



Semiregular and strongly irregular boundary points for p -harmonic functions on unbounded sets in metric spaces

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Received: 16 July 2020 / Accepted: 15 February 2021 / Published online: 28 July 2021
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Abstract

The trichotomy between regular, semiregular, and strongly irregular boundary points for p -harmonic functions is obtained for unbounded open sets in complete metric spaces with a doubling measure supporting a p -Poincaré inequality, $1 < p < \infty$. We show that these are local properties. We also deduce several characterizations of semiregular points and strongly irregular points. In particular, semiregular points are characterized by means of capacity, p -harmonic measures, removability, and semibarriers.

Keywords Boundary regularity · Metric space · p -Harmonic function · Semibarrier · Semiregular boundary point · Strongly irregular boundary point

Mathematics Subject Classification Primary 31E05 · Secondary 30L99 · 35J66 · 35J92 · 49Q20

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set and let $f \in C(\partial\Omega)$. The Perron method provides us with a unique function Pf that is harmonic in Ω and takes the boundary values f in a weak sense, i.e., Pf is a solution of the Dirichlet problem for the Laplace equation $\Delta u = 0$. It was introduced on \mathbb{R}^2 in 1923 by Perron [25] and independently by Remak [26]. A point $x_0 \in \partial\Omega$ is *regular* if $\lim_{\Omega \ni y \rightarrow x_0} Pf(y) = f(x_0)$ for every $f \in C(\partial\Omega)$. Wiener [28] characterized regular boundary points by means of the *Wiener criterion* in 1924. In the same year Lebesgue [22] gave a different characterization using barriers.

This definition of boundary regularity can be paraphrased in the following way: The point $x_0 \in \partial\Omega$ is regular if the following two conditions hold:

- (i) For all $f \in C(\partial\Omega)$ the limit $\lim_{\Omega \ni y \rightarrow x_0} Pf(y)$ exists.
- (ii) For all $f \in C(\partial\Omega)$ there is a sequence $\Omega \ni y_j \rightarrow x_0$ such that $\lim_{j \rightarrow \infty} Pf(y_j) = f(x_0)$.

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Perhaps surprisingly, it is the case that for irregular boundary points *exactly one* of these two properties fails; one might have guessed that both can fail at the same time but this can in fact never happen. A boundary point $x_0 \in \partial\Omega$ is *semiregular* if the first condition holds but not the second; and *strongly irregular* if the second condition holds but not the first.

For the Laplace equation it is well known that all boundary points are either regular, semiregular, or strongly irregular, and this trichotomy (in an abstract linear setting) was developed in detail in Lukeš–Malý [23]. Key examples of semiregular and strongly irregular points are Zaremba's punctured ball and the Lebesgue spine, respectively, see Examples 13.3 and 13.4 in [6].

A nonlinear analogue is to consider the Dirichlet problem for p -harmonic functions, which are solutions of the p -Laplace equation $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$, $1 < p < \infty$. This leads to a nonlinear potential theory that has been studied since the 1960s. Initially, it was developed for \mathbb{R}^n , but it has also been extended to weighted \mathbb{R}^n , Riemannian manifolds, and other settings. In more recent years, it has been generalized to metric spaces, see, e.g., the monograph Björn–Björn [6] and the references therein. The Perron method was extended to such metric spaces by Björn–Björn–Shanmugalingam [9] for bounded open sets and Hansevi [16] for unbounded open sets.

Boundary regularity for p -harmonic functions on metric spaces was first studied by Björn [13] and Björn–MacManus–Shanmugalingam [14], and a rather extensive study was undertaken by Björn–Björn [5] on bounded open sets. Recently this theory was generalized to unbounded open sets by Björn–Hansevi [11]; see also Björn–Björn–Li [7]. For further references and a historical discussion on regularity for p -harmonic functions we refer the interested reader to the introduction in [11].

For p -harmonic functions on \mathbb{R}^n and metric spaces the trichotomy was obtained by Björn [4] for bounded open sets. It was also obtained for unbounded sets in certain Ahlfors regular metric spaces by Björn–Björn–Li [7]. Adamowicz–Björn–Björn [1] obtained the trichotomy for $p(\cdot)$ -harmonic functions on bounded open sets in \mathbb{R}^n .

In this paper we obtain the trichotomy in the following form, where regularity is defined using upper Perron solutions (Definition 5.1). (We use upper Perron solutions as it is not known whether continuous functions are resolutive with respect to unbounded p -hyperbolic sets.)

Theorem 1.1 (Trichotomy) *Assume that X is a complete metric space equipped with a doubling measure supporting a p -Poincaré inequality, $1 < p < \infty$. Let $\Omega \subset X$ be a nonempty (possibly unbounded) open set with the capacity $C_p(X \setminus \Omega) > 0$. If Ω is unbounded, then the point at infinity, ∞ , is considered to belong to the boundary $\partial\Omega$ in the one-point compactification $X^* := X \cup \{\infty\}$.*

Let $x_0 \in \partial\Omega \setminus \{\infty\}$. Then x_0 is either regular, semiregular, or strongly irregular for functions that are p -harmonic in Ω . Moreover,

- x_0 is strongly irregular if and only if $x_0 \in \overline{R} \setminus R$, where

$$R := \{x \in \partial\Omega \setminus \{\infty\} : x \text{ is regular}\}$$

and \overline{R} is the closure of R .

- *The relatively open set*

$$S := \{x \in \partial\Omega \setminus \{\infty\} : \text{there is } r > 0 \text{ such that } C_p(B(x, r) \cap \partial\Omega) = 0\} \quad (1.1)$$

consists exactly of all semiregular boundary points of $\partial\Omega \setminus \{\infty\}$.

The importance of the distinction between semiregular and strongly irregular boundary points is perhaps best illustrated by the equivalent characterizations given in Theorems 6.4 and 6.5. Semiregular points are in some ways not seen by Perron solutions.

Our contribution here is to extend the results in [4] to unbounded open sets. In order to do so there are extra complications, most notably the fact that it is not known whether continuous functions are resolutive with respect to unbounded p -hyperbolic sets. We will also rely on the recent results by Björn–Hansevi [11] on regularity for p -harmonic functions on unbounded sets in metric spaces. Most of our results are new also on unweighted \mathbb{R}^n .

2 Notation and preliminaries

We assume that (X, d, μ) is a metric measure space (which we simply refer to as X) equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for every ball $B \subset X$. It follows that X is second countable. For balls $B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ and $\lambda > 0$, we let $\lambda B = \lambda B(x_0, r) := B(x_0, \lambda r)$. The σ -algebra on which μ is defined is the completion of the Borel σ -algebra. We also assume that $1 < p < \infty$. Later we will impose further requirements on the space and on the measure. We will keep the discussion short, see the monographs Björn–Björn [6] and Heinonen–Koskela–Shanmugalingam–Tyson [18] for proofs, further discussion, and references on the topics in this section.

The measure μ is *doubling* if there exists a constant $C \geq 1$ such that

$$0 < \mu(2B) \leq C\mu(B) < \infty$$

for every ball $B \subset X$. A metric space is *proper* if all bounded closed subsets are compact, and this is in particular true if the metric space is complete and the measure is doubling.

We say that a property holds for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e., there exists a nonnegative $\rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for every curve $\gamma \in \Gamma$. For us, a curve in X is a rectifiable nonconstant continuous mapping from a compact interval into X , and it can thus be parametrized by its arc length ds .

Following Koskela–MacManus [21] we make the following definition, see also Heinonen–Koskela [17].

Definition 2.1 A measurable function $g : X \rightarrow [0, \infty]$ is a *p -weak upper gradient* of the function $u : X \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ if

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g \, ds$$

for p -almost every curve $\gamma : [0, l_\gamma] \rightarrow X$, where we use the convention that the left-hand side is ∞ whenever at least one of the terms on the left-hand side is infinite.

One way of controlling functions by their p -weak upper gradients is to require a Poincaré inequality to hold.

Definition 2.2 We say that X supports a p -Poincaré inequality if there exist constants, $C > 0$ and $\lambda \geq 1$ (the dilation constant), such that for all balls $B \subset X$, all integrable functions u on X , and all p -weak upper gradients g of u ,

$$\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C \operatorname{diam}(B) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p d\mu \right)^{1/p}, \tag{2.1}$$

where $u_B := \frac{1}{\mu(B)} \int_B u d\mu$.

Shanmugalingam [27] used p -weak upper gradients to define so-called Newtonian spaces.

Definition 2.3 The Newtonian space on X , denoted $N^{1,p}(X)$, is the space of all extended real-valued functions $u \in L^p(X)$ such that

$$\|u\|_{N^{1,p}(X)} := \left(\int_X |u|^p d\mu + \inf_g \int_X g^p d\mu \right)^{1/p} < \infty,$$

where the infimum is taken over all p -weak upper gradients g of u .

The quotient space $N^{1,p}(X)/\sim$, where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, is a Banach space, see Shanmugalingam [27].

Definition 2.4 The Dirichlet space on X , denoted $D^p(X)$, is the space of all measurable extended real-valued functions on X that have a p -weak upper gradient in $L^p(X)$.

In this paper we assume that functions in $N^{1,p}(X)$ and $D^p(X)$ are defined everywhere (with values in $\overline{\mathbb{R}}$), not just up to an equivalence class. This is important, in particular for the definition of p -weak upper gradients to make sense.

A measurable set $A \subset X$ can itself be considered to be a metric space (with the restriction of d and μ to A) with the Newtonian space $N^{1,p}(A)$ and the Dirichlet space $D^p(A)$ given by Definitions 2.3 and 2.4, respectively. If X is proper and $\Omega \subset X$ is open, then $u \in N^{1,p}_{\text{loc}}(\Omega)$ if and only if $u \in N^{1,p}(V)$ for every open V such that \overline{V} is a compact subset of Ω , and similarly for $D^p_{\text{loc}}(\Omega)$. In general, the local spaces $N^{1,p}_{\text{loc}}(\Omega)$ and $D^p_{\text{loc}}(\Omega)$ are not equal. However, under the assumptions given at the beginning of Sect. 3, it is true that $N^{1,p}_{\text{loc}}(\Omega) = D^p_{\text{loc}}(\Omega)$, by Proposition 4.14 in [6].

If $u \in D^p_{\text{loc}}(X)$, then there exists a minimal p -weak upper gradient $g_u \in L^p_{\text{loc}}(X)$ of u such that $g_u \leq g$ a.e. for all p -weak upper gradients $g \in L^p_{\text{loc}}(X)$ of u .

Definition 2.5 The (Sobolev) capacity of a set $E \subset X$ is the number

$$C_p(E) := \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E .

A property that holds for all points except for those in a set of capacity zero is said to hold *quasieverywhere* (q.e.).

The capacity is countably subadditive, and it is the correct gauge for distinguishing between two Newtonian functions: If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in N_{\text{loc}}^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e.

Continuous functions will be assumed to be real-valued unless otherwise stated, whereas semicontinuous functions are allowed to take values in $\overline{\mathbb{R}}$. We use the common notation $u_+ = \max\{u, 0\}$, let χ_E denote the characteristic function of the set E , and consider all neighbourhoods to be open.

3 The obstacle problem and p -harmonic functions

We assume from now on that $1 < p < \infty$, that X is a complete metric measure space supporting a p -Poincaré inequality, that μ is doubling, and that $\Omega \subset X$ is a nonempty (possibly unbounded) open subset with $C_p(X \setminus \Omega) > 0$.

Definition 3.1 A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is a *minimizer* in Ω if

$$\int_{\varphi \neq 0} g_u^p d\mu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu \quad \text{for all } \varphi \in N_0^{1,p}(\Omega),$$

where $N_0^{1,p}(\Omega) = \{u|_{\Omega} : u \in N^{1,p}(X) \text{ and } u = 0 \text{ in } X \setminus \Omega\}$. Moreover, a function is *p -harmonic* if it is a continuous minimizer.

Kinnunen–Shanmugalingam [20, Proposition 3.3 and Theorem 5.2] used De Giorgi's method to show that every minimizer u has a Hölder continuous representative \tilde{u} such that $\tilde{u} = u$ q.e. Björn–Marola [12, p. 362] obtained the same conclusions using Moser iterations. See alternatively Theorems 8.13 and 8.14 in [6].

The following obstacle problem is an important tool. In this generality, it was considered by Hansevi [15].

Definition 3.2 Let $V \subset X$ be a nonempty open subset with $C_p(X \setminus V) > 0$. For $\psi : V \rightarrow \overline{\mathbb{R}}$ and $f \in D^p(V)$, let

$$\mathcal{K}_{\psi,f}(V) = \{v \in D^p(V) : v - f \in D_0^p(V) \text{ and } v \geq \psi \text{ q.e. in } V\},$$

where $D_0^p(V) = \{u|_V : u \in D^p(X) \text{ and } u = 0 \text{ in } X \setminus V\}$. We say that $u \in \mathcal{K}_{\psi,f}(V)$ is a *solution of the $\mathcal{K}_{\psi,f}(V)$ -obstacle problem (with obstacle ψ and boundary values f)* if

$$\int_V g_u^p d\mu \leq \int_V g_v^p d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(V).$$

When $V = \Omega$, we usually denote $\mathcal{K}_{\psi,f}(\Omega)$ by $\mathcal{K}_{\psi,f}$.

The $\mathcal{K}_{\psi,f}$ -obstacle problem has a unique (up to sets of capacity zero) solution whenever $\mathcal{K}_{\psi,f} \neq \emptyset$, see Hansevi [15, Theorem 3.4]. Furthermore, there is a unique lsc-regularized solution of the $\mathcal{K}_{\psi,f}$ -obstacle problem, by Theorem 4.1 in [15]. A function u is *lsc-regularized* if $u = u^*$, where the *lsc-regularization* u^* of u in Ω is defined by

$$u^*(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y) := \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{B(x,r)} u, \quad x \in \Omega.$$

If $\psi : \Omega \rightarrow [-\infty, \infty)$ is continuous as an extended real-valued function, and $\mathcal{K}_{\psi,f} \neq \emptyset$, then the lsc-regularized solution of the $\mathcal{K}_{\psi,f}$ -obstacle problem is continuous, by Theorem 4.4 in [15]. Hence the following generalization of Definition 3.3 in Björn–Björn–Shanmugalingam [8] (and Definition 8.31 in [6]) to Dirichlet functions and to unbounded sets makes sense. It was first used by Hansevi [15, Definition 4.6].

Definition 3.3 Let $V \subset X$ be a nonempty open set with $C_p(X \setminus V) > 0$. The p -harmonic extension $H_{V,f}$ of $f \in D^p(V)$ to V is the continuous solution of the $\mathcal{K}_{-\infty,f}(V)$ -obstacle problem. When $V = \Omega$, we usually write Hf instead of $H_{\Omega}f$.

Definition 3.4 A function $u : \Omega \rightarrow (-\infty, \infty]$ is *superharmonic* in Ω if

- (i) u is lower semicontinuous;
- (ii) u is not identically ∞ in any component of Ω ;
- (iii) for every nonempty open set V such that \bar{V} is a compact subset of Ω and all $v \in \operatorname{Lip}(V)$, we have $H_V v \leq u$ in V whenever $v \leq u$ on ∂V .

A function $u : \Omega \rightarrow [-\infty, \infty)$ is *subharmonic* if $-u$ is superharmonic.

There are several other equivalent definitions of superharmonic functions, see, e.g., Theorem 6.1 in Björn [2] (or Theorem 9.24 and Propositions 9.25 and 9.26 in [6]).

An lsc-regularized solution of the obstacle problem is always superharmonic, by Proposition 3.9 in [15] together with Proposition 7.4 in Kinnunen–Martio [19] (or Proposition 9.4 in [6]). On the other hand, superharmonic functions are always lsc-regularized, by Theorem 7.14 in Kinnunen–Martio [19] (or Theorem 9.12 in [6]).

4 Perron solutions

In addition to the assumptions given at the beginning of Sect. 3, from now on we make the convention that if Ω is unbounded, then the point at infinity, ∞ , belongs to the boundary $\partial\Omega$. Topological notions should therefore be understood with respect to the one-point compactification $X^ := X \cup \{\infty\}$.*

Note that this convention does not affect any of the definitions in Sect. 2 or 3, as ∞ is not added to X (it is added solely to $\partial\Omega$).

Since continuous functions are assumed to be real-valued, every function in $C(\partial\Omega)$ is bounded even if Ω is unbounded. Note that since X is second countable, so is X^* , and hence X^* is metrizable by Urysohn’s metrization theorem, see, e.g., Munkres [24, Theorems 32.3 and 34.1].

We will only consider Perron solutions and p -harmonic measures with respect to Ω and therefore omit Ω from the notation below.

Definition 4.1 Given a function $f : \partial\Omega \rightarrow \bar{\mathbb{R}}$, let \mathcal{U}_f be the collection of all functions u that are superharmonic in Ω , bounded from below, and such that

$$\liminf_{\Omega \ni y \rightarrow x} u(y) \geq f(x) \quad \text{for all } x \in \partial\Omega.$$

The upper Perron solution of f is defined by

$$\overline{P}f(x) = \inf_{u \in \mathcal{U}_f} u(x), \quad x \in \Omega.$$

The lower Perron solution can be defined similarly using subharmonic functions, or by letting $\underline{P}f = -\overline{P}(-f)$. If $\overline{P}f = \underline{P}f$, then we denote the common value by Pf . Moreover, if Pf is real-valued, then f is said to be *resolutive* (with respect to Ω).

An immediate consequence of the definition is that $\overline{P}f \leq \overline{P}h$ whenever $f \leq h$ on $\partial\Omega$. Moreover, if $\alpha \in \mathbb{R}$ and $\beta \geq 0$, then $\overline{P}(\alpha + \beta f) = \alpha + \beta \overline{P}f$. Corollary 6.3 in Hansevi [16] shows that $\underline{P}f \leq \overline{P}f$. In each component of Ω , $\overline{P}f$ is either p -harmonic or identically $\pm\infty$, by Theorem 4.1 in Björn–Björn–Shanmugalingam [9] (or Theorem 10.10 in [6]); the proof is local and applies also to unbounded Ω .

Definition 4.2 Assume that Ω is unbounded. Then Ω is p -parabolic if for every compact $K \subset \Omega$, there exist functions $u_j \in N^{1,p}(\Omega)$ such that $u_j \geq 1$ on K for all $j = 1, 2, \dots$, and

$$\int_{\Omega} g_{u_j}^p d\mu \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Otherwise, Ω is p -hyperbolic.

For examples of p -parabolic sets, see, e.g., Hansevi [16]. The main reason for introducing p -parabolic sets in [16] was to be able to obtain resolutive results, and in particular, establishing the following resolutive and invariance result for p -parabolic unbounded sets. The first such invariance result for p -harmonic functions was obtained, for bounded sets, by Björn–Björn–Shanmugalingam [9].

Theorem 4.3 ([9, Theorem 6.1] and [16, Theorem 7.8]) *Assume that Ω is bounded or p -parabolic. Let $h : \partial\Omega \rightarrow \overline{\mathbb{R}}$ be 0 q.e. on $\partial\Omega \setminus \{\infty\}$ and $f \in C(\partial\Omega)$. Then f and $f + h$ are resolutive and $P(f + h) = Pf$.*

Resolutivity of continuous functions is not known for unbounded p -hyperbolic sets, but it is rather trivial to show that constant functions are resolutive. We shall show that a similar invariance result as in Theorem 4.3 can be obtained for constant functions on unbounded p -hyperbolic sets. This fact will be an important tool when characterizing semiregular boundary points.

We first need to define p -harmonic measures, which despite the name are (usually) not measures, but nonlinear generalizations of the harmonic measure.

Definition 4.4 The upper and lower p -harmonic measures of $E \subset \partial\Omega$ are

$$\overline{\omega}(E) := \overline{P}\chi_E \quad \text{and} \quad \underline{\omega}(E) := \underline{P}\chi_E,$$

respectively.

Proposition 4.5 *Let $E \subset \partial\Omega \setminus \{\infty\}$, $a \in \mathbb{R}$, and $f : \partial\Omega \rightarrow \overline{\mathbb{R}}$ be such that $C_p(E) = 0$ and $f(x) = a$ for all $x \in \partial\Omega \setminus E$. Then $Pf \equiv a$.*

In particular, $\overline{\omega}(E) = \underline{\omega}(E) \equiv 0$.

Proof Without loss of generality we may assume that $a = 0$. As the capacity C_p is an outer capacity, by Corollary 1.3 in Björn–Björn–Shanmugalingam [10] (or [6, Theorem 5.31]), we can find open sets $G'_j \supset E$ such that $C_p(G'_j) < 2^{-j-1}$, $j = 1, 2, \dots$. From the decreasing sequence $\{\bigcup_{k=j}^\infty G'_k\}_{j=1}^\infty$, we can choose a decreasing subsequence of open sets G_k with $C_p(G_k) < 2^{-kp}$, $k = 1, 2, \dots$. By Lemma 5.3 in Björn–Björn–Shanmugalingam [9] (or [6, Lemma 10.17]), there is a decreasing sequence $\{\psi_j\}_{j=1}^\infty$ of nonnegative functions such that $\lim_{j \rightarrow \infty} \|\psi_j\|_{N^{1,p}(X)} = 0$ and $\psi_j \geq k - j$ in G_k whenever $k > j$. In particular, $\psi_j = \infty$ on E for each $j = 1, 2, \dots$.

Let u_j be the lsc-regularized solution of the $\mathcal{H}_{\psi_j,0}(\Omega)$ -obstacle problem, $j = 1, 2, \dots$. As u_j is lsc-regularized and $u_j \geq \psi_j$ q.e., we see that $u_j \geq k - j$ everywhere in G_k whenever $k > j$, and also that $u_j \geq 0$ everywhere in Ω . In particular, $\liminf_{\Omega \ni y \rightarrow x} u_j(y) = \infty$ for $x \in E$, which shows that $u_j \in \mathcal{U}_f(\Omega)$ and thus $u_j \geq \overline{P}f$.

On the other hand, Theorem 3.2 in Hansevi [16] shows that the sequence u_j decreases q.e. to 0, and hence $\overline{P}f \leq 0$ q.e. in Ω . Since $\overline{P}f$ is continuous, we get that $\overline{P}f \leq 0$ everywhere in Ω . Applying this to $-f$ shows that $\underline{P}f = -\overline{P}(-f) \geq 0$ everywhere in Ω , which together with the inequality $\underline{P}f \leq \overline{P}f$ shows that $\underline{P}f = \overline{P}f \equiv 0$. In particular, $\underline{\omega}(E) = \underline{P}\chi_E \equiv 0$ and $\overline{\omega}(E) = \overline{P}\chi_E \equiv 0$. □

We will also need the following result.

Proposition 4.6 *If $f : \partial\Omega \rightarrow [-\infty, \infty)$ is an upper semicontinuous function, then*

$$\overline{P}f = \inf_{C(\partial\Omega) \ni \varphi \geq f} \overline{P}\varphi.$$

Proof Let $\mathcal{F} = \{\varphi \in C(\partial\Omega) : \varphi \geq f\}$. Then \mathcal{F} is downward directed, i.e., for each pair of functions $u, v \in \mathcal{F}$ there is a function $w \in \mathcal{F}$ such that $w \leq \min\{u, v\}$. Because f is upper semicontinuous, $\partial\Omega$ is compact, and X^* is metrizable, it follows from Proposition 1.12 in [6] that $f = \inf_{\varphi \in \mathcal{F}} \varphi$. Hence by Lemma 10.31 in [6] (whose proof is valid also for unbounded Ω) $\overline{P}f = \inf_{\varphi \in \mathcal{F}} \overline{P}\varphi$. □

5 Boundary regularity

It is not known whether continuous functions are resolutive also with respect to unbounded p -hyperbolic sets. We therefore define regular boundary points in the following way.

Definition 5.1 We say that a boundary point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}f(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

This can be paraphrased in the following way: A point $x_0 \in \partial\Omega$ is regular if the following two conditions hold:

- (I) For all $f \in C(\partial\Omega)$ the limit

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}f(y) \quad \text{exists.}$$

(II) For all $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^\infty$ in Ω such that

$$\lim_{j \rightarrow \infty} y_j = x_0 \quad \text{and} \quad \lim_{j \rightarrow \infty} \bar{P}f(y_j) = f(x_0).$$

Furthermore, we say that a boundary point $x_0 \in \partial\Omega$ is *semiregular* if (I) holds but not (II); and *strongly irregular* if (II) holds but not (I).

We do not require Ω to be bounded in this definition, but if it is, then it follows from Theorem 6.1 in Björn–Björn–Shanmugalingam [9] (or Theorem 10.22 in [6]) that our definition coincides with the definitions of regularity in Björn–Björn–Shanmugalingam [8, 9], and Björn–Björn [5, 6], where regularity is defined using Pf or Hf . Thus we can use the boundary regularity results from these papers when considering bounded sets.

Since $\bar{P}f = -\underline{P}(-f)$, the same concept of regularity is obtained if we replace the upper Perron solution by the lower Perron solution in Definition 5.1.

Boundary regularity for p -harmonic functions on unbounded sets in metric spaces was recently studied by Björn–Hansevi [11]. We will need some of the characterizations obtained therein. For the reader’s convenience we state these results here. We will not discuss regularity of the point ∞ in this paper. One of the important results we will need from [11] is the Kellogg property.

Theorem 5.2 (The Kellogg property) *If I is the set of irregular points in $\partial\Omega \setminus \{\infty\}$, then $C_p(I) = 0$.*

Definition 5.3 A function u is a *barrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if

- (i) u is superharmonic in Ω ;
- (ii) $\lim_{\Omega \ni y \rightarrow x_0} u(y) = 0$;
- (iii) $\liminf_{\Omega \ni y \rightarrow x} u(y) > 0$ for every $x \in \partial\Omega \setminus \{x_0\}$.

Superharmonic functions satisfy the strong minimum principle, i.e., if u is superharmonic and attains its minimum in some component G of Ω , then $u|_G$ is constant (see Theorem 9.13 in [6]). This implies that a barrier is always nonnegative, and furthermore, that a barrier is positive if $\partial G \setminus \{x_0\} \neq \emptyset$ for every component $G \subset \Omega$.

The following result is a collection of the key facts we will need from Björn–Hansevi [11, Theorems 5.2, 5.3, 6.2, and 9.1].

Theorem 5.4 *Let $x_0 \in \partial\Omega \setminus \{\infty\}$ and $\delta > 0$. Also define $d_{x_0} : X^* \rightarrow [0, 1]$ by*

$$d_{x_0}(x) = \begin{cases} \min\{d(x, x_0), 1\} & \text{if } x \neq \infty, \\ 1 & \text{if } x = \infty. \end{cases} \tag{5.1}$$

Then the following are equivalent:

- (a) *The point x_0 is regular.*
- (b) *There is a barrier at x_0 .*
- (c) *There is a positive continuous barrier at x_0 .*
- (d) *The point x_0 is regular with respect to $\Omega \cap B(x_0, \delta)$.*
- (e) *It is true that*

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}f(y) = f(x_0)$$

for all $f : \partial\Omega \rightarrow \mathbb{R}$ that are bounded on $\partial\Omega$ and continuous at x_0 .

(f) It is true that

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}d_{x_0}(y) = 0.$$

(g) The continuous solution u of the $\mathcal{K}_{d_{x_0}, d_{x_0}}$ -obstacle problem, satisfies

$$\lim_{\Omega \ni y \rightarrow x_0} u(y) = 0.$$

(h) If $f \in C(\overline{\Omega}) \cap D^p(\Omega)$, then the continuous solution u of the $\mathcal{K}_{f, f}$ -obstacle problem, satisfies

$$\lim_{\Omega \ni y \rightarrow x_0} u(y) = f(x_0).$$

6 Semiregular and strongly irregular points

We are now ready to start our discussion of semiregular and strongly irregular boundary points. We begin by proving Theorem 1.1.

Proof of Theorem 1.1 We consider two complementary cases.

Case 1: There exists $r > 0$ such that $C_p(B \cap \partial\Omega) = 0$, where $B := B(x_0, r)$.

Let G be the component of B containing x_0 . Since X is quasiconvex, by, e.g., Theorem 4.32 in [6], and thus locally connected, it follows that G is open. Let $F = G \setminus \Omega$. Then

$$C_p(G \cap \partial F) = C_p(G \cap \partial\Omega) \leq C_p(B \cap \partial\Omega) = 0,$$

and hence $C_p(F) = 0$, by Lemma 8.6 in Björn–Björn–Shanmugalingam [9] (or Lemma 4.5 in [6]).

Let $f \in C(\partial\Omega)$. Then the Perron solution $\overline{P}f$ is bounded (as f is bounded), and thus $\overline{P}f$ has a p -harmonic extension U to $\Omega \cup G$, by Theorem 6.2 in Björn [3] (or Theorem 12.2 in [6]). Since U is continuous, it follows that

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}f(y) = \lim_{\Omega \ni y \rightarrow x_0} U(y) = U(x_0),$$

i.e., condition (I) in Definition 5.1 holds, and hence x_0 is either regular or semiregular.

To show that x_0 must be semiregular, we let $f(x) = (1 - d_{x_0}(x)/\min\{r, 1\})_+$ on $\partial\Omega$, where d_{x_0} is defined by (5.1). Then $f = 0$ q.e. on $\partial\Omega$, and Proposition 4.5 shows that $Pf \equiv 0$. Since

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}f(y) = 0 \neq 1 = f(x_0),$$

x_0 is not regular, and hence must be semiregular.

Case 2: For all $r > 0$, $C_p(B(x_0, r) \cap \partial\Omega) > 0$.

For every $j = 1, 2, \dots$, $C_p(B(x_0, 1/j) \cap \partial\Omega) > 0$, and by the Kellogg property (Theorem 5.2) there exists a regular boundary point $x_j \in B(x_0, 1/j) \cap \partial\Omega$. (We do not require the x_j to be distinct.)

Let $f \in C(\partial\Omega)$. Because x_j is regular, there is $y_j \in B(x_j, 1/j) \cap \Omega$ so that $|\overline{P}f(y_j) - f(x_j)| < 1/j$. It follows that $y_j \rightarrow x_0$ and $\overline{P}f(y_j) \rightarrow f(x_0)$ as $j \rightarrow \infty$, i.e., condition (II) in Definition 5.1 holds, and hence x_0 must be either regular or strongly irregular.

As there are no strongly irregular points in case 1, it follows that $x_0 \in \partial\Omega \setminus \{\infty\}$ is strongly irregular if and only if $x_0 \in \overline{R} \setminus R$, where $R := \{x \in \partial\Omega \setminus \{\infty\} : x \text{ is regular}\}$ and \overline{R} is the closure of R . And since there are no semiregular points in case 2, the set S in (1.1) consists exactly of all semiregular boundary points of $\partial\Omega \setminus \{\infty\}$. □

In fact, in case 2 it is possible to improve upon the result above. The sequence $\{y_j\}_{j=1}^\infty$ can be chosen independently of f , see the characterization (c) in Theorem 6.4.

We will characterize semiregular points by a number of equivalent conditions in Theorem 6.4. But first we obtain the following characterizations of relatively open sets of semiregular points.

Theorem 6.1 *Let $V \subset \partial\Omega \setminus \{\infty\}$ be relatively open. Then the following statements are equivalent:*

- (a) *The set V consists entirely of semiregular points.*
- (b) *The set V does not contain any regular point.*
- (c) *The capacity $C_p(V) = 0$.*
- (d) *The upper p -harmonic measure $\overline{\omega}(V) \equiv 0$.*
- (e) *The lower p -harmonic measure $\underline{\omega}(V) \equiv 0$.*
- (f) *The set $\Omega \cup V$ is open in X , $C_p(X \setminus (\Omega \cup V)) > 0$, $\mu(V) = 0$, and every function that is bounded and superharmonic in Ω has a superharmonic extension to $\Omega \cup V$.*
- (g) *The set $\Omega \cup V$ is open in X , $C_p(X \setminus (\Omega \cup V)) > 0$, and every function that is bounded and p -harmonic in Ω has a p -harmonic extension to $\Omega \cup V$.*

If moreover Ω is bounded or p -parabolic, then also the following statements are equivalent to the statements above.

- (h) *For every function f on $\partial\Omega$ that is continuous on $\partial\Omega \setminus V$, the Perron solution Pf depends only on $f|_{\partial\Omega \setminus V}$ (i.e., if f_1 and f_2 are functions on $\partial\Omega$ that are continuous on $\partial\Omega \setminus V$ and such that $f_1 = f_2$ on $\partial\Omega \setminus V$, then $Pf_1 \equiv Pf_2$).*
- (i) *For every $f \in C(\partial\Omega)$, the Perron solution Pf depends only on $f|_{\partial\Omega \setminus V}$.*

Note that there are examples of sets with positive capacity and even positive measure which are removable for bounded p -harmonic functions, see Section 9 in Björn [3] (or [6, Section 12.3]). For superharmonic functions it is not known whether such examples exist. This motivates the formulations of (f) and (g).

The following example shows that the condition $C_p(X \setminus (\Omega \cup V)) > 0$ cannot be dropped from (g), nor from (j) in Theorem 6.4 below. We do not know whether the conditions $C_p(X \setminus (\Omega \cup V)) > 0$ and $\mu(V) = 0$ can be dropped from (f), but they are needed for our proof. Similarly they are needed in (l) in Theorem 6.4 below.

The condition $C_p(X \setminus (\Omega \cup V)) > 0$ was unfortunately overlooked in Björn [4] and in Björn–Björn [6]: It should be added to conditions (d') and (e') in [4, Theorem 3.1], to (h) and (i) in [4, Theorem 3.3], to (f') and (g') in [6, Theorem 13.5], and to (j) and (l) in [6, Theorem 13.10].

Example 6.2 Let $X = [0, 1]$ be equipped with the Lebesgue measure, and let $1 < p < \infty$, $\Omega = (0, 1]$ and $V = \{0\}$. Then $C_p(V) > 0$. In this case the p -harmonic functions on Ω are just the constant functions, and these trivially have p -harmonic extensions to X . Thus the condition $C_p(X \setminus (\Omega \cup V)) > 0$ cannot be dropped from (g).

On the other hand, the set V is not removable for bounded superharmonic functions on Ω , see Example 9.1 in Björn [3] or Example 12.17 in [6].

Proof of Theorem 6.1 (b) \Rightarrow (c) This follows from the Kellogg property (Theorem 5.2).

(c) \Rightarrow (d) This follows directly from Proposition 4.5.

(d) \Rightarrow (e) This is trivial.

(e) \Rightarrow (b) Suppose that $x \in V$ is regular. Because χ_V is continuous at x , this yields a contradiction, as it follows from Theorem 5.4 that

$$0 = \lim_{\Omega \ni y \rightarrow x} \underline{\omega}(V)(y) = \lim_{\Omega \ni y \rightarrow x} P \chi_V(y) = - \lim_{\Omega \ni y \rightarrow x} \bar{P}(-\chi_V)(y) = \chi_V(x) = 1.$$

Thus V does not contain any regular point.

(c) \Rightarrow (f) Suppose that $C_p(V) = 0$. Then $C_p(X \setminus (\Omega \cup V)) = C_p(X \setminus \Omega) > 0$ and $\mu(V) = 0$. Let $x \in V$ and let G be a connected neighbourhood of x such that $G \cap \partial\Omega \subset V$. Sets of capacity zero cannot separate space, by Lemma 4.6 in Björn–Björn [6], and hence $G \setminus \partial\Omega$ must be connected, i.e., $G \subset \bar{\Omega}$, from which it follows that $\Omega \cup V$ is open in X . The superharmonic extension is now provided by Theorem 6.3 in Björn [3] (or Theorem 12.3 in [6]).

(f) \Rightarrow (g) Let u be a bounded p -harmonic function on Ω . Then, by assumption, u has a superharmonic extension U to $\Omega \cup V$. Moreover, as $-u$ is also bounded and p -harmonic, there is a superharmonic extension W of $-u$ to $\Omega \cup V$. Now, as $-W$ is clearly a subharmonic extension of u to $\Omega \cup V$, Proposition 6.5 in Björn [3] (or Proposition 12.5 in [6]) asserts that $U = -W$ is p -harmonic (it is here that we use that $\mu(V) = 0$).

(g) \Rightarrow (a) Let $x_0 \in V$. Since $\Omega \cup V$ is open in X , we see that $V \cap \partial(\Omega \cup V) = \emptyset$, and hence $x_0 \notin \partial(\Omega \cup V)$. Let

$$h(x) = \left(1 - \frac{d_{x_0}(x)}{\text{dist}_{x_0}(\partial(\Omega \cup V))_+} \right), \quad x \in \partial\Omega,$$

where d_{x_0} is defined by (5.1), and $\text{dist}_{x_0}(E) := \inf_{y \in E} d_{x_0}(y)$, $E \subset X^*$. Then $\bar{P}h$ is bounded and has a p -harmonic extension U to $\Omega \cup V$, and hence the Kellogg property (Theorem 5.2) implies that

$$\lim_{\Omega \cup V \ni y \rightarrow x} U(y) = \lim_{\Omega \ni y \rightarrow x} \bar{P}h(y) = h(x) = 0 \quad \text{for q.e. } x \in \partial(\Omega \cup V) \setminus \{\infty\}. \tag{6.1}$$

Let G be the component of $\Omega \cup V$ containing x_0 . Then

$$C_p(X \setminus G) \geq C_p(X \setminus (\Omega \cup V)) > 0.$$

It then follows from Lemma 4.3 in Björn–Björn [5] (or Lemma 4.5 in [6]) that $C_p(\partial G) > 0$. In particular, it follows from (6.1) that $U \not\equiv 1$ in G , and thus, by the strong maximum principle (see Corollary 6.4 in Kinnunen–Shanmugalingam [20] or [6, Theorem 8.13]), that $U(x_0) < 1$. Therefore

$$\lim_{\Omega \ni y \rightarrow x_0} \bar{P}h(y) = U(x_0) < 1 = h(x_0),$$

and hence x_0 must be irregular.

However, if $f \in C(\partial\Omega)$, then $\bar{P}f$ has a p -harmonic extension W to $\Omega \cup V$. Since W is continuous in $\Omega \cup V$, it follows that

$$\lim_{\Omega \ni y \rightarrow x_0} \bar{P}f(y) = W(x_0),$$

and hence the limit on the left-hand side always exists. Thus x_0 is semiregular.

(a) \Rightarrow (b) This is trivial.

We now assume that Ω is bounded or p -parabolic.

(c) \Rightarrow (h) Suppose that the functions f_1 and f_2 on $\partial\Omega$ are continuous on $\partial\Omega \setminus V$ and that $f_1 = f_2$ on $\partial\Omega \setminus V$. Since V is relatively open in $\partial\Omega \setminus \{\infty\}$, by Urysohn's lemma, see, e.g., Munkres [24, Theorem 33.1], there exists $f \in C(\partial\Omega)$ such that $f = f_1 = f_2$ on $\partial\Omega \setminus V$. Let $h_1 = f_1 - f$ and $h_2 = f_2 - f$. Then h_1 and h_2 are equal to 0 q.e. on $\partial\Omega$, as $C_p(V) = 0$. By Theorem 4.3, it follows that $Pf_1 = P(f + h_1) = Pf = P(f + h_2) = Pf_2$.

(h) \Rightarrow (i) This is trivial.

(i) \Rightarrow (e) As $-\chi_V : \partial\Omega \rightarrow \mathbb{R}$ is upper semicontinuous, it follows from Proposition 4.6, and (h), that

$$0 \leq \underline{\omega}(V) = -\bar{P}(-\chi_V) = - \inf_{\substack{\varphi \in C(\partial\Omega) \\ -\chi_V \leq \varphi \leq 0}} \bar{P}\varphi = 0,$$

and hence $\underline{\omega}(V) = 0$. □

Definition 6.3 A function u is a *semibarrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if

- (i) u is superharmonic in Ω ;
- (ii) $\liminf_{\Omega \ni y \rightarrow x_0} u(y) = 0$;
- (iii) $\liminf_{\Omega \ni y \rightarrow x} u(y) > 0$ for every $x \in \partial\Omega \setminus \{x_0\}$.

Moreover, we say that u is a *weak semibarrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if u is a positive superharmonic function such that (ii) holds.

Now we are ready to characterize the semiregular points by means of capacity, p -harmonic measures, removable singularities, and semibarriers. In particular, we show that semiregularity is a local property.

Theorem 6.4 Let $x_0 \in \partial\Omega \setminus \{\infty\}$, $\delta > 0$, and $d_{x_0} : X^* \rightarrow [0, 1]$ be defined by (5.1). Then the following statements are equivalent:

- (a) The point x_0 is semiregular.
- (b) The point x_0 is semiregular with respect to $G := \Omega \cap B(x_0, \delta)$.
- (c) There is no sequence $\{y_j\}_{j=1}^\infty$ in Ω such that $y_j \rightarrow x_0$ as $j \rightarrow \infty$ and

$$\lim_{j \rightarrow \infty} \bar{P}f(y_j) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

- (d) The point x_0 is neither regular nor strongly irregular.
- (e) It is true that $x_0 \notin \{x \in \partial\Omega : x \text{ is regular}\}$.
- (f) There is a neighbourhood V of x_0 such that $C_p(V \cap \partial\Omega) = 0$.
- (g) There is a neighbourhood V of x_0 such that $C_p(V \setminus \Omega) = 0$.

- (h) *There is a neighbourhood V of x_0 such that $\overline{\omega}(V \cap \partial\Omega) \equiv 0$.*
- (i) *There is a neighbourhood V of x_0 such that $\underline{\omega}(V \cap \partial\Omega) \equiv 0$.*
- (j) *There is a neighbourhood $V \subset \overline{\Omega}$ of x_0 , with $C_p(X \setminus (\Omega \cup V)) > 0$, such that every function that is bounded and p -harmonic in Ω has a p -harmonic extension to $\Omega \cup V$.*
- (k) *There is a neighbourhood V of x_0 such that every function that is bounded and p -harmonic in Ω has a p -harmonic extension to $\Omega \cup V$, and moreover x_0 is irregular.*
- (l) *There is a neighbourhood V of x_0 , with $C_p(X \setminus (\Omega \cup V)) > 0$ and $\mu(V \setminus \Omega) = 0$, such that every function that is bounded and superharmonic in Ω has a superharmonic extension to $\Omega \cup V$.*
- (m) *It is true that*

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}d_{x_0}(y) > 0.$$

- (n) *It is true that*

$$\liminf_{\Omega \ni y \rightarrow x_0} \overline{P}d_{x_0}(y) > 0.$$

- (o) *There is no weak semibarrier at x_0 .*
- (p) *There is no semibarrier at x_0 .*
- (q) *The continuous solution of the $\mathcal{H}_{d_{x_0}, d_{x_0}}$ -obstacle problem is not a semibarrier at x_0 .*

If moreover Ω is bounded or p -parabolic, then also the following statements are equivalent to the statements above.

- (r) *There is a neighbourhood V of x_0 such that for every function f on $\partial\Omega$ that is continuous on $\partial\Omega \setminus V$, the Perron solution Pf depends only on $f|_{\partial\Omega \setminus V}$ (i.e., if f_1 and f_2 are functions on $\partial\Omega$ that are continuous on $\partial\Omega \setminus V$ and such that $f_1 = f_2$ on $\partial\Omega \setminus V$, then $Pf_1 \equiv Pf_2$).*
- (s) *There is a neighbourhood V of x_0 such that for every $f \in C(\partial\Omega)$, the Perron solution Pf depends only on $f|_{\partial\Omega \setminus V}$.*

Proof (e) \Leftrightarrow (f) \Leftrightarrow (h) \Leftrightarrow (i) \Rightarrow (a) This follows directly from Theorem 6.1, with V therein corresponding to $V \cap \partial\Omega$ here.

(a) \Rightarrow (m) Since x_0 is semiregular, the limit

$$\alpha := \lim_{\Omega \ni y \rightarrow x_0} \overline{P}d_{x_0}(y)$$

exists. If $\alpha = 0$, then x_0 must be regular by Theorem 5.4, which is a contradiction. Hence $\alpha > 0$.

(m) \Rightarrow (n) \Rightarrow (d) \Rightarrow (c) These implications are trivial.

\neg (e) \Rightarrow \neg (c) Suppose that $x_0 \in \{x \in \partial\Omega : x \text{ is regular}\}$. For each integer $j \geq 2$, there exists a regular point $x_j \in B(x_0, 1/j) \cap \partial\Omega$. Define $f_j \in C(\partial\Omega)$ by letting

$$f_j(x) = (j d_{x_0}(x) - 1)_+, \quad j = 2, 3, \dots$$

Because x_j is regular, there is $y_j \in B(x_j, 1/j) \cap \Omega$ such that

$$|\overline{P}f_j(y_j)| = |f_j(x_j) - \overline{P}f_j(y_j)| < 1/j.$$

Hence $y_j \rightarrow x_0$ and $\overline{P}f_j(y_j) \rightarrow 0$ as $j \rightarrow \infty$.

Let $f \in C(\partial\Omega)$ and $\alpha := f(x_0)$. Let $\varepsilon > 0$. Then we can find an integer $k \geq 2$ such that $|f - \alpha| \leq \varepsilon$ on $B(x_0, 2/k) \cap \partial\Omega$. Choose m such that $|f - \alpha| \leq m$. It follows that $f - \alpha \leq mf_j + \varepsilon$ for every $j \geq k$, and thus

$$\limsup_{j \rightarrow \infty} \overline{P}f(y_j) \leq \limsup_{j \rightarrow \infty} \overline{P}(mf_j + \alpha + \varepsilon)(y_j) = m \lim_{j \rightarrow \infty} \overline{P}f_j(y_j) + \alpha + \varepsilon = \alpha + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ shows that $\limsup_{j \rightarrow \infty} \overline{P}f(y_j) \leq \alpha$.

Applying this to $\tilde{f} = -f$ yields $\limsup_{j \rightarrow \infty} \overline{P}\tilde{f}(y_j) \leq -\alpha$. It follows that

$$\liminf_{j \rightarrow \infty} \overline{P}f(y_j) \geq \liminf_{j \rightarrow \infty} \underline{P}f(y_j) = -\limsup_{j \rightarrow \infty} \overline{P}\tilde{f}(y_j) \geq \alpha,$$

and hence $\lim_{j \rightarrow \infty} \overline{P}f(y_j) = f(x_0)$.

(f) \Leftrightarrow (b) Observe that (f) is equivalent to the existence of a neighbourhood U of x_0 with $C_p(U \cap \partial G) = 0$, which is equivalent to (b), by the already proved equivalence (f) \Leftrightarrow (a) applied to G instead of Ω .

(f) \Rightarrow (g) Let V be a neighbourhood of x_0 such that $C_p(V \cap \partial\Omega) = 0$. By Theorem 6.1, (c) \Rightarrow (f), the set $U := \Omega \cup (V \cap \partial\Omega)$ is open and $C_p(U \setminus \Omega) = 0$.

(g) \Rightarrow (f) This is trivial.

(g) \Leftrightarrow (j) \Leftrightarrow (l) In all three statements it follows directly that $V \subset \overline{\Omega}$. Thus their equivalence follows directly from Theorem 6.1, with V in Theorem 6.1 corresponding to $V \cap \partial\Omega$ here.

(j) \Rightarrow (k) We only have to show the last part, i.e., that x_0 is irregular, but this follows from the already proved implication (j) \Rightarrow (a).

(k) \Rightarrow (a) Let $f \in C(\partial\Omega)$. Then $\overline{P}f$ has a p -harmonic extension U to $\Omega \cup V$ for some neighbourhood V of x_0 , and hence

$$\lim_{\Omega \ni y \rightarrow x_0} \overline{P}f(y) = U(x_0).$$

Since x_0 is irregular it follows that x_0 must be semiregular.

(l) \Rightarrow (o) Let u be a positive superharmonic function on Ω . Then $\min\{u, 1\}$ is superharmonic by Lemma 9.3 in Björn–Björn [6], and hence has a superharmonic extension U to $\Omega \cup V$. As U is lsc-regularized (see Sect. 3) and $\mu(V \setminus \Omega) = 0$, it follows that $U \geq 0$ in $\Omega \cup V$. Suppose that $U(x_0) = 0$. Then the strong minimum principle [6, Theorem 9.13] implies that $U \equiv 0$ in the component of $\Omega \cup V$ that contains x_0 . But this is in contradiction with u being positive in Ω , and thus

$$\liminf_{\Omega \ni y \rightarrow x_0} u(y) \geq U(x_0) > 0.$$

Thus there is no weak semibarrier at x_0 .

\neg (p) \Rightarrow \neg (o) Let u be a semibarrier at x_0 . If $u > 0$ in all of Ω , then u is a weak semibarrier at x_0 . On the other hand, assume that there exists $x \in \Omega$ such that $u(x) = 0$ (in this case u is not a weak semibarrier). Then the strong minimum principle [6, Theorem 9.13] implies that $u \equiv 0$ in the component $G \subset \Omega$ that contains x , and hence x_0 must be the only boundary point of G , because u is a semibarrier. As $C_p(X \setminus G) \geq C_p(X \setminus \Omega) > 0$, Lemma 4.3 in Björn–Björn [5] (or Lemma 4.5 in [6]) implies that $C_p(\{x_0\}) = C_p(\partial G) > 0$. By the Kellogg property (Theorem 5.2), x_0 is regular, and hence Theorem 5.4 asserts that there is a positive barrier v at x_0 , and thus v is a weak semibarrier.

(p) \Rightarrow (q) This is trivial.

\neg (e) \Rightarrow \neg (q) Let u be the continuous solution of the $\mathcal{K}_{d_{x_0}, d_{x_0}}$ -obstacle problem, which is superharmonic (see Sect. 3). Moreover, it is clear that

$$\liminf_{\Omega \ni y \rightarrow x} u(y) > 0 \quad \text{whenever } x \in \partial\Omega \setminus \{x_0\},$$

and thus u satisfies (i) and (iii) in Definition 6.3.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of regular boundary points such that $d_{x_0}(x_j) < 1/j$. By Theorem 5.4, $\lim_{\Omega \ni y \rightarrow x_j} u(y) = d_{x_0}(x_j)$. Thus we can find $y_j \in B(x_j, 1/j) \cap \Omega$ so that $u(y_j) < 2/j$. Hence u satisfies (ii) in Definition 6.3 as

$$0 \leq \liminf_{\Omega \ni y \rightarrow x_0} u(y) \leq \liminf_{j \rightarrow \infty} u(y_j) = 0.$$

We now assume that Ω is bounded or p -parabolic.

(e) \Leftrightarrow (r) \Leftrightarrow (s) This follows directly from Theorem 6.1, with V therein corresponding to $V \cap \partial\Omega$ here. □

We conclude our description of boundary points with some characterizations of strongly irregular points. As for regular and semiregular points, strong irregularity is a local property.

Theorem 6.5 *Let $x_0 \in \partial\Omega \setminus \{\infty\}$, $\delta > 0$, and $d_{x_0} : X^* \rightarrow [0, 1]$ be defined by (5.1). Then the following are equivalent:*

- (a) *The point x_0 is strongly irregular.*
- (b) *The point x_0 is strongly irregular with respect to $G := \Omega \cap B(x_0, \delta)$.*
- (c) *The point x_0 is irregular and there exists a sequence $\{y_j\}_{j=1}^\infty$ in Ω such that $y_j \rightarrow x_0$ as $j \rightarrow \infty$, and*

$$\lim_{j \rightarrow \infty} \bar{P}f(y_j) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

- (d) *It is true that $x_0 \in \bar{R} \setminus R$, where $R := \{x \in \partial\Omega : x \text{ is regular}\}$.*
- (e) *It is true that*

$$\liminf_{\Omega \ni y \rightarrow x_0} \bar{P}d_{x_0}(y) = 0 < \limsup_{\Omega \ni y \rightarrow x_0} \bar{P}d_{x_0}(y).$$

- (f) *There exists $f \in C(\partial\Omega)$ such that*

$$\lim_{\Omega \ni y \rightarrow x_0} \bar{P}f(y)$$

does not exist.

- (g) *The continuous solution u of the $\mathcal{H}_{d_{x_0}, d_{x_0}}$ -obstacle problem satisfies*

$$\liminf_{\Omega \ni y \rightarrow x_0} u(y) = 0 < \limsup_{\Omega \ni y \rightarrow x_0} u(y).$$

- (h) *There is a semibarrier (or equivalently there is a weak semibarrier) but no barrier at x_0 .*

The trichotomy property (Theorem 1.1) shows that a boundary point is either regular, semiregular, or strongly irregular. We will use this in the following proof.

Proof (a) \Leftrightarrow (b) By Theorems 5.4 and 6.4, regularity and semiregularity are local properties, and hence this must be true also for strong irregularity.

(a) \Leftrightarrow (c) \Leftrightarrow (d) This follows from Theorem 6.4 (a) \Leftrightarrow (c) \Leftrightarrow (e).

(a) \Rightarrow (e) Since x_0 is strongly irregular and $\bar{P}d_{x_0}$ is nonnegative, it follows that

$$\liminf_{\Omega \ni y \rightarrow x_0} \bar{P}d_{x_0}(y) = 0 \leq \limsup_{\Omega \ni y \rightarrow x_0} \bar{P}d_{x_0}(y).$$

If $\limsup_{\Omega \ni y \rightarrow x_0} \bar{P}d_{x_0}(y) = 0$, then x_0 must be regular by Theorem 5.4, which is a contradiction. Thus

$$\limsup_{\Omega \ni y \rightarrow x_0} \bar{P}d_{x_0}(y) > 0.$$

(e) \Rightarrow (f) This is trivial.

(f) \Rightarrow (a) By definition, x_0 is neither regular nor semiregular, and hence must be strongly irregular.

(a) \Leftrightarrow (g) Theorem 5.4 shows that x_0 is regular if and only if $\lim_{\Omega \ni y \rightarrow x_0} u(y) = 0$. On the other hand, Theorem 6.4 implies that x_0 is semiregular if and only if $\liminf_{\Omega \ni y \rightarrow x_0} u(y) > 0$. The equivalence follows by combining these two facts.

(a) \Leftrightarrow (h) By Theorem 6.4, x_0 is semiregular if and only if there is no (weak) semibarrier at x_0 . On the other hand, by Theorem 5.4, there is a barrier at x_0 if and only if x_0 is regular. Combining these two facts gives the equivalence. \square

Acknowledgements The first author was supported by the Swedish Research Council, Grant 2016-03424.

Funding Open access funding provided by Linköping University.

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