Structure of the group of automorphisms of the spectral 2-ball

Łukasz Kosiński

Abstract We show that the group generated by triangular and diagonal conjugations is dense in $\mathrm{Aut}(\Omega_2)$ (in the compact-uniform topology). Moreover, it is shown that any automorphism of Ω_2 is a local holomorphic conjugation (it extends the results from (Rostand, Studia Math 155:207–230, 2003, Thomas, Collect Math 59(3):321–324, 2008)).

1 Introduction and statement of the result

Let Ω_n denote the spectral ball in $\mathbb{C}^{n^2}$, that is a domain composed of $n \times n$ complex matrices whose spectral radius is <1.

The natural question that arises is to classify its group of automorphisms. It may be easily checked that among them there are the following three forms:

(i) *Transposition*: $\tau : x \mapsto x^t$,
(ii) *Möbius maps*: $m_{\alpha,\gamma} : x \mapsto \gamma(x - \alpha)(1 - \overline{\alpha}x)^{-1}$, where α lies in the unit disc and $|\gamma| = 1$,
(iii) *Conjugations:* $\mathcal{J}_u : x \mapsto u(x)^{-1}xu(x)$, where $u : \Omega \to \mathcal{M}_n^{-1}$ is a conjugate invariant holomorphic map, i.e. $u(q^{-1}xq) = u(x)$ for each $x \in \Omega$ and $q \in \mathcal{M}_n^{-1}$

(throughout the paper $\mathcal{M}_n^{-1}$ denotes the group of invertible $n \times n$ complex matrices).

Ransford and White have asked in [9] whether the compositions of the three above forms generate the whole $\mathrm{Aut}(\Omega)$—the group of automorphisms of the spectral ball. In [6] we have shown that the answer to this question is negative. Nevertheless the question about the description of $\mathrm{Aut}(\Omega)$ remains open. In this note we deal with this problem.

Let us introduce some notation. $\mathcal{M}_n$ denotes the algebra of $n \times n$ matrices with complex coefficients. Moreover, $\mathcal{T}_n$ is the set of non-cyclic matrices lying in Ω_n. Recall that a matrix

Ł. Kosiński (✉)
WydziałMatematyki i Informatyki, Uniwersytet Jagielloński, Ul. Prof. St. Łojasiewicza 6,
30-348 Kraków, Poland
e-mail: lukasz.kosinski@gazeta.pl

M is cyclic if there exists a cyclic vector $v \in \mathbb{C}^n$ for M, i.e. v such that $(v, Mv, \ldots, M^{n-1}v)$ spans $\mathbb{C}^n$. It is well known that M is cyclic if and only if for any $\lambda \in \mathbb{C}$

$$\dim \operatorname{Ker}(M - \lambda) \leq 1.$$

In particular, any matrix having n distinct eigenvalues is cyclic.

In the case $n = 2$ we shall simply write $\Omega = \Omega_2$, $\mathcal{M} = \mathcal{M}_2$ and $\mathcal{T} = \mathcal{T}_2$.

Definition 1 Let U be an open subset of the spectral ball Ω_n. We shall say that a mapping $\varphi : U \to \mathcal{M}_n$ is a *holomorphic conjugation* if there is a holomorphic mapping $p : U \mapsto \mathcal{M}_n^{-1}$ such that $\varphi(x) = p(x)xp(x)^{-1}$, $x \in U$.

Moreover, we shall say that φ is a *local holomorphic conjugation* if any $x \in U$ has a neighborhood V such that φ restricted to it is a holomorphic conjugation.

The paper is organized as follows. Recall that we presented in [6] two counterexamples to the question on the description of the group of automorphisms of the spectral ball. We focus on them in the sense that we classify all automorphisms of the spectral ball of the form

$$\Omega \ni x \mapsto u(x)xu(x)^{-1} \in \Omega,$$

where $u \in \mathcal{O}(\Omega, \mathcal{M}^{-1})$ is such that $u(x)$ is either diagonal or triangular, $x \in \Omega$ (we shall call them *diagonal* and *triangular* conjugations respectively).

It is known that any automorphism of the spectral ball fixing the origin preserves the spectrum (see [9]). Therefore, trying to describe the group $\operatorname{Aut}(\Omega)$ it is natural to investigate the behavior of automorphisms restricted to the fibers of Ω, i.e. sets of the form $\mathcal{F}_{(\lambda_1, \lambda_2)} := \{x \in \Omega : \sigma(x) = \{\lambda_1, \lambda_2\}\}$, $\lambda_1, \lambda_2 \in \mathbb{D}$, where $\sigma(x)$ denotes the spectrum of $x \in \mathcal{M}$. If $\lambda_1 \neq \lambda_2$, the fiber $\mathcal{F}_{(\lambda_1, \lambda_2)}$ forms a submanifold known as the *Danielewski surfaces*. Recall that the Danielewski surface associated with a polynomial $p \in \mathbb{C}[z]$ is given by

$$D_p := \{(x, y, z) \in \mathbb{C}^3 : xy = p(z)\}$$

and its complex structure is naturally induced from $\mathbb{C}^3$. Algebraic properties of Danielewski surfaces have been intensively studied in the literature. Anyway little is known about their holomorphic automorphisms. It was lastly shown (see [5,7]) that the group generated by shears and overshears (for the definition see e.g. [8]) is dense in the group of holomorphic automorphisms. It was a little surprise to us that shears are just the restriction to the fiber of diagonal and triangular conjugations.

Following the idea from [8] (we recall all details for the convenience of the reader) we shall show that the spectral ball satisfies the property obtained by Andersén and Lempert for holomorphic automorphisms of $\mathbb{C}^n$ (see [1,2,4]). To be more precise we shall show that the group generated by triangular and diagonal conjugations is dense in $\operatorname{Aut}(\Omega)$ in the local-uniform topology (see Theorem 4).

Finally, in Lemma 6 we shall show that the uniform limit of conjugations is a local conjugation in a neighborhood of $\mathcal{T}$. This, together with the density of the the group generated by triangular and diagonal conjugations and results obtained by Thomas and Rostand (see [10, 11]) imply that any automorphism of the spectral ball is a local conjugation (see Theorem 7).

The approach presented in the proof seems to be new and we do believe that it is an important step toward solving the problem of finding the description of the group of automorphism of the spectral unit ball.

2 Diagonal and triangular conjugations

Throughout the paper $\mathbb{G}_2$ denotes the symmetrized bidisc, i.e. a domain in $\mathbb{C}^2$ given by the formula $\mathbb{G}_2 := \{(\operatorname{tr} x, \det x) : x \in \Omega\}$ (see [3] for its basic properties).

Let us focus on conjugations of the following form:

$$\tilde{D}_a : x \mapsto \begin{pmatrix} a(x) & 0 \\ 0 & 1/a(x) \end{pmatrix} x \begin{pmatrix} a(x) & 0 \\ 0 & 1/a(x) \end{pmatrix}^{-1}, \tag{1}$$

and

$$T_b : x \mapsto \begin{pmatrix} 1 & 0 \\ b(x) & 1 \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ b(x) & 1 \end{pmatrix}^{-1}. \tag{2}$$

Our aim is to describe holomorphic functions a and b such that the above mappings are automorphisms of the spectral unit ball.

The case (1) is easy. First note that the simply-connectedness of Ω imply that there is $\tilde{a} \in \mathcal{O}(\Omega)$ such that

$$a = \exp(\tilde{a}/2).$$

Some easy computations give

$$\tilde{D}_a(x) = \begin{pmatrix} x_{11} & \exp(\tilde{a}(x))x_{12} \\ \exp(-\tilde{a}(x))x_{21} & x_{22} \end{pmatrix}$$

Therefore, for fixed x_{11} and x_{22} the mapping $(x_{12}, x_{21}) \mapsto (\exp(\tilde{a}(x))x_{12}, \exp(-\tilde{a}(x))x_{21})$ is injective on its domain. In particular, putting $t = x_{21}x_{12}$ we see that the mapping

$$z \mapsto \exp(\tilde{a}(x_{11}, z, t/z, x_{22}))z$$

is an automorphism of $\mathbb{C}_*$. Thus $z \mapsto \tilde{a}(x_{11}, z, t/z, x_{22})$ is constant, whence $\tilde{a}$ depends only on x_{11}, x_{22} and $x_{12}x_{21}$.

Now we focus our attention on (2). We want to find b such that

$$x \mapsto \begin{pmatrix} x_{11} - b(x)x_{12} & x_{12} \\ b(x)x_{11} + x_{21} - b^2(x)x_{12} - b(x)x_{22} & b(x)x_{12} + x_{22} \end{pmatrix}$$

is an automorphism of Ω. Put $(s, p) := (\operatorname{tr} x, \det x) \in \mathbb{G}_2$. Looking at the automorphism restricted to the fibers (it is obvious that conjugations preserve fibers) intersected with $\{x_{12} \neq 0\}$ we get that the mapping

$$(x_{11}, x_{12}) \mapsto \left(x_{11} - x_{12}b\left(\begin{pmatrix} x_{11} & x_{12} \\ (x_{11}(s - x_{11}) - p)x_{12}^{-1} & s - x_{11} \end{pmatrix} \right), x_{12} \right)$$

is an automorphism of $\mathbb{C} \times \mathbb{C}_*$. It is quite easy to observe that if $(x, y) \mapsto (x - f(x, y), y)$ is an automorphism of $\mathbb{C} \times \mathbb{C}_*$, iff $f(x, y) = x(1 - c(y)) - \gamma(y)$, $x \in \mathbb{C}$, $y \in \mathbb{C}_*$, where $c \in \mathcal{O}(\mathbb{C}_*, \mathbb{C}_*)$ and $\gamma \in \mathcal{O}(\mathbb{C}_*, \mathbb{C})$.

Applying this reasoning to b we find that there are $c \in \mathcal{O}(\mathbb{C}_* \times \mathbb{G}_2, \mathbb{C}_*)$ and $\gamma \in \mathcal{O}(\mathbb{C}_* \times \mathbb{G}_2)$ such that

$$b(x) = x_{11} \frac{1 - c(x_{12}, \operatorname{tr} x, \det x)}{x_{12}} + \frac{\gamma(x_{12}, \operatorname{tr} x, \det x)}{x_{12}}.$$

Putting $x_{11} = 0$ we see that $\gamma(x_{12}, s, p) = x_{12}\beta(x_{12}, s, p)$, where $\beta \in \mathcal{O}(\mathbb{C} \times \mathbb{G}_2)$. Using this we see that c may be extended holomorphically through $x_{12} = 0$. Moreover, one may easily check that $c(x_{12}, s, p) = e^{x_{12}\alpha(x_{12}, \mathrm{tr}\, x, \det x)}$, where $\alpha \in \mathcal{O}(\mathbb{C} \times \mathbb{G}_2)$. Thus

$$b(x) = x_{11} \frac{1 - \exp(x_{12}\alpha(x_{12}, \mathrm{tr}\, x, \det x))}{x_{12}} + \beta(x_{12}, \mathrm{tr}\, x, \det x). \tag{3}$$

Summarizing we get the following

Proposition 2 *Let $a \in \mathcal{O}(\Omega, \mathbb{C})$. Then the conjugation $\tilde{D}_a$ is an automorphism of the spectral 2-ball if and only if there is a function $\tilde{a} \in \mathcal{O}(\Omega)$ depending only on x_{11}, x_{22} and $x_{12}x_{21}$ such that*

$$a = \exp(\tilde{a}/2).$$

Similarly, if $b \in \mathcal{O}(\Omega)$ that the conjugation T_b is an automorphism of Ω if and only if there are functions $\alpha, \beta \in \mathcal{O}(\mathbb{C} \times \mathbb{G}_2)$ such that

$$b(x) = x_{11} \frac{1 - \exp(x_{12}\alpha(x_{12}, \mathrm{tr}\, x, \det x))}{x_{12}} + \beta(x_{12}, \mathrm{tr}\, x, \det x), \quad x \in \Omega.$$

Remark 3 Note that T_b is generated by T_β and T_{β_1}, where $\beta_1(x) = x_{11} \frac{1 - \exp(x_{12}\alpha(x_{12}, \mathrm{tr}\, x, \det x))}{x_{12}}$, $x \in \Omega$, and α and β are as in Proposition 2.

3 Vector fields generated by triangular and diagonal conjugations and relations between them

To describe vector fields generated by (1) and (2) it is convenient to introduce the following notation:

$D_a := \tilde{D}_{\exp(a/2)}$.

$T'_\alpha := T_b$, where $b(x) = x_{11} \frac{1 - \exp(x_{12}\alpha(x_{12}, \mathrm{tr}\, x, \det x))}{x_{12}}$.

Note that the mappings D_a, T_β and S_α generate the following vector fields $\frac{d}{dt}\Phi_t(\Phi_{t_0}^{-1}(x))|_{t=t_0}$, where $\{\Phi_t\}$ is one of one-parameter groups $\{D_{ta}\}$, $\{T_{t\beta}\}$, $\{S_{t\alpha}\}$ (such vector fields are sometimes called infinitesimal generators):

$$HD_a = ax_{12}\frac{\partial}{\partial x_{12}} - ax_{21}\frac{\partial}{\partial x_{21}},$$

$$HT_\beta = -\beta x_{12}\frac{\partial}{\partial x_{11}} + (\beta x_{11} - \beta x_{22})\frac{\partial}{\partial x_{21}} + \beta x_{12}\frac{\partial}{\partial x_{22}},$$

$$HS_\alpha = x_{11}x_{12}\alpha\frac{\partial}{\partial x_{11}} + (x_{22} - x_{11})x_{11}\alpha\frac{\partial}{\partial x_{21}} - x_{11}x_{21}\alpha\frac{\partial}{\partial x_{22}}.$$

Additionally $\tau \circ T_\beta \circ \tau$ and $\tau \circ S_\alpha \circ \tau$ (where $\tau(x) = x^t$ is a transposition) generate

$$\tilde{H}T_\beta = -\beta x_{21}\frac{\partial}{\partial x_{11}} + (\beta x_{11} - \beta x_{22})\frac{\partial}{\partial x_{12}} + \beta x_{21}\frac{\partial}{\partial x_{22}},$$

$$\tilde{H}S_\alpha = x_{11}x_{21}\alpha\frac{\partial}{\partial x_{11}} + (x_{22} - x_{11})x_{11}\alpha\frac{\partial}{\partial x_{12}} - x_{11}x_{22}\alpha\frac{\partial}{\partial x_{22}}.$$

Let us write the above vector fields in "spectral" coordinates

$$(x, y, s, p) = (x_{11}, x_{12}, \mathrm{tr}\, x, \det x).$$

Observe that for any vector field V on Ω orthogonal to $\det x$ and $\operatorname{tr} x$ its divergence $\operatorname{div} V$ (in Euclidean coordinates) is equal to $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} - \frac{v_2}{y}$, where $V = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$. We get:

$$HD_a = ay\frac{\partial}{\partial y}, \quad HT_\beta = -\beta y\frac{\partial}{\partial x} \quad \text{and} \quad HS_\beta = xy\beta\frac{\partial}{\partial x}.$$

A straightforward calculation leads to:

$$[HD_a, HT_b] = ya(yb)'_y\frac{\partial}{\partial x} - y^2 a'_x b\frac{\partial}{\partial y}, \operatorname{div}[HD_a, HT_b] = 0,$$

$$[HD_a, HS_b] = xya(yb)'_y\frac{\partial}{\partial x} - xy^2 a'_x b\frac{\partial}{\partial y}, \operatorname{div}[HD_a, HS_b] = ay(yb)'_y.$$

$$\tilde{H}S_b = x\frac{x(s-x)-p}{y}b\frac{\partial}{\partial x} + (s-2x)xb\frac{\partial}{\partial y}.$$

Putting $b = 1$ we see moreover that $[[HD_a, HT_1], \tilde{H}S_1] = -a(p - x(s-x))\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ for some v.

4 Density of triangular and diagonal conjugations

Theorem 4 *The group generated by finite compositions of the transposition, Möbius maps, T_b and D_a, where a, b are described above is dense in the group of holomorphic automorphisms of the spectral unit ball.*

Proof Let φ be an automorphism of Ω. Composing it, if necessary, with a Möbius map one may assume that $\varphi(0) = 0$ (see [3,9]). Then $\varphi'(0)$ is also an automorphism of the spectral ball. By [9] there is an invertible matrix a such that either $\varphi'(0)(x) = axa^{-1}, x \in \Omega$ or $\varphi'(0)(x) = ax^t a^{-1}, x \in \Omega$. Losing no generality assume that the first possibility holds. Since any invertible matrix may be represented as a finite product of triangular and diagonal matrices $\varphi'(0)$ satisfies trivially the assertion.

Note that $t \mapsto \Phi_t := t^{-1}\varphi(t\cdot)$ is a well defined mapping from $\mathbb{R}$ into $\operatorname{Aut}(\Omega)$, $\Phi_0 = \varphi'(0)$. Since any automorphism of Ω fixing the origin preserves the spectrum we see that $\sigma(\Phi_t(x)) = \sigma(x), t \in \mathbb{R}, x \in \Omega$ ($\sigma(x)$ denotes the spectrum of a matrix x).

Consider the following time-dependent vector field:

$$X_{t_0}(x) := \frac{d}{dt}(\Phi_t(\Phi_{t_0}^{-1}(x)))|_{t=t_0}, \quad x \in \Omega.$$

It is clear that $\Phi_{t_0} \circ \Phi_0^{-1}$ is obtained by integrating X_t from 0 to t_0. Since Φ_t preserves the spectrum it follows that $X_{t_0}(\det x) = 0$ and $X_{t_0}(\operatorname{tr} x) = 0$ (in other words X_{t_0} is orthogonal to $\operatorname{tr} x$ and $\det x$).

We proceed as in [2] and [8]. X_t may be approximated on compact sets by integrating the time dependent vector field $X_{k/N}$ from time k/N to $(k + 1)/N$. Note that every $X_{k/N}$ may approximated by polynomial vector fields X' such that $X'(\det x) = 0$ and $X'(\operatorname{tr} x) = 0$. Actually, write $X_{k/N}$ as

$$w_1\frac{\partial}{\partial x_{11}} + w_2\frac{\partial}{\partial x_{12}} + w_3\frac{\partial}{\partial x_{21}} + w_4\frac{\partial}{\partial x_{22}}.$$

Since $X_{k/N}$ is orthogonal to $\operatorname{tr} x$ we find that $w_1 + w_4 = 0$. Similarly, the orthogonality of $\det x$ means that $w_1(x_{22} - x_{11}) = w_2 x_{21} + w_3 x_{12}$. Now observe that the spectral ball

is a pseudoconvex balanced domain, so any holomorphic function on it may be expanded into a series of homogenous polynomials (in particular the spectral ball is a Runge domain). Expanding coefficients of $X_{k/N}$ in a series of homogenous polynomials and looking at degrees of the polynomials appearing there one can easily get the desired claim.

We shall show that X' is a *Lie combination* of polynomial vector fields generated by diagonal and triangular conjugations and the transposition. Recall that a Lie combination of the subset S is an element which can be written as a finite sum of terms of the form $[..[[a_1, a_2], a_3], \dots, a_\mu]$, where $a_1, \dots, a_\mu \in S$. Then the standard argument (sometimes called *Euler's method*) will imply that we can approximate $\Phi_1 \circ \Phi_0^{-1}$ by compositions of the transposition and diagonal and triangular conjugations. Whence we will be able to approximate Φ_1, as well.

Therefore it remains to show that X' is a sum of vector fields appearing in Sect. 3. Let us denote

$$X' = v_1 \frac{\partial}{\partial x_{11}} + v_2 \frac{\partial}{\partial x_{12}} + v_3 \frac{\partial}{\partial x_{21}} + v_4 \frac{\partial}{\partial x_{22}}.$$

Recall that assumptions on X_{t_0} imply that

$$v_4 = -v_1$$

and

$$v_1(x_{22} - x_{11}) = v_2 x_{21} + v_3 x_{12} \tag{4}$$

(use the orthogonality of tr x and det x). Note that v_1 may be written as

$$v_1 = \sum_{j=1}^{n} x_{12}^j f_j(x_{11}, x_{22}, x_{12}x_{21}) + \sum_{j=1}^{n} x_{21}^j g_j(x_{11}, x_{22}, x_{12}x_{21}) + \varphi(x_{11}, x_{22}, x_{12}x_{21}) \tag{5}$$

for some polynomials $f_j, g_j, \varphi \in \mathbb{C}[x_{11}, x_{22}, x_{12}x_{21}]$.

Using (4) it is easily seen that $\varphi(x_{11}, x_{22}, 0) = 0$, so $\varphi(x_{11}, x_{22}, x_{12}x_{21}) = x_{12}x_{21}\alpha$ $(x_{11}, \text{tr } x, \det x)$ for some polynomial α. Adding to X', if necessary, $[[HD_a, HT_1], \tilde{H}S_1]$ with suitable chosen a we may assume that $\varphi = 0$.

Now adding vector fields of the form $[HD_a, HS_b]$ and $[HD_a, \tilde{H}S_b]$ we may assume that div $X' = \psi(x_{12}x_{21}, x_{11}, x_{22})$ for some polynomial ψ (more precisely, adding $[HD_a, HS_b]$ we remove terms of the form $a x_{11}^{j_1} x_{12}^{j_2} x_{21}^{j_3} x_{22}^{j_4}$ with $j_2 > j_3$, $a \in \mathbb{C}$, and adding $[HD_a, \tilde{H}S_b]$ we remove terms of the form $a x_{11}^{j_1} x_{12}^{j_2} x_{21}^{j_3} x_{22}^{j_4}$ with $j_2 < j_3$, $a \in \mathbb{C}$).

Again, adding to X', if necessary, vector fields of the forms $[HD_a, HT_b]$ and $[HD_a, \tilde{H}T_b]$ (note that they are of zero divergence) we may assume that $v_1 = 0$.

Thus, up to adding Lie combinations of the vector fields generated by triangular and diagonal conjugations we may assume that

$$X' = v_2 \frac{\partial}{\partial x_{12}} + v_3 \frac{\partial}{\partial x_{21}}$$

and $X'(\det x) = 0$ and div $X' = \psi(x_{12}x_{21}, x_{11}, x_{22})$. The second condition means that $x_{21}v_2 + x_{12}v_3 = 0$ so $v_2 = x_{12}w$, $v_3 = -x_{21}w$ for some polynomial w. In particular, div $X' = x_{12}\frac{\partial w}{\partial x_{12}} - x_{21}\frac{\partial w}{\partial x_{21}}$. Let us write w as $w = \sum x_{12}^j x_{21}^k \zeta_{j,k}(x_{11}, x_{22})$. Then it is straightforward to see that $\sum x_{12}^j x_{21}^k (j - k)\zeta_{j,k}(x_{11}, x_{22}) = \psi(x_{12}x_{21}, x_{11}, x_{22})$. Therefore $\psi = 0$ and $\zeta_{j,k} = 0$ whenever $j \neq k$ and $w = w(x_{12}x_{21}, x_{11}, x_{22}) = w(x, s, p)$. In particular X' is of the form HD_a with a depending only on $(x_{12}x_{21}, x_{11}, x_{22})$. This finishes the proof. □

5 Limit of conjugations

Remark 5 Note that the holomorphic mapping p occurring in Definition 1 is defined up to a multiplication with $x \mapsto a(x) + b(x)x$. More precisely, $p(x)xp(x)^{-1} = q(x)xq(x)^{-1}$, $x \in U$, where $p, q \in \mathcal{O}(U, \mathcal{M}^{-1})$ if and only if there are holomorphic functions $a, b : U \to \mathbb{C}$ such that $p(x) = (a(x) + b(x)x)q(x)$ and $\det(a(x) + b(x)x) \neq 0$, $x \in U$. The proof of this fact is quite elementary and we leave it to the reader. We would like to point out that the comutant of x coincides with the polynomials in x for a dense subset of matrices. Moreover, those polynomials can be taken of degree no more than 1 (recall that $n = 2$).

Lemma 6 *Let $p_n \in \mathcal{O}(W, \mathcal{M}^{-1})$, $\mathcal{T} \subset W \subset \Omega$, $\det p_n = 1$ be a sequence of holomorphic mappings such that $\varphi_n(x) := p_n(x)xp_n(x)^{-1}$, $x \in W$ is convergent locally uniformly to $\psi \in \mathcal{O}(W, \Omega)$. Then there is a neighborhood U of $\mathcal{T}$ and diagonal mappings $a_n, b_n \in \mathcal{O}(U, \mathcal{M})$ such that $p_n(x)(a_n(x) + xb_n(x))$ is locally uniformly convergent on U to $u \in \mathcal{O}(U, \mathcal{M}^{-1})$ and $\det(a_n(x) + xb_n(x)) = 1$.*

The proof presented below is elementary and relies upon purely analytic methods. We do not know whether the lemma would follow from more general algebraic properties. Note that the main difficulty lies in the fact that standard methods do not work for non-cyclic matrices.

Proof To simplify the notation we will omit subscript n. Composing φ with Möbius maps from both sides (more precisely taking $m_{-a} \circ \varphi \circ m_a$, where $m_a = m_{a,1}$) we easily see that $q(a, x) := p(m_a(x))xp(m_a(x))^{-1}$ is convergent locally uniformly with respect to (a, x) in a neighborhood of $\mathbb{D} \times \mathcal{T}$. Therefore $\frac{\partial q}{\partial x}(a, x)(h)$ converges locally uniformly in an open neighborhood of $\mathbb{D} \times \mathcal{T}$ for any h. Putting $x = 0$ we find that

$$\frac{\partial q}{\partial x}(a, 0)(h) = p(-a)hp(-a)^{-1},$$

whence $p(a)$ is convergent locally uniformly with respect to $a \in \mathbb{D}$. Therefore, replacing φ with

$$x \mapsto p^{-1}\left(\begin{pmatrix} \operatorname{tr} x/2 & 0 \\ 0 & \operatorname{tr} x/2 \end{pmatrix}\right) \varphi(x) p^{-1}\left(\begin{pmatrix} \operatorname{tr} x/2 & 0 \\ 0 & \operatorname{tr} x/2 \end{pmatrix}\right)$$

we may assume that $p(x) = 1$ for all non-cyclic matrices $x \in \Omega$.

Put $\Omega' := \left\{ \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{11} \end{pmatrix} : x_{11} \in \mathbb{D}, x_{21} \in \mathbb{C} \right\}$. First we show that there are diagonal mappings a', b' defined on a U' neighborhood of $\mathcal{T}$ in Ω' such that $p(x)(a'(x) + xb'(x))$ is convergent and $\det(a'(x) + xb'(x)) = 1$ for $x \in \Omega'$ lying in a neighborhood of $\mathcal{T}$. Multiplying $p(x)xp(x)^{-1}$ out we get

$$\begin{pmatrix} x_{11} + p_{12}p_{22}x_{21} & -p_{12}^2 x_{21} \\ p_{22}^2 x_{21} & x_{11} - p_{12}p_{22}x_{21} \end{pmatrix}.$$

Using the fact that $p(x)xp(x)^{-1}$ converges locally uniformly on $W \cap \Omega'$, we get that p_{21} and p_{22} converge uniformly on compact subsets of $W \cap \Omega'$. Since $p_{21} \equiv 0$ on $\mathcal{T}$ we get that there is $q \in \mathcal{O}(W \cap \Omega')$ such that $p_{21}(x) = x_{21}q(x)$. Moreover, $p_{22} \equiv 1$ on $\mathcal{T}$, so there is a neighborhood U_0 of $\mathcal{T}$ in Ω', uniform with respect to n, on which p_{22} does not vanish. Put $b'(x) := -q(x)/p_{22}(x)$, $a'(x) := 1 - b'(x)x_{11}$, $x \in U$. Direct calculations show that a' and b' satisfy the desired claim.

Now we shall show that there are diagonal mappings a'', b'' on a neighborhood U'' of $\mathcal{T}$ in $\Omega'' := \left\{ \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} : x_{11}, x_{22} \in \mathbb{D}, x_{21} \in \mathbb{C} \right\}$ such that $p(x)(a'(x) + xb'(x))$ is convergent

locally uniformly and $\det(a'(x) + xb'(x)) = 1$ for $x \in \Omega''$ in a small neighborhood of $\mathcal{T}$. Let us consider the following projection

$$j_1 : \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} (x_{11} + x_{22})/2 & 0 \\ x_{21} & (x_{11} + x_{22})/2 \end{pmatrix}$$

and note that $V'' := j_1^{-1}(U')$ is a neighborhood of $\mathcal{T}$ in Ω''.

It follows from the previous step that

$$u(x) := p(j_1(x))(a'(j_1(x)) + b'(j_1(x))j_1(x)), \quad x \in V''$$

is convergent locally uniformly on V'' and $\det u = 1$ there. Therefore, replacing φ with $x \mapsto u^{-1}(x)\varphi(x)u(x)$ we may assume that $\varphi(x) = x$ for $x \in V'' \cap \Omega' = U'$. Multiplying $p(x)xp(x)^{-1}$ out we get

$$\begin{pmatrix} x_{11} + p_{12}p_{21}(x_{11} - x_{22}) + p_{12}p_{22}x_{21} & -p_{11}p_{12}(x_{11} - x_{22}) - p_{12}^2 x_{21} \\ p_{21}p_{22}(x_{11} - x_{22}) + p_{22}^2 x_{21} & x_{22} - p_{12}p_{21}(x_{11} - x_{22}) - p_{12}p_{22}x_{21} \end{pmatrix}.$$

Since $p(x)xp(x)^{-1} = x$ on U' we deduce from the formula above that $p_{12} = 0$ and $p_{22}^2 = 1$ and thus $p_{11} = p_{22} = 1$ on U'.

In particular, there is a holomorphic function q_{12} on V'' such that $p_{12}(x) = (x_{11} - x_{22})q_{12}(x)$. Similarly, $\tilde{b}(x) := (p_{22}(x) - p_{11}(x))(x_{11} - x_{22})^{-1}$, $x \in V'$ is a well defined holomorphic function on V'. Let us put $\tilde{a}(x) := p_{11}(x) + q_{12}(x)x_{21} - \tilde{b}(x)x_{22}$. It is quite elementary to verify that $p(x)(\tilde{a}(x) + \tilde{b}(x)x)$ is convergent in V'' (actually, to check it observe that $p(x)(\tilde{a}(x) + \tilde{b}(x)x)(x_{11} - x_{22})$ converges locally uniformly). Let us compute it. Moreover, $\det(\tilde{a}(x) + \tilde{b}(x)x) = (p_{11}(x) + q_{12}(x)x_{21})(p_{22}(x) + q_{12}(x)x_{21})$, so it is equal to 1 when $x \in \mathcal{T}$. Therefore there is a simply-connected neighborhood U'' of $\mathcal{T}$ in Ω'' (uniform with respect to subscripts n), $U'' \subset V''$ such that $\det(\tilde{a}(x) + x\tilde{b}(x))$ does not vanish there. Let us take the branch of the square root $s(x) := \det(\tilde{a}(x) + x\tilde{b}(x))^{1/2}$ preserving 1. Observe that $a'' := \tilde{a}/s$, $b'' := \tilde{b}/s$ satisfy the claim.

Now we prove the existence of a and b satisfying the assertion of the lemma. Put

$$j_2 : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix}.$$

Note that $V := \Omega \cap j_2^{-1}(U'')$ is a neighborhood of $\mathcal{T}$ in Ω so we may repeat the previous reasoning: since $v := p \circ j_2 \cdot (a'' \circ j_2 + j_2 \cdot b'' \circ j_2)$ is convergent on V and $\det v = 1$ there, replacing φ with $v^{-1}\varphi v$ we may assume that $\varphi(x) = x$ for $x \in U''$. Comparing the coefficients of $p(x)xp(x)^{-1}$ for $x \in U''$:

$$\begin{pmatrix} p_{11}p_{22}(x_{11} - x_{22}) + p_{12}p_{22}x_{21} + x_{22} & -p_{11}p_{12}(x_{11} - x_{22}) - p_{12}^2 x_{21} \\ p_{21}p_{22}(x_{11} - x_{22}) + -p_{22}^2 x_{12} & -p_{11}p_{22}(x_{11} - x_{22}) - p_{12}p_{22}x_{21} + x_{11} \end{pmatrix}$$

and the ones of x we get that $p_{12}(p_{11}(x_{11} - x_{22}) + p_{12}x_{21}) = 0$, so by the identity principle either $p_{12} = 0$ or $p_{11}(x_{11} - x_{22}) + p_{12}x_{21} = 0$. If the second possibility held, then comparing the elements lying in the first column and the first row we would get that $x_{11} = p_{11}p_{22}(x_{11} - x_{22}) + p_{12}p_{22}x_{21} + x_{22} = x_{22}$, a contradiction. Therefore $p_{12} = 0$. Then, in particular, $p_{12} = 0$ on U''. Thus $p_{12}(x) = x_{12}q(x)$, $x \in V$, for some holomorphic function q. Put $b := -q$ and $a := p_{11} - bx_{22}$. Then $p(x)(a(x) + b(x)x)$ is convergent locally uniformly in V. In particular, $x \mapsto \det(a(x) + b(x)x)$ is convergent. Moreover $\det(a(x) + b(x)x) = 1$ on $\mathcal{T}$, therefore shrinking, if necessary, V and dividing a and b by a proper non-vanishing holomorphic mapping we finish the proof. $\square$

6 Local form of the automorphisms of the spectral unit ball

It is well known (see e.g. [3]) that for any $\varphi \in \text{Aut}(\Omega)$ there is a Möbius map m such that $\sigma(\varphi(x)) = \sigma(m(x))$. Therefore, composing φ with a Möbius map, a transposition and a linear automorphism of Ω we may always assume that φ is *normalized*, i.e. $\sigma(\varphi(x)) = \sigma(x)$, $x \in \Omega$ and $\varphi'(0) = id$.

As a consequence of our considerations we get the following

Theorem 7 *Let φ be a normalized automorphism of Ω. Then for any $x \in \Omega$ there is $u \in \mathcal{O}(U, \mathcal{M}_{2\times2}^{-1})$ defined in an open neighborhood U of x such that $\varphi(x) = u(x)xu(x)^{-1}$ on U.*

In other words, any normalized automorhism of Ω is a local holomorphic conjugation.

Proof It follows from Theorem 4 that there is a sequence $(p_n) \subset \mathcal{O}(\Omega, \mathcal{M}^{-1})$ such that $p_n(x)xp_n(x)^{-1}$ converges locally uniformly to $\varphi(x)$. It follows from Lemma 6 that there is a neighborhood U of $\mathcal{T}$ and $u \in \mathcal{O}(U, \mathcal{M}^{-1})$ such that $\varphi(x) = u(x)xu(x)^{-1}$, $x \in U$.

On the other hand it is well known (see [11]) that φ is a local conjugation on $\Omega \setminus \mathcal{T}$ (we would like to note that this fact may also deduced from Theorem 4). □

Remark 8 It is very natural to ask whether a local conjugation on a domain U satisfying "nice" topological properties (for example $H^1(U, \mathcal{O}) = H^2(U, \mathbb{Z}) = 0$) is a holomorphic conjugation. Note, that this is equivalent to finding a solution of the following problem which may be viewed as a counterpart of the meromorphic Cousin problem:

given a covering $\{\Omega_\alpha\}$ and $a_{\alpha\beta}, b_{\alpha\beta} \in \mathcal{O}(\Omega_\alpha \cap U_\beta)$, $\det(a_{\alpha\beta}(x) + xb_{\alpha\beta}(x)) \neq 0$, $x \in \Omega_\alpha \cap \Omega_\beta$, such that

$$(a_{\alpha\beta}(x) + xb_{\alpha\beta}(x))(a_{\beta\gamma}(x) + xb_{\beta\gamma}(x)) = a_{\alpha\gamma}(x) + xb_{\alpha\gamma}(x), \quad x \in \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma$$

and

$$(a_{\alpha\beta}(x) + xb_{\alpha\beta}(x))(a_{\beta\alpha}(x) + xb_{\beta\alpha}(x)) = 1, \quad x \in \Omega_\alpha \cap \Omega_\beta$$

find $a_\alpha, b_\alpha \in \mathcal{O}(\Omega_\alpha)$ such that $a_{\alpha\beta}(x) + xb_{\alpha\beta}(x) = (a_\alpha(x) + xb_\alpha(x))(a_\beta(x) + xb_\beta(x))^{-1}$ on $\Omega_\alpha \cap \Omega_\beta$ and $\det(a_\alpha(x) + xb_\alpha(x))$ does not vanish on Ω_α.

Actually, assume that the problem stated above has a solution. A local conjugation φ gives a data for the above problem in the following way: We may locally expand φ a a holomorphic conjugation, i.e. there are u_α and Ω_α such that $\varphi(x) = u_\alpha(x)xu_\alpha(x)^{-1}$, where $u_\alpha \in \mathcal{O}(\Omega_\alpha, \mathcal{M}^{-1})$ and $\{\Omega_\alpha\}$ is an open covering of U. Then, it follows from Remark 5 that $u_{\alpha,\beta} := u_\alpha u_\beta^{-1}$ are data for the problem stated above. Solving it we find that $u_\alpha(x)v_\beta(x)^{-1} = (a_\alpha(x)+xb_\alpha(x))(a_\beta(x)+xb_\beta(x))^{-1}$ on $\Omega_\alpha \cap \Omega_\beta$. Putting $w(x) := u_\alpha(x)(a_\alpha(x)+xb_\alpha(x))$, $x \in \Omega_\alpha$ we get a well defined holomorphic mapping on U such that $\varphi(x) = w(x)xw(x)^{-1}$.

On the other hand, suppose that $\{\Omega_{\alpha\beta}, a_{\alpha\beta}, b_{\alpha,\beta}\}$ are data for the above problem. Then, solving the second Cousin problem for matrices we get $u_\alpha \in \mathcal{O}(\Omega_\alpha, \mathcal{M}^{-1})$ such that $a_{\alpha\beta}(x)+xb_{\alpha\beta}(x) = u_\alpha(x)u_\beta(x)$ on $\Omega_\alpha \cap \Omega_\beta$. Putting $\varphi(x) := u_\alpha(x)xu_\alpha(x)^{-1}$ we get a local holomorphic conjugation on U. If it were a holomorphic conjugation, we would get $u \in \mathcal{O}(U, \mathcal{M}^{-1})$ such that $\varphi(x) = u(x)xu(x)^{-1}$. Making use of Remark 5 again we get a_α, b_β such that $u_\alpha(x) = (a_\alpha(x) + xb_\alpha(x))u(x)$, $x \in \Omega_\alpha$. Then it is a direct to observe that a_α, b_α solve the above problem.

Finally, we present a list of open questions:

Open problems

- Does the similar results holds for $n \geq 3$? We conjecture that the group of autmorphism generated by conjugations $x \mapsto a(x)xa(x)^{-1}$, where $a(x)$ is an elementary matrix, is dense in $\text{Aut}(\Omega_n)$.

- More generaly, one may conjecture that the higher-dimensional counterparts of the Danielewski surfaces, i.e. sets of the form

$$\{x \in \mathcal{M}_{n \times n}(\mathbb{C}) : \sigma(x) = \{\lambda_1, \ldots, \lambda_n\}\},$$

  where $\lambda_i \neq \lambda_j$, $i \neq j$, satisfy the density property.
- Is every automorphism of Ω a holomorphic conjugation?
- In the case $n = 2$ the affirmative answer to the question stated above would follow from the following one: Is a local holomorphic conjugation a holomorphic conjugation?
- Do Danielewski surfaces have the Alexander property i.e. is every proper holomorphic selfmapping of D_P an automorphism? Note that a similar result holds for Ω (see [12]).

References

1. Andersén, E.: Volume-preserving automorphisms of $\mathbb{C}^n$. Complex Var. Theory Appl. **14**(1–4), 223–235 (1990)
2. Andersén, E., Lempert, L.: On the group of holomorphic automorphisms of $\mathbb{C}^n$. Invent. Math. **110**(2), 371–388 (1992)
3. Edigarian, A., Zwonek, W.: Geometry of the symmetrized polydisc. Arch. Math. (Basel) **84**(4), 364–374 (2005)
4. Forstneric, F., Rosay, J.P.: Approximations of biholomorphic mappings by automorphisms of $\mathbb{C}^n$. Invent. Math. **112**(2), 323–349 (1993)
5. Kaliman, S., Kutzschebauch, F.: Density property for hypersurfaces $UV = P(\bar{X})$. Math. Z. **258**(1), 115–131 (2008)
6. Kosiński, Ł.: The group of automorphisms of the spectral ball. Proc. AMS **140**, 2029–2031 (2012)
7. Kutzschebauch, F., Lind, A.: Holomorphic automorphisms of Danielewski surfaces I, density of the group of overshears. Proc. AMS **139**(11), 3915–3927 (2011)
8. Lind, A.: Holomorphic automorphims of Danielewski surfaces. PhD thesis (2009)
9. Ransford, T.J., White, M.C.: Holomorphic self-maps of the spectral unit ball. Bull. Lond. Math. Soc. **23**, 256–262 (1991)
10. Rostand, J.: On the automorphisms of the spectral unit ball. Studia Math. **155**, 207–230 (2003)
11. Thomas, P.J.: A local form for the automorphisms of the spectral unit ball. Collect. Math. **59**(3), 321–324 (2008)
12. Zwonek, W.: Proper holomorphic mappings of the spectral unit ball. Proc. Am. Math. Soc. **136**(8), 2869–2874 (2008)