

Viable periodic solutions in state-dependent impulsive problems

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Abstract Periodic solutions of state-dependent impulsive ODEs in a prescribed set of constraints are examined. The so-called impulsive index is introduced, as a topological tool, and its properties are studied. Some sufficient conditions for its homotopy property are discussed in detail. In a construction of the impulsive index the fixed point index on ANRs is applied to an induced discrete semidynamical system on a barrier where jumps occur. Several illustrative examples are added.

Keywords Periodic solution · Viable solution · Fixed point index · State-dependent impulsive problems · Impulse function

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1 Introduction

Impulsive ordinary, or partial, differential equations became a strongly growing field of research in a last half of a century. Some results in this field are simple modifications and generalizations of analogous results for non-impulsive problems but, beyond them, there are essentially new, nontrivial and interesting questions requiring new methods and proof arguments. We meet this especially in *state-dependent impulsive problems* where jumps occur in variable times. A study in this direction was initiated by Mil'man and Myshkis in [18]. Problems with impulses in variable times are met in e.g. physical and biological models (cf. [3]), where concerned quantities rapidly change when they attain some barriers or some critical levels in a phase space or in an extended phase space. In an analysis of solutions of such problems, in contrast to those with fixed impulse times, one also has to

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take into account a geometry or topology of barriers and their relationships with dynamics of the problem. This area is still poorly investigated, especially in the context of applications of known or construction of new appropriate topological tools.

In the paper we focus our attention on detecting a periodic behavior of some of trajectories for state-dependent impulsive ODEs under an additional requirement that these trajectories are *viable* in a prescribed closed subset $K \subset \mathbb{R}^n$ of a state space. In other words, we are going to find a solution to the following periodic problem

$$\begin{cases} \dot{x}(t) = f(x(t)) & \text{for a.e. } t \geq 0, x \in \mathbb{R}^n, \\ x(t) \in K, \\ x(t^+) := \lim_{s \rightarrow t^+} x(s) = I(x(t)) & \text{for } x(t) \in M \subset \partial K, \\ x(0) = x(T_x), & \text{for some } T_x > 0, \end{cases} \quad (1.1)$$

where $f : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^n$ is sufficiently smooth to generate a semiflow, and $I : M \rightarrow \mathbb{R}^n$ is a continuous *impulse function*. More specific assumptions on f , M and I will be given later. The notion of viability used above is motivated by biological models where some species survives only if it does not leave a region of safety K given as a set of constraints.

There are only a few papers concerning problem (1.1) (see, e.g. [8, 9, 14, 17]). In most of them additional Nagumo-type tangency conditions are assumed to ensure that from every point of K there starts at least one trajectory, even with impulses taken into account, remaining in K for some nontrivial time interval. Then the fixed point theory can be applied for a suitably defined Poincaré–Krasnoselskii operator.

If tangency conditions do not hold on the whole boundary, we allow trajectories for a non-impulsive problem, even all of them, to leave K through a so-called *exit set* K^- . Of course, we want to prevent this escape, so we place a barrier M in K^- trying to return at least some of them to the set K via the impulse function I (see [14]). In some biological models it means an external impact (e.g. restocking the lake). In [14] the impulse function was assumed to satisfy $I(M) \subset K$. Analysis of models in mathematical biology shows that it is worth considering a weaker assumption $I(M) \subset \mathbb{R}^n$, in practice at least $I(M) \cap K \neq \emptyset$. This situation is more complicated and needs deeper local topological tools to study problem (1.1). In the paper we construct a so-called *impulsive index*, which is a fixed point index of suitably defined continuous map in an exit set K^- . Its nontriviality will imply the existence of viable periodic trajectories in K .

The paper is organized as follows. In Sect. 2 at first we give basic assumptions, and define the impulsive index. Then properties of the index are presented, discussed and proved. Several examples are given to illustrate the results. In Sect. 3 we present geometric sufficient conditions for a homotopy property of the index with some results, concerning exit sets, interesting in themselves. The Poincaré map technique is adapted to compute the impulsive index in a neighborhood of a periodic point in Sect. 4 (see also [14] where this technique was proposed). The last section (Sect. 5) contains several concluding remarks concerning an alternative construction of the impulsive index. Advantages and disadvantages of this alternative technique are discussed.

2 Impulsive index

Let $K \subset \mathbb{R}^n$ be an arbitrary closed subset, $\Omega \supset K$ an open neighborhood in \mathbb{R}^n , and $f : \Omega \rightarrow \mathbb{R}^n$ be a sufficiently smooth vector field such that the equation $\dot{x} = f(x)$ generates a semiflow π on Ω . We are interested in viable in K solutions so, without any loss of generality, we can assume in the sequel that $\Omega = \mathbb{R}^n$. Let us recall that by a solution of problem (1.1) we

mean a left-continuous function $x : [0, \infty) \rightarrow \mathbb{R}^n$, with a discrete set of discontinuity points, absolutely continuous between these points, and satisfying $\dot{x}(t) = f(x(t))$ in a.e. $t \geq 0$, and $x(t^+) = I(x(t))$ in discontinuity points. The set of functions from $[0, \infty)$ to \mathbb{R}^n which are continuous between discontinuity points taken from a discrete set, and left-continuous in these points will be denoted below by $PC([0, \infty), \mathbb{R}^n)$.

In each point $x \in K$ we can consider the Bouligand tangent cone

$$T_K(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{h \rightarrow 0^+} \text{dist}(x + hv, K)/h = 0 \right\}.$$

Obviously, if the following tangency condition

$$f(x) \in T_K(x) \quad \text{for every } x \in K$$

is satisfied, then all solutions starting in K remain in this set forever (see the first results of this type in [19] and [6]). If it is not satisfied, the following *exit set* for K

$$K^- = K^-(f) := \{x_0 \in K \mid \forall \varepsilon > 0 : \pi(\{x_0\} \times (0, \varepsilon)) \not\subset K\}$$

appears. If topological properties of the exit set sufficiently differs from properties of K , then still there exists at least one viable trajectory in K . This was firstly proved in the celebrated Ważewski paper [20]. More precisely, if K^- is closed and it is not a strong deformation retract of K , then a viable solution exists (comp. [11]). In the paper we are interested in even worse case:

$$\text{All absolutely continuous solutions to } \dot{x} = f(x) \text{ leave the set } K. \quad (2.1)$$

It means, in consequence, that K is strongly deformed via homotopy onto K^- . Namely, the exit function $\tau_K : K \rightarrow [0, \infty)$, $\tau_K(x_0) := \sup\{t \geq 0 \mid \pi(\{x_0\} \times [0, t]) \subset K\}$ is continuous, and we can define a homotopy $h : K \times [0, 1] \rightarrow K$, $h(x_0, \lambda) := \pi(x_0, \lambda \tau_K(x_0))$. Putting $r(x_0) := h(x_0, 1)$ we obtain a (continuous) retraction $r : K \rightarrow K^-$.

Hence, in the whole paper we assume that

$$h(x, \alpha) := \pi(x, \alpha \tau_K(x)), x \in K, \text{ is a deformation of } K \text{ onto } K^-. \quad (2.2)$$

Of course, (2.2) implies that K^- is a closed subset of K . When we consider a family π_λ of semiflows, we will assume, analogously, that deformations h_λ are given, see below.

Let $M \subset K^-$ be an *impulse set*, i.e., there is an *impulse function* $I : M \rightarrow \mathbb{R}^n$ moving instantly each trajectory reaching the set M in a point $p \in M$ to a point $I(p) \in \mathbb{R}^n$. In practice, we will be interested in impulse functions satisfying $I(M) \cap K \neq \emptyset$ which means that some trajectories are moving back to K . In our considerations we assume that

- (A1) K^- is a closed ANR,
- (A2) $M = cl_{K^-}(int_K M)$,
- (A3) I is a compact map, i.e. $I(M)$ is relatively compact,
- (A4) $\text{dist}(I(M), M) > 0$,

where $int_K A$, $cl_{K^-} A$ and, in the sequel, $\partial_{K^-} A$ denote the relative interior, closure and boundary of a set $A \subset K^-$ in K^- , respectively. Note that, if M is compact, which is often the case, (A4) is implied by $I(M) \cap M = \emptyset$. This condition or (A4) is assumed in many papers concerning impulsive semidynamical systems (see, e.g., [1, 5, 10]). It implies, in particular, that for every $x_0 \in K \setminus M$ the sequence $\sigma_0(x_0) = 0$, $\sigma_1(x_0) = \phi(x_0)$, $\sigma_{m+1}(x_0) = \sigma_m(x_0) + \phi((I \circ r)^m(x_0))$ satisfies the condition $\lim_{m \rightarrow \infty} \sigma_m(x_0) = \infty$, where $\phi(x) = \inf\{s > 0 \mid \pi(\{x\} \times [0, s]) \cap M = \emptyset \text{ and } \pi(x, s) \in M\}$.

We define $M^I := I^{-1}(K)$ and

$$g : M^I \rightarrow K^-, \quad g(x) := r(I(x)) \text{ for } x \in M^I. \quad (2.3)$$

Denote

$$\mathcal{P} := \{x \in PC([0, \infty), \mathbb{R}^n) \mid x \text{ is a viable periodic solution to (1.1)}\}.$$

Notice that, since every absolutely continuous solution to $\dot{x} = f(x)$ leaves the set K , there is a correspondence between elements of \mathcal{P} and fixed points of iterations of the map g defined in (2.3). Indeed, let $x \in \mathcal{P}$ start from $x_0 \in K$. Then $z = \pi(x_0, \tau_K(x_0)) \in M$, and there is a minimal $m \in \mathbb{N}$ such that $x_0 = \pi(I(g^m(z)), s)$ for some $s \in [0, \tau_K(I(g^m(z)))]$. It implies that $z = \pi(x_0, \tau_K(x_0)) = g^{m+1}(z)$, so z is a fixed point of g^{m+1} . On the other hand, if $z = g^m(z)$, then the map $x(0) := z, x(t) := \pi(I(g^k(z)), t - \sigma_k(I(z)))$ for $t \in (\sigma_k(I(z)), \sigma_{k+1}(I(z))]$, $k \geq 0$, is a viable periodic solution to problem (2.2) starting from z .

In particular, each fixed point of g generates a periodic viable trajectory in K . This motivates us to construct a fixed point index type topological tool detecting fixed points of g .

Let V be an arbitrary open subset of K^- . Let us define

$$U_V := \text{int}_{K^-}(I^{-1}(K) \cap V) \subset V, \quad (2.4)$$

and assume that

$$\text{Fix } g \cap U_V \text{ is a compact set.}$$

We define an *impulsive index* of π on V as

$$\text{ind}_K(\pi, I, V) := \text{ind}(g, U_V), \quad (2.5)$$

where $\text{ind}(g, U_V)$ is a fixed point index for compact maps on ANRs (see [16]).

Below we show the following main properties of the impulsive index.

Theorem 2.1 (Existence) *If $\text{ind}_K(\pi, I, V) \neq 0$, then there exists a viable periodic solution to (1.1) reaching the set V .*

Theorem 2.2 (Additivity) *Let $K^- \supset V = V_1 \cup V_2$, V_i open and $V_1 \cap V_2 = \emptyset$. If $\text{Fix } g \cap (U_V)$ is a compact set, then*

$$\text{ind}_K(\pi, I, V) = \text{ind}_K(\pi, I, V_1) + \text{ind}_K(\pi, I, V_2).$$

Theorem 2.3 (Homotopy I) *Let $\pi_\lambda, \lambda \in [0, 1]$, be a continuous family of semiflows, i.e. the map $\mathbb{R}^n \times [0, \infty) \times [0, 1] \ni (x, t, \lambda) \mapsto \pi_\lambda(x, t)$ is continuous. Assume that, for every $\lambda \in [0, 1]$, the exit set K_λ^- for π_λ is a closed ANR, there are retractions $r_\lambda : K \rightarrow K_\lambda^-$ along trajectories, i.e., $r_\lambda(x) = \pi_\lambda(x, \tau_K(x))$, and $M \subset K_\lambda^-$ satisfies $M = \text{cl}_{K_\lambda^-}(\text{int}_{K_\lambda^-} M)$. Moreover, assume that $X = \bigcup_{\lambda \in [0, 1]} K_\lambda^- \in \text{ANR}$, and the set $\bigcup_{\lambda \in [0, 1]} r_\lambda(I(M) \cap K)$ is relatively compact in X .*

Let $V \subset \bigcap_{\lambda \in [0, 1]} K_\lambda^-$ be open in every K_λ^- , and such that $U_{V_{\lambda_0}}$ is an open subset of X for some $\lambda_0 \in [0, 1]$, and $\text{Fix } g_\lambda \cap U_{V_\lambda}$ is compact, where $g_\lambda := r_\lambda \circ I : M^I \rightarrow K_\lambda^-$.

Then the numbers $\text{ind}_K(\pi_\lambda, I, V)$ are well defined, and

$$\text{ind}_K(\pi_0, I, V) = \text{ind}_K(\pi_1, I, V).$$

Theorem 2.4 (Homotopy II) *Let $\pi_\lambda, \lambda \in [0, 1]$, be a continuous family of semiflows, and assume that, for every $\lambda \in [0, 1]$, the exit set K_λ^- for π_λ is a closed ANR, there are retractions $r_\lambda : K \rightarrow K_\lambda^-$ along trajectories, impulse sets $M_\lambda \subset K_\lambda^-$ satisfying $M_\lambda = \text{cl}_{K_\lambda^-}(\text{int}_{K_\lambda^-} M_\lambda)$,*

Fig. 1 A system with $V \cap M = \emptyset$

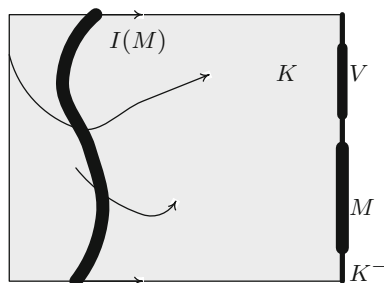
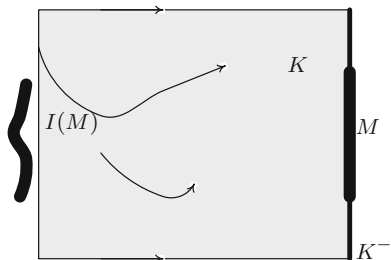


Fig. 2 A system with $I(M) \cap K = \emptyset$



and impulse functions $I_\lambda \rightarrow \mathbb{R}^n$. Moreover, assume that, for every pair $0 \leq a \leq b \leq 1$, the set $X_a^b = \bigcup_{\lambda \in [a,b]} K_\lambda^-$ is an ANR, and the set $\bigcup_{\lambda \in [a,b]} r_\lambda(I_\lambda(M_\lambda) \cap K)$ is relatively compact in X_a^b .

Let $V_\lambda \subset K_\lambda^-$ be such that $\bigcup_{\lambda \in [0,1]} U_{V_\lambda} \times \{\lambda\}$ is an open subset of X_0^1 , and such that $F := \bigcup_{\lambda \in [0,1]} ((\text{Fix } g_\lambda \cap U_{V_\lambda}) \times \{\lambda\})$ is compact, where $g_\lambda := r_\lambda \circ I_\lambda : M_\lambda^I \rightarrow K_\lambda^-$ and $M_\lambda^I := I_\lambda^{-1}(I_\lambda(M_\lambda) \cap K)$. Then

$$\text{ind}_K(\pi_0, I_0, V_0) = \text{ind}_K(\pi_1, I_1, V_1).$$

Before proofs, let us comment some of assumptions in the above theorems.

Remark 2.5 If $\text{Fix } g \cap \partial_{K^-}(U_V) = \emptyset$ in (2.5) and Theorem 2.1, then $\text{Fix } g \cap U_V$ is a compact set. Indeed, it follows from the continuity of g and compactness of the impulse map in the following way. We have $I(M^I) \subset K$, K is closed in \mathbb{R}^n , and $\text{cl}(I(M))$ is compact. Hence, $\text{cl}(I(M^I)) \subset K$ is also compact. Now, $\text{cl}(r(\text{cl}(I(M^I)))) \subset \text{cl}(r(\text{cl}(I(M^I)))) = r(\text{cl}(I(M^I))) \subset K^-$, and $r(\text{cl}(I(M^I)))$ is compact because r is continuous.

If $I(V \cap M) \subset K$ in Theorem 2.1, then $U_V = \text{int}_{K^-}(I^{-1}(K \cap V)) = \text{int}_{K^-}(V \cap M)$, and

$$\text{ind}_K(\pi, I, V) = \text{ind}(g, \text{int}_{K^-}(V \cap M)).$$

In particular, if $V \subset M$ is an open subset of K^- , then

$$\text{ind}_K(\pi, I, V) = \text{ind}(g, V),$$

as in paper [14].

If $V \cap M = \emptyset$ (see Fig. 1) or $I(M) \cap K = \emptyset$ (see Fig. 2), then $U_V = \emptyset$, and we trivially obtain $\text{ind}_K(\pi, I, V) = 0$.

The assumption $M \subset \bigcap_{\lambda \in [0,1]} K_\lambda^-$ in Theorem 2.3 is sensible. One knows that

$$K_{\Rightarrow}^\lambda := \{x \in \partial K; f_\lambda(x) \notin T_K(x)\} \subset K_\lambda^- \subset \text{cl } K_{\Rightarrow}^\lambda,$$

if semiflows are generated by equations $\dot{x} = f_\lambda(x)$ ([7], Lemma 5.1). Assume that $M \subset \{x \in \partial K; f_{\lambda_0}(x) \notin T_K(x)\}$ and (f_λ) is a continuous family of maps. Then, for λ sufficiently near λ_0 , $M \subset \{x \in \partial K; f_\lambda(x) \notin T_K(x)\} \subset K_\lambda^-$.

The existence of a family of retractions r_λ in Theorem 2.3 is also natural. Indeed, if K is bounded and π_{λ_0} does not have a viable trajectory in K for some $\lambda_0 \in [0, 1]$, then π_λ does not have a viable trajectory in K for λ sufficiently close to λ_0 . To check this, let us assume, by contrary, that there is a sequence $\lambda_n \rightarrow \lambda_0$ and a viable trajectory $\pi_{\lambda_n}(\cdot, x_n)$ in K for every $n \geq 1$. By the compactness of K we can assume, without any loss of generality, that $x_n \rightarrow x \in K$. From the continuity of a family of semiflows it follows that $\pi_{\lambda_0}(\cdot, x)$ is viable in K ; a contradiction.

Observe that in Theorem 2.4 we have a family of maps g_λ with possibly different domains and codomains, and, in fact, we consider an otopy introduced in [4] in the context of vector fields on manifolds.

Proof of Theorem 2.1. It is sufficient to notice that $0 \neq \text{ind}_K(\pi, I, V) = \text{ind}(g, U_V)$ implies $\text{Fix } g \cap U_V \neq \emptyset$ so, there is a point $p \in U_V \subset V$ with $r(I(p)) = p$. This means that a solution of (1.1) starting from $I(p)$ is viable in K and periodic, and touches the barrier in $p \in V$. \square

Proof of Theorem 2.2. Notice that

$$\begin{aligned} U_V &= \text{int}_{K^-}(I^{-1}(K) \cap V) \\ &= \text{int}_{K^-}(I^{-1}(K) \cap (V_1 \cup V_2)) \\ &= \text{int}_{K^-}([I^{-1}(K) \cap V_1] \cup [I^{-1}(K) \cap V_2]) \end{aligned}$$

Denote $A := I^{-1}(K) \cap V_1$ and $B = I^{-1}(K) \cap V_2$. We are going to check that $U_V = U_{V_1} \cup U_{V_2}$, i.e., $\text{int}_{K^-}(A \cup B) = \text{int}_{K^-}A \cup \text{int}_{K^-}B$. Of course, ' \supset ' is always true. It is sufficient to prove that each open set contained in $A \cup B$ can be represented as a sum of open subsets in A and B , respectively.

Let W be an open subset of K^- contained in $A \cup B = I^{-1}(K) \cap V$. Define $W_1 := W \cap A$ and $W_2 := W \cap B$. Since $W_1 \subset V_1$ and V_1 is open in K^- , for an arbitrary $x \in W_1$ there exists $r_0 > 0$ such that $B(x, r_0) \subset V_1$, where $B(x, r_0)$ is an open ball in K^- . The set W is open so there is $0 < r \leq r_0$ such that $B(x, r) \subset W \subset I^{-1}(K) \cap V$. Therefore, $B(x, r) \subset V_1 \cap I^{-1}(K) = A$. Similarly we check that W_2 is an open set contained in B .

Now, from the assumption it follows that $\text{Fix } g \cap (U_{V_1} \cup U_{V_2})$ is a compact set. Moreover, $U_{V_1} \cup U_{V_2} = \emptyset$ so $\text{Fix } g \cap U_{V_i}$ is also compact for $i = 1, 2$. By the additivity property of the fixed point index on ANRs we obtain

$$\begin{aligned} \text{ind}_K(\pi, I, V) &= \text{ind}(g, U_V) = \text{ind}(g, U_{V_1}) + \text{ind}(g, U_{V_2}) \\ &= \text{ind}_K(\pi, I, V_1) + \text{ind}_K(\pi, I, V_2). \end{aligned}$$

\square

Proof of Theorem 2.3. At first we prove that $U_{V_\lambda} = U_{V_{\lambda_0}}$ for every $\lambda \in [0, 1]$. To do this we take any open set $W \subset K_\lambda^-$ contained in $I^{-1}(K) \cap V$. Let $\lambda' \in [0, 1]$ be arbitrary. Since V is open in K_λ^- and in $K_{\lambda'}^-$, there is $r_0 > 0$ such that for every $0 < r \leq r_0$ we have $B_{K_\lambda^-}(x, r) \cup B_{K_{\lambda'}^-}(x, r) \subset V$, and consequently, $B_V(x, r) = B_{K_\lambda^-}(x, r) = B_{K_{\lambda'}^-}(x, r)$, where these balls are in spaces $V, K_\lambda^-, K_{\lambda'}^-$, respectively. But W is open in K_λ^- so there is $0 < \delta \leq r_0$ such that $B_{K_\lambda^-}(x, \delta) \subset W$ which implies that $B_{K_{\lambda'}^-}(x, \delta) = B_{K_\lambda^-}(x, \delta) \subset W$. Hence, W is open in $K_{\lambda'}^-$.

Consider the maps g_λ as $g_\lambda : M^I \rightarrow X$, i.e., with a codomain X . By the assumption, the set $U_{V_{\lambda_0}}$ is open in X . Therefore, we have a homotopy $g_\lambda : U_{V_{\lambda_0}} \rightarrow X$. Now, we use the contraction and homotopy properties of the fixed point index on ANRs to obtain

$$\begin{aligned}\operatorname{ind}_K(\pi_0, I, V) &= \operatorname{ind}(g_0, U_{V_{\lambda_0}}) = \operatorname{ind}_X(g_0, U_{V_{\lambda_0}}) \\ &= \operatorname{ind}_X(g_1, U_{V_{\lambda_0}}) = \operatorname{ind}_K(\pi_1, I, V),\end{aligned}$$

where, to distinguish, ind_X stands for the fixed point index on the ANR X . \square

Remark 2.6 It is not obvious that $U_{V_{\lambda_0}}$, which is open in every K_λ^- , is open in X , simultaneously. Some sufficient conditions for this property are presented in a separate section (see Sect. 3).

Proof of Theorem 2.4. We show that the index $\operatorname{ind}_K(\pi_\lambda, I_\lambda, V_\lambda)$ is locally constant, i.e., for every $\lambda_0 \in [0, 1]$ there exists $\delta > 0$ such that $\operatorname{ind}_K(\pi_\lambda, I_\lambda, V_\lambda) = \operatorname{ind}_K(\pi_{\lambda_0}, I_{\lambda_0}, V_{\lambda_0})$ for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$.

Fix any $\lambda_0 \in [0, 1]$. From the assumption on the set F it follows that $F_{\lambda_0} := \operatorname{Fix} g_{\lambda_0} \cap U_{V_{\lambda_0}}$ is a compact set as an image of F by the projection $p : \partial K \times [0, 1] \rightarrow \partial K$, $p(x, \lambda) = x$. For every point $x \in F_{\lambda_0}$ there exists $r(x) > 0$ such that $B_X(x, r(x)) \times (\lambda_0 - r(x), \lambda_0 + r(x)) \subset \bigcup_{\lambda \in [0, 1]} U_{V_\lambda} \times \{\lambda\}$. Hence, we can take a finite covering $\{B_X(x_k, r(x_k)) \times (\lambda_0 - r(x_k), \lambda_0 + r(x_k))\}_{k=1}^m$ of $F_{\lambda_0} \times \{\lambda_0\}$, and its Lebesgue number l . Then, denoting $I_{\lambda_0} := (\lambda_0 - l, \lambda_0 + l) \cap [0, 1]$, we obtain

$$F_{\lambda_0} \times \{\lambda_0\} \subset \bigcup_{x \in F_{\lambda_0}} (B_X(x, l) \times I_{\lambda_0}) = \left(\bigcup_{x \in F_{\lambda_0}} B_X(x, l) \right) \times I_{\lambda_0} \subset \bigcup_{\lambda \in I_{\lambda_0}} (U_{V_\lambda} \times \{\lambda\}).$$

Denote $U := \bigcup_{x \in F_{\lambda_0}} B_X(x, l)$. Then $U \subset U_{V_\lambda}$ for every $\lambda \in I_{\lambda_0}$, and all assumptions of Theorem 2.3 are satisfied with $V := U$ and $X := X_a^b$, $a := \max\{\lambda_0 - l, 0\}$, $b := \min\{\lambda_0 + l, 1\}$. Hence, for every $\lambda \in I_{\lambda_0}$,

$$\operatorname{ind}_K(\pi_\lambda, I_\lambda, V_\lambda) = \operatorname{ind}(g_\lambda, U_{V_\lambda}) = \operatorname{ind}(g_\lambda, U) = \operatorname{const}.$$

By the compactness of $[0, 1]$ we get $\operatorname{ind}_K(\pi_0, I_0, V_0) = \operatorname{ind}_K(\pi_1, I_1, V_1)$. \square

3 Sufficient conditions for the homotopy property

As noted in Remark 2.6, conditions which are sufficient for the homotopy property (see Theorem 2.3) need a deeper study. We start with a rather obvious observation:

Remark 3.1 If U is an open subset of each A_i , $i \in \{1, \dots, m\}$, then U is open in $X := \bigcup_{i=1}^m A_i$.

Indeed, in a simple proof for every $x \in U$ and $i \in \{1, \dots, m\}$ we find $r_i > 0$ such that $U \supset B_{A_i}(x, r_i) = B_X(x, r_i) \cap A_i$. Taking $r := \min\{r_1, \dots, r_m\}$ we immediately obtain

$$B_X(x, r) = \bigcup_{i=1}^m (B_X(x, r) \cap A_i) = \bigcup_{i=1}^m B_{A_i}(x, r) \subset U,$$

which means that U is open in X .

It is easily seen that the above arguments fail in the case of infinitely many sets A_i .

Below we present several results which give, aside from the main aim of the section, interesting properties of exit sets.

Proposition 3.2 *If (f_λ) , $\lambda \in [0, 1]$, is a continuous family of vector fields in \mathbb{R}^n generating, via equations $\dot{x} = f_\lambda(x)$, a continuous family of semiflows π_λ , and K is a closed subset of \mathbb{R}^n with compact exit sets K_λ^- , then the map $\phi^- : [0, 1] \rightarrow \mathbb{R}^n$, $\phi^-(\lambda) = K_\lambda^-$ is lsc (lower semicontinuous).*

Proof Suppose that ϕ^- is not lsc. Then there are $\lambda_0 \in [0, 1]$ and $\varepsilon > 0$ such that for every $\delta > 0$ we can find $\lambda \in [0, 1]$ with $|\lambda - \lambda_0| < \delta$ and $K_{\lambda_0}^- \not\subset N_\varepsilon(K_\lambda^-)$, where $N_\varepsilon(\cdot)$ denotes an ε -neighborhood of a set. Taking $\delta = \frac{1}{m}$, $m \geq 1$, we obtain, for every $m \geq 1$, a number $\lambda_m \in (\lambda_0 - \frac{1}{m}, \lambda_0 + \frac{1}{m}) \cap [0, 1]$ and a point $x_m \in K_{\lambda_0}^-$ such that $\text{dist}(x_m, K_{\lambda_m}^-) \geq \varepsilon$.

By the compactness of K_{λ_0} we obtain, up to subsequence, that $x_m \rightarrow x_0$ for some $x_0 \in K_{\lambda_0}^-$. Consider a family of Cauchy problems

$$\begin{cases} \dot{x} = f_{\lambda_m}(x), \\ x(0) = x_0 \end{cases} \quad (3.1)$$

with solutions y_m defined on a common interval $[0, T]$, $T > 0$. From the convergence $\lambda_m \rightarrow \lambda_0$ it follows that $y_m \rightarrow y$, where y is a solution to

$$\begin{cases} \dot{x} = f_{\lambda_0}(x), \\ x(0) = x_0. \end{cases} \quad (3.2)$$

We know that, for every $\eta > 0$, there is a positive time $s \in (0, \rho)$ with $y(s) \notin K$. Take s such that $|x_0 - y(t)| < \frac{\varepsilon}{3}$ for every $t \in [0, s]$. By the convergence $y_m \rightarrow y$ and closedness of the set K we get that $y_m(s) \notin K$ for sufficiently big $m \geq 1$. It implies that $y_m(s_m) \in K_{\lambda_m}^-$ for some $s_m \in [0, s]$.

We can assume that m is so big that $|y_m(t) - y(t)| < \frac{\varepsilon}{3}$ for every $t \in [0, s]$. Then

$$|y_m(s_m) - x_m| \leq |y_m(s_m) - y(s_m)| + |y(s_m) - x_0| + |x_0 - x_m| < \varepsilon,$$

which contradicts the inequality $\text{dist}(x_m, K_{\lambda_m}^-) \geq \varepsilon$. \square

Under some regularity assumptions on the set K one obtains the following

Proposition 3.3 *Assume that $K \subset \mathbb{R}^n$ is a closed C^1 -manifold with a boundary ∂K , (f_λ) is as in Proposition 3.2, and $\text{int}_{\partial K} K_\lambda^- = K_\lambda^-$ for every $\lambda \in [0, 1]$ (see Remark 2.5.4). Then the map $\phi^+ : [0, 1] \rightarrow \mathbb{R}^n$, $\phi^+(\lambda) = \text{cl}(\partial K \setminus K_\lambda^-)$ has a closed graph.*

Proof Assume that $[0, 1] \ni \lambda_m \rightarrow \lambda$ and $y_m \rightarrow y$, where $y_m \in \text{cl}(\partial K \setminus K_{\lambda_m}^-)$. Take a sequence (x_m) such that $x_m \in \partial K \setminus K_{\lambda_m}^-$ and $|x_m - y_m| < \frac{1}{n}$, for every $n \geq 1$. Then, obviously, $x_m \rightarrow y$. We shall show that $y \in \text{cl}(\partial K \setminus K_{\lambda_0}^-)$.

Suppose it is not true. Then $y \in \text{int}_{\partial K} K_{\lambda_0}^-$ which implies, by the assumption, that $\langle n(y), f_{\lambda_0}(y) \rangle =: d > 0$, where $n(y)$ stands for the outer normal vector to K in y . Using the continuity of the family (f_λ) , the map $n(\cdot)$ and an inner product, we get, for m big enough,

$$\begin{aligned} \langle n(x_m), f_{\lambda_m}(x_m) \rangle &= \langle n(x_m) - n(y), f_{\lambda_m}(x_m) \rangle + \langle n(y), f_{\lambda_0}(y) \rangle \\ &\quad + \langle n(y), f_{\lambda_m}(x_m) - f_{\lambda_0}(y) \rangle \\ &\geq -|n(x_m) - n(y)|(|f_{\lambda_0}(y)| + 1) + d - |n(y)||f_{\lambda_m}(x_m) - f_{\lambda_0}(y)|. \end{aligned}$$

It implies that $\langle n(x_m), f_{\lambda_m}(x_m) \rangle > 0$ for sufficiently big m . Hence, $x_m \in K_{\lambda_m}^-$; a contradiction. \square

We use Propositions 3.2 and 3.3 to prove:

Proposition 3.4 *Let assumptions of Proposition 3.3 be satisfied with compact exit sets K_λ^- . Assume that $U \subset \text{int}_{\partial K} K_{\lambda_0}^-$ is open in $K_{\lambda_0}^-$. Then*

$$\forall x \in U \exists r > 0 \exists \delta > 0 \forall \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1] : \text{dist}(x, \partial_{\partial K} K_\lambda^-) > r.$$

Proof Suppose that for some $x \in U$ we have

$$\forall r > 0 \forall \delta > 0 \exists \lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1] : \text{dist}(x, \partial_{\partial K} K_\lambda^-) \leq r.$$

Take $r = \frac{1}{m}$ and $\delta = \frac{1}{m}$. Then, for every $m \geq 1$, we can find $\lambda_m \in (\lambda_0 - \frac{1}{m}, \lambda_0 + \frac{1}{m}) \cap [0, 1]$ such that $\text{dist}(x, \partial_{\partial K} K_{\lambda_m}^-) \leq \frac{1}{m}$ which implies that there exists $x_m \in \partial_{\partial K} K_{\lambda_m}^-$ with $|x - x_m| \leq \frac{1}{m}$.

From Proposition 3.2 it follows that for every $m \geq 1$ there exists $\eta > 0$ such that $K_{\lambda_0}^- \subset N_{\frac{1}{m}}(K_\lambda^-)$ for each $\lambda \in [0, 1]$ with $|\lambda - \lambda_0| < \eta$. Therefore, we can assume that $K_{\lambda_0}^- \subset N_{\frac{1}{m}}(K_{\lambda_m}^-)$. By Proposition 3.3 the map $\phi^+ : [0, 1] \rightarrow \mathbb{R}^n$, $\phi^+(\lambda) = cl(\partial K \setminus K_\lambda^-)$ has a closed graph. Hence, since $\lambda_m \rightarrow \lambda_0$, $x_m \rightarrow x$ and $x_m \in \partial_{\partial K} K_{\lambda_m}^- \subset cl(\partial K \setminus K_{\lambda_m}^-)$, we obtain that $x \in cl(\partial K \setminus K_{\lambda_0}^-)$. But we know that $x \in K_{\lambda_0}^-$, so $x \in \partial_{\partial K} K_{\lambda_0}^-$ which contradicts the assumption that $x \in U \subset \text{int}_{\partial K} K_{\lambda_0}^-$. \square

Assumptions of Proposition 3.4 are sufficient for one of the main proof arguments in the homotopy property of the impulsive index (see Remark 2.6) as we can see in the following:

Proposition 3.5 *Under assumptions of Proposition 3.4 every set $U \subset \text{int}_{\partial K} K_{\lambda_0}^-$, which is open in $K_{\lambda_0}^-$ and contained in each K_λ^- , is also open in $X = \bigcup_{\lambda \in [0, 1]} K_\lambda^-$.*

Proof It is easy to see that U is open in ∂K . Moreover, $U \subset X \subset \partial K$. This completes the proof. \square

Open problem. Is the following property true?

If $(f_\lambda), \lambda \in [0, 1]$, is a continuous family of vector fields in \mathbb{R}^n generating a continuous family of semiflows π_λ , and K is a closed subset of \mathbb{R}^n with compact exit sets K_λ^- , then the map $\phi^+ : [0, 1] \rightarrow \mathbb{R}^n$, $\phi^+(\lambda) = cl(\partial K \setminus K_\lambda^-)$ has a closed graph.

In other words, is Proposition 3.3 true without C^1 -regularity of the set and an additional assumption $\text{int}_{\partial K} K_\lambda^- = K_\lambda^\lambda$ on the exit set?

4 Computation of the index via Poincaré-type maps

In the section we briefly remind the technique of computation of the fixed point index $\text{ind}(g_\lambda, U)$ and, consequently, the impulsive index $\text{ind}_K(\pi, I, V)$, proposed firstly in [14]. The idea is partially based on the Poincaré map and Poincaré section technique used in the study of periodic trajectories of ordinary smooth flows.

Let K be a closed subset of \mathbb{R}^n , and we are given a problem (1.1) with the differential equation generating a closed exit set K^- which is a strong deformation retract of K via the homotopy along trajectories (see Sect. 2). Let $M \subset K^-$ be a barrier, and $V \subset K^-$ be any subset open in K^- .

In what follows we assume that the vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the impulse function $I : M \rightarrow \mathbb{R}^n$ are of class C^1 , at least. Let $p \in M \cap V$ be a fixed point of $g = rI : M^I \rightarrow K^-$. Denote $q := I(p) \in K$ and the time $T := \tau_K(q)$. Obviously, $p = \pi(q, T)$, where π is a flow generated by f . We shall also use the assumption that M is a level set $\{x \in \mathbb{R}^n \mid \gamma(x) = 0\}$ for some smooth function $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$.

Finally assume that $p \in \int_{\partial K} (M \cap V)$, the vector $f(p)$ is transversal to ∂K , and $f(q)$ is transversal to $I(M)$. Then, for some $N \subset M \cap V$ with $p \in \text{int}_K N$ there exists a $\delta > 0$ such that for every $y \in \{\pi(x, t) \mid x \in N, |t| < \delta\}$ one can find a unique $\delta(y) \in (-\delta, \delta)$ with $\pi(y, \delta(y)) \in N$. Such a neighborhood N is a *Poincaré section* for π .

Under the above assumptions we can check that the exit time function τ_K is smooth in a neighborhood of q . Without any loss of generality we can assume that $I(N)$ is this neighborhood. Since $g(x) = \pi(I(x), \tau_K(I(x)))$, we get

$$g_x(x) = \pi_x(I(x), \tau_K(I(x)))I_x(x) + \pi_t(I(x), \tau_K(I(x)))(\tau_K)_x(I(x))I_x(x), \quad (4.1)$$

where, e.g., $g_x(x)$ stands for the state variable derivative of g . Remind that (one easily checks it) $t \mapsto f(\pi(q, t))$ is a solution on $[0, T]$ to the linear differential equation $\dot{y} = f_x(\pi(q, t))y$ with an initial condition $y(0) = f(q)$. It is seen that $y(T) = f(p)$. We know also that $t \mapsto \pi_x(q, t)$ is a fundamental matrix solution to the equation $\dot{Y} = f_x(\pi(q, t))Y$ with $Y(0) = \pi_x(q, 0)$. Hence, we get

$$Y(T)f(q) = \pi_x(q, T)f(q) = f(\pi(q, T)) = f(p).$$

Now we choose in \mathbb{R}^n two bases $\mathcal{B}_1 := \{f(p), a_2, \dots, a_n\}$ and $\mathcal{B}_2 := \{f(q), b_2, \dots, b_n\}$ such that a_i are parallel to M at p (they form a basis in a tangent space to M at p) and b_i are parallel to $I(M)$ at q . We consider a linear map $\pi_x(q, T)$ in bases \mathcal{B}_2 and \mathcal{B}_1 obtaining the matrix

$$\pi_x(q, T) = \begin{bmatrix} 1 & A \\ 0 & B \end{bmatrix}, \quad \text{where } A \in M_{1 \times (n-1)} \quad \text{and } B \in M_{(n-1) \times (n-1)}.$$

If $w \in \mathbb{R}^n$ is parallel to M at p , then it has the form $w = \begin{bmatrix} 0 \\ w_M \end{bmatrix}$. Moreover, $I_x(p) \begin{bmatrix} 0 \\ w_M \end{bmatrix} = \begin{bmatrix} 0 \\ w_{IM} \end{bmatrix}$, where $\begin{bmatrix} 0 \\ w_{IM} \end{bmatrix}$ is a vector parallel to $I(M)$ at q .

Now, using (4.1) we can compute a derivative of a Poincaré map. Let w be a vector parallel to M at p . Then

$$\begin{aligned} Dg(p)w &= \pi_x(q, T)I_x(p) \begin{bmatrix} 0 \\ w_M \end{bmatrix} + \pi_t(q, T)(\tau_K)_x(q)I_x(p) \begin{bmatrix} 0 \\ w_M \end{bmatrix} \\ &= \pi_x(q, T) \begin{bmatrix} 0 \\ w_{IM} \end{bmatrix} + f(p)(\tau_K)_x(q) \begin{bmatrix} 0 \\ w_{IM} \end{bmatrix} \\ &= \begin{bmatrix} Aw_{IM} \\ Bw_{IM} \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix}, \end{aligned}$$

for some $c \in \mathbb{R}$. Since g maps M into $K^-(f)$, we obtain in the basis \mathcal{B}_1

$$Dg(p)w = \begin{bmatrix} 0 \\ Bw_{IM} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B \cdot T_p I \end{bmatrix} w,$$

where $T_p I : T_p M \rightarrow T_q I(M)$ is the derivative of I treated as a map between manifolds. This is a very good information because in a computation of the degree one uses eigenvalues

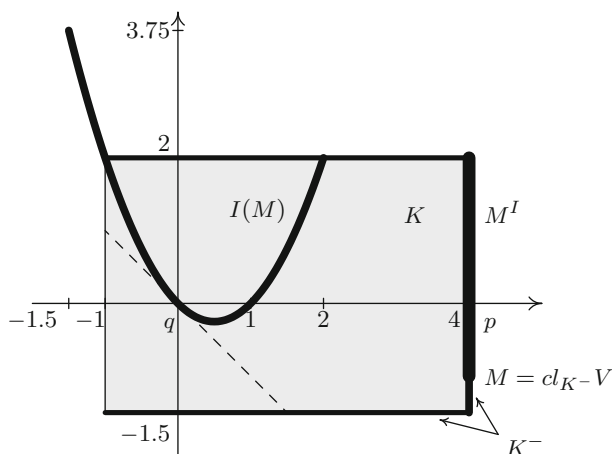


Fig. 3 Illustration to Example 4.1

of the derivative. Here, eigenvalues of $Dg(p)$ correspond to those of the matrix $B \cdot T_p I \in M_{(n-1) \times (n-1)}$.

Taking a sufficiently small neighborhood U of p we obtain

$$\text{ind}(g, p) := \text{ind}(g, U) = \deg(i - g, U) = (-1)^\beta,$$

where β is the sum of the multiplicities of the negative real eigenvalues of $\text{id}_{T_p M} - B \cdot T_p I$.

To illustrate how the above technique works in practice we present an example which is a modification of Example 4.1 in [14].

Example 4.1 We have a system of equations

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = y \end{cases}$$

generating a global flow $\pi((x, y), t) = (x + t, ye^t)$ on \mathbb{R}^2 . For the set of constraints $K := [-1, 4] \times [-3/2, 2]$ we obtain the exit set $K^- = K_1 \cup K_2 \cup K_3$, where K_i are the faces $x = 4$, $y = 2$ and $y = -3/2$ of K , respectively (see Fig. 3). Let $M := \{(4, y) \mid y \in [-3/2, 2]\} = \text{cl}_K - \{(4, y) \mid y \in (-3/2, 2)\}$ and $I : M \rightarrow \mathbb{R}^n$ be an impulse function given by $I(x, y) := (y, y(y - 1))$. For simplicity, take $V = \{(4, y) \mid y \in (-3/2, 2)\}$ which is open in K^- .

It is easy to check that $M^I = I^{-1}(I(M) \cap K) = \{(4, y) \mid y \in [-1, 2]\}$ and $U_V = \text{int}_{K^-}(I^{-1}(I(V \cap M) \cap K) \cap V) = \{(4, y) \mid y \in (-1, 2)\}$.

Since $g(4, -1) = I(4, -1) = (-1, 2) \in K^-$ and $g(4, 2) = I(4, 2) = (2, 2) \in K^-$, we have $\text{ind}(g, U_V) = 0$. On the other hand, it is evident that $p := (4, 0)$ is a fixed point of g with $q := I(p) = (0, 0)$. Now, as in [14], we take two bases $\mathcal{B}_1 := \{(1, 0), (0, 1)\}$ and $\mathcal{B}_2 := \{(1, 0), (1, -1)\}$ of \mathbb{R}^2 , and notice that $\pi_x(q, T) = \begin{bmatrix} 1 & 1 \\ 0 & -e^T \end{bmatrix}$ in bases \mathcal{B}_2 and \mathcal{B}_1 , i.e., $\pi_x(q, T) : (\mathbb{R}^2, \mathcal{B}_2) \rightarrow (\mathbb{R}^2, \mathcal{B}_1)$, where $T = \tau_K(q) = 4$.

Since $DI(p) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in bases \mathcal{B}_1 and \mathcal{B}_2 , for any vector $w = \begin{bmatrix} 0 \\ v \end{bmatrix}$ parallel to M we get

$$Dg(p)(w) = Dg(p) \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ -e^T v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -e^T \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

This implies that $\text{ind}(g, p) := \text{ind}(g, W) = \deg(i - g, W) = \text{sgn}(1 + e^T) = 1$ for some small neighborhood W of p in U_V . From the additivity property of the fixed point index we obtain $\text{ind}(g, U_V \setminus \text{cl}_{K-(f)} W) = \text{ind}(g, U_V) - \text{ind}(g, W) = -1 \neq 0$ which implies the existence of another periodic trajectory.

5 Concluding remarks

We start the section with a discussion on a possible alternative definition of the impulsive index. Instead of conditions (A1)–(A4) from Sect. 2 we assume

- (Z1) ∂K is a closed ANR, and the exit set K^- is closed,
- (Z2) $M = \text{cl}_{\partial K}(\text{int}_{\partial K} M)$,
- (Z3) I is a compact map, i.e. $I(M)$ is relatively compact,
- (Z4) $\text{dist}(I(M), M) > 0$.

As before, we assume (2.1) with the retraction $r : K \rightarrow K^-$, $r(x) = \pi(x, \tau_K(x))$. Define $M^I := I^{-1}(K)$ and $g : M^I \rightarrow \partial K$, $g(x) := r(I(x))$ for $x \in M^I$. Notice that a codomain is different from the one in (2.3). Of course, $g(M^I) \subset K^-$.

Let V be an arbitrary open subset of ∂K . We define

$$U_V := \text{int}_{\partial K}(I^{-1}(K) \cap V) \subset V, \quad (5.1)$$

and assume that

$$\text{Fix } g \cap U_V \text{ is a compact set.}$$

Now we put

$$\text{ind}'_K(\pi, I, V) := \text{ind}(g, U_V), \quad (5.2)$$

where $\text{ind}(g, U_V)$ is a fixed point index in an ANR ∂K , not in K^- .

This index has standard properties (see Sect. 2) with the homotopy property under easier assumptions. We give, for instance, an analogue to Theorem 2.3 as the following

Theorem 5.1 (Homotopy I) *Let π_λ , $\lambda \in [0, 1]$, be a continuous family of semiflows. Assume that, for every $\lambda \in [0, 1]$, the exit set K_λ^- for π_λ is closed, there are retractions $r_\lambda : K \rightarrow K_\lambda^-$, $r_\lambda(x) = \pi_\lambda(x, \tau_K(x))$, and $M = \text{cl}_{\partial K}(\text{int}_{\partial K} M)$. Moreover, assume that the set $\bigcup_{\lambda \in [0, 1]} r_\lambda(I(M) \cap K)$ is relatively compact.*

Let V be an open subset of ∂K , and $\text{Fix } g_\lambda \cap U_{V_\lambda}$ is compact.

Then the numbers $\text{ind}'_K(\pi_\lambda, I, V)$ are well defined, and

$$\text{ind}'_K(\pi_0, I, V) = \text{ind}'_K(\pi_1, I, V).$$

Proof Notice that $U_{V_\lambda} = \text{int}_{\partial K}(I^{-1}(K) \cap V) = U_{V_0}$ is independent of the choice of λ . So, we have a homotopy $g_\lambda : U_{V_0} \rightarrow \partial K$ with assumptions allowing to apply the homotopy property of the index on ANRs. This finishes the proof. \square

Fig. 4 $\text{ind}_K(\pi, I, V) = 1$ for $V = M = K^-$

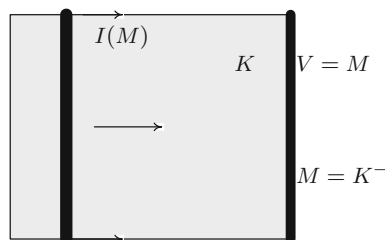


Fig. 5 $\text{ind}'_K(\pi, I, V_0)$ is not defined

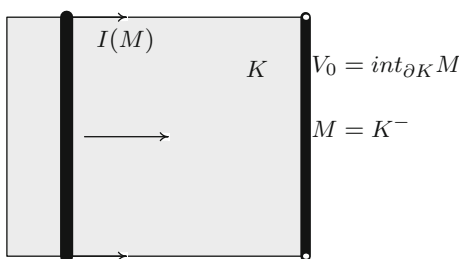
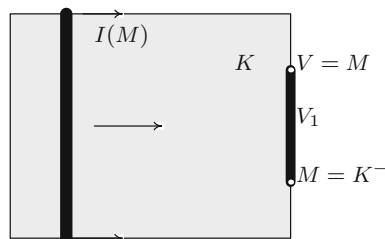


Fig. 6 $\text{ind}_K(\pi, I, V_1) = \text{ind}'_K(\pi, I, V_1) = -1$ and $\text{ind}_K(\pi, I, V) = 1$



One would think that the index ind'_K is better because it simplifies considerations. However, it does not cover some important cases. Indeed, the set V cannot be equal to K^- since K^- is closed and usually not open in ∂K . The following concrete example shows the difference between the results obtained by the use of ind_K and ind'_K .

Example 5.2 Let K be a rectangle in \mathbb{R}^2 , for instance $K = [0, 3] \times [0, 2]$ and π is generated by the equation $(\dot{x}, \dot{y}) = (g(t), 0)$ with $g(t) > 0$. Then, obviously, $K^- = \{3\} \times [0, 2]$. Put $M = K^-$, see Figs. 4, 5, 6 and 7.

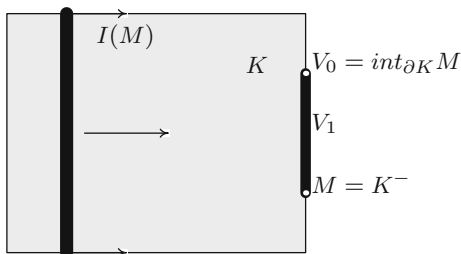
At first, assume that $I(x, y) := (1, y)$ and $V = M = K^-$ which is allowed in Sect. 2 (Fig. 4). Then $U_V = K^-$ and $\text{ind}_K(\pi, I, V) = \text{ind}(id, K^-) = \chi(K^-) = 1$, where $\chi(K^-)$ is the Euler characteristic of the set K^- (equal to the Lefschetz number of id).

Notice that, for every open subset V_0 of ∂K , the set U_{V_0} constructed in (5.1), even the largest one, is not equal to K^- . It implies that the index $\text{ind}'_K(\pi, I, V_0)$ for no V_0 (see Fig. 5) is well defined, because $\text{Fix } id \cap U_{V_0} = U_{V_0}$ is not compact.

Now, let M, V, V_0 be as above, and $I(x, y) := (1, \sqrt[3]{y-1} + 1)$. Consider $V_1 := \{3\} \times (1/2, 3/2)$ (see Figs. 6, 7). Since $U_{V_1} = V_1$ in both approaches [see (2.4) and (5.1)], and $g(3, y) = (3, \sqrt[3]{y-1} + 1)$, we easily obtain $\text{ind}_K(\pi, I, V_1) = \text{ind}'_K(\pi, I, V_1) = -1$ and $\text{ind}_K(\pi, I, V) = 1$, while $\text{ind}'_K(\pi, I, V_0) = -1$.

Our final remarks concerns possible generalizations of the impulsive index. One knows that a differential inclusion $\dot{x} \in F(x)$ taken instead of $\dot{x} = f(x)$ in (1.1), where F is upper semicontinuous (usc) convex compact valued with a sublinear growth, generates a

Fig. 7 $\text{ind}_K(\pi, I, V_0) =$
 $\text{ind}_K(\pi, I, V_0) = -1$



multivalued dynamical system. Moreover, the solution map $\mathbb{R}^n \ni x_0 \mapsto S_F(x_0)$, where $S_F(x_0) \in C([0, \infty), \mathbb{R}^n)$ is a solution set for the Cauchy problem with $x(0) = x_0$, is usc with compact R_δ -values. Under suitable assumptions on the set of constraints K and an exit set, one can obtain results on the existence of viable and stationary trajectories (see, e.g., [12, 13]). If we suppose that there is no viable solution in K , it is sensible to assume that there exists a multivalued retraction Φ of K onto $K_e(F)$, where

$$K_e(F) := \{x_0 \in \partial K \mid \exists x \in S_F(x_0) : x \text{ leaves } K \text{ immediately}\}$$

is a bigger exit set (see [14], or [12] where a discussion on two exit sets $K^-(F)$ and $K_e(F)$ is presented). Now, if $I : M \multimap \mathbb{R}^n$ is a multivalued impulse map, then the construction of the fixed point index of the composition $\Phi \circ I$ is still possible under suitable geometric assumptions on $K_e(F)$ and regularity assumptions on I . In particular, if $K_e(F)$ is a compact ANR and I is usc with R_δ -values, then the index proposed in [2] could be used (see also [15] for other versions of the index). Details of the construction and properties of the impulsive index for multivalued flows and jumps we leave for further considerations.

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