Compact differences of weighted composition operators

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Abstract

Compact differences of two weighted composition operators acting from the weighted Bergman space A_ω^p to another weighted Bergman space A_ν^q , where $0 < p \leq q < \infty$ and ω, ν belong to the class $\mathcal{D}$ of radial weights satisfying two-sided doubling conditions, are characterized. On the way to the proof a new description of q -Carleson measures for A_ω^p , with $\omega \in \mathcal{D}$, in terms of pseudohyperbolic discs is established. This last-mentioned result generalizes the well-known characterization of q -Carleson measures for the classical weighted Bergman space A_α^p with $-1 < \alpha < \infty$ to the setting of doubling weights.

Keywords Bergman space · Doubling weight · Weighted composition operator · Compactness

Mathematics Subject Classification 47B33 · 30H20

1 Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a nonnegative function $\omega \in L^1([0, 1))$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. For $0 < p < \infty$ and a radial weight ω , the weighted Bergman space A_ω^p consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{A_\omega^p}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

where $dA(z) = \frac{dx dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. As usual, A_α^p stands for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1 - |z|^2)^\alpha$, where $-1 < \alpha < \infty$.

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For a radial weight ω , write $\hat{\omega}(z) = \int_{|z|}^1 \omega(s) ds$ for all $z \in \mathbb{D}$. In this paper we always assume $\hat{\omega}(z) > 0$, for otherwise $A_{\omega}^p = \mathcal{H}(\mathbb{D})$ for each $0 < p < \infty$. A weight ω belongs to the class $\hat{\mathcal{D}}$ if there exists a constant $C = C(\omega) \geq 1$ such that $\hat{\omega}(r) \leq C\hat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}(1 - \frac{1-r}{K})$ for all $0 \leq r < 1$, then we write $\omega \in \check{\mathcal{D}}$. In other words, $\omega \in \check{\mathcal{D}}$ if there exists $K = K(\omega) > 1$ and $C' = C'(\omega) > 0$ such that

$$\hat{\omega}(r) \leq C' \int_r^{1-\frac{1-r}{K}} \omega(t) dt, \quad 0 \leq r < 1.$$

The intersection $\hat{\mathcal{D}} \cap \check{\mathcal{D}}$ is denoted by $\mathcal{D}$, and this is the class of weights that we mainly work with.

Each analytic self-map φ of $\mathbb{D}$ induces the composition operator C_{φ} on $\mathcal{H}(\mathbb{D})$ defined by $C_{\varphi}f = f \circ \varphi$. The weighted composition operator induced by $u \in \mathcal{H}(\mathbb{D})$ and φ is uC_{φ} and sends $f \in \mathcal{H}(\mathbb{D})$ to $u \cdot f \circ \varphi \in \mathcal{H}(\mathbb{D})$. These operators have been extensively studied in a variety of function spaces (see for example [4–8, 22, 23, 25, 26]).

If now ψ is another analytic self-map of $\mathbb{D}$, the pair (φ, ψ) induces the operator $C_{\varphi} - C_{\psi}$. One of the most important problem considering these operators is to characterize compact differences in Hardy spaces. Shapiro and Sundberg [24] studied this problem in 1990. Very recently, Choe, Choi, Koo and Yang [3] have solved this problem. For more about difference operators, see [2, 8, 10, 11, 20]. Moorhouse [11, 12] obtain some important results on this operator in weighted Bergman spaces. He showed [11], among other things, that $C_{\varphi} - C_{\psi}$ is compact on A_{α}^2 if and only if

$$\lim_{|z| \rightarrow 1^-} |\delta_1(z)| \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) = 0, \quad (1)$$

where

$$\delta_1(z) = \frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)}, \quad z \in \mathbb{D}.$$

Saukko [20, 21] generalized this result by showing that if either $1 < p \leq q$, or $p > q \geq 1$, then $C_{\varphi} - C_{\psi} : A_{\alpha}^p \rightarrow A_{\beta}^q$ is compact if and only if the operators $\delta_1 C_{\varphi}$ and $\delta_1 C_{\psi}$ are both compact from A_{α}^p to L_{β}^q . Very recently, Acharyya and Wu [1] characterized the compact differences of two weighted composition operators $uC_{\varphi} - vC_{\psi}$ between different weighted Bergman spaces A_{α}^p and A_{β}^q , where $0 < p \leq q < \infty$ and $-1 < \alpha, \beta < \infty$. Their result states that, if $\frac{\alpha+2}{p} \leq \frac{\beta+2}{q}$ and $u, v \in \mathcal{H}(\mathbb{D})$ satisfy

$$\sup_{z \in \mathbb{D}} (|u(z)| + |v(z)|)(1 - |z|^2)^{\frac{\beta+2}{q} - \frac{\alpha+2}{p}} < \infty,$$

then $uC_{\varphi} - vC_{\psi} : A_{\alpha}^p \rightarrow A_{\beta}^q$ is compact if and only if

$$\lim_{|z| \rightarrow 1^-} |\delta_1(z)| \left(|u(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0 \quad (2)$$

and

$$\lim_{|z| \rightarrow 1^-} (1 - |\delta_1(z)|^2)^{\frac{2+\alpha}{p}} |u(z) - v(z)| \left(\frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0. \quad (3)$$

In this paper we characterize compact differences of two weighted composition operators from the weighted Bergman space A_ω^p to another weighted Bergman space A_ν^q with $0 < p \leq q < \infty$ and $\omega, \nu \in \mathcal{D}$. To state the result, write

$$\delta_2(z) = \frac{\psi(z) - \varphi(z)}{1 - \overline{\psi(z)}\varphi(z)}, \quad z \in \mathbb{D},$$

and observe that $|\delta_1| = |\delta_2|$ on $\mathbb{D}$. Our main result reads as follows.

Theorem 1 *Let $\omega, \nu \in \mathcal{D}$ and $0 < p \leq q < \infty$ such that $\widehat{\nu}(z)(1 - |z|) \lesssim (\widehat{\omega}(z)(1 - |z|))^{\frac{q}{p}}$ for all $z \in \mathbb{D}$. Further, let $u, v \in \mathcal{H}(\mathbb{D})$ and φ and ψ analytic self-maps of $\mathbb{D}$ such that*

$$\sup_{z \in \mathbb{D}} (|u(z)| + |v(z)|) \frac{(\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}}}{(\widehat{\omega}(z)(1 - |z|))^{\frac{1}{p}}} < \infty. \quad (4)$$

Then there exists $\gamma = \gamma(\omega, p) > 0$ with the following property: $uC_\varphi - vC_\psi : A_\omega^p \rightarrow A_\nu^q$ is compact if and only if

$$\lim_{|z| \rightarrow 1^-} |\delta_1(z)| \left(|u(z)| \frac{(\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}}}{(\widehat{\omega}(\varphi(z))(1 - |\varphi(z)|))^{\frac{1}{p}}} + |v(z)| \frac{(\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}}}{(\widehat{\omega}(\psi(z))(1 - |\psi(z)|))^{\frac{1}{p}}} \right) = 0 \quad (5)$$

and

$$\lim_{|z| \rightarrow 1^-} \left(\frac{|1 - \overline{\varphi(z)}\delta_1(z)|^\gamma}{(\widehat{\omega}(\varphi(z))(1 - |\varphi(z)|))^{\frac{1}{p}}} + \frac{|1 - \overline{\psi(z)}\delta_2(z)|^\gamma}{(\widehat{\omega}(\psi(z))(1 - |\psi(z)|))^{\frac{1}{p}}} \right) \cdot |u(z) - v(z)| (\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}} = 0. \quad (6)$$

If $\omega(z) = (1 - |z|)^\alpha$ and $\nu(z) = (1 - |z|)^\beta$ for $-1 < \alpha, \beta < \infty$, then $\widehat{\omega}(z) \asymp (1 - |z|)^{\alpha+1}$ and $\widehat{\nu}(z) \asymp (1 - |z|)^{\beta+1}$ for all $z \in \mathbb{D}$. Therefore (5) reduces to (2). Moreover, the Proof of Theorem 1 shows that the only requirement for $\gamma = \gamma(\omega, p) > 0$ appearing in the statement is that

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1 - \overline{a}z|^{p\gamma}} dA(z) \leq C \frac{\widehat{\omega}(a)}{(1 - |a|)^{\gamma p - 1}}, \quad a \in \mathbb{D},$$

for some constant $C = C(\omega, p, \gamma) > 0$. If $\omega(z) = (1 - |z|)^\alpha$, any $\gamma > \frac{\alpha+2}{p}$ is acceptable, and the choice $\gamma = 2\frac{\alpha+2}{p}$ converts (6) to (3), as a simple computation shows. Therefore Theorem 1 indeed generalizes [1, Theorem 1] for weights in $\mathcal{D}$.

We need two specific tools for the Proof of Theorem 1. The first one concerns continuous embeddings $A_\omega^p \subset L_\mu^q$. Recall that a positive Borel measure μ on $\mathbb{D}$ is a q -Carleson measure for A_ω^p if the identity operator $I_d : A_\omega^p \rightarrow L_\mu^q$ is bounded. A complete characterization of such measures in the case $\omega \in \widehat{\mathcal{D}}$ can be found in [15], see also [13, 17]. In particular, it is known that if $q \geq p$ and $\omega \in \widehat{\mathcal{D}}$, then μ is a q -Carleson measure for A_ω^p if and only if

$$\sup_{a \in \mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}} < \infty.$$

Here and from now on $S(a) = \{z : 1 - |a| < |z| < 1, |\arg z - \arg a| < (1 - |a|)/2\}$ is the Carleson square induced by the point $a \in \mathbb{D} \setminus \{0\}$, $S(0) = \mathbb{D}$ and $\omega(E) = \int_E \omega dA$ for each measurable set $E \subset \mathbb{D}$. We will need a variant of this result and its “compact” counterpart for $\omega \in \mathcal{D}$ where the Carleson squares are replaced by pseudohyperbolic discs. To this end, denote $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a, z \in \mathbb{D}$. The pseudohyperbolic distance between two points a and b in $\mathbb{D}$ is $\rho(a, b) = |\varphi_a(b)|$. For $a \in \mathbb{D}$ and $0 < r < 1$, the pseudohyperbolic disc of center a and of radius r is $\Delta(a, r) = \{z \in \mathbb{D} : \rho(a, z) < r\}$. It is well known that $\Delta(a, r)$ is an Euclidean disk centered at $(1 - r^2)a/(1 - r^2|a|^2)$ and of radius $(1 - |a|^2)r/(1 - r^2|a|^2)$.

Theorem 2 *Let $0 < p \leq q < \infty$, $\omega \in \mathcal{D}$ and μ a positive Borel measure on $\mathbb{D}$. Then there exists $r = r(\omega) \in (0, 1)$ such that the following statements hold:*

- (i) μ is a q -Carleson measure for A_ω^p if and only if

$$\sup_{a \in \mathbb{D}} \frac{\mu(\Delta(a, r))}{(\omega(\Delta(a, r)))^{\frac{q}{p}}} < \infty. \quad (7)$$

Moreover, if μ is a q -Carleson measure for A_ω^p , then the identity operator satisfies

$$\|I_d\|_{A_\omega^p \rightarrow L_\mu^q}^q \asymp \sup_{a \in \mathbb{D}} \frac{\mu(\Delta(a, r))}{(\omega(\Delta(a, r)))^{\frac{q}{p}}}.$$

- (ii) The identity operator $I_d : A_\omega^p \rightarrow L_\mu^q$ is compact if and only if

$$\lim_{|a| \rightarrow 1^-} \frac{\mu(\Delta(a, r))}{(\omega(\Delta(a, r)))^{\frac{q}{p}}} = 0. \quad (8)$$

Another result needed is a lemma that allows us to estimate the distance between images of two points, say z and a , under f sufficiently accurately whenever z is close to a in the sense that $z \in \Delta(a, r)$, and $f \in A_\omega^p$ with $\omega \in \mathcal{D}$. For the statement, denote $\tilde{\omega}(z) = \hat{\omega}(z)/(1 - |z|)$ for all $z \in \mathbb{D}$.

Lemma 1 *Let $\omega \in \mathcal{D}$, $0 < p \leq q < \infty$ and $0 < r < R < 1$. Then there exists a constant $C = C(\omega, p, q, r, R) > 0$ such that*

$$|f(z) - f(a)|^q \leq C \frac{\rho(z, a)^q}{(\hat{\omega}(a)(1 - |a|))^{\frac{q}{p}}} \int_{\Delta(a, R)} |f(\zeta)|^p \tilde{\omega}(\zeta) dA(\zeta), \quad a \in \mathbb{D}, \quad z \in \Delta(a, r), \quad (9)$$

for all $f \in A_\omega^p$ with $\|f\|_{A_\omega^p} \leq 1$.

This lemma plays an important role in the Proof of Theorem 1 when we show that (5) and (6) are sufficient conditions for the compactness. By [18, Proposition 5] we know that

$$\|f\|_{A_{\omega}^p} \asymp \|f\|_{A_{\tilde{\omega}}^p}, \quad f \in \mathcal{H}(\mathbb{D}), \quad (10)$$

provided $\omega \in \mathcal{D}$. This explains the appearance of the weight $\tilde{\omega}$ on the right hand side of (9). It is worth observing that, despite of (10), the strictly positive weight $\tilde{\omega}$ cannot be replaced by ω in the statement because $\omega \in \mathcal{D}$ may vanish in pseudohyperbolic discs of fixed radius that tend to the boundary.

The rest of the paper contains the proofs of the results stated above. We first prove Lemma 1 in the next section. The proof of the result on Carleson measures, Theorem 2, is given in Sect. 3, and finally, Theorem 1 is proved in Sect. 4.

To this end, couple of words about the notation used in the sequel. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable.

2 Proof of Lemma 1

It is known that if $\omega \in \mathcal{D}$, then there exist constants $0 < \alpha = \alpha(\omega) \leq \beta(\omega) < \infty$ and $C = C(\omega) \geq 1$ such that

$$\frac{1}{C} \left(\frac{1-r}{1-t} \right)^{\alpha} \leq \frac{\hat{\omega}(r)}{\hat{\omega}(t)} \leq C \left(\frac{1-r}{1-t} \right)^{\beta}, \quad 0 \leq r \leq t < 1. \quad (11)$$

In fact, this pair of inequalities characterizes the class $\mathcal{D}$ because the right hand inequality is satisfied if and only if $\omega \in \hat{\mathcal{D}}$ by [17, Lemma 2.1] while the left hand inequality describes the class $\hat{\mathcal{D}}$ in an analogous way, see [14, (2.27)]. The chain of inequalities (11) will be frequently used in the sequel.

To prove the lemma, let $a \in \mathbb{D}$, $0 < r < 1$ and $z \in \Delta(a, r)$. Then

$$\begin{aligned} |f(z) - f(a)|^p &= |f(\varphi_a(\varphi_a(z))) - f(\varphi_a(0))|^p = \left| \int_0^{\varphi_a(z)} (f \circ \varphi_a)'(\xi) d\xi \right|^p \\ &\leq \max_{\xi \in D(0, |\varphi_a(z)|)} |(f \circ \varphi_a)'(\xi)|^p |\varphi_a(z)|^p \leq \max_{\xi \in D(0, r)} |(f \circ \varphi_a)'(\xi)|^p |\varphi_a(z)|^p. \end{aligned} \quad (12)$$

Let $R \in (r, 1)$ and set $R' = \frac{r+R}{2}$. Further, let $0 < s < 1$. Then the Cauchy integral formula for the derivative and the subharmonicity of $|f|^p$ yield

$$\begin{aligned} |(f \circ \varphi_a)'(\xi)|^p &= \left| \frac{1}{2\pi} \int_{|w|=R'} \frac{(f \circ \varphi_a)(w)}{(w - \xi)^2} dw \right|^p \leq \left(\frac{2}{R-r} \right)^{2p} (R')^p \max_{|w|=R'} |f(\varphi_a(w))|^p \\ &\lesssim \frac{4}{\pi s^2} \max_{|w|=R'} \frac{1}{(1 - |\varphi_a(w)|)^2} \int_{\Delta(\varphi_a(w), s)} |f(\zeta)|^p dA(\zeta) \\ &\lesssim \max_{|w|=R'} \frac{1}{(1 - |a|)^2} \int_{\Delta(\varphi_a(w), s)} |f(\zeta)|^p dA(\zeta), \quad \xi \in \overline{D(0, r)}. \end{aligned} \quad (13)$$

Fix now $s = s(r, R) \in (0, 1)$ sufficiently small such that $\Delta(\varphi_a(w), s) \subset \Delta(a, R)$ for all w such that $|w| = R'$. Further, an application of the right hand inequality in (11) shows that $\hat{\omega}(\zeta) \asymp \hat{\omega}(a)$ for all $\zeta \in \Delta(a, R)$. Therefore, by combining (12) and (13) we deduce

$$\begin{aligned}
|f(z) - f(a)|^p &\lesssim \frac{|\varphi_a(z)|^p}{(1 - |a|)^2} \int_{\Delta(a, R)} |f(\zeta)|^p dA(\zeta) \\
&\lesssim \frac{|\varphi_a(z)|^p}{\widehat{\omega}(a)(1 - |a|)} \int_{\Delta(a, R)} |f(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta), \quad a \in \mathbb{D}, \quad z \in \Delta(a, r).
\end{aligned}$$

This proves the case $p = q$ because $|\varphi_a(z)| = \rho(z, a)$ for all $a, z \in \mathbb{D}$. This part of the proof is valid for all $f \in \mathcal{H}(\mathbb{D})$ if $\omega \in \widehat{\mathcal{D}}$.

Let now $q > p$, and observe that trivially $|f(z) - f(a)|^q = (|f(z) - f(a)|^p)^{\frac{q}{p}}$. An application of the case $q = p$ implies

$$\begin{aligned}
|f(z) - f(a)|^q &\lesssim \frac{\rho(z, a)^q}{(\widehat{\omega}(a)(1 - |a|))^{\frac{q}{p}}} \left(\int_{\Delta(a, R)} |f(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta) \right)^{\frac{q}{p}} \\
&\leq \frac{\rho(z, a)^q \|f\|_{A_{\widehat{\omega}}^p}^{q-p}}{(\widehat{\omega}(a)(1 - |a|))^{\frac{q}{p}}} \int_{\Delta(a, R)} |f(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta).
\end{aligned}$$

But (10) guarantees $\|f\|_{A_{\widehat{\omega}}^p} \approx \|f\|_{A_{\omega}^p} \leq 1$, and thus the assertion in the case $q > p$ follows from the above estimate.

3 Proof of Theorem 2

To prove (i), assume first (7) and let $0 < r < 1$. The fact that $|f|^p$ is subharmonic in $\mathbb{D}$ together with Minkowski's inequality in continuous form (Fubini's theorem in the case $q = p$) and (7) imply

$$\begin{aligned}
\|f\|_{L_{\mu}^q}^q &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(z, r)} \frac{|f(\zeta)|^p}{(1 - |\zeta|)^2} dA(\zeta) \right)^{\frac{q}{p}} d\mu(z) \\
&\leq \left(\int_{\mathbb{D}} |f(\zeta)|^p \frac{\mu(\Delta(\zeta, r))^{\frac{p}{q}} (1 - |\zeta|)^2}{(1 - |\zeta|)^2} dA(\zeta) \right)^{\frac{q}{p}} \\
&\lesssim \left(\int_{\mathbb{D}} |f(\zeta)|^p \frac{\omega(\Delta(\zeta, r))}{(1 - |\zeta|)^2} dA(\zeta) \right)^{\frac{q}{p}}, \quad f \in \mathcal{H}(\mathbb{D}).
\end{aligned}$$

Since $\omega \in \mathcal{D}$ by the hypothesis, we may apply the right hand inequality in (11) to deduce

$$\omega(\Delta(\zeta, r)) \lesssim \widehat{\omega}(\zeta)(1 - |\zeta|), \quad \zeta \in \mathbb{D}. \quad (14)$$

It follows that $\|f\|_{L_{\mu}^q} \lesssim \|f\|_{A_{\widehat{\omega}}^p}$, and hence $\|f\|_{L_{\mu}^q} \lesssim \|f\|_{A_{\omega}^p}$ for all $f \in \mathcal{H}(\mathbb{D})$ by (10). Thus μ is a q -Carleson measure A_{ω}^p .

Conversely, assume that μ is a q -Carleson measure A_{ω}^p . For each $a \in \mathbb{D}$, consider the function

$$f_a(z) = \left(\frac{1 - |a|^2}{1 - \bar{a}z} \right)^{\gamma} \frac{1}{(\widehat{\omega}(a)(1 - |a|))^{\frac{1}{p}}} = \frac{(1 - \bar{a}\varphi_a(z))^{\gamma}}{(\widehat{\omega}(a)(1 - |a|))^{\frac{1}{p}}}, \quad z \in \mathbb{D}, \quad (15)$$

induced by ω and $0 < \gamma, p < \infty$. Then [17, Lemma 2.1] implies that for all $\gamma = \gamma(\omega, p) > 0$ sufficiently large we have $\|f_a\|_{A_\omega^p} \asymp 1$ for all $a \in \mathbb{D}$. Therefore the assumption yields

$$1 \asymp \|f_a\|_{A_\omega^p}^q \gtrsim \|f_a\|_{L_\mu^q}^q \gtrsim \frac{\mu(\Delta(a, r))}{(\hat{\omega}(a)(1 - |a|))^{\frac{q}{p}}}, \quad a \in \mathbb{D},$$

that is, $\mu(\Delta(a, r)) \lesssim (\hat{\omega}(a)(1 - |a|))^{\frac{q}{p}}$ for all $a \in \mathbb{D}$. Since $\omega \in \mathcal{D} \subset \tilde{\mathcal{D}}$ by the hypothesis, there exists $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $0 \leq r < 1$ by the definition. Fix now $r = r(K) \in (0, 1)$ sufficiently large such that

$$\Delta(a, r) \supset \left\{ te^{i\theta} : |a| \leq t \leq 1 - \frac{1 - |a|}{K}, |\arg a - \theta| \leq \frac{1 - |a|}{2} \left(1 - \frac{1}{K}\right) \right\}.$$

Then, as $\omega \in \mathcal{D} \subset \hat{\mathcal{D}}$, the right hand inequality in (11) yields

$$\begin{aligned} \omega(\Delta(a, r)) &\geq (1 - |a|) \left(1 - \frac{1}{K}\right) |a| \int_{|a|}^{1 - \frac{1 - |a|}{K}} \omega(s) ds \\ &\geq (C - 1)(1 - |a|) \left(1 - \frac{1}{K}\right) |a| \hat{\omega} \left(1 - \frac{1 - |a|}{K}\right) \\ &\gtrsim \hat{\omega}(a)(1 - |a|)|a|, \quad a \in \mathbb{D}, \end{aligned}$$

and therefore

$$\mu(\Delta(a, r)) \lesssim (\hat{\omega}(a)(1 - |a|))^{\frac{q}{p}} \lesssim \left(\frac{\omega(\Delta(a, r))}{|a|} \right)^{\frac{q}{p}}, \quad a \in \mathbb{D} \setminus \{0\}.$$

The claim (7) now follows from these estimates for all $r = r(\omega) \in (0, 1)$ sufficiently large.

To prove (ii), assume first that $I_d : A_\omega^p \rightarrow L_\mu^q$ is compact. An application of [17, Lemma 2.1] and the right hand inequality in (11) ensure that we may choose $\gamma = \gamma(p, \omega) > 0$ sufficiently large such that $\|f_a\|_{A_\omega^p} \asymp 1$ for all $a \in \mathbb{D}$, and $f_a \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, as $|a| \rightarrow 1^-$. Therefore the closure of the set $\{f_a : a \in \mathbb{D}\}$ is compact in L_μ^q . Since for each $\varepsilon > 0$ the open balls $B(f_a, \varepsilon) = \{f \in L_\mu^q : \|f_a - f\|_{L_\mu^q} < \varepsilon\}$ cover $\{f_a : a \in \mathbb{D}\}$, there exists a finite subcover $\{B(f_{a_n}, \varepsilon) : n = 1, \dots, N = N(\varepsilon)\}$. Let now $a \in \mathbb{D}$ be arbitrary, and let $j = j(a) \in \{1, \dots, N\}$ such that $f_a \in B(f_{a_n}, \varepsilon)$. Then, for each $R \in (0, 1)$, we have

$$\begin{aligned} \int_{\mathbb{D} \setminus D(0, R)} |f_a(z)|^q d\mu(z) &\lesssim \int_{\mathbb{D} \setminus D(0, R)} |f_a(z) - f_{a_j}(z)|^q d\mu(z) + \int_{\mathbb{D} \setminus D(0, R)} |f_{a_j}(z)|^q d\mu(z) \\ &\leq \|f_a - f_{a_j}\|_{L_\mu^q}^q + \max_{n=1, \dots, N} \int_{\mathbb{D} \setminus D(0, R)} |f_{a_n}(z)|^q d\mu(z). \end{aligned}$$

By fixing $R \in (0, 1)$ sufficiently large, and taking into account that $\varepsilon > 0$ was arbitrary, we deduce

$$\lim_{R \rightarrow 1^-} \int_{\mathbb{D} \setminus D(0, R)} |f_a(z)|^q d\mu(z) = 0$$

uniformly in a . This together with the uniform convergence yield

$$0 = \lim_{|a| \rightarrow 1^-} \|f_a\|_{L_\mu^q}^q \geq \lim_{|a| \rightarrow 1^-} \int_{\Delta(a,r)} |f_a(z)|^q d\mu(z) \gtrsim \lim_{|a| \rightarrow 1^-} \frac{\mu(\Delta(a,r))}{(\widehat{\omega}(a)(1-|a|))^{\frac{q}{p}}}$$

for all $r \in (0, 1)$. Now fix $r = r(\omega)$ as in the case (i) to have $\widehat{\omega}(a)(1-|a|) \lesssim \omega(\Delta(a,r))$ for all $a \in \mathbb{D}$. Then we obtain (8).

Conversely, assume (8). Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence in A_ω^p such that $\sup_{k \in \mathbb{N}} \|f_k\|_{A_\omega^p} = M < \infty$. Then it is easy to see that $\{f_k\}_{k \in \mathbb{N}}$ is uniformly bounded on compact subsets of $\mathbb{D}$ – this follows, for example, from (18) below. Therefore $\{f_k\}_{k \in \mathbb{N}}$ constitutes a normal family by Montel's theorem, and hence we may extract a subsequence $\{f_{k_j}\}_{j \in \mathbb{N}}$ that converges uniformly on compact subsets of $\mathbb{D}$ to a function f which belongs to $\mathcal{H}(\mathbb{D})$ by Weierstrass' theorem. Fatou's lemma now shows that $f \in A_\omega^p$. For $r \in (0, 1)$, fix an r -lattice $\{a_n\}_{n \in \mathbb{N}}$. Since $|a_n| \rightarrow 1$, as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{\mu(\Delta(a_n, r))}{\omega(\Delta(a_n, r))^{\frac{q}{p}}} = 0$$

by the hypothesis. Therefore, for each $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\frac{\mu(\Delta(a_n, r))}{\omega(\Delta(a_n, r))^{\frac{q}{p}}} < \varepsilon, \quad n \geq N.$$

Hence, as in the case (i), Minkowski's inequality in continuous form (Fubini's theorem in the case $q = p$), (14), (11) and (10) yield

$$\begin{aligned} & \sum_{n=N}^{\infty} \int_{\Delta(a_n, r)} |f(z) - f_{k_j}(z)|^q d\mu(z) \\ & \lesssim \sum_{n=N}^{\infty} \int_{\Delta(a_n, r)} \left(\int_{\Delta(z, R)} \frac{|f(\zeta) - f_{k_j}(\zeta)|^p}{(1-|\zeta|)^2} dA(\zeta) \right)^{\frac{q}{p}} d\mu(z) \\ & \leq \sum_{n=N}^{\infty} \left(\int_{\{\zeta : \Delta(a_n, r) \cap \Delta(\zeta, R) \neq \emptyset\}} \frac{|f(\zeta) - f_{k_j}(\zeta)|^p}{(1-|\zeta|)^2} \mu(\Delta(a_n, r))^{\frac{p}{q}} dA(\zeta) \right)^{\frac{q}{p}} \\ & \leq \varepsilon \sum_{n=N}^{\infty} \left(\int_{\{\zeta : \Delta(a_n, r) \cap \Delta(\zeta, R) \neq \emptyset\}} \frac{|f(\zeta) - f_{k_j}(\zeta)|^p}{(1-|\zeta|)^2} \omega(\Delta(a_n, r)) dA(\zeta) \right)^{\frac{q}{p}} \\ & \lesssim \varepsilon \left(\sum_{n=N}^{\infty} \int_{\{\zeta : \Delta(a_n, r) \cap \Delta(\zeta, R) \neq \emptyset\}} |f(\zeta) - f_{k_j}(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta) \right)^{\frac{q}{p}} \\ & \lesssim \varepsilon \|f - f_{k_j}\|_{A_\omega^p}^q \asymp \varepsilon \|f - f_{k_j}\|_{A_\omega^p}^q \lesssim M^q \varepsilon. \end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \sum_{n=1}^{N-1} \int_{\Delta(a_n, r)} |f(z) - f_{k_j}(z)|^q d\mu(z) = 0$$

by the uniform convergence in compact subsets, we deduce

$$\begin{aligned} \limsup_{j \rightarrow \infty} \int_{\mathbb{D}} |f(z) - f_{k_j}(z)|^q d\mu(z) &\leq \limsup_{j \rightarrow \infty} \left(\sum_{n=1}^{N-1} \int_{\Delta(a_n, r)} |f(z) - f_{k_j}(z)|^q d\mu(z) \right. \\ &\quad \left. + \sum_{n=N}^{\infty} \int_{\Delta(a_n, r)} |f(z) - f_{k_j}(z)|^q d\mu(z) \right) \lesssim \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\limsup_{j \rightarrow \infty} \int_{\mathbb{D}} |f(z) - f_{k_j}(z)|^q d\mu(z) = 0,$$

and hence $I_d : A_{\omega}^p \rightarrow L_{\mu}^q$ is compact. This completes the proof of the theorem.

4 Proof of Theorem 1

With the auxiliary results proved in the previous sections we are ready for the proof of the main result. We will follow the arguments used in [1] with appropriate modifications. The following two propositions will prove Theorem 1. The first one gives necessary conditions for $uC_{\varphi} - vC_{\psi} : A_{\omega}^p \rightarrow A_{\nu}^q$ to be compact.

Proposition 1 *Let $\omega, \nu \in \mathcal{D}$, $0 < p, q < \infty$, $u, v \in \mathcal{H}(\mathbb{D})$ and φ and ψ be analytic self-maps of $\mathbb{D}$. If $uC_{\varphi} - vC_{\psi} : A_{\omega}^p \rightarrow A_{\nu}^q$ is compact, then*

$$\lim_{|z| \rightarrow 1^-} |\delta_1(z)| \left(|u(z)| \frac{(\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}}}{(\widehat{\omega}(\varphi(z))(1 - |\varphi(z)|))^{\frac{1}{p}}} + |v(z)| \frac{(\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}}}{(\widehat{\omega}(\psi(z))(1 - |\psi(z)|))^{\frac{1}{p}}} \right) = 0$$

and there exists $\gamma = \gamma(\omega, p) > 0$ such that

$$\lim_{|z| \rightarrow 1^-} \left(\frac{|1 - \overline{\varphi}(z)\delta_1(z)|^{\gamma}}{(\widehat{\omega}(\varphi(z))(1 - |\varphi(z)|))^{\frac{1}{p}}} + \frac{|1 - \overline{\psi}(z)\delta_2(z)|^{\gamma}}{(\widehat{\omega}(\psi(z))(1 - |\psi(z)|))^{\frac{1}{p}}} \right) |u(z) - v(z)| (\widehat{\nu}(z)(1 - |z|))^{\frac{1}{q}} = 0.$$

Proof Consider the test functions f_a defined in (15), and set $F_a(z) = \varphi_a(z)f_a(z)$ for all $a, z \in \mathbb{D}$. Obviously, $\|F_a\|_{A_{\omega}^p} \leq \|f_a\|_{A_{\omega}^p}$ for all $a \in \mathbb{D}$. Further, by the Proof of Theorem 2, both f_a and F_a tend to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \rightarrow 1^-$, and $\|f_a\|_{A_{\omega}^p} \asymp 1$ for all $a \in \mathbb{D}$ if $\gamma = \gamma(\omega, p) > 0$ is sufficiently large. Since $uC_{\varphi} - vC_{\psi} : A_{\omega}^p \rightarrow A_{\nu}^q$ is compact by the hypothesis, we therefore have

$$\lim_{|a| \rightarrow 1^-} \|uC_{\varphi}(f_a) - vC_{\psi}(f_a)\|_{A_{\nu}^q} = 0 \quad (16)$$

and

$$\lim_{|a| \rightarrow 1^-} \|uC_{\varphi}(F_a) - vC_{\psi}(F_a)\|_{A_{\nu}^q} = 0. \quad (17)$$

Also, if $\lim_{|a| \rightarrow 1^-}$ is replaced by $\sup_{a \in \mathbb{D}}$ in the above formulas, then the corresponding quantities are bounded.

We next observe that for each $\omega \in \widehat{\mathcal{D}}$ and $0 < q < \infty$, there exists a positive bounded function C_{ω} on $[0, 1)$ such that

$$|f(z)| \leq \frac{C_\omega(|z|)}{(\widehat{\omega}(z)(1 - |z|))^{\frac{1}{q}}} \|f\|_{A_\omega^q}, \quad f \in A_\omega^q, \quad z \in \mathbb{D}, \quad (18)$$

and $C_\omega(z) \rightarrow 0$ as $|z| \rightarrow 1^-$. Namely, for each $z \in \mathbb{D}$ we have

$$\begin{aligned} \|f\|_{A_\omega^q}^q &\geq \int_{\mathbb{D} \setminus D(0, \frac{1+|z|}{2})} |f(\zeta)|^q \omega(\zeta) dA(\zeta) \geq M_q^q \left(\frac{1+|z|}{2}, f \right) \int_{\frac{1+|z|}{2}}^1 r \omega(r) dr \\ &\geq \frac{1}{2} M_q^q \left(\frac{1+|z|}{2}, f \right) \widehat{\omega} \left(\frac{1+|z|}{2} \right) \gtrsim M_q^q \left(\frac{1+|z|}{2}, f \right) \widehat{\omega}(z), \end{aligned}$$

which combined with the well-known inequality $M_\infty(|z|, f) \lesssim M_q \left(\frac{1+|z|}{2}, f \right) (1 - |z|)^{-\frac{1}{q}}$, valid for all $f \in \mathcal{H}(\mathbb{D})$, yields (18) because

$$\int_{\mathbb{D} \setminus D(0, \frac{1+|z|}{2})} |f(\zeta)|^q \omega(\zeta) dA(\zeta) \rightarrow 0, \quad |z| \rightarrow 1^-,$$

for each $f \in A_\omega^q$.

By combining (16) and (17) with (18), we deduce

$$\lim_{\max(|a|, |z|) \rightarrow 1^-} |uC_\varphi(f_a)(z) - vC_\psi(f_a)(z)| (\widehat{v}(z)(1 - |z|))^{\frac{1}{q}} = 0$$

and

$$\lim_{\max(|a|, |z|) \rightarrow 1^-} |uC_\varphi(F_a)(z) - vC_\psi(F_a)(z)| (\widehat{v}(z)(1 - |z|))^{\frac{1}{q}} = 0.$$

By choosing $a = \varphi(z)$, we obtain

$$\lim_{|z| \rightarrow 1^-} |u(z)f_{\varphi(z)}(\varphi(z)) - v(z)f_{\varphi(z)}(\psi(z))| (\widehat{v}(z)(1 - |z|))^{\frac{1}{q}} = 0 \quad (19)$$

and

$$\lim_{|z| \rightarrow 1^-} |\delta_1(z)| |v(z)| |f_{\varphi(z)}(\psi(z))| (\widehat{v}(z)(1 - |z|))^{\frac{1}{q}} = 0. \quad (20)$$

Since $|\delta_1(z)| < 1$ for all $z \in \mathbb{D}$, and

$$|u(z)| |f_{\varphi(z)}(\varphi(z))| \leq |u(z)f_{\varphi(z)}(\varphi(z)) - v(z)f_{\varphi(z)}(\psi(z))| + |v(z)| |f_{\varphi(z)}(\psi(z))|,$$

by combining (19) and (20) we deduce

$$\begin{aligned} &\lim_{|z| \rightarrow 1^-} \frac{|\delta_1(z)| |u(z)| (\widehat{v}(z)(1 - |z|))^{\frac{1}{q}}}{(\widehat{\omega}(\varphi(z))(1 - |\varphi(z)|))^{\frac{1}{p}}} \\ &= \lim_{|z| \rightarrow 1^-} |\delta_1(z)| |u(z)| |f_{\varphi(z)}(\varphi(z))| (\widehat{v}(z)(1 - |z|))^{\frac{1}{q}} = 0. \end{aligned} \quad (21)$$

Further, we claim that for each $0 < \gamma < \infty$ and each bounded set $\Omega \subset \mathbb{C}$, there exists a constant $C = C(\gamma, \Omega) > 0$ such that

$$|1 - z^\gamma| \leq C|1 - z|, \quad z \in \Omega, \quad (22)$$

the proof of which is postponed for a moment. By using this and the fact that $1 - \frac{1-|a|^2}{1-\bar{a}b} = \bar{a}\varphi_a(b)$ for all $a, b \in \mathbb{D}$, we deduce

$$\begin{aligned} |f_{\varphi(z)}(\varphi(z)) - f_{\varphi(z)}(\psi(z))| &= |f_{\varphi(z)}(\varphi(z))| \left| 1 - \left(\frac{1 - |\varphi(z)|^2}{1 - \overline{\varphi(z)}\psi(z)} \right)^\gamma \right| \\ &\lesssim |f_{\varphi(z)}(\varphi(z))| \left| 1 - \frac{1 - |\varphi(z)|^2}{1 - \overline{\varphi(z)}\psi(z)} \right| \leq |f_{\varphi(z)}(\varphi(z))| |\delta_1(z)|, \end{aligned}$$

and hence

$$\begin{aligned} |u(z) - v(z)| |f_{\varphi(z)}(\psi(z))| &\leq |u(z)f_{\varphi(z)}(\varphi(z)) - v(z)f_{\varphi(z)}(\psi(z))| \\ &\quad + |u(z)| |f_{\varphi(z)}(\varphi(z)) - f_{\varphi(z)}(\psi(z))| \\ &\lesssim |u(z)f_{\varphi(z)}(\varphi(z)) - v(z)f_{\varphi(z)}(\psi(z))| \\ &\quad + |\delta_1(z)| |u(z)| |f_{\varphi(z)}(\varphi(z))|, \quad z \in \mathbb{D}. \end{aligned}$$

Therefore, by (19) and (21), we finally obtain

$$\begin{aligned} &\lim_{|z| \rightarrow 1^-} \left(\frac{|1 - \overline{\varphi}(z)\delta_1(z)|^\gamma}{(\widehat{\omega}(\varphi(z))(1 - |\varphi(z)|))^\frac{1}{p}} \right) |u(z) - v(z)| (\widehat{v}(z)(1 - |z|))^\frac{1}{q} \\ &= \lim_{|z| \rightarrow 1^-} |u(z) - v(z)| |f_{\varphi(z)}(\psi(z))| (\widehat{v}(z)(1 - |z|))^\frac{1}{q} = 0. \end{aligned}$$

By following the reasoning above, but with the choice $a = \psi(z)$, we obtain

$$\lim_{|z| \rightarrow 1^-} \frac{|\delta_1(z)| |v(z)| (\widehat{v}(z)(1 - |z|))^\frac{1}{q}}{(\widehat{\omega}(\psi(z))(1 - |\psi(z)|))^\frac{1}{p}} = 0$$

as an analogue of (21), and then eventually

$$\lim_{|z| \rightarrow 1^-} \left(\frac{|1 - \overline{\psi}(z)\delta_2(z)|^\gamma}{(\widehat{\omega}(\psi(z))(1 - |\psi(z)|))^\frac{1}{p}} \right) |u(z) - v(z)| (\widehat{v}(z)(1 - |z|))^\frac{1}{q} = 0.$$

Therefore to finish the proof of the proposition, it remains to establish (22). If $z \in \Omega \setminus \{z : |1 - z| < 1/2\}$, then

$$|1 - z^\gamma| \leq 1 + \sup_{z \in \Omega} |z|^\gamma \leq 2 \left(1 + \sup_{z \in \Omega} |z|^\gamma \right) |1 - z|,$$

while if $z \in \Omega \cap \{z : |1 - z| < 1/2\}$, we have

$$|1 - z^\gamma| = \left| \int_z^1 \gamma \zeta^{\gamma-1} d\zeta \right| \leq \gamma \int_z^1 |\zeta|^{\gamma-1} |d\zeta| \leq \frac{\gamma \max\{1, 3^{\gamma-1}\}}{2^{\gamma-1}} |1 - z|.$$

This proves (22), and completes the proof of the proposition. $\square$

Sufficient conditions for the compactness of $uC_\varphi - vC_\psi : A_\omega^p \rightarrow A_v^q$ are given in the next result.

Proposition 2 *Let $\omega, v \in \mathcal{D}$ and $0 < p \leq q < \infty$ such that $\widehat{v}(z)(1 - |z|) \lesssim (\widehat{\omega}(z)(1 - |z|))^{\frac{q}{p}}$ for all $z \in \mathbb{D}$. Further, let $u, v \in \mathcal{H}(\mathbb{D})$ and φ and ψ analytic self-maps of $\mathbb{D}$ such that (4) is satisfied. If there exists $\gamma > 0$ such that (5) and (6) are satisfied, then $uC_\varphi - vC_\psi : A_\omega^p \rightarrow A_v^q$ is compact.*

Proof It suffices to show that for any norm bounded sequence $\{f_n\}$ in A_ω^p which tends to zero uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$, we have $\|(uC_\varphi - vC_\psi)(f_n)\|_{A_v^q} \rightarrow 0$ as $n \rightarrow \infty$. For simplicity, assume $\|f_n\|_{A_\omega^p} \leq 1$ for all n . Fix $0 < r < R < 1$, and denote $E = \{z \in \mathbb{D} : |\delta_1(z)| < r\}$ and $E' = \mathbb{D} \setminus E$. Write

$$(uC_\varphi - vC_\psi)(f_n) = (uC_\varphi - vC_\psi)(f_n)\chi_{E'} + (u - v)C_\psi(f_n)\chi_E + u(C_\varphi - C_\psi)(f_n)\chi_E,$$

and observe that it is enough to prove that each of the three quantities

$$\|(uC_\varphi - vC_\psi)(f_n)\chi_{E'}\|_{A_v^q}, \quad \|(u - v)C_\psi(f_n)\chi_E\|_{A_v^q} \quad \text{and} \quad \|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{A_v^q} \quad (23)$$

tends to zero as $n \rightarrow \infty$.

We begin with considering the first two quantities in (23). By the definition of the set E we have the estimates

$$|(uC_\varphi - vC_\psi)(f_n)\chi_{E'}| \leq \frac{1}{r}(|\delta_1 u C_\varphi(f_n)| + |\delta_1 v C_\psi(f_n)|)$$

and

$$|(u - v)C_\psi(f_n)\chi_E| \leq \left(\frac{1}{1 - r}\right)^\gamma |1 - \overline{\psi}\delta_2|^\gamma |u - v| |C_\psi(f_n)|$$

on $\mathbb{D}$. Therefore it suffices to prove that $\delta_1 u C_\varphi$, $\delta_1 v C_\psi$ and $(1 - \overline{\psi}\delta_2)^\gamma (u - v)C_\psi$ are compact operators from A_ω^p to L_v^q . We show in detail that $\delta_1 u C_\varphi$ is compact - the same argument shows the compactness of the other two operators.

Let μ be a finite nonnegative Borel measure on $\mathbb{D}$ and h a measurable function on $\mathbb{D}$. For an analytic self-map φ of $\mathbb{D}$, the weighted pushforward measure is defined by

$$\varphi_*(h, \mu)(M) = \int_{\varphi^{-1}(M)} h d\mu \quad (24)$$

for each measurable set $M \subset \mathbb{D}$. If μ is the Lebesgue measure, we omit the measure in the notation and write $\varphi_*(h)(M)$ for the left hand side of (24). By the measure theoretic change of variable [9, Section 39], we have $\|\delta_1 u C_\varphi(f)\|_{L_v^q} = \|f\|_{L_{\varphi_*(|\delta_1 u|^q \nu)}^q}$ for each $f \in A_\omega^p$. Therefore Theorem 2 shows that $\delta_1 u C_\varphi : A_\omega^p \rightarrow L_v^q$ is compact if and only if

$$\frac{\varphi_*(|\delta_1 u|^q \nu)(\Delta(a, r))}{\omega(\Delta(a, r))^{\frac{q}{p}}} = \frac{\int_{\varphi^{-1}(\Delta(a, r))} |\delta_1(z)u(z)|^q \nu(z) dA(z)}{\omega(\Delta(a, r))^{\frac{q}{p}}} \rightarrow 0, \quad |a| \rightarrow 1^-.$$

This is what we prove next. Define

$$W_{a,r} = \sup_{z \in \varphi^{-1}(\Delta(a,r))} \left| (\delta_1(z)u(z))^q \frac{\widehat{v}(z)(1-|z|)}{(\widehat{\omega}(\varphi(z))(1-|\varphi(z)|))^{\frac{q}{p}}} \right|.$$

Then $W_{a,r} \rightarrow 0$, as $|a| \rightarrow 1^-$, by the hypothesis (5). Moreover, for $a \in \mathbb{D}$ and $z \in \varphi^{-1}(\Delta(a,r))$, (11) yields

$$|\delta_1(z)u(z)|^q \lesssim W_{a,r} \frac{(\widehat{\omega}(a)(1-|a|))^{\frac{q}{p}}}{\widehat{v}(z)(1-|z|)},$$

and therefore, for each $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} \varphi_*(|\delta_1 u|^q \nu)(\Delta(a,r)) &= \int_{\varphi^{-1}(\Delta(a,r))} |\delta_1(z)u(z)|^q \nu(z) dA(z) \\ &\lesssim \sup_{z \in \mathbb{D}} \left| (\delta_1(z)u(z))^q \frac{\widehat{v}(z)(1-|z|)}{(\widehat{\omega}(z)(1-|z|))^{\frac{q}{p}}} \right|^{1-\varepsilon} W_{a,r}^\varepsilon (\widehat{\omega}(a)(1-|a|))^{\frac{\varepsilon q}{p}} \\ &\quad \cdot \int_{\varphi^{-1}(\Delta(a,r))} \frac{(\widehat{\omega}(z)(1-|z|))^{(1-\varepsilon)\frac{q}{p}} \nu(z)}{\widehat{v}(z)(1-|z|)} dA(z), \quad a \in \mathbb{D}. \end{aligned} \quad (25)$$

Before proceeding further, we indicate how to get to this point with the operator $(1 - \overline{\psi}\delta_2)^\gamma(u - v)C_\psi$. After the measure theoretic change of variable and an application of Theorem 2, consider

$$V_{a,r} = \sup_{z \in \psi^{-1}(\Delta(a,r))} \left| (1 - \overline{\psi}(z)\delta_2(z))^{\gamma q} (u(z) - v(z))^q \frac{\widehat{v}(z)(1-|z|)}{(\widehat{\omega}(\psi(z))(1-|\psi(z)|))^{\frac{q}{p}}} \right|$$

instead of $W_{a,r}$. Then $V_{a,r} \rightarrow 0$, as $|a| \rightarrow 1^-$, by the hypothesis (6), and moreover, for $a \in \mathbb{D}$ and $z \in \psi^{-1}(\Delta(a,r))$, (11) yields

$$|1 - \overline{\psi}(z)\delta_2(z)|^{\gamma q} |u(z) - v(z)|^q \lesssim V_{a,r} \frac{(\widehat{\omega}(a)(1-|a|))^{\frac{q}{p}}}{\widehat{v}(z)(1-|z|)}.$$

Therefore, for each $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} \psi_* (|1 - \overline{\psi}\delta_2|^{\gamma q} |u - v|^q \nu)(\Delta(a,r)) &\lesssim \sup_{z \in \mathbb{D}} \left| (1 - \overline{\psi}(z)\delta_2(z))^{\gamma q} (u(z) - v(z))^q \frac{\widehat{v}(z)(1-|z|)}{(\widehat{\omega}(z)(1-|z|))^{\frac{q}{p}}} \right|^{1-\varepsilon} V_{a,r}^\varepsilon (\widehat{\omega}(a)(1-|a|))^{\frac{\varepsilon q}{p}} \\ &\quad \cdot \int_{\psi^{-1}(\Delta(a,r))} \frac{(\widehat{\omega}(z)(1-|z|))^{(1-\varepsilon)\frac{q}{p}} \nu(z)}{\widehat{v}(z)(1-|z|)} dA(z), \quad a \in \mathbb{D}. \end{aligned}$$

To estimate this last integral, which is the same as the one appearing in (25), we first show that there exists $\varepsilon = \varepsilon(\omega, \nu, q, p) \in (0, 1)$ sufficiently small such that the function

$$\mu(z) = \mu_{\omega, \nu, \varepsilon, q, p}(z) = \frac{(\widehat{\omega}(z)(1 - |z|))^{(1-\varepsilon)\frac{q}{p}} \nu(z)}{\widehat{\nu}(z)(1 - |z|)}, \quad z \in \mathbb{D},$$

is a weight and belongs to $\mathcal{D}$. To see that $\mu \in \widehat{\mathcal{D}}$, for $n \in \mathbb{N} \cup \{0\}$, define ρ_n by $\widehat{\nu}(\rho_n) = \frac{\widehat{\nu}(0)}{K^n}$, where $K > 1$. Let $0 \leq r < 1$ and fix $M \in \mathbb{N} \cup \{0\}$ such that $\rho_M \leq r < \rho_{M+1}$. Set $\alpha = (1 - \varepsilon)\frac{q}{p}$ for short. Since $\omega, \nu \in \mathcal{D}$ by the hypothesis, (11) yields

$$\begin{aligned} \widehat{\omega}(\rho_{j+N+1}) &\leq \widehat{\omega}(\rho_j) \lesssim \widehat{\omega}(\rho_{j+N+1}) \left(\frac{1 - \rho_j}{1 - \rho_{j+N+1}} \right)^{\beta(\omega)} \lesssim \widehat{\omega}(\rho_{j+N+1}) \left(\frac{\widehat{\nu}(\rho_j)}{\widehat{\nu}(\rho_{j+N+1})} \right)^{\frac{\beta(\omega)}{\alpha(\nu)}} \\ &= \widehat{\omega}(\rho_{j+N+1}) K^{(N+1)\frac{\beta(\omega)}{\alpha(\nu)}} \asymp \widehat{\omega}(\rho_{j+N+1}) \end{aligned}$$

and $1 - \rho_j \asymp 1 - \rho_{j+1}$ for all j and for each fixed $N \in \mathbb{N}$. Therefore

$$\begin{aligned} \int_r^1 \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt &\leq \sum_{j=M}^{\infty} \int_{\rho_j}^{\rho_{j+1}} \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt \leq \sum_{j=M}^{\infty} \frac{(\widehat{\omega}(\rho_j))^\alpha}{(1-\rho_{j+1})^{1-\alpha}} \int_{\rho_j}^{\rho_{j+1}} \frac{\nu(t)}{\widehat{\nu}(t)} dt \\ &= \sum_{j=M}^{\infty} \frac{(\widehat{\omega}(\rho_j))^\alpha}{(1-\rho_{j+1})^{1-\alpha}} \int_{\rho_{j+N}}^{\rho_{j+N+1}} \frac{\nu(t)}{\widehat{\nu}(t)} dt \asymp \int_{\rho_{M+N}}^1 \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt. \end{aligned}$$

Another application of (11) shows that there exists $N = N(\nu) \in \mathbb{N}$ such that $\rho_{M+N} \geq \frac{1+\rho_{M+1}}{2}$. Namely, the right hand inequality implies

$$\frac{1 - \rho_{M+1}}{1 - \rho_{M+N}} \geq \left(\frac{\widehat{\nu}(\rho_{M+1})}{C\widehat{\nu}(\rho_{M+N})} \right)^{\frac{1}{\beta}} = \left(\frac{K^{N-1}}{C} \right)^{\frac{1}{\beta}} \geq 2$$

for sufficiently large N giving what we want. Therefore

$$\int_r^1 \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt \lesssim \int_{\frac{1+\rho_{M+1}}{2}}^1 \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt \leq \int_{\frac{1+r}{2}}^1 \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt,$$

which shows that $\mu \in \widehat{\mathcal{D}}$, provided μ is a weight.

We next show that μ is a weight in $\check{\mathcal{D}}$. By using (11), with β in place of α , we obtain

$$\begin{aligned} \int_r^1 \frac{(\widehat{\omega}(t))^\alpha}{(1-t)^{1-\alpha}} \frac{\nu(t)}{\widehat{\nu}(t)} dt &\lesssim \int_r^1 \frac{\left(\widehat{\omega}(r) \left(\frac{1-t}{1-r} \right)^\beta (1-t) \right)^\alpha}{\widehat{\nu}(t)(1-t)} \nu(t) dt \\ &\lesssim \frac{\widehat{\omega}(r)^\alpha}{(1-r)^{\beta\alpha}} \int_r^1 \frac{(1-t)^{(1+\beta)\alpha}}{\widehat{\nu}(t)(1-t)} \nu(t) dt. \end{aligned}$$

Now fix $\varepsilon \in (0, 1)$ sufficient small such that $\sigma = (1 + \beta)\alpha = (1 + \beta)(1 - \varepsilon)\frac{q}{p} > 1$. By [19, Lemma 3], the last expression above is dominated by a constant times

$$\frac{\widehat{\omega}(r)^\alpha}{(1-r)^{\beta\alpha}} (1-r)^{\sigma-1} = \frac{(\widehat{\omega}(r)(1-r))^\alpha}{1-r}.$$

Since $\nu \in \check{\mathcal{D}}$ by the hypothesis, there exists $K = K(\nu) > 1$ such that

$$\begin{aligned} \frac{(\widehat{\omega}(r)(1-r))^\alpha}{1-r} &\asymp \frac{(\widehat{\omega}(r)(1-r))^\alpha}{\widehat{\nu}(r)(1-r)} \int_r^{1-\frac{1-r}{K}} \nu(t) dt \\ &\asymp \int_r^{1-\frac{1-r}{K}} \frac{(\widehat{\omega}(t)(1-t))^\alpha}{\widehat{\nu}(t)(1-t)} \nu(t) dt, \end{aligned}$$

where the last step is a consequence of (11), applied to both weights $\omega, \nu \in \mathcal{D}$. This reasoning shows that μ is a weight in $\check{\mathcal{D}}$, and thus $\mu \in \mathcal{D}$.

We return to estimate the last integral in (25). By [16, Proposition 18], the operator $C_\varphi : A_\mu^p \rightarrow A_\mu^p$ is bounded for each $0 < p < \infty$. By the measure theoretic change of variable, this is equivalent to saying that $I_d : A_\mu^p \rightarrow L_{\varphi_*(\mu)}^p$ is bounded. Since we just proved that $\mu \in \mathcal{D}$, this is in turn equivalent to $\varphi_*(\mu)(\Delta(a, r)) \lesssim \mu(\Delta(a, r))$ by Theorem 2. By the definition of these two measures and the poof of the fact $\mu \in \check{\mathcal{D}}$ above, we have

$$\int_{\varphi^{-1}(\Delta(a, r))} \mu(z) dA(z) \lesssim \int_{\Delta(a, r)} \mu(z) dA(z) \lesssim (\widehat{\omega}(a)(1-|a|))^{(1-\varepsilon)\frac{q}{p}}, \quad a \in \mathbb{D}.$$

This combined with (25) gives

$$\varphi_*(|\delta_1 u|^q \nu)(\Delta(a, r)) \lesssim \sup_{z \in \mathbb{D}} \left| \left(\delta_1(z) u(z) \right)^q \frac{\widehat{\nu}(z)(1-|z|)}{(\widehat{\omega}(z)(1-|z|))^\frac{q}{p}} \right|^{1-\varepsilon} W_{a, r}^\varepsilon (\widehat{\omega}(a)(1-|a|))^\frac{q}{p}.$$

Since the supremum above is bounded by the hypothesis (4), and $\widehat{\omega}(a)(1-|a|) \lesssim \omega(\Delta(a, r))$ for $r = r(\omega) \in (0, 1)$ sufficiently large by the Proof of Theorem 2, we deduce via Theorem 2 that $\delta_1 u C_\varphi : A_\omega^p \rightarrow L_\nu^q$ is compact. As mentioned already, $\delta_1 \nu C_\psi$ and $(1 - \overline{\psi} \delta_2)^\gamma (u - \nu) C_\psi$ can be treated in the same way.

It remains to deal with the third term in (23). By Lemma 1, Fubini's theorem and (11), we have

$$\begin{aligned} \|u(C_\varphi - C_\psi)(f_n) \chi_E\|_{A_\nu^q}^q &= \int_E |u(z)|^q |f_n(\varphi(z)) - f_n(\psi(z))|^q \nu(z) dA(z) \\ &\lesssim \int_E \frac{|u(z) \delta_1(z)|^q}{(\widehat{\omega}(\varphi(z))(1-|\varphi(z)|))^\frac{q}{p}} \int_{(\varphi(z), R)} |f_n(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta) \nu(z) dA(z) \\ &\leq \int_{\mathbb{D}} |f_n(\zeta)|^p \widetilde{\omega}(\zeta) \\ &\quad \cdot \left(\int_{\varphi^{-1}(\Delta(\zeta, R)) \cap E} \frac{|u(z) \delta_1(z)|^q}{(\widehat{\omega}(\varphi(z))(1-|\varphi(z)|))^\frac{q}{p}} \nu(z) dA(z) \right) dA(\zeta) \\ &\asymp \int_{\mathbb{D}} |f_n(\zeta)|^p \widetilde{\omega}(\zeta) \left(\int_{\varphi^{-1}(\Delta(\zeta, R))} \frac{|u(z) \delta_1(z)|^q}{(\widehat{\omega}(\zeta)(1-|\zeta|))^\frac{q}{p}} \nu(z) dA(z) \right) dA(\zeta) \\ &= \int_{\mathbb{D}} |f_n(\zeta)|^p \frac{\varphi_*(|\delta_1 u|^q \nu)(\Delta(\zeta, R))}{(\widehat{\omega}(\zeta)(1-|\zeta|))^\frac{q}{p}} \widetilde{\omega}(\zeta) dA(\zeta). \end{aligned}$$

Since the identity operator from A_ω^p to $L_{\varphi_*(|\delta_1 u|^q \nu)}^q$ is compact, it is also bounded. This and Theorem 2 yield

$$\begin{aligned}
\|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{A_v^q}^q &\lesssim \sup_{\zeta \in D(0,r)} \frac{\varphi_*(|\delta_1 u|^q v)(\Delta(\zeta, R))}{(\widehat{\omega}(\zeta)(1 - |\zeta|))^{\frac{q}{p}}} \int_{D(0,r)} |f_n(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta) \\
&\quad + \sup_{\zeta \in \mathbb{D} \setminus D(0,r)} \frac{\varphi_*(|\delta_1 u|^q v)(\Delta(\zeta, R))}{(\widehat{\omega}(\zeta)(1 - |\zeta|))^{\frac{q}{p}}} \int_{\mathbb{D} \setminus D(0,r)} |f_n(\zeta)|^p \widetilde{\omega}(\zeta) dA(\zeta) \\
&\lesssim \sup_{\zeta \in D(0,r)} |f_n(\zeta)|^p + \sup_{\zeta \in \mathbb{D} \setminus D(0,r)} \frac{\varphi_*(|\delta_1 u|^q v)(\Delta(\zeta, R))}{(\widehat{\omega}(\zeta)(1 - |\zeta|))^{\frac{q}{p}}}, \quad 0 < r < 1.
\end{aligned}$$

By choosing $0 < r < 1$ sufficiently large, the last term can be made smaller than a pregiven $\varepsilon > 0$. For such fixed r , the first term tends to zero as $n \rightarrow \infty$ by the uniform convergence. Therefore

$$\|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{A_v^q}^q \rightarrow 0, \quad n \rightarrow \infty,$$

and hence also the last term in (23) tends to zero. This finishes the proof of the proposition. $\square$

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