(\Omega)$. By Lemma 4.2, passing to a subsequence if necessary, we may assume that (u_i) converges pointwise everywhere, except in a set E with $M_{\varphi}(\Gamma_E) = |E|_n = 0$. Let $\tilde{u}(x) := \liminf_{i \rightarrow \infty} u_i(x)$ for every $x \in \Omega$. Since the functions u_i are continuous, it follows that $\tilde{u}$ is a Borel function. Since $u_i(x)$ converges for every $x \in \Omega \setminus E$, it follows that

$\tilde{u}(x) = \lim_{i \rightarrow \infty} u_i(x)$ for $x \in \Omega \setminus E$. By Lemma 2.3, $u_i \rightarrow \tilde{u}$ in $L^\varphi(\Omega)$, and it follows that $\tilde{u} = u$ almost everywhere.

Since $u_i \rightarrow u$ in $W^{1,\varphi}(\Omega)$ we may assume, considering a subsequence if necessary, that

$$\|\nabla u_{i+1} - \nabla u_i\|_\varphi < 2^{-i}$$

for every $i \in \mathbb{N}$. Since

$$u_i = u_1 + \sum_{j=1}^{i-1} (u_{j+1} - u_j),$$

we have $|\nabla u_i| \leq g_i$ for every $i \in \mathbb{N}$, where

$$g_i = |\nabla u_1| + \sum_{j=i}^{i-1} |\nabla u_{j+1} - \nabla u_j|.$$

Since the sequence $(g_i(x))$ is increasing for every $x \in \Omega$, the limit $g(x) := \lim_{i \rightarrow \infty} g_i(x)$ (possibly ∞) exists. Since the functions g_i are continuous, g is a Borel function. For every $m > n$ we have

$$\|g_m - g_n\|_\varphi = \left\| \sum_{j=n}^{m-1} |\nabla u_{j+1} - \nabla u_j| \right\|_\varphi \lesssim \sum_{j=n}^{\infty} \|\nabla u_{j+1} - \nabla u_j\|_\varphi < \sum_{j=n}^{\infty} 2^{-j} < 2^{-n+1},$$

i.e. (g_i) is a Cauchy sequence in $L^\varphi(\Omega)$. Lemma 2.3 implies that $g_i \rightarrow g$ in $L^\varphi(\Omega)$.

Let

$$\Gamma_1 := \left\{ \gamma \in \Gamma_{\text{rect}}(\Omega) : \int_\gamma g \, ds = \infty \right\}.$$

Since g/j is Γ_1 -admissible for every $j \in \mathbb{N}$, we find that $M_\varphi(\Gamma_1) = 0$. By Lemma 4.1, passing to a subsequence if necessary, we find an exceptional set $\Gamma_2 \subset \Gamma_{\text{rect}}(\Omega)$, such that

$$\lim_{i \rightarrow \infty} \int_\gamma g - g_i \, ds = 0$$

for every $\gamma \in \Gamma_{\text{rect}}(\Omega) \setminus \Gamma_2$. The set Γ_2 has the following property: if $\gamma \in \Gamma_{\text{rect}}(\Omega) \setminus \Gamma_2$ and $0 \leq a \leq b \leq \ell(\gamma)$, then $\gamma|_{[a,b]} \in \Gamma_{\text{rect}}(\Omega) \setminus \Gamma_2$. The reason is that, since $g - g_i \geq 0$, we have

$$\int_\gamma g - g_i \, ds \geq \int_{\gamma|_{[a,b]}} g - g_i \, ds \geq 0,$$

and since the first term tends to zero, the middle term must also tend to zero. Let $\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_E$. By Lemma 3.2(b) $M_\varphi(\Gamma) = 0$.

We complete the proof by showing that $\tilde{u} \circ \gamma$ is absolutely continuous for every $\gamma \in \Gamma_{\text{rect}}(\Omega) \setminus \Gamma$. Let $k \in \mathbb{N}$ and for $j \in \{1, 2, \dots, k\}$, let $(a_j, b_j) \subset [0, \ell(\gamma)]$ be disjoint intervals. Since $\text{Im}(\gamma)$ does not intersect E , and $u_i \in C^1(\Omega)$ for every i , we have

$$\begin{aligned} \sum_{j=1}^k |\tilde{u}(\gamma(b_j)) - \tilde{u}(\gamma(a_j))| &= \lim_{i \rightarrow \infty} \sum_{j=1}^k |u_i(\gamma(b_j)) - u_i(\gamma(a_j))| \\ &\leq \limsup_{i \rightarrow \infty} \sum_{j=1}^k \int_{\gamma|_{[a_j, b_j]}} |\nabla u_i| \, ds. \end{aligned}$$

Using first the fact that $|\nabla u_i| \leq g_i$, and then the fact that $\gamma|_{[a_j, b_j]} \notin \Gamma_2$, we get

$$\limsup_{i \rightarrow \infty} \sum_{j=1}^k \int_{\gamma|_{[a_j, b_j]}} |\nabla u_i| \, ds \leq \limsup_{i \rightarrow \infty} \sum_{j=1}^k \int_{\gamma|_{[a_j, b_j]}} g_i \, ds = \sum_{j=1}^k \int_{\gamma|_{[a_j, b_j]}} g \, ds.$$

Thus

$$\sum_{j=1}^k |\tilde{u}(\gamma(b_j)) - \tilde{u}(\gamma(a_j))| \leq \sum_{j=1}^k \int_{\gamma|_{[a_j, b_j]}} g \, ds$$

Since $\gamma \notin \Gamma_1$, we have $g \circ \gamma \in L^1[0, \ell(\gamma)]$, which together with the inequality above implies that $\tilde{u} \circ \gamma$ is absolutely continuous on $[0, \ell(\gamma)]$. $\square$

We can combine Theorem 5.7 with Lemmas 5.2 and 5.4 to get the following corollary:

Corollary 5.8 *Let $\varphi \in \Phi_w(\Omega)$ satisfy (5.1). If $C^1(\Omega)$ -functions are dense in $W^{1,\varphi}(\Omega)$, then*

$$ACC^\varphi(\Omega) \cap L^\varphi(\Omega) = ACL^\varphi(\Omega) \cap L^\varphi(\Omega) = W^{1,\varphi}(\Omega).$$

As was noted at the end of Sect. 2, $C^\infty(\Omega)$ functions are dense in $W^{1,\varphi}(\Omega)$ if φ satisfies (A0), (A1), (A2) and (aDec). By Remark 5.5, (A0) implies (5.1). Thus Corollary 5.8 also holds with assumptions (A0), (A1), (A2) and (aDec), instead of (5.1) and density of $C^1(\Omega)$ -functions.

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