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The structure of fixed-point sets of uniformly lipschitzian semigroups

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Abstract In this paper, by asymptotic center techniques, we shown that the set of fixed points of a uniformly k-lipschitzian semigroup (one-parameter or left reversible semi-topological) in a uniformly convex Banach space is a retract of the domain if k is close to 1. The results presented in this paper includes (among others, in the discrete situation) many known results as special cases.

Keywords One-parameter semigroup · Left reversible semigroup · Uniformly lipschitzian semigroup · Retraction · Asymptotic center · Fixed point · Uniformly convex Banach space

Mathematics Subject Classification (2000) Primary 47H10 · 47H20; Secondary 47H09 · 54C15

1 Introduction

We will consider a Banach spaces E over the real field. Our notation and terminology are standard. Let C be a nonempty bounded closed convex subset of E. We say that a mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\| \quad \text{for every } x, y \in C.$$

The result of Bruck [2] asserts that if a nonexpansive mapping $T : C \to C$ has a fixed point in every nonempty closed convex subset of C which is invariant under T and if C is convex and weakly compact, then $F(T) = \{x \in C : Tx = x\}$, the set of fixed points, is a nonexpansive retract of C (that is, there exists a nonexpansive mapping $R : C \to F(T)$ such that $R_{|F(T)} = I$). A few years ago, the Bruck result was extended by Domínguez Benavides

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and Lorenzo Ramírez [4] to the case of asymptotically nonexpansive mappings if the space E was sufficiently regular.

On the other hand it is known, the set of fixed points of k-lipschitzian mapping can be very irregular for any $k > 1$.

Example 1 ([11]) Let F be a nonempty closed subset of C. Fix $z \in F$, $0 < \varepsilon < 1$ and put

$$Tx = x + \varepsilon \cdot dist(x, F) \cdot (z - x), \quad x \in C.$$

It is not difficult to see that $F(T) = F$ and the Lipschitz constant of T tends to 1 if $\varepsilon \downarrow 0$.

In 1973, Goebel and Kirk [5] introduced the class of uniformly k-lipschitzian mappings and stated a relationship between the existence of fixed point for uniformly k-lipschitzian mappings and the Clarkson modulus of convexity δ_E. Recall that a mapping $T : C \to C$ is *uniformly k-lipschitzian*, $k \geqslant 0$, if

$$\|T^n x - T^n y\| \leqslant k\|x - y\| \quad \text{for every } x, y \in C \quad \text{and} \quad n \in \mathbb{N}.$$

Theorem 2 *Let E be a uniformly convex Banach space with modulus of convexity δ_E and let C be a nonempty bounded closed convex subset of E. Suppose $T : C \to C$ is uniformly k-lipschitzian map and $k < \gamma$, where $\gamma > 1$ is the unique solution of the equation*

$$\gamma \left(1 - \delta_E\left(\frac{1}{\gamma}\right)\right) = 1. \tag{1}$$

Then $F(T) \neq \varnothing$ (note that in a Hilbert space, $k < \gamma = \frac{1}{2}\sqrt{5}$, in L^p-spaces ($2 \leqslant p < \infty$), $k < \gamma = (1 + 2^{-p})^{\frac{1}{p}}$), and $F(T)$ is not only connected but even a retract of C (see [11]).

In this paper we establish some results on the structure of fixed point sets for one-parameter uniformly k-lipschitzian semigroups and semi-topological uniformly k-lipschitzian semigroups in uniformly convex Banach spaces when k is less than a constant bigger than the constant from Theorem 2.

2 Uniformly convex Banach spaces

Recall that the *modulus of convexity* δ_E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\right\}$$

and that the space E is *uniformly convex* if $\delta_E(\varepsilon) > 0$ for $\varepsilon > 0$. A Hilbert space H is uniformly convex. This fact is a direct consequence of parallelogram identity. It is well known that δ_E is continuous on $[0, 2)$ and strictly increasing in uniformly convex Banach spaces [6, Lemma 5.1].

Recall the concept and the notion of *asymptotic center* due to Edelstein, see [1,6]. Let C be a nonempty closed convex subset of a Banach space E, and G an unbounded subset of $[0, +\infty)$ such that $t + h \in G$ for all $t, h \in G$ and $t - h \in G$ for all $t, h \in G$ with $t \geqslant h$ (i.e., $G = [0, +\infty)$, $G = [0, +\infty) \cap \mathbb{Q}$ or $G = \mathbb{N}_0$, the set of nonnegative integers), and let $\{x_t : t \in G\}$ be a bounded family of elements of E. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t \in G}$ with respect to C are the number

$$r(C, \{x_t\}) := \inf_{y \in C}\left(\limsup_{G \ni t \to \infty} \|y - x_t\|\right)$$

and the (possibly empty) set

$$A(C, \{x_t\}) := \left\{ y \in C : \limsup_{G \ni t \to \infty} \|y - x_t\| = r\left(C, \{x_t\}\right) \right\},$$

respectively. It is well known that if E is reflexive, then $A(C, \{x_t\})$ is bounded closed convex and nonempty, and if E is uniformly convex, then $A(C, \{x_t\})$ consist only a single point, $\{z\} = A(C, \{x_t\})$, i.e., other words $z \in C$ is the unique point which minimizes functional

$$\limsup_{G \ni t \to \infty} \|y - x_t\|$$

over y in C.

Suppose $\mathcal{F} = \{T_s : s \in G\}$ is a *one-parameter uniformly k-lipschitzian semigroup* on C, i.e., a family of self-mappings on C satisfying the conditions:

1. $T_{s+h}x = T_s T_h x$ for all $s, h \in G$ and $x \in C$,
2. for each $x \in C$, the mapping $s \to T_s x$ from G into C is continuous when G has the relative topology of $[0, +\infty)$,
3. for each $s \in G$, $\|T_s x - T_s y\| \leqslant k\|x - y\|$ for all $x, y \in C$.

Let $A : C \to C$ denote a mapping which associates with a given $x \in C$ a unique $z \in A(C, \{T_t x\})$, that is, $z = Ax$. Now we generalize to uniformly k-lipschitzian semigroups the lemma due to Sędłak and Wiśnicki [11]. This lemma is crucial to our results.

Lemma 3 *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then the mapping $A : C \to C$ is continuous.*

Proof On the contrary, suppose that there exists $x_0 \in C$ and $\varepsilon_0 > 0$ such that:
for all $\eta > 0$ there exists $x_1 \in C$ such that $\|x_1 - x_0\| < \eta$ and $\|z_1 - z_0\| \geqslant \varepsilon_0$, where $\{z_0\} = A(C, \{T_t x_0\})$, $\{z_1\} = A(C, \{T_t x_1\})$.

Fix $\eta > 0$ and take $x_1 \in C$ such that

$$\|x_1 - x_0\| < \eta \quad \text{and} \quad \|z_1 - z_0\| \geqslant \varepsilon_0.$$

Let

$$R_0 = r\left(C, \{T_t x_0\}\right) = \inf_{y \in C} \left(\limsup_{G \ni t \to +\infty} \|y - T_t x_0\| \right),$$

$$R_1 = r\left(C, \{T_t x_1\}\right) = \inf_{y \in C} \left(\limsup_{G \ni t \to +\infty} \|y - T_t x_1\| \right)$$

and

$$R = \limsup_{G \ni t \to +\infty} \|z_1 - T_t x_0\|.$$

Notice that

$$R_0 < R.$$

Choose $\varepsilon > 0$. Then exists $s(\varepsilon) \in G$ that

$$\begin{cases} \|z_1 - T_t x_0\| < R + \varepsilon, \\ \|z_0 - T_t x_0\| < R_0 + \varepsilon < R + \varepsilon, \\ \|z_0 - z_1\| \geqslant \varepsilon_0, \end{cases} \tag{2}$$

for all $t \in G$ and $t \geqslant s(\varepsilon)$.

It follows by (2) and the properties of δ_E that for $G \ni t \geqslant s(\varepsilon)$,

$$\left\| T_t x_0 - \frac{z_1 + z_0}{2} \right\| \leqslant \left(1 - \delta_E\left(\frac{\varepsilon_0}{R + \varepsilon}\right)\right)(R + \varepsilon)$$

and hence

$$R_0 < \limsup_{G \ni t \to +\infty} \left\| T_t x_0 - \frac{z_1 + z_0}{2} \right\| \leqslant \left(1 - \delta_E\left(\frac{\varepsilon_0}{R + \varepsilon}\right)\right)(R + \varepsilon). \tag{3}$$

Moreover for $t \in G$, from triangle inequality we have

$$\|T_t x_0 - z_1\| \leqslant \|T_t x_o - T_t x_1\| + \|T_t x_1 - z_1\| \leqslant k\|x_0 - x_1\| + R_1 + \varepsilon,$$

and hence

$$R = \limsup_{G \ni t \to +\infty} \|T_t x_0 - z_1\| \leqslant k\eta + R_1 + \varepsilon. \tag{4}$$

Similarly,

$$R_1 < \limsup_{G \ni t \to +\infty} \|T_t x_1 - z_0\| \leqslant k\eta + R_1 + \varepsilon. \tag{5}$$

From (4) and (5), we have

$$R \leqslant k\eta + R_1 + \varepsilon < 2k\eta + 2\varepsilon + R. \tag{6}$$

Combining (6) with (3) and applying the monotonicity of δ_E, we obtain

$$R_0 < \left(1 - \delta_E\left(\frac{\varepsilon_0}{2k\eta + 3\varepsilon + R_0}\right)\right)(2k\eta + 3\varepsilon + R_0).$$

Letting $\eta, \varepsilon \downarrow 0$, and using the continuity of δ_E, we conclude that

$$1 \leqslant \left(1 - \delta_E\left(\frac{\varepsilon_0}{R_0}\right)\right) < 1.$$

This contradiction proves the continuity of the mapping A. □

This result can be extend to left reversible semigroups. Now let J be a semi-topological semigroup, i.e., J is a semigroup with a Hausdorff topology such that for each $a \in J$ the mapping $s \to a \cdot s$ and $s \to s \cdot a$ from J to J are continuous. A semi-topological semigroup J is said to be *left reversible* if any two closed right ideals have non-void intersection. (This latter is automatically fulfilled if, for example J is commutative, and in particular if $J = [0, +\infty)$.) In this case $(J, \leqslant)$ is a directed system when the binary relation "$\leqslant$" on J is defined by $a \leqslant b$ if and only if $\{a\} \cup \overline{aJ} \supseteq \{b\} \cup \overline{bJ}$.

Let $\{x_a : a \in J\}$ be a bounded net in uniformly convex Banach space E and let C a nonempty closed convex subset of E. For a fixed $p > 1$, let us set

$$r(x) = \inf_{b \in J} \sup_{a \geqslant b} \|x_a - x\|^p \quad \text{and} \quad r = \inf_{x \in C} r(x).$$

Then we have a unique point $z \in C$ (called the asymptotic center of the net $\{x_a\}$ in C) such that $r(z) = r$.

Let C be a nonempty bounded closed convex subset of a Banach space E, let J be a left reversible semi-topological semigroup, and let $\mathcal{T} = \{T_s : s \in J\}$ be a family of self-mappings of C into itself. Then $\mathcal{T}$ is said to be a *left reversible semi-topological uniformly k-lipschitzian semigroup* on C if the following conditions are satisfied:

1′. $T_{ts}x = T_t T_s x$ for all $t, s \in J$ and $x \in C$,
2′. the mapping $(s, x) \to T_s x$ from $J \times C$ into C is continuous when $J \times C$ has the product topology,
3′. for each $s \in J$, $\|T_s x - T_s y\| \leqslant k\|x - y\|$ for all $x, y \in C$.

Remark 4 For such a family of mappings Lemma 3 remains true.

Normal structure plays essential role in some problems of metric fixed point theory. Let C be a nonempty bounded set in a Banach space E. We put

$$r(C) = \inf_{x \in C}\left(\sup_{y \in C}\|x - y\|\right).$$

This number is called the Chebyshev radius of A.

A Banach space E is said to have *uniformly normal structure* (UNS) if for some $c \in (0, 1)$ and every bounded closed convex subset $C \subset E$ with $\operatorname{diam} C > 0$, it has

$$r(C) \leqslant c \cdot \operatorname{diam} C.$$

The *normal structure coefficient* (also called the Jung constant) was defined by Bynum [3] in the following way

$$N(E) := \inf\left\{\frac{\operatorname{diam} C}{r(C)}\right\}$$

where the infimum is taken over all bounded closed convex sets $C \subset E$ with $\operatorname{diam} C > 0$. Clearly, the condition $N(E) > 1$ characterizes spaces E with UNS. It is well known that all uniformly convex Banach spaces possess UNS [10, Theorem 5.12]. It is difficult to calculate the normal structure coefficient in an arbitrary Banach space. However, for a Hilbert space, $N(H) = \sqrt{2}$, and $N(l^p) = N(L^p) = \min\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}$ for $1 < p < \infty$, see [1,10].

The following lemma can be proved in exactly the same way as Lim [8, Theorem 1] for sequences and the proof is thus omitted here.

Lemma 5 *Let E be a Banach space with UNS. Then for every bounded family $\{x_t\}_{t \in G}$ of elements of E there exists y in $\overline{\operatorname{conv}}\{x_t : t \in G\}$ such that*

$$\limsup_{G \ni t \to +\infty}\|y - x_t\| \leqslant \frac{1}{N(E)} \cdot \lim_{G \ni t \to +\infty}(\sup\{\|x_i - x_j\| : t \leqslant i, j \in G\}).$$

Now we improve the fixed point theorem due to Tan and Xu [12, Theorem 3.5].

Theorem 6 *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Suppose $\mathcal{F} = \{T_s : s \in G\}$ is a one-parameter uniformly k-lipschitzian semigroup on C with $k < \alpha$, where $\alpha > 1$ is the unique solution of the equation*

$$\alpha^2 \cdot \delta_E^{-1}\left(1 - \frac{1}{\alpha}\right) \cdot \frac{1}{N(E)} = 1 \tag{7}$$

$\big($*in a Hilbert space* $\alpha = (\sqrt{3} - 1)^{-\frac{1}{2}} > \frac{1}{2}\sqrt{5}\big)$. *Then*

$$F(\mathcal{F}) = \{x \in C : T_s x = x \quad \textit{for all } s \in G\} \neq \varnothing$$

and $F(\mathcal{F})$ is a retract of C.

Suppose E is uniformly convex Banach space and $\alpha > 1$ is the unique solution of Eq. (7). Then $\gamma < \alpha$, where $\gamma > 1$ is the unique solution of Eq. (1), see [12, Lemma 3.3].

Proof The proof of existence z in C such that $T_s z = z$ for all $s \in G$, based on the Lemma 5, is given in [12, Theorem 3.5] or in [7, Theorem 4]. In this proof by induction we define a sequence $\{x_n\}_{n=0,1,2,\dots}$ in C in the following manner

$$x_0 = x \quad \text{and} \quad x_{n+1} = A(C, \{T_t x_n\}) = A^{n+1}x, \quad n = 0, 1, \dots$$

($z = \lim_{n\to+\infty} x_n$). Thus by the inequalities

$$d(x_n) = \sup_{t\in G} \|T_t x_n - x_n\| \leqslant d(x_{n-1}) \leqslant B^n d(x),$$

$$\|x_{n+1} - x_n\| \leqslant \left(\frac{k}{N(E)} + 1\right) B^n d(x) \to 0 \quad \text{as } n \to +\infty,$$

where $B = \frac{k^2}{N(E)} \delta_E^{-1}\left(1 - \frac{1}{k}\right) < 1$, we have

$$\|A^{n+1}x - A^n x\| \leqslant \left(\frac{k}{N(E)} + 1\right) B^n d(x) \leqslant \left(\frac{k}{N(E)} + 1\right) B^n \operatorname{diam} C$$

for $n = 1, 2, \dots$ So

$$\sup_{x\in C} \|A^i x - A^m x\| \leqslant \left(\frac{k}{N(E)} + 1\right) \frac{B^m}{1 - B} \operatorname{diam} C \to 0 \quad \text{if } i, m \to +\infty,$$

which implies that the sequence $\{A^m x\}_{m=1,2,\dots}$ converges uniformly to a function

$$Rx = \lim_{m\to\infty} A^m x, \quad x \in C.$$

It follows from Lemma 3, that $R : C \to C$ is continuous. Moreover,

$$\begin{aligned}
&\|Rx - T_s Rx\| \\
&\quad\leqslant \|Rx - A^m x\| + \|A^m x - T_{s+h} A^m x\| + \|T_{s+h} A^m x - T_s A^m x\| + \|T_s A^m x - T_s Rx\| \\
&\quad\leqslant (1 + k)\|Rx - A^m x\| + \|A^m x - T_{s+h} A^m x\| + k\|T_h A^m x - A^m x\| \\
&\quad\leqslant (1 + k)\|Rx - A^m x\| + (1 + k) \cdot d(A^m x) \\
&\quad\leqslant (1 + k)\|Rx - A^m x\| + (1 + k) \cdot B^m \cdot \operatorname{diam} C \to 0 \quad \text{as } m \to +\infty
\end{aligned}$$

and $Rx = T_s Rx$ for all $s \in G$ and $x \in C$. Thus R is a retraction of C onto $F(\mathcal{F})$. □

This result can be sharpened in some uniformly convex Banach spaces, for example in a Hilbert space and in L^p-spaces $(1 < p < \infty)$.

3 *p*-Uniformly convex Banach spaces

Let $p > 1$ be a real number. A Banach space E is said to be *p-uniformly convex* (or E is said to have the modulus of convexity of power type p) if there exists a constant $d > 0$ such that the modulus of convexity $\delta_E(\varepsilon) \geqslant d \cdot \varepsilon^p$ for $0 \leqslant \varepsilon \leqslant 2$. We note that a Hilbert space is 2-uniformly convex (indeed, $\delta_H(\varepsilon) = 1 - \sqrt{1 - (\frac{\varepsilon}{2})^2} \geqslant \frac{1}{8}\varepsilon^2$) and an L^p-space $(1 < p < \infty)$ is $\max\{p, 2\}$-uniformly convex.

In [9, 14] the following result was proved.

Theorem 7 *Let $p > 1$ be a real number and let E be a p-uniformly convex Banach space. Then there exists a constant $c_p > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x-y\|^p$$

for all $x, y \in E, 0 \leqslant \lambda \leqslant 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Let H be a Hilbert space, then

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H, 0 \leqslant \lambda \leqslant 1$.

When E is an L^p-space, we have the following

Theorem 8 *Suppose E is an L^p-space.*

(a) *If $1 < p \leqslant 2$, then*

$$\|\lambda x + (1-\lambda)y\|^2 \leqslant \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (p-1)\cdot\lambda\cdot(1-\lambda)\cdot\|x-y\|^2$$

for all $x, y \in E$ and $0 \leqslant \lambda \leqslant 1$ ($c_p = p-1$);

(b) *If $2 < p < \infty$, then*

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x-y\|^p$$

for all $x, y \in E, 0 \leqslant \lambda \leqslant 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$ and

$$c_p = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}} = (p-1)(1+t_p)^{2-p}$$

with t_p being the unique solution of the equation

$$(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0, \quad 0 < t < 1.$$

All constant appeared in the above inequalities are the best possible.

In the following theorem we improve the fixed point theorem due to Xu [14] from point of view of the structure of the set of fixed points.

Theorem 9 *Let $p > 1$ be a real number and let E be a p-uniformly convex Banach space, C a nonempty bounded closed convex subset of E. Suppose $\mathcal{F} = \{T_s : s \in G\}$ is a one-parameter uniformly k-lipschitzian semigroup on C with $k < k_p$, where $k_p > 1$ is the unique solution of the equation*

$$(t^p)^2 - t^p - [N(E)]^p \cdot c_p = 0, \quad t \in (0, +\infty),$$

i.e., $k_p = \left[\frac{1}{2}\left(1+\sqrt{1+4\cdot c_p\cdot[N(E)]^p}\right)\right]^{\frac{1}{p}}$. Then $F(\mathcal{F}) \neq \varnothing$ and $F(\mathcal{F})$ is a retract of C.

Proof We may assume that $k \geqslant 1$ since if $k < 1$, the well known Banach Contraction Principle guarantees a fixed point of $\mathcal{F}$.

For an $x = x_0 \in C$, we can inductively define a sequence $\{x_m\}_{m=1,2,\ldots}$ in C in the following way: x_{m+1} is the asymptotic center of the sequence $\{T_t x_m\}_{t\in G}$, that is, x_{m+1} is the unique point in C that minimizes the functional

$$\limsup_{G\ni t\to+\infty} \|y - T_t x_m\|$$

over y in C. For each $m \geqslant 0$, we set

$$r_m = \limsup_{G \ni t \to +\infty} \|x_{m+1} - T_t x_m\| \quad \text{and} \quad d(x_m) = \sup_{t \in G} \|x_m - T_t x_m\|.$$

Then by Lemma 5, we have

$$\begin{aligned} r_m &\leqslant \frac{1}{N(E)} \cdot \lim_{G \ni t \to +\infty} (\sup\{\|T_{s+h} x_m - T_s x_m\| : t \leqslant s \in G \quad \text{and} \quad h \in G\}) \\ &\leqslant \frac{k}{N(E)} \cdot \sup_{t \in G} \|T_t x_m - x_m\| = \frac{k}{N(E)} \cdot d(x_m) \end{aligned} \tag{8}$$

for $m = 0, 1, 2, \ldots$

Now from Theorem 7 for each fixed $m \geqslant 0$ and $s, h \in G$, we have

$$\begin{aligned} &\|\lambda x_{m+1} + (1-\lambda) T_s x_{m+1} - T_{s+h} x_m\|^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T_s x_{m+1}\|^p \\ &\leqslant \lambda \|x_{m+1} - T_{s+h} x_m\|^p + (1-\lambda)\|T_{s+h} x_m - T_s x_{m+1}\|^p \\ &\leqslant \lambda \|x_{m+1} - T_{s+h} x_m\|^p + (1-\lambda) \cdot k^p \cdot \|T_h x_m - x_{m+1}\|^p. \end{aligned}$$

Taking the limit superior as $G \in h \to +\infty$ and nothing that x_{m+1} is the asymptotic center of the sequence $\{T_t x_m\}_{t \in G}$, we obtain for each $s \in G$,

$$r_m^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T_s x_{m+1}\|^p \leqslant [\lambda + (1-\lambda) \cdot k^p] \cdot r_m^p,$$

and

$$\|x_{m+1} - T_s x_{m+1}\|^p \leqslant \frac{(1-\lambda)(k^p - 1)}{c_p \cdot W_p(\lambda)} \cdot r_m^p.$$

It then follows that

$$[d(x_{m+1})]^p \leqslant \frac{(1-\lambda)(k^p-1)}{c_p \cdot W_p(\lambda)} \cdot r_m^p \leqslant \frac{(1-\lambda)(k^p-1)}{c_p \cdot W_p(\lambda)} \cdot \frac{k^p}{[N(E)]^p} \cdot [d(x_m)]^p.$$

Letting $\lambda \uparrow 1$, we get

$$[d(x_{m+1})]^p \leqslant \frac{k^p(k^p-1)}{c_p \cdot [N(E)]^p} \cdot [d(x_m)]^p,$$

and

$$d(x_{m+1}) \leqslant \left(\frac{k^p(k^p-1)}{c_p \cdot [N(E)]^p}\right)^{\frac{1}{p}} \cdot d(x_m) = B_p \cdot d(x_m), \quad m = 0, 1, 2, \ldots,$$

where $B_p = \left(\frac{k^p(k^p-1)}{c_p \cdot [N(E)]^p}\right)^{\frac{1}{p}} < 1$ by assumption of the theorem. In a similar way, we obtain

$$d(x_{m+1}) \leqslant B_p \cdot d(x_m) \leqslant \cdots \leqslant (B_p)^{m+1} \cdot d(x). \tag{9}$$

Since

$$\|x_{m+1} - x_m\| \leqslant \|x_{m+1} - T_t x_m\| + \|T_t x_m - x_m\|,$$

so taking the limit superior as $G \ni t \to +\infty$, we get by (8), (9),

$$\begin{aligned} \|x_{m+1} - x_m\| \leqslant r_m + d(x_m) &\leqslant \left(\frac{k}{N(E)} + 1\right) d(x_m) \\ &\leqslant \left(\frac{k}{N(E)} + 1\right) \cdot (B_p)^{m+1} \cdot d(x) \to 0 \quad \text{as } m \to +\infty, \end{aligned}$$

and we see that $\{x_m\}$ is norm Cauchy and hence strong convergent. Let $z = \lim_{m\to\infty} x_m$. Then we have

$$\begin{aligned}\|T_s z - z\| &\leqslant \|T_s z - T_s x_m\| + \|T_s x_m - x_m\| + \|x_m - z\|\\ &\leqslant (1+k)\|x_m - z\| + d(x_m)\\ &\leqslant (1+k)\|x_m - z\| + (B_p)^m \cdot d(x) \to 0 \quad \text{as } m \to +\infty,\end{aligned}$$

and $T_s z = z$ for all $s \in G$.

The proof of the retraction $R : C \to F(\mathcal{F})$ can be proved in exactly the same way as in the proof of Theorem 6. □

Corollary 10 *Let H be a Hilbert space, C a nonempty bounded closed convex subset of H and $\mathcal{F} = \{T_s : s \in G\}$ be a one-parameter uniformly k-lipschitzian semigroup on C with $k < \sqrt{2}$. Then $F(\mathcal{F}) \neq \varnothing$ and $F(\mathcal{F})$ is a retract of C.*

Corollary 11 *Let C be a nonempty bounded closed convex subset of L^p-space $(1 < p < \infty)$ and $\mathcal{F} = \{T_s : s \in G\}$ be a one-parameter uniformly k-lipschitzian semigroup on C. Suppose $k < \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + (p-1)4^{2-\frac{1}{p}}}}$ if $1 < p \leqslant 2$ (in particular, in L^2-space, $k < \sqrt{2}$) and $k < \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8 \cdot c_p}\right)^{\frac{1}{p}}$ if $2 < p < +\infty$ (here c_p is as in Theorem 8(b)). Then $F(\mathcal{F}) \neq \varnothing$ and $F(\mathcal{F})$ is a retract of C.*

For left reversible semi-topological semigroup we have the following [13, Lemma 3].

Lemma 12 *Let E be p-uniformly convex Banach space for some $p > 1$, C a nonempty bounded closed convex subset of E. Let J be a left reversible semi-topological semigroup and $\{x_a : a \in J\}$ be a net in C. Let us set*

$$r(x) = \inf_{b\in J} \sup_{a\geqslant b} \|x_a - x\|^p \quad \textit{and} \quad r = \inf_{x\in C} r(x).$$

Then we have a unique point $z \in C$ (called the asymptotic center of the net $\{x_a\}$ in C) such that $r(z) = r$ and

$$r(z) \leqslant r(x) - c_p\|x - z\|^p$$

for all $x \in C$, where the constant c_p is as in Theorem 7.

Theorem 13 *Let $p > 1$ be a real number and let E be p-uniformly convex Banach space, C a nonempty bounded closed convex subset of E. Suppose $\mathcal{T} = \{T_s : s \in J\}$ is a left reversible semi-topological uniformly k-lipschitzian semigroup on C with $k < (1 + c_p)^{\frac{1}{p}}$, where the constant c_p is as in Theorem 7. Then*

$$F(\mathcal{T}) = \{x \in C : T_s x = x \quad \textit{for all } s \in J\} \neq \varnothing$$

and $F(\mathcal{T})$ is a retract of C.

Proof We may assume that $k \geqslant 1$ since if $k < 1$, the well known Banach Contraction Principle guarantees a fixed point of $\mathcal{T}$.

Define a sequence $\{x_n\} \subset C$ in the following way: x_{n+1} is the asymptotic center of the net $\{T_s x_n\}_{s\in J}$ in C. Then, by Lemma 12, we have for $x \in C$ and $n = 1, 2, \ldots$

$$c_p\|x - x_{n+1}\|^p \leqslant \inf_s \sup_{t\geqslant s} \|T_t x_n - x\|^p - \inf_s \sup_{t\geqslant s} \|T_t x_n - x_{n+1}\|^p. \tag{10}$$

Noting the inequality

$$\inf_s \sup_{t\geqslant s} \|T_t y - x\|^p \leqslant \inf_s \sup_{t\geqslant s} \|T_{at} y - x\|^p$$

is valid for all $x, y \in C$ and every $a \in J$. Putting $x = T_a x_{n+1}$ into (10) we get

$$\begin{aligned}
c_p \|T_a x_{n+1} - x_{n+1}\|^p &\leqslant \inf_s \sup_{t\geqslant s} \|T_t x_n - T_a x_{n+1}\|^p - \inf_s \sup_{t\geqslant s} \|T_t x_n - x_{n+1}\|^p \\
&\leqslant \inf_s \sup_{t\geqslant s} \|T_{at} x_n - T_a x_{n+1}\|^p - \inf_s \sup_{t\geqslant s} \|T_t x_n - x_{n+1}\|^p \\
&\leqslant (k^p - 1) \inf_s \sup_{t\geqslant s} \|T_t x_n - x_{n+1}\|^p \\
&\leqslant (k^p - 1) \inf_s \sup_{t\geqslant s} \|T_t x_n - x_n\|^p
\end{aligned}$$

and hence

$$\begin{aligned}
\|T_a x_{n+1} - x_{n+1}\|^p &\leqslant \frac{k^p - 1}{c_p} \inf_s \sup_{t\geqslant s} \|T_t x_n - x_n\|^p \\
&\leqslant M^{n+1} \inf_s \sup_{t\geqslant s} \|T_t x_0 - x_0\|^p, \qquad (11)
\end{aligned}$$

where $M = \frac{k^p-1}{c_p} < 1$ by assumption of the theorem. Inserting $x = T_a x_{n-1}$ into (10) and in a similar way to above, we obtain

$$\|T_a x_n - x_{n+1}\|^p \leqslant \frac{k^p}{c_p} \inf_s \sup_{t\geqslant s} \|T_t x_n - x_n\|^p. \qquad (12)$$

Combining (11) and (12) it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\|^p &\leqslant (\|x_{n+1} - T_a x_n\| + \|T_a x_n - x_n\|)^p \\
&\leqslant 2^{p-1}(\|x_{n+1} - T_a x_n\|^p + \|T_a x_n - x_n\|^p) \\
&\leqslant 2^{p-1} \cdot M^n \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot \inf_s \sup_{t\geqslant s} \|T_t x_0 - x_0\|^p, \qquad (13)
\end{aligned}$$

which shows that $\{x_n\}$ is Cauchy. Let $z = \lim_{n\to+\infty} x_n$. Then for each $a \in J$ we have

$$\begin{aligned}
\|z - T_a z\|^p &\leqslant (\|z - x_n\| + \|x_n - T_a x_n\| + \|T_a x_n - T_a z\|)^p \\
&\leqslant ((1+k)\|z - x_n\| + \|x_n - T_a x_n\|)^p \\
&\leqslant 2^{p-1}[(1+k)^p \|z - x_n\|^p + \|x_n - T_a x_n\|^p] \\
&\leqslant 2^{p-1}[(1+k)^p \|z - x_n\|^p + M^n \cdot \inf_s \sup_{t\geqslant s} \|T_t x_0 - x_0\|^p] \to 0
\end{aligned}$$

as $n \to +\infty$. Therefore $T_a z = z$ for all $a \in J$.

Note that if $x_0 = x$ is a arbitrary point in C, then $x_m = A^m x$ for $m = 1, 2, \ldots$ and by (13)

$$\begin{aligned}
\|A^{m+1} x - A^m x\| &\leqslant 2^{p-1} \cdot M^m \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot \inf_s \sup_{t\geqslant s} \|T_t x - x\|^p \\
&\leqslant 2^{p-1} \cdot M^m \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot (\operatorname{diam} C)^p
\end{aligned}$$

for $m = 1, 2, \ldots$ Thus

$$\sup_{x\in C} \|A^i x - A^m x\| \leqslant 2^{p-1} \cdot \frac{M^m}{1 - M} \cdot \left(\frac{k^p}{c_p} + 1\right) \cdot (\operatorname{diam} C)^p \to 0$$

if $i, m \to +\infty$, which implies that the sequence $\{A^m x\}$ converges uniformly to a function

$$Rx = \lim_{m \to +\infty} A^m x, \quad x \in C.$$

It follows from Lemma 3 and Remark 4 that $R : C \to C$ is continuous. Moreover

$$\begin{aligned} \|Rx - T_a Rx\|^p &\leqslant (\|Rx - A^m x\| + \|A^m x - T_a A^m x\| + \|T_a A^m x - T_a Rx\|)^p \\ &\leqslant ((1+k)\|Rx - A^m x\| + \|A^m x - T_a A^m x\|)^p \\ &\leqslant 2^{p-1}[(1+k)^p \cdot \|Rx - A^m x\|^p + \|A^m x - T_a A^m x\|^p] \\ &\leqslant 2^{p-1}[(1+k)^p \cdot \|Rx - A^m x\|^p + M^m \cdot \inf_s \sup_{t \geqslant s} \|T_t x - x\|^p] \to 0 \end{aligned}$$

as $m \to +\infty$, and $Rx = T_a Rx$ for all $a \in J$ and $x \in C$. Thus R is a retraction C onto $F(\mathcal{T})$. □

Corollary 14 *Let H be a Hilbert space, C a nonempty bounded closed convex subset of H and $\mathcal{T} = \{T_s : s \in J\}$ be a left reversible semi-topological uniformly k-lipschitzian semigroup on C with $k < \sqrt{2}$. Then $F(\mathcal{T}) \neq \varnothing$ and $F(\mathcal{T})$ is a retract of C.*

Corollary 15 *Let C be a nonempty bounded closed convex subset of L^p-space* $(1 < p < \infty)$ *and $\mathcal{T} = \{T_s : s \in J\}$ be a left reversible semi-topological uniformly k-lipschitzian semigroup on C. Suppose $k < \sqrt{p}$ if $1 < p \leqslant 2$, and $k < (1 + c_p)^{\frac{1}{p}}$ if $2 < p < +\infty$ (here c_p is as in Theorem 8(b)). Then $F(\mathcal{T}) \neq \varnothing$ and $F(\mathcal{T})$ is a retract of C.*

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