Modular Frobenius pseudo-varieties

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Received: 23 February 2021 / Accepted: 27 September 2021 / Published online: 28 October 2021
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Abstract

If $m \in \mathbb{N} \setminus \{0, 1\}$ and A is a finite subset of $\bigcup_{k \in \mathbb{N} \setminus \{0, 1\}} \{1, \dots, m-1\}^k$, then we denote by

$$\mathcal{C}(m, A) = \{S \in \mathcal{S}_m \mid s_1 + \dots + s_k - m \in S \text{ if } (s_1, \dots, s_k) \in S^k \text{ and } (s_1 \bmod m, \dots, s_k \bmod m) \in A\}.$$

In this work we prove that $\mathcal{C}(m, A)$ is a Frobenius pseudo-variety. We also show algorithms that allows us to establish whether a numerical semigroup belongs to $\mathcal{C}(m, A)$ and to compute all the elements of $\mathcal{C}(m, A)$ with a fixed genus. Moreover, we introduce and study three families of numerical semigroups, called of second-level, thin and strong, and corresponding to $\mathcal{C}(m, A)$ when $A = \{1, \dots, m-1\}^3$, $A = \{(1, 1), \dots, (m-1, m-1)\}$, and $A = \{1, \dots, m-1\}^2 \setminus \{(1, 1), \dots, (m-1, m-1)\}$, respectively.

Keywords Modular pseudo-varieties · Second-level numerical semigroups · Thin numerical semigroups · Strong numerical semigroups · Tree associated (with a modular pseudo-variety)

Mathematics Subject Classification 20M14

1 Introduction

Let $\mathbb{N}$ be the set of non-negative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ that is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. The *Frobenius number* of S , denoted by

A. M. Robles-Pérez, J. C. Rosales: Both authors are supported by the project MTM2017-84890-P (funded by Ministerio de Economía, Industria y Competitividad and Fondo Europeo de Desarrollo Regional FEDER) and by the Junta de Andalucía Grant Number FQM-343.

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$F(S)$, is the greatest integer that does not belong to S . The cardinality of $\mathbb{N} \setminus S$, denoted by $g(S)$, is the *genus* of S .

A *Frobenius pseudo-variety* is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfils the following conditions.

1. $\mathcal{P}$ has a maximum element (with respect to the inclusion order).
2. If $S, T \in \mathcal{P}$, then $S \cap T \in \mathcal{P}$.
3. If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \{F(S)\} \in \mathcal{P}$.

The *multiplicity* of a numerical semigroup S , denoted by $m(S)$, is the least positive integer that belongs to S . If m is a positive integer, then we denote by $\mathcal{S}_m = \{S \mid S \text{ is a numerical semigroup with } m(S) = m\}$.

Let $m \in \mathbb{N} \setminus \{0, 1\}$ and let A be a finite subset of $\bigcup_{k \in \mathbb{N} \setminus \{0, 1\}} \{1, \dots, m-1\}^k$ (where $X^k = X \times \dots \times X = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in X\}$). We denote by

$$\mathcal{C}(m, A) = \{S \in \mathcal{S}_m \mid s_1 + \dots + s_k - m \in S \text{ if } (s_1, \dots, s_k) \in S^k \text{ and } (s_1 \bmod m, \dots, s_k \bmod m) \in A\}.$$

Our main purpose in this work is to study the set $\mathcal{C}(m, A)$.

In Sect. 2 we show that $\mathcal{C}(m, A)$ is a Frobenius pseudo-variety with maximum element given by $\Delta(m) = \{0, m, \rightarrow\}$ (where the symbol $\rightarrow$ means that every integer greater than m belongs to $\Delta(m)$). Thus, we call the pseudo-varieties that arise in this way *modular Frobenius pseudo-varieties*. Also, we give an algorithm that allows us to establish whether a numerical semigroup belongs to $\mathcal{C}(m, A)$. In addition, with the help of the results in [9], we can arrange $\mathcal{C}(m, A)$ in a rooted tree and we find an algorithm to compute all the elements of $\mathcal{C}(m, A)$ with a fixed genus.

If X is a non-empty subset of $\mathbb{N}$, then we denote by $\langle X \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by X , that is,

$$\langle X \rangle = \{\lambda_1 x_1 + \dots + \lambda_n x_n \mid n \in \mathbb{N} \setminus \{0\}, x_1, \dots, x_n \in X, \lambda_1, \dots, \lambda_n \in \mathbb{N}\}.$$

It is well known (see Lemma 2.1 of [11]) that $\langle X \rangle$ is a numerical semigroup if and only if $\gcd(X) = 1$. If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle X \rangle$, then we say that X is a *system of generators* of M . Moreover, if $M \neq \langle Y \rangle$ for any subset $Y \subsetneq X$, then we say that X is a *minimal system of generators* of M . In Corollary 2.8 of [11] it is shown that each submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators and that such a system is finite. We denote by $\text{msg}(M)$ the minimal system of generators of M . The cardinality of $\text{msg}(M)$, denoted by $e(M)$, is the *embedding dimension* of M .

By applying Proposition 2.10 of [11], if S is a numerical semigroup, then we know that $e(S) \leq m(S)$. A numerical semigroup S has *maximal embedding dimension* if $e(S) = m(S)$. This family of numerical semigroups has been extensively studied (for instance, see [2] and [11]). Let us denote by $\mathcal{M}_m$ the set formed by the numerical semigroups that have maximal embedding dimension and with multiplicity m . It is easy to see that $\mathcal{M}_m = \mathcal{C}(m, \{1, \dots, m-1\}^2)$.

In Sects. 3, 4, and 5 we study the family $\mathcal{C}(m, A)$ for

- $A = \{1, \dots, m-1\}^3$,
- $A = \{(1, 1), \dots, (m-1, m-1)\}$,
- $A = \{1, \dots, m-1\}^2 \setminus \{(1, 1), \dots, (m-1, m-1)\}$,

respectively. Observe that, in a certain sense, these families are generalisation of $\mathcal{M}_m$.

To finish this introduction, we are going to comment several ideas (see [3,4,14]) that motivate the study of modular Frobenius pseudo-varieties.

First of all, observe that modular Frobenius pseudo-varieties are related to the non-homogeneous patterns with positive coefficients that involve in their constant parameter the multiplicity of the numerical semigroup (see [4]).

The notion of non-homogeneous pattern was introduced in [4] as a generalisation of the notion of homogeneous pattern [3]. Thus, a *linear pattern* $p(x_1, \dots, x_n)$ is an expression of the form $a_1x_1 + \dots + a_nx_n + a_0$, with $a_0 \in \mathbb{Z}$ (as usual, $\mathbb{Z}$ is the set of integers numbers) and $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$. In particular, the (linear) pattern p is *homogeneous* if $a_0 = 0$, and *non-homogeneous* if $a_0 \neq 0$.

On the other hand, it is said that a numerical semigroup S admits the homogeneous pattern p if $p(s_1, \dots, s_n) \in S$ for every non-increasing sequence $(s_1, \dots, s_n) \in S^n$. The corresponding definition for non-homogeneous patterns is a bit different: a numerical semigroup S admits the non-homogeneous pattern p if $p(s_1, \dots, s_n) \in S$ for every non-increasing sequence $(s_1, \dots, s_n) \in (S \setminus \{0\})^n$.

Having in mind that $M(S) = S \setminus \{0\}$ is an ideal of the numerical semigroup S (in fact, $M(S)$ is the maximal ideal of S), in [14] the concepts of the above paragraph were extended in the following way: an ideal I of a numerical semigroups S admits the pattern p if $p(s_1, \dots, s_n) \in S$ for every non-increasing sequence $(s_1, \dots, s_n) \in I^n$.

At this point we have the main difference between the above-mentioned papers and our proposal in this work: we do not keep the non-increasing condition (on the sequences in which we evaluate the pattern) in mind. In addition, we take sequences in sets A without structure (that is, A does not have to be an ideal of a numerical semigroup).

Now, let us denote by $\mathcal{S}_m(p)$ the family of numerical semigroups with multiplicity m that admit the pattern p . If we take the patterns $p_1 = 2x_1 + x_2 - m$ and $p_2 = x_1 + 2x_2 - m$, then $\mathcal{S}_m(p_1, p_2) = \mathcal{S}_m(p_1) \cap \mathcal{S}_m(p_2)$ (that is, the family of numerical semigroups which admit p_1 and p_2 simultaneously) is equal to $\mathcal{C}(m, A)$ for

$$A = \{(a_1, a_2, a_3) \in \{1, \dots, m-1\}^3 \mid a_1 \equiv a_2 \pmod{m}\},$$

being this one an easy example of the connection between modular Frobenius pseudo-varieties and families of numerical semigroups defined by non-homogeneous patterns (as considered in [4]).

A first question studied in [4,14] is the next one: if $p = a_1x_1 + \dots + a_nx_n + a_0$ is a non-homogeneous pattern, for which values of a_0 we have that the family $\mathcal{S}(p)$, of numerical semigroups that admit the pattern p , is non-empty? If the answer is affirmative, then it is said that p is an *admissible pattern*. In our case, we have imposed that $a_0 = -m$, where m will be the multiplicity of all the numerical semigroups, in order to get a relevant advantage: we want to build in a (more or less) easy and explicit way the Apéry set of the numerical semigroups belonging to $\mathcal{C}(m, A)$. As a consequence of this fact, we will be able to obtain extra information about such families of numerical semigroups.

Of course, in addition to $-m$, it is possible to consider other values that maintain the pseudo-variety structure. Thus, for $p = x_1 + \dots + x_n + a_0$, we have that, independently of the chosen set A , every numerical semigroup S , such that $a_0 \in S$, admits the pattern p .

Now, let us observe that, if $A_1 \subseteq A_2$, then $\mathcal{C}(m, A_2) \subseteq \mathcal{C}(m, A_1)$. This fact allow us to have sufficient conditions on the patterns $p = a_1x_1 + \dots + a_nx_n - km$, with $k \in \mathbb{N} \setminus \{0, 1\}$, from the results in [4,14]. For example, from [4, Theorem 4.1] or [14, Proposition 20], we can assert that $\mathcal{C}(m, A) \neq \emptyset$ if $n - k \geq 1$.

A second question that appears in [3,14] is about the *equivalence of patterns*. Briefly, the pattern p_1 induces the pattern p_2 if $\mathcal{S}(p_2) \subseteq \mathcal{S}(p_1)$ and, moreover, p_1 and p_2 are equivalent

patterns if they induce each other. Maybe the first result of this type is the equivalence between the homogeneous patterns $2x_1 - x_2$ and $x_1 + x_2 - x_3$, which correspond to the family of Arf numerical semigroups (see [5]). Since the set A has an important role (not to say the main role) in the families $\mathcal{C}(m, A)$, the large number of possibilities in the choice of A leads us to believe that this is an issue that deserves a new work.

By the way, in Sect. 4 we study the family of thin numerical semigroups, denoted by $\mathcal{T}_m$, and in Sect. 5 we consider the family of strong numerical semigroups, denoted by $\mathcal{R}_m$. These families are associated with the patterns $2x - m$ and $x + y - m$, respectively. However, the set A is different in each case and, as a consequence, we have that there is not a inclusion relation between both of them (see Examples 4.6 and 5.6). This fact may give an idea about the difficulty of obtaining similar results to those seen in [3, 14].

In any case, we can show simple results about the equivalence (or, at least, the inclusions) of the families $\mathcal{C}(m, A)$.

Remark 1.1 All the patterns $p = x_1 + \cdots + x_n - m$ are equivalent (independently of the chosen set A) if $n \geq m$, in which case we have that $\mathcal{C}(m, A) = \mathcal{S}_m$. Indeed, applying the pigeonhole principle, if $s_1, \dots, s_n$ are elements of $S \in \mathcal{S}_m$ such that $s_i \not\equiv 0 \pmod{m}$, $1 \leq i \leq n$, then there exist $i, j \in \{1, \dots, n\}$, with $i < j$, such that $s_i + \cdots + s_j = km$ for some $k \in \mathbb{N} \setminus \{0\}$. Consequently, $S \in \mathcal{C}(m, A)$.

Remark 1.2 Let us set $A = \{(1, 1), (3, 4)\}$, $B = \{(1, 1, 2), (1, 3, 1), (1, 3, 4)\}$, and $m \geq 5$. Then $\mathcal{C}(m, A) \subseteq \mathcal{C}(m, B)$. In order to verify this inclusion, we take $a_1, a_2, a_3 \in S \in \mathcal{C}(m, A)$.

- If $a_1 \equiv 1 \pmod{m}$, $a_2 \equiv 1 \pmod{m}$, $a_3 \equiv 2 \pmod{m}$, then $a_1 + a_2 + a_3 - m = (a_1 + a_2 - m) + a_3 \in S$.
- If $a_1 \equiv 1 \pmod{m}$, $a_2 \equiv 3 \pmod{m}$, $a_3 \equiv 1 \pmod{m}$, then $a_1 + a_2 + a_3 - m = (a_1 + a_3 - m) + a_2 \in S$.
- If $a_1 \equiv 1 \pmod{m}$, $a_2 \equiv 3 \pmod{m}$, $a_3 \equiv 4 \pmod{m}$, then $a_1 + a_2 + a_3 - m = (a_2 + a_3 - m) + a_1 \in S$.

More generally, let us suppose that, for each $b \in B$, there exists $a \in A$ such that b is obtained by adding coordinates to a . Then $\mathcal{C}(m, A) \subseteq \mathcal{C}(m, B)$.

As a final recommendation, it is worth mentioning that in [4, Section 2] and in [14, Introduction] there are several motivating examples, and references, as to why it is interesting to study (non-homogeneous) patterns of numerical semigroups.

2 The pseudo-variety $\mathcal{C}(m, A)$

In this section, m is an integer greater than or equal to 2 and A is a finite subset of $\bigcup_{k \in \mathbb{N} \setminus \{0, 1\}} \{1, \dots, m-1\}^k$. Moreover, recall that

$$\mathcal{C}(m, A) = \{S \in \mathcal{S}_m \mid s_1 + \cdots + s_k - m \in S \text{ if } (s_1, \dots, s_k) \in S^k \text{ and } (s_1 \bmod m, \dots, s_k \bmod m) \in A\},$$

where $x \bmod m$ denotes the remainder after division of x by m .

If S is a numerical semigroup and $x \in S \setminus \{0\}$, then the *Apéry set of x in S* (see [1]) is $\text{Ap}(S, x) = \{w(0) = 0, w(1), \dots, w(x-1)\}$, where $w(i)$ is the least element of S that is congruent with i modulus x . Observe that an integer s belongs to S if and only if there exists $t \in \mathbb{N}$ such that $s = w(s \bmod x) + tx$.

Proposition 2.1 *Let $S \in \mathcal{S}_m$ and $\text{Ap}(S, m) = \{w(0) = 0, w(1), \dots, w(m-1)\}$. Then the following conditions are equivalent.*

1. $S \in \mathcal{C}(m, A)$.
2. $w(i_1) + \dots + w(i_k) - m \in S$ for all $(i_1, \dots, i_k) \in A$.

Proof (1. $\Rightarrow$ 2.) Since $w(i_1), \dots, w(i_k) \in S$ and $(w(i_1) \bmod m, \dots, w(i_k) \bmod m) = (i_1, \dots, i_k) \in A$, then $w(i_1) + \dots + w(i_k) - m \in S$.

(2. $\Rightarrow$ 1.) Let $s_1, \dots, s_k \in S$ such that

$$(s_1 \bmod m, \dots, s_k \bmod m) = (i_1, \dots, i_k) \in A.$$

Then there exist $t_1, \dots, t_k \in \mathbb{N}$ such that $s_j = w(i_j) + t_j m$, $1 \leq j \leq k$, and thus, $s_1 + \dots + s_k = (w(i_1) + \dots + w(i_k) - m) + (t_1 + \dots + t_k)m \in S$. $\square$

By using the function `AperyListOfNumericalSemigroupWRTElement(S, m)` of [6], we can compute $\text{Ap}(S, m)$ from a system of generators of S . Thereby, we have the following algorithm to decide if a numerical semigroup S belongs or not to $\mathcal{C}(m, A)$.

Algorithm 2.2 INPUT: A finite subset G of positive integers.

OUTPUT: $\langle G \rangle \in \mathcal{C}(m, A)$ or $\langle G \rangle \notin \mathcal{C}(m, A)$.

- (1) If $\min(G) \neq m$, return $\langle G \rangle \notin \mathcal{C}(m, A)$.
- (2) If $\gcd(G) \neq 1$, return $\langle G \rangle \notin \mathcal{C}(m, A)$.
- (3) Compute $\text{Ap}(\langle G \rangle, m) = \{w(0), w(1), \dots, w(m-1)\}$.
- (4) If $w(i_1) + \dots + w(i_k) - m \in S$ for all $(i_1, \dots, i_k) \in A$, return $\langle G \rangle \in \mathcal{C}(m, A)$.
- (5) Return $\langle G \rangle \notin \mathcal{C}(m, A)$.

Let us illustrate the working of the previous algorithm through an example.

Example 2.3 Let us make use of Algorithm 2.2 with $G = \{5, 7, 9\}$, $m = 5$, and $A = \{(1, 3), (2, 2)\}$.

- $\min(G) = 5$.
- $\gcd(G) = 1$.
- $\text{Ap}(\langle G \rangle, 5) = \{w(0) = 0, w(1) = 16, w(2) = 7, w(3) = 18, w(4) = 19\}$.
- $w(1) + w(3) - 5 = 29 \in \langle G \rangle$ and $w(2) + w(2) - 5 = 9 \in \langle G \rangle$.

Therefore, $\langle G \rangle = \langle 5, 7, 9 \rangle \in \mathcal{C}(5, \{(1, 3), (2, 2)\})$.

Recall that, if m is an integer greater than or equal to 2, then we denote by $\Delta(m) = \{0, m, \rightarrow\}$. It is clear that $\Delta(m) \in \mathcal{S}_m$ and that, if $s_1, \dots, s_k \in \Delta(m) \setminus \{0\}$ and $k \geq 2$, then $s_1 + \dots + s_k - m \in \Delta(m)$. Therefore, we have the next result.

Lemma 2.4 *Let $m \in \mathbb{N} \setminus \{0, 1\}$ and $A \subseteq \bigcup_{k \in \{2, \rightarrow\}} \{1, \dots, m-1\}^k$ (A finite). Then $\Delta(m) \in \mathcal{C}(m, A)$.*

Let $m \in \mathbb{N} \setminus \{0, 1\}$. Then it is easy to show that $S \cap T \in \mathcal{S}_m$ for all $S, T \in \mathcal{S}_m$. Moreover, $\Delta(m)$ is the maximum of $\mathcal{S}_m$ and, if $S \in \mathcal{S}_m$ and $S \neq \Delta(m)$, then $S \cup \{F(S)\} \in \mathcal{S}_m$. From all this, we conclude the following result.

Lemma 2.5 *Let $m \in \mathbb{N} \setminus \{0, 1\}$. Then $\mathcal{S}_m$ is a Frobenius pseudo-variety with $\Delta(m)$ as maximum element.*

Proposition 2.6 *$\mathcal{C}(m, A)$ is a Frobenius pseudo-variety.*

Proof From Lemmas 2.4 and 2.5, we have that $\Delta(m)$ is the maximum of $\mathcal{C}(m, A)$. It is also easy to see that, if $S, T \in \mathcal{C}(m, A)$, then $S \cap T \in \mathcal{C}(m, A)$. In order to finish the proof, let us see that, if $S \in \mathcal{C}(m, A)$ and $S \neq \Delta(m)$, then $S \cup \{F(S)\} \in \mathcal{C}(m, A)$. For that, we have to show that, if $s_1, \dots, s_k \in S \cup \{F(S)\}$ and $(s_1 \bmod m, \dots, s_k \bmod m) \in A$, then $s_1 + \dots + s_k - m \in S \cup \{F(S)\}$. In effect, if $F(S) \notin \{s_1, \dots, s_k\}$, then the result is true because $S \in \mathcal{C}(m, A)$. On the other hand, if $F(S) \in \{s_1, \dots, s_k\}$, then $s_1 + \dots + s_k - m \geq F(S)$ and, consequently, $s_1 + \dots + s_k - m \in S \cup \{F(S)\}$. $\square$

Our next purpose in this section is to show an algorithm that allows us to build all the elements of $\mathcal{C}(m, A)$ that have a fixed genus. To do that, we use the concept of rooted tree.

A graph G is a pair (V, E) where V is a non-empty set (whose elements are called *vertices* of G) and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$ (whose elements are called *edges* of G). A path (of length n) connecting the vertices x and y of G is a sequence of different edges $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

We say that a graph G is a (rooted) tree if there exists a vertex r (known as the *root* of G) such that, for any other vertex x of G , there exists a unique path connecting x and r . If there exists a path connecting x and y , then we say that x is a *descendant* of y . In particular, if (x, y) is an edge of the tree, then we say that x is a *child* of y . (See [12].)

We define the graph $G(\mathcal{C}(m, A))$ in the following way: $\mathcal{C}(m, A)$ is the set of vertices and $(S, S') \in \mathcal{C}(m, A) \times \mathcal{C}(m, A)$ is an edge if $S \cup \{F(S)\} = S'$. The following result is a consequence of Lemma 4.2 and Theorem 4.3 of [9].

Theorem 2.7 $G(\mathcal{C}(m, A))$ is a tree with root $\Delta(m)$. Moreover, the children set of $S \in \mathcal{C}(m, A)$ is $\{S \setminus \{x\} \mid x \in \text{msg}(S), S \setminus \{x\} \in \mathcal{C}(m, A), x > F(S)\}$.

Let S be a numerical semigroup and let $x \in S$. Then it is clear that $S \setminus \{x\}$ is a numerical semigroup if and only if $x \in \text{msg}(S)$. Moreover, let us observe that, if $x \in \text{msg}(S)$ and $m \in S \setminus \{0, x\}$, then

$$\text{Ap}(S \setminus \{x\}, m) = (\text{Ap}(S, m) \setminus \{x\}) \cup \{x + m\}.$$

In the following result we characterise the children of $S \in \mathcal{C}(m, A)$.

Proposition 2.8 Let $S \in \mathcal{C}(m, A)$, $\text{Ap}(S, m) = \{w(0), w(1), \dots, w(m-1)\}$ and $x \in \text{msg}(S) \setminus \{m\}$. Then $S \setminus \{x\} \in \mathcal{C}(m, A)$ if and only if $w(i_1) + \dots + w(i_k) \neq m + x$ for all $(i_1, \dots, i_k) \in A$.

Proof Let us suppose that $\text{Ap}(S \setminus \{x\}, m) = (\text{Ap}(S, m) \setminus \{x\}) \cup \{x + m\} = \{w'(0), w'(1), \dots, w'(m-1)\}$.

(Necessity.) If $w(i_1) + \dots + w(i_k) = m + x$, then $m + x \notin \{w(i_1), \dots, w(i_k)\}$ and, therefore, $w(i_1) = w'(i_1), \dots, w(i_k) = w'(i_k)$. Moreover, $w(i_1) + \dots + w(i_k) - m = x \notin S \setminus \{x\}$ and, consequently, $S \setminus \{x\} \notin \mathcal{C}(m, A)$.

(Sufficiency.) In order to prove that $S \setminus \{x\} \in \mathcal{C}(m, A)$, by Proposition 2.1, it is enough to see that, if $(i_1, \dots, i_k) \in A$, then $w'(i_1) + \dots + w'(i_k) - m \in S \setminus \{x\}$.

Since $S \in \mathcal{C}(m, A)$, we easily deduce that $w'(i_1) + \dots + w'(i_k) - m \in S$. Now, if $w'(i_1) + \dots + w'(i_k) - m \notin S \setminus \{x\}$, then $w'(i_1) + \dots + w'(i_k) = m + x$. Thus, $w(i_1) = w'(i_1), \dots, w(i_k) = w'(i_k)$ and $w(i_1) + \dots + w(i_k) = m + x$, where the last equality is in contradiction with $S \in \mathcal{C}(m, A)$. $\square$

Let us observe that a tree can be built in a recurrent way starting from its root and connecting each vertex with its children. Let us also observe that the elements of $\mathcal{C}(m, A)$ with genus equal to $g + 1$ are precisely the children of the elements of $\mathcal{C}(m, A)$ with genus equal to g .

We are ready to show the above announced algorithm.

Algorithm 2.9 INPUT: A positive integer g .

OUTPUT: $\{S \in \mathcal{C}(m, A) \mid g(S) = g\}$.

- (1) If $g < m - 1$, return $\emptyset$.
- (2) $X = \{\Delta(m)\}$ and $i = m - 1$.
- (3) If $i = g$, return X .
- (4) For each $S \in X$, compute the set

$$\mathcal{B}_S = \{x \in \text{msg}(S) \mid x > F(S), x \neq m, S \setminus \{x\} \in \mathcal{C}(m, A)\}.$$

- (5) If $\bigcup_{S \in X} \mathcal{B}_S = \emptyset$, return $\emptyset$.
- (6) $X = \bigcup_{S \in X} \{S \setminus \{x\} \mid x \in \mathcal{B}_S\}$, $i = i + 1$, and go to (3).

For computing item (4) in the above algorithm, we use Proposition 2.8 and the following Lemma 2.10, which is a reformulation of Corollary 18 of [8], and is useful to obtain the minimal system of generators of $S \setminus \{x\}$ starting from the minimal system of generators of S .

Lemma 2.10 Let S be a numerical semigroup with $\text{msg}(S) = \{n_1 < n_2 < \dots < n_e\}$. If $i \in \{2, \dots, e\}$ and $n_i > F(S)$, then

$$\text{msg}(S \setminus \{n_i\}) = \begin{cases} \{n_1, \dots, n_e\} \setminus \{n_i\}, & \text{if there exists } j \in \{2, \dots, i-1\} \\ & \text{such that } n_i + n_1 - n_j \in S; \\ (\{n_1, \dots, n_e\} \setminus \{n_i\}) \cup \{n_i + n_1\}, & \text{in other case.} \end{cases}$$

We illustrate the functioning of Algorithm 2.9 with an example.

Example 2.11 Let us compute all the elements of $\mathcal{C}(5, \{(1, 1), (1, 2)\})$ with genus equal to 6.

- $X = \{\langle 5, 6, 7, 8, 9 \rangle\}$ and $i = 4$.
- $B_{\langle 5, 6, 7, 8, 9 \rangle} = \{6, 9\}$.
- $X = \{\langle 5, 7, 8, 9, 11 \rangle, \langle 5, 6, 7, 8 \rangle\}$ and $i = 5$.
- $B_{\langle 5, 7, 8, 9, 11 \rangle} = \{7, 8, 9, 11\}$ and $B_{\langle 5, 6, 7, 8 \rangle} = \emptyset$.
- $X = \{\langle 5, 8, 9, 11, 12 \rangle, \langle 5, 7, 9, 11, 13 \rangle, \langle 5, 7, 8, 11 \rangle, \langle 5, 7, 8, 9 \rangle\}$ and $i = 6$.
- Return $\{\langle 5, 8, 9, 11, 12 \rangle, \langle 5, 7, 9, 11, 13 \rangle, \langle 5, 7, 8, 11 \rangle, \langle 5, 7, 8, 9 \rangle\}$.

Taking advantage of the above example, we finish this section building the first three levels of the tree $G(\mathcal{C}(5, \{(1, 1), (1, 2)\}))$.

Observe that the number which appear next to each edge (S' , S) is the minimal generator x of S such that $S' = S \setminus \{x\}$.

3 Second-level numerical semigroups

We say that a numerical semigroup S is of *second-level* if $x + y + z - m(S) \in S$ for all $(x, y, z) \in (S \setminus \{0\})^3$. We denote by $\mathcal{L}_{2,m}$ the set of all the second-level numerical semigroups with multiplicity equal to m .

Proposition 3.1 *Let S be a numerical semigroup with minimal system of generators given by $\{m = n_1 < n_2 < \dots < n_e\}$. Then the following two conditions are equivalents.*

1. $S \in \mathcal{L}_{2,m}$.
2. If $(i, j, k) \in \{2, \dots, e\}^3$, then $n_i + n_j + n_k - m \in S$.

Proof (1. $\Rightarrow$ 2.) It is evident from the definition of second-level numerical semigroup.

(2. $\Rightarrow$ 1.) Let $(x, y, z) \in (S \setminus \{0\})^3$. If $0 \in \{x \bmod m, y \bmod m, z \bmod m\}$, then it is clear that $x + y + z - m \in S$. Now, if $0 \notin \{x \bmod m, y \bmod m, z \bmod m\}$, then we easily deduce that there exist $(i, j, k) \in \{2, \dots, e\}^3$ and $(s_1, s_2, s_3) \in S^3$ such that $(x, y, z) = (n_i, n_j, n_k) + (s_1, s_2, s_3)$. Therefore, $x + y + z - m = (n_i + n_j + n_k - m) + s_1 + s_2 + s_3 \in S$. Consequently, $S \in \mathcal{L}_{2,m}$. $\square$

The above proposition allows us to easily decide whether a numerical semigroup is of second-level or not.

Example 3.2 Let us see that $S = \langle 5, 7, 16 \rangle \in \mathcal{L}_{2,m}$. In effect, it is clear that $\{7 + 7 + 7 - 5, 7 + 7 + 16 - 5, 7 + 16 + 16, 16 + 16 + 16 - 5\} = \{16, 25, 34, 43\} \subseteq S$. Therefore, by Proposition 3.1, we have that $S \in \mathcal{L}_{2,m}$.

Now our intention is to show that $\mathcal{L}_{2,m}$ is a modular Frobenius pseudo-variety.

Proposition 3.3 *Let $m \in \mathbb{N} \setminus \{0, 1\}$. Then $\mathcal{L}_{2,m} = \mathcal{C}(m, \{1, \dots, m-1\}^3)$.*

Proof Let $S \in \mathcal{L}_{2,m}$ and $\text{Ap}(S, m) = \{w(0), w(1), \dots, w(m-1)\}$. If $(i, j, k) \in \{1, \dots, m-1\}^3$, then $(w(i), w(j), w(k)) \in (S \setminus \{0\})^3$ and, therefore, $w(i) + w(j) + w(k) - m \in S$. By applying Proposition 2.1, we have that $S \in \mathcal{C}(m, \{1, \dots, m-1\}^3)$.

In order to see the other inclusion, let $S \in \mathcal{C}(m, \{1, \dots, m-1\}^3)$ and $(x, y, z) \in (S \setminus \{0\})^3$. Firstly, if $0 \in \{x \bmod m, y \bmod m, z \bmod m\}$, then it is clear that $x + y + z - m \in S$. Secondly, if $0 \notin \{x \bmod m, y \bmod m, z \bmod m\}$, then there exist $(i, j, k) \in \{1, \dots, m-1\}^3$ and $(p, q, r) \in \mathbb{N}^3$ such that $x = w(i) + pm$, $y = w(j) + qm$, and $z = w(k) + rm$. Thus, $x + y + z - m = (w(i) + w(j) + w(k) - m) + (p + q + r)m \in S$ and, therefore, $S \in \mathcal{L}_{2,m}$. $\square$

As an immediate consequence of Propositions 2.6 and 3.3, we have the following result.

Corollary 3.4 *Let $m \in \mathbb{N} \setminus \{0, 1\}$. Then $\mathcal{L}_{2,m}$ is a modular pseudo-Frobenius variety and, in addition, $\Delta(m)$ is the maximum of $\mathcal{L}_{2,m}$.*

Our next step is to build the tree associated with the pseudo-variety $\mathcal{L}_{2,m}$. To do this, we should characterise the possible children of each element in $\mathcal{L}_{2,m}$.

Let S be a numerical semigroup with $\text{msg}(S) = \{n_1, \dots, n_e\}$. If $s \in S$, then we denote by

$$L_S(s) = \max\{a_1 + \dots + a_e \mid (a_1, \dots, a_e) \in \mathbb{N}^e \text{ and } a_1 n_1 + \dots + a_e n_e = s\}.$$

Proposition 3.5 *Let $m \in \mathbb{N} \setminus \{0, 1\}$, $S \in \mathcal{L}_{2,m}$, and $x \in \text{msg}(S) \setminus \{m\}$. Then $S \setminus \{x\} \in \mathcal{L}_{2,m}$ if and only if $L_{S \setminus \{x\}}(x + m) \leq 2$.*

Proof (Necessity.) Let us suppose that $L_{S \setminus \{x\}}(x + m) \geq 3$. Then there exists $(a, b, c) \in (S \setminus \{0, x\})^3$ such that $x + m = a + b + c$. Thus, $a + b + c - m = x \notin S \setminus \{x\}$ and, therefore, $S \setminus \{x\} \notin \mathcal{L}_{2,m}$.

(Sufficiency.) If $(a, b, c) \in (S \setminus \{0, x\})^3$, then $a + b + c - m \in S$, since $S \in \mathcal{L}_{2,m}$. Now, if $a + b + c - m = x$, then $L_{S \setminus \{x\}}(x + m) \geq 3$. Therefore, $a + b + c - m \neq x$ and, consequently, $a + b + c - m \in S \setminus \{x\}$. Thus, we conclude that $S \setminus \{x\} \in \mathcal{L}_{2,m}$. $\square$

By applying Theorem 2.7, Propositions 3.3 and 3.5, and Lemma 2.10, we can easily build the tree $G(\mathcal{L}_{2,m})$.

Example 3.6 The first four levels of $G(\mathcal{L}_{2,4})$ appear in the following figure.

The Frobenius problem (see [7]) consists in finding formulas that allow us to compute the Frobenius number and the genus of a numerical semigroup in terms of the minimal system of generators of such a numerical semigroup. This problem was solved in [15] for numerical semigroups with embedding dimension two. At present, the Frobenius problem is open for embedding dimension greater than or equal to 3. However, if we know the Apéry set $\text{Ap}(S, x)$ for some $x \in S \setminus \{0\}$, then we have solved the Frobenius problem for S because we have the following result from [13].

Lemma 3.7 Let S be a numerical semigroup and let $x \in S \setminus \{0\}$. Then

1. $F(S) = (\max(\text{Ap}(S, x))) - x$,
2. $g(S) = \frac{1}{x} (\sum_{w \in \text{Ap}(S, x)} w) - \frac{x-1}{2}$.

The knowledge of $\text{Ap}(S, x) = \{w(0), w(1), \dots, w(x-1)\}$ also allows us to determine the membership of an integer to the numerical semigroup S . In fact, if $n \in \mathbb{N}$, then $n \in S$ if and only if $n \geq w(n \bmod x)$.

Now our purpose is to show that, if $S \in \mathcal{L}_{2,m}$, then is rather easy to compute $\text{Ap}(S, m)$. We need the following easy result.

Lemma 3.8 Let $m \in \mathbb{N} \setminus \{0, 1\}$, $S \in \mathcal{L}_{2,m}$, $\text{msg}(S) = \{n_1 = m, n_2, \dots, n_e\}$. Then $\{0, n_2, \dots, n_e\} \subseteq \text{Ap}(S, m) \subseteq \{0, n_2, \dots, n_e\} \cup \{n_i + n_j \mid (i, j) \in \{2, \dots, e\}^2\}$.

As an immediate consequence of the above lemma we can formulate the following result.

Proposition 3.9 Let $m \in \mathbb{N} \setminus \{0, 1\}$, $S \in \mathcal{L}_{2,m}$, $\text{msg}(S) = \{n_1 = m, n_2, \dots, n_e\}$. Then $\text{Ap}(S, m) = \{w(0), w(1), \dots, w(m-1)\}$ where $w(i)$ is the least element of $\{0, n_2, \dots, n_e\} \cup \{n_i + n_j \mid (i, j) \in \{2, \dots, e\}^2\}$ that is congruent to i modulo m .

Corollary 3.10 Let $m \in \mathbb{N} \setminus \{0, 1\}$ and $S \in \mathcal{L}_{2,m}$. Then $m = m(S) \leq \frac{e(S)(e(S)+1)}{2}$.

Following the notation introduced in [10], we say that $x \in \mathbb{Z} \setminus S$ is a *pseudo-Frobenius number* of S if $x + s \in S$ for all $s \in S \setminus \{0\}$. We denote by $\text{PF}(S)$ the set of all the pseudo-Frobenius numbers of S . The cardinality of $\text{PF}(S)$ is an important invariant of S (see [2]) that is the so-called *type* of S and it is denoted by $t(S)$.

Let S be a numerical semigroup. Then we define the following binary relation over $\mathbb{Z}$: $a \leq_S b$ if $b - a \in S$. In [11] it is shown that $\leq_S$ is a partial order (that is, reflexive, transitive, and antisymmetric). Moreover, Proposition 2.20 of [11] is the next result.

Proposition 3.11 *Let S be a numerical semigroup and $x \in S \setminus \{0\}$. Then*

$$\text{PF}(S) = \{w - x \mid w \in \text{Maximals}_{\leq_S}(\text{Ap}(S, x))\}$$

Let us observe that, if $w, w' \in \text{Ap}(S, x)$, then $w' - w \in S$ if and only if $w' - w \in \text{Ap}(S, x)$. Therefore, $\text{Maximals}_{\leq_S}(\text{Ap}(S, x))$ is the set

$$\{w \in \text{Ap}(S, x) \mid w' - w \notin \text{Ap}(S, x) \setminus \{0\} \text{ for all } w' \in \text{Ap}(S, x)\}.$$

We finish this section with an example that illustrates the above results.

Example 3.12 Having in mind that $S = \langle 5, 6, 13 \rangle$ is a second-level numerical semigroup, it is easy to compute $\text{Ap}(S, 5)$. In fact, by applying Lemma 3.8, we have that $\text{Ap}(S, 5) \subseteq \{0, 6, 13\} \cup \{12, 19, 26\}$ and, by Proposition 3.9, we conclude that $\text{Ap}(S, 5) = \{0, 6, 12, 13, 19\}$. On the other hand, by Lemma 3.7, we know that $F(S) = 19 - 5 = 14$ and $g(S) = \frac{1}{5}(6 + 12 + 13 + 19) - \frac{5-1}{2} = 8$. Finally, since $\text{Maximals}_{\leq_S}(\text{Ap}(S, 5)) = \{12, 19\}$, Proposition 3.11 allows us to claim that $\text{PF}(S) = \{7, 12\}$ and $t(S) = 2$.

Remark 3.13 The definition of second-level numerical semigroup can be easily generalised to greater levels. Thus, we say that a numerical semigroup is of *n th-level* if $x_1 + \dots + x_{n+1} - m(S) \in S$ for all $(x_1, \dots, x_{n+1}) \in (S \setminus \{0\})^{n+1}$ and denote by $\mathcal{L}_{n,m}$ the set of all the n th-level numerical semigroups with multiplicity equal to m .

It is clear that for n th-level we obtain similar results to those of second-level. In particular, $\mathcal{L}_{n,m} = \mathcal{C}(m, \{1, \dots, m-1\}^{n+1})$ and Proposition 3.5 remains true taking $L_{S \setminus \{x\}}(x+m) \leq n$.

On the other hand, having in mind that $\mathcal{L}_{1,m}$ is the family of numerical semigroups with maximal embedding dimension and that (by Remark 1.1) $\mathcal{L}_{m-1,m} = \mathcal{S}_m$, we can observe that

$$\mathcal{L}_{1,m} \subseteq \mathcal{L}_{2,m} \subseteq \dots \subseteq \mathcal{L}_{m-1,m} = \mathcal{S}_m,$$

where all the inclusions are strict, as we can deduce from the following example.

Example 3.14 Let us set $m = 5$ and

$$S_1 = \langle 5, 6 \rangle = \{0, 5, 6, 10, 11, 12, 15, 16, 17, 18, 20, \rightarrow\}.$$

Then,

- $S_1 \in \mathcal{L}_{4,5} \setminus \mathcal{L}_{3,5} = \mathcal{S}_5 \setminus \mathcal{L}_{3,5}$, because $4 \times 6 - 5 \notin S_1$ and $s_1 + \dots + s_5 - 5 \geq 5 \times 6 - 5 \geq F(S_1) + 1$;
- $S_2 = S_1 \cup \{19\} = \langle 5, 6, 19 \rangle \in \mathcal{L}_{3,5} \setminus \mathcal{L}_{2,5}$, because $3 \times 6 - 5 \notin S_2$ and $s_1 + \dots + s_4 - 5 \geq 4 \times 6 - 5 \geq F(S_2) + 1$;
- $S_3 = S_2 \cup \{13, 14\} = \langle 5, 6, 13, 14 \rangle \in \mathcal{L}_{2,5} \setminus \mathcal{L}_{1,5}$, because $2 \times 6 - 5 \notin S_2$ and $s_1 + s_2 + s_3 - 5 \geq 3 \times 6 - 5 \geq F(S_3) + 1$.

Generalising this example to other values of m is trivial.

Remark 3.15 Let us observe that the chain obtained in Remark 3.13 is reminiscent, in some sense, of the chain associated with subtraction patterns (see [3, Section 6]).

4 Thin numerical semigroups

We say that a numerical semigroup S is *thin* if $2x - m(S) \in S$ for all $x \in S \setminus \{0\}$. We denote by $\mathcal{T}_m$ the set of all the thin numerical semigroups with multiplicity equal to m .

Proposition 4.1 *Let S be a numerical semigroup with minimal system of generators given by $\{m = n_1 < n_2 < \dots < n_e\}$. Then the following two conditions are equivalents.*

1. $S \in \mathcal{T}_m$.
2. $2n_i - m \in S$ for all $i \in \{2, \dots, e\}$.

Proof (1. $\Rightarrow$ 2.) It follows from the definition of thin numerical semigroup.

(2. $\Rightarrow$ 1.) Let $x \in S \setminus \{0\}$. If $x \equiv 0 \pmod{m}$, then it is clear that $2x - m \in S$. Now, if $x \not\equiv 0 \pmod{m}$, then we have that there exist $i \in \{2, \dots, e\}$ and $s \in S$ such that $x = n_i + s$. Therefore, $2x - m = (2n_i - m) + 2s \in S$ and, consequently, $S \in \mathcal{T}_m$. $\square$

The above proposition allows us to easily decide whether a numerical semigroup is thin or not.

Example 4.2 Let us see that $S = \langle 4, 6, 7 \rangle \in \mathcal{T}_4$. In effect, it is clear that $\{2 \cdot 6 - 4, 2 \cdot 7 - 4\} = \{8, 10\} \subseteq S$. Therefore, by Proposition 4.1, we have that $S \in \mathcal{T}_m$.

Now we want to show that $\mathcal{T}_m$ is a modular Frobenius pseudo-variety.

Proposition 4.3 $\mathcal{T}_m = \mathcal{C}(m, \{(1, 1), (2, 2), \dots, (m-1, m-1)\})$ for all $m \in \mathbb{N} \setminus \{0, 1\}$.

Proof If $S \in \mathcal{T}_m$ and $\text{Ap}(S, m) = \{w(0), w(1), \dots, w(m-1)\}$, then it is clear that $\{w(1) + w(1) - m, \dots, w(m-1) + w(m-1) - m\} \subseteq S$. Therefore, by applying Proposition 2.1, we have that $S \in \mathcal{C}(m, \{(1, 1), (2, 2), \dots, (m-1, m-1)\})$.

To see the other inclusion, let $S \in \mathcal{C}(m, \{(1, 1), (2, 2), \dots, (m-1, m-1)\})$ and $x \in S \setminus \{0\}$. On the one hand, if $x \equiv 0 \pmod{m}$, then it is clear that $2x - m \in S$. On the other hand, if $x \not\equiv 0 \pmod{m}$, then there exist $i \in \{1, \dots, m-1\}$ and $t \in \mathbb{N}$ such that $x = w(i) + tm$. Therefore, $2x - m = (w(i) + w(i) - m) + 2tm \in S$ and in consequence $S \in \mathcal{T}_m$. $\square$

From Propositions 2.6 and 4.3, we get the following result.

Corollary 4.4 *Let $m \in \mathbb{N} \setminus \{0, 1\}$. Then $\mathcal{T}_m$ is a modular pseudo-Frobenius variety and, in addition, $\Delta(m)$ is the maximum of $S \in \mathcal{T}_m$.*

In order to build the tree associated with the pseudo-variety $S \in \mathcal{T}_m$, we are going to characterise the possible children of each $S \in \mathcal{T}_m$.

Proposition 4.5 *Let $m \in \mathbb{N} \setminus \{0, 1\}$, $S \in \mathcal{T}_m$, and $x \in \text{msg}(S) \setminus \{m\}$. Then $S \setminus \{x\} \in \mathcal{T}_m$ if and only if $\frac{x+m}{2} \notin S$.*

Proof (Necessity.) If $\frac{x+m}{2} \in S$, then $\frac{x+m}{2} \in S \setminus \{0, x\}$ and $2\frac{x+m}{2} - m = x \notin S \setminus \{x\}$. Therefore, $S \setminus \{x\} \notin \mathcal{T}_m$.

(Sufficiency.) If $a \in S \setminus \{0, x\}$, then $2a - m \in S$ because $S \in \mathcal{T}_m$. Now, if $2a - m = x$, then $\frac{x+m}{2} = a \in S$. Therefore, $2a - m \neq x$ and, consequently, $2a - m \in S \setminus \{x\}$. Thereby, $S \setminus \{x\} \in \mathcal{T}_m$. $\square$

By applying Theorem 2.7, Propositions 4.3 and 4.5, and Lemma 2.10, we can build the tree $G(\mathcal{T}_m)$. Let us see an example.

Example 4.6 In the next figure we have the first four levels of $G(\mathcal{T}_4)$.

We finish this section describing the Apéry set for $S \in \mathcal{T}_m$.

Proposition 4.7 Let $m \in \mathbb{N} \setminus \{0\}$, $S \in \mathcal{T}_m$, $\text{msg}(S) = \{m = n_1, n_2, \dots, n_e\}$. Then $\text{Ap}(S, m) = \{w(0), w(1), \dots, w(m-1)\}$, where $w(i)$ is the least element of the set $\{a_2 n_2 + \dots + a_e n_e \mid (a_2, \dots, a_e) \in \{0, 1\}^{e-1}\}$ that is congruent to i modulo m .

Corollary 4.8 Let $m \in \mathbb{N} \setminus \{0\}$ and $S \in \mathcal{T}_m$. Then $m = m(S) \leq 2^{e(S)-1}$.

Remark 4.9 Let us observe that we can generalise the concept of thin numerical semigroups in the following way: setting $n \in \mathbb{N} \setminus \{0, 1\}$, we say that a numerical semigroup S is n -thin if $nx - m(S) \in S$ for all $x \in S \setminus \{0\}$, and denote by $\mathcal{T}_{n,m}$ the set of all k -thin numerical semigroups with multiplicity equal to m . It is clear that $\mathcal{T}_m = \mathcal{T}_{2,m}$.

Again, $\mathcal{T}_{n,m}$ is a modular pseudo-variety, Proposition 4.5 is valid for the condition $\frac{x+m}{n} \notin S$, and we can build the chain

$$\mathcal{T}_{2,m} \subsetneq \mathcal{T}_{3,m} \subsetneq \dots \subsetneq \mathcal{T}_{m,m} = \mathcal{S}_m.$$

Note that Example 3.14 also gives us the construction that ensures the strict inclusions in this case.

5 Strong numerical semigroups

We say that a numerical semigroup S is *strong* if $x + y - m(S) \in S$ for all $(x, y) \in (S \setminus \{0\})^2$ such that $x \not\equiv y \pmod{m(S)}$. We denote by $\mathcal{R}_m$ the set of all the strong numerical semigroups with multiplicity equal to m .

Proposition 5.1 Let S be a numerical semigroup with minimal system of generators given by $\{m = n_1 < n_2 < \dots < n_e\}$. Then the following two conditions are equivalents.

1. $S \in \mathcal{R}_m$.
2. $\{n_i + n_j - m, 3n_i - m\} \subseteq S$ for all $i \in \{2, \dots, e\}$ and for all $(i, j) \in \{2, \dots, e\}^2$ such that $i \neq j$.

Proof ($1. \Rightarrow 2.$) It is enough to observe that $n_i \not\equiv n_j \pmod{m}$ and that $2n_i \not\equiv n_i \pmod{m}$.

($2. \Rightarrow 1.$) Let $x, y \in S \setminus \{0\}$ such that $x \not\equiv y \pmod{m}$. If $x \equiv 0 \pmod{m}$ or $y \equiv 0 \pmod{m}$, then it is clear that $x + y - m \in S$. Now, if $x \not\equiv 0 \pmod{m}$ and $y \not\equiv 0 \pmod{m}$, then there exists $(i, j) \in \{1, \dots, m-1\}^2$, with $i \neq j$, and there exists $(p, q) \in \mathbb{N}^2$ such that $x = w(i) + pm$ and $y = w(j) + qm$. Moreover, if there exists $(a, b) \in \{2, \dots, e\}^2$ such that $a \neq b$ and $w(i) - na, w(j) - nb \in S$, then it is easy to see that $x + y - m \in S$. In other case, there exists $(r, t) \in (\mathbb{N} \setminus \{0\})^2$ such that $w(i) = r \cdot n_a$ and $w(j) = t \cdot n_a$ for some $a \in \{2, \dots, e\}$. Then, since $i \neq j$, we deduce that $r + t \geq 3$ and, consequently, $x + y - m \in S$. In conclusion, $S \in \mathcal{R}_m$. $\square$

The above proposition allows us to easily decide whether a numerical semigroup is strong or not.

Example 5.2 If $S = \langle 4, 5, 7 \rangle$, then $\{5+7-4, 3 \cdot 5-4, 3 \cdot 7-4\} = \{8, 11, 17\} \subseteq S$. Therefore, by Proposition 5.1, we have that $S \in \mathcal{R}_m$.

Now we want to show that $\mathcal{R}_m$ is a modular Frobenius pseudo-variety. Let us denote by $A = \{1, \dots, m-1\}^2 \setminus \{(1, 1), (2, 2), \dots, (m-1, m-1)\}$.

Proposition 5.3 $\mathcal{R}_m = \mathcal{C}(m, A)$ for all $m \in \mathbb{N} \setminus \{0, 1\}$.

Proof Let $S \in \mathcal{R}_m$, $\text{Ap}(S, m) = \{w(0), w(1), \dots, w(m-1)\}$, and $(i, j) \in A$. Then $\{(w(i), w(j)) \in (S \setminus \{0\})^2$ and $w(i) \not\equiv w(j) \pmod{m}$. Therefore, $w(i) - w(j) - m \in S$ and, by applying Proposition 2.1, we have that $S \in \mathcal{C}(m, A)$.

To see the other inclusion, let $S \in \mathcal{C}(m, A)$ and $(x, y) \in (S \setminus \{0\})^2$. On the one hand, if $0 \in \{x \bmod m, y \bmod m\}$, then it is clear that $x + y - m \in S$. On the other hand, if $0 \notin \{x \bmod m, y \bmod m\}$, then there exist $(i, j) \in A$ and $(p, q) \in \mathbb{N}^2$ such that $x = w(i) + pm$ and $y = w(j) + qm$. Therefore, $x + y - m = (w(i) + w(j) - m) + (p + q)m \in S$ and, consequently, $S \in \mathcal{R}_m$. $\square$

From Propositions 2.6 and 5.3, we get the following result.

Corollary 5.4 Let $m \in \mathbb{N} \setminus \{0, 1\}$. Then $\mathcal{R}_m$ is a modular pseudo-Frobenius variety and, in addition, $\Delta(m)$ is the maximum of $S \in \mathcal{R}_m$.

We are now interested in the description of the tree associated with the pseudo-variety $S \in \mathcal{R}_m$. In order to do that, we are going to characterise the children of an arbitrary $S \in \mathcal{R}_m$.

Proposition 5.5 Let $m \in \mathbb{N} \setminus \{0, 1\}$, $S \in \mathcal{R}_m$, and $x \in \text{msg}(S) \setminus \{m\}$. Then $S \setminus \{x\} \in \mathcal{R}_m$ if and only if $x + m \notin \{a + b \mid a, b \in \text{msg}(S) \setminus \{m, x\}, a \neq b\} \cup \{3a \mid a \in \text{msg}(S) \setminus \{m, x\}\}$.

Proof (Necessity.) If $a, b \in \text{msg}(S) \setminus \{m, x\}$ and $a \neq b$, then $(a, b) \in (S \setminus \{0, x\})^2$ and $a \not\equiv b \pmod{m}$. Since $S \setminus \{x\} \in \mathcal{R}_m$, we have that $a + b - m \in S \setminus \{x\}$ and, therefore, $a + b - x \neq x$. Thus, $x + m \notin \{a + b \mid a, b \in \text{msg}(S) \setminus \{m, x\}, a \neq b\}$.

On the other hand, if $a \in \text{msg}(S) \setminus \{m, x\}$, then $(a, 2a) \in (S \setminus \{0, x\})^2$ and $a \not\equiv 2a \pmod{m}$. Once again, since $S \setminus \{x\} \in \mathcal{R}_m$, we have that $3a - m \in S \setminus \{x\}$ and, therefore, $a + b - x \neq x$. Thus, $x + m \notin \{3a \mid a \in \text{msg}(S) \setminus \{m, x\}\}$.

(Sufficiency.) Let $a, b \in S \setminus \{0, x\}$ such that $a \neq b$. Since $S \in \mathcal{R}_m$, we have that $a + b - m \in S$ and $3a - m \in S$. Now, by Lemma 2.10, we know that $\text{msg}(S) \setminus \{x\} \subseteq \text{msg}(S \setminus \{x\}) \subseteq (\text{msg}(S) \setminus \{x\}) \cup \{x + m\}$. Thus, from this fact and the hypothesis, it easily follows that $a + b - m \neq x$ and $3a - m \neq x$, that is, $a + b - m, 3a - m \in S \setminus \{x\}$. By applying Proposition 5.1, we conclude that $S \setminus \{x\} \in \mathcal{R}_m$. $\square$

From Theorem 2.7, Propositions 5.3 and 5.5, and Lemma 2.10, we can build the tree $G(\mathcal{R}_m)$. Let us see an example.

Example 5.6 In the next figure we have the first four levels of $G(\mathcal{R}_4)$.

We finish this section describing the Apéry set for $S \in \mathcal{R}_m$.

Proposition 5.7 *Let $m \in \mathbb{N} \setminus \{0, 1\}$, $S \in \mathcal{R}_m$, $\text{msg}(S) = \{m = n_1, n_2, \dots, n_e\}$. Then $\text{Ap}(S) = \{w(0), w(1), \dots, w(n)\}$, where $w(i)$ is the least element of the set $\{0, n_2, \dots, n_e\} \cup \{2n_2, \dots, 2n_e\}$ that is congruent to i modulo m .*

Corollary 5.8 *Let $m \in \mathbb{N} \setminus \{0, 1\}$ and $S \in \mathcal{R}_m$. Then $m = m(S) \leq 2e(S) - 1$.*

It is interesting to observe that, under the hypotheses of Proposition 5.7, $n_i \in \text{Maximals}_{\leq_S}(\text{Ap}(S, m))$ if and only if $2n_i \notin \text{Ap}(S, m)$. In addition, $2n_i \in \text{Ap}(S, m)$ if and only if $2n_i \in \text{Maximals}_{\leq_S}(\text{Ap}(S, m))$. Therefore, from Propositions 3.11 and 5.7, we get the next result.

Corollary 5.9 *Let $m \in \mathbb{N} \setminus \{0, 1\}$ and $S \in \mathcal{R}_m$. Then $t(S) = e(S) - 1$.*

It is well known that, if S is a numerical semigroup, then $e(S) \leq m(S)$ and $t(S) \leq m(S) - 1$ (see Proposition 2.10 and Corollary 2.23 in [11]). Combining these facts with Corollaries 5.8 and 5.9, we have the next result.

Corollary 5.10 *Let $m \in \mathbb{N} \setminus \{0, 1\}$ and $S \in \mathcal{R}_m$. Then $\frac{m(S)-1}{2} \leq t(S) \leq m(S) - 1$ (or, equivalently, $\frac{m(S)+1}{2} \leq e(S) \leq m(S)$).*

Remark 5.11 Contrary to what happens in Sects. 3 and 4, the concept of strong numerical semigroup does not have a natural generalisation. In fact, we have, at least, two possibilities.

1. $x_1 + \dots + x_n - m(S) \in S$ for all $(x_1, \dots, x_n) \in (S \setminus \{0\})^n$ such that $x_i \not\equiv x_j \pmod{m(S)}$, for all $i \neq j$. In this case, $(x_1, \dots, x_n) \in A_n$, where

$$A_n = \{(\alpha_1, \dots, \alpha_n) \in \{1, \dots, m-1\}^n \mid \alpha_i \neq \alpha_j \text{ for all } i \neq j\}.$$

2. $x_1 + \dots + x_n - m(S) \in S$ for all $(x_1, \dots, x_n) \in B_n$, where

$$B_n = \{1, \dots, m-1\}^n \setminus \{(1, \dots, 1), (2, \dots, 2), \dots, (m-1, \dots, m-1)\}.$$

It is clear that, if $n \geq 3$, then $A_n \subsetneq B_n$ and, in consequence, $\mathcal{C}(m, B_n) \subseteq \mathcal{C}(m, A_n)$. In addition, it is guessed that, if $n < m$, then the second inclusion will be strict.

Acknowledgements The authors thank the referee for the helpful comments and suggestions which have improved this paper.

Funding Funding for open access charge: Universidad de Granada / CBUA

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