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Rolling reductive homogeneous spaces

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Abstract

Rollings of reductive homogeneous spaces are investigated. More precisely, for a reductive homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we consider rollings of $\mathfrak{m}$ over G/H without slip and without twist, where G/H is equipped with an invariant covariant derivative. To this end, an intrinsic point of view is taken, meaning that a rolling is a curve in the configuration space Q which is tangent to a certain distribution. By considering an H-principal fiber bundle $\overline{\pi}\colon \overline{Q} \to Q$ over the configuration space equipped with a suitable principal connection, rollings of $\mathfrak{m}$ over G/H can be expressed in terms of horizontally lifted curves on $\overline{Q}$. The total space of $\overline{\pi}\colon \overline{Q} \to Q$ is a product of Lie groups. In particular, for a given control curve, this point of view allows for characterizing rollings of $\mathfrak{m}$ over G/H as solutions of an explicit, time-variant ordinary differential equation (ODE) on $\overline{Q}$, the so-called kinematic equation. An explicit solution for the associated initial value problem is obtained for rollings with respect to the canonical invariant covariant derivative of first and second kind if the development curve in G/H is the projection of a one-parameter subgroup in G. Lie groups and Stiefel manifolds are discussed as examples.

Keywords Distributions · Frame bundles · Horizontal lifts · Reductive homogeneous spaces · Rolling without slip and without twist · Stiefel manifolds

Mathematics Subject Classification 53A17 · 53B05 · 53C30 · 58A30

1 Introduction

Meanwhile, there is a vast literature on rolling manifolds without slip and without twist. First, we mention some works, where concrete expressions for extrinsic rollings of certain submanifolds of (pseudo-)Euclidean vector spaces over their affine tangent

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spaces are derived. Using the definition from [1, Ap. B] as starting point, extrinsic rollings of spheres $\mathrm{S}^n \subseteq \mathbb{R}^{n+1}$, real Grassmann manifolds $\mathrm{Gr}_{n,k} \subseteq \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and special orthogonal groups $\mathrm{SO}(n) \subseteq \mathbb{R}^{n\times n}$ over their affine tangent spaces are studied in [2]. In a similar context, the Stiefel manifold $\mathrm{St}_{n,k} \subseteq \mathbb{R}^{n\times k}$, endowed with the Euclidean metric, is investigated in [3] while rollings of pseudo-orthogonal groups are considered in [4]. For these works, the need to solve interpolation problems on these submanifolds in various applications seems to serve as a motivation. Indeed, the rolling and unwrapping technique from [2], see also the more recent work [5], is a method to compute a $\mathscr{C}^2$-curve connecting a finite number of given points on the manifolds S^n, $\mathrm{Gr}_{n,k}$ and $\mathrm{SO}(n)$, where the velocities at the initial and final point are prescribed. This algorithm relies on having an explicit expression for the rolling of the manifold over its affine tangent space along a curve joining the initial point with the final point.

Beside these works, there is the paper [6], where a notion of intrinsic rolling of an oriented Riemannian manifold M over another oriented Riemannian manifold $\widehat{M}$ is introduced assuming $\dim(M) = \dim(\widehat{M})$. In [7], this notion of intrinsic rolling is generalized to pseudo-Riemannian manifolds. A further generalization can be found in [8, Sec. 7] and [9, p. 35], where the Levi-Civita covariant derivatives coming from the pseudo-Riemannian metrics on M and $\widehat{M}$ are replaced by arbitrary covariant derivatives on M and $\widehat{M}$, respectively.

In this text, we investigate the following situation. Let G be a Lie group and $H \subseteq G$ a closed subgroup such that G/H is a reductive homogeneous space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then G/H can be equipped with an invariant covariant derivative corresponding to an invariant affine connection from [10]. Motivated by the study of rollings of (pseudo-Riemannian) symmetric spaces over flat spaces in [11], we consider rollings of $\mathfrak{m}$ over G/H. Here we generalize the above mentioned definition proposed in [8] and [9, p. 35] slightly in order take additional structures of the involved manifolds into account. In particular, this definition allows for considering rollings of *not* necessarily oriented manifolds.

Moreover, if one is interested in getting rather simple formulas describing the rollings, it might be convenient to consider rollings of $\mathfrak{m}$ over G/H with respect to the canonical covariant derivative of first or second kind on G/H. These covariant derivatives can be defined independently of a pseudo-Riemannian metric although they are in some sense similar the Levi-Civita covariant derivatives on naturally reductive homogeneous spaces or pseudo-Riemannian symmetric spaces, respectively.

We now give an overview of this text. In Sect. 2, we start with introducing some notations and recalling some definitions and well-known facts related to Lie groups and principal fiber bundles. Moreover, we recall some facts on reductive homogeneous spaces with an emphasis on invariant covariant derivatives.

In Sect. 3, we briefly recall the notion of rolling intrinsically a manifold M over another manifold $\widehat{M}$ of equal dimension from the literature. More precisely, as already announced above, a slightly generalized definition of intrinsic rolling is introduced.

As preparation to determine the configuration space for the intrinsic rollings considered in Sect. 5, an explicit description of the frame bundle of a reductive homogeneous space G/H is needed. Therefore frame bundles of reductive homogeneous spaces are investigated in Sect. 4. Here we first consider a more general situation. The frame bundle of a vector bundle associated to an H-principal fiber bundle $P \to M$ is identified with an other fiber bundle associated to $P \to M$. Afterwards, reductive homogeneous spaces are treated as a special case.

In Sect. 5, we turn our attention to rollings of a reductive homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We consider the intrinsic rolling of $\mathfrak{m}$ over G/H with respect to an invariant covariant derivative ∇^α. To this end, the configuration space $Q \to \mathfrak{m} \times G/H$ is investigated in detail. Here we determine an H-principal fiber bundle $\overline{\pi} \colon \overline{Q} \to Q$ over Q which is equipped with a suitable principal connection. Its total space is given by $\overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$, where $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ is a closed subgroup, i.e. the manifold $\overline{Q}$ is a product of Lie groups.

For a fixed invariant covariant derivative ∇^α on G/H defined by an $\mathrm{Ad}(H)$-invariant billinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, we determine a distribution $\overline{D^\alpha}$ on $\overline{Q}$ that projects to a distribution D^α on Q with the following property. A curve $q \colon I \to Q$ is horizontal with respect to D^α iff it is a rolling of $\mathfrak{m}$ over G/H with respect to ∇^α. Moreover, horizontal lifts of curves on Q with respect to the principal connection on $\overline{\pi} \colon \overline{Q} \to Q$ mentioned above are horizontal with respect to $\overline{D^\alpha}$ iff they are horizontal with respect to D^α. In particular, this fact allows for characterizing rollings of $\mathfrak{m}$ over G/H in terms of an ODE on $\overline{Q}$. More precisely, for a prescribed control curve $u \colon I \to \mathfrak{m}$, we obtain an explicit, time-variant ODE on $\overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$ whose solutions projected to Q are rollings of $\mathfrak{m}$ over G/H with respect to ∇^α. This ODE can be seen as a generalization of the kinematic equation for rollings of oriented pseudo-Riemannian symmetric spaces over flat spaces from [11, Sec. 4.2].

In Sect. 5.4, we turn our attention to rollings of $\mathfrak{m}$ over G/H with respect to the canonical covariant derivative of first and second kind such that the development curve is of the form $I \ni t \mapsto \mathrm{pr}(\exp(t\xi)) \in G/H$ with some $\xi \in \mathfrak{g}$, i.e. a projection of a *not* necessarily horizontal one-parameter subgroup in G. For this special case, an explicit solution of the kinematic equation is obtained.

We end this text by discussing intrinsic rollings of Lie groups and Stiefel manifolds as examples.

2 Notations, terminology and background

In this section, we introduce the notation and terminology that is used throughout this text. Moreover, some facts concerning Lie groups and principal fiber bundles are recalled. We end this section by discussing reductive homogeneous spaces with an emphasis on invariant covariant derivatives.

2.1 Notations and terminology

We start with introducing some notations and terminology concerning differential geometry. This subsection is based on [12, Sec. 2] partially copied word by word.

Notation 2.1 *Throughout this text we follow the convention in [13, Chap. 2]. A scalar product is defined as a non-degenerated symmetric bilinear form. An inner product is a positive definite symmetric bilinear form.*

Next we introduce some notations concerning differential geometry. Let M be a smooth (finite-dimensional) manifold. We denote by TM and T^*M the tangent and cotangent bundle of M, respectively. A smooth vector subbundle D of the tangent bundle TM is called a regular distribution on M. For a smooth map $f\colon M \to N$ between manifolds M and N, the tangent map of f is denoted by $Tf\colon TM \to TN$. We write $\mathscr{C}^\infty(M)$ for the algebra of smooth real-valued functions on M.

Let $E \to M$ be a vector bundle over M with typical fiber V. The smooth sections of E are denoted by $\Gamma^\infty(E)$. We write $\operatorname{End}(E) \cong E^* \otimes E$ for the endomorphism bundle of E. Moreover, we denote by $E^{\otimes k}$, $\mathrm{S}^k E$ and $\Lambda^k E$ the k-th tensor power, the k-th symmetrized tensor power and the k-th anti-symmetrized tensor power of E. If $\omega \in \Gamma^\infty\big(\Lambda^k(T^*N)\big) \otimes V$ is a differential form taking values in a finite dimensional $\mathbb{R}$-vector space V, its pull-back by $f\colon M \to N$ is denoted by $f^*\omega$. Next let $S_1 \times \cdots \times S_k$ be a product of sets and let $i \in \{1, \ldots, k\}$. Then $\mathrm{pr}_i\colon S_1 \times \cdots \times S_k \to S_i$ denotes the projection onto the i-th factor.

We now recall a well-known fact on surjective submersions. This is the next lemma, see e.g. [14, Thm. 4.29], which is used frequently without referencing it explicitly.

Lemma 2.2 *Let* $\mathrm{pr}\colon P \to M$ *be a surjective submersion and let* N *be some manifold. Let* $f\colon M \to N$ *be a map. Then* f *is smooth iff* $f \circ \mathrm{pr}\colon P \to N$ *is smooth.*

Concerning the regularity of curves on manifolds, we use the following convention.

Notation 2.3 *Whenever* $c\colon I \to M$ *denotes a curve in a manifold* M *defined on an interval* $I \subseteq \mathbb{R}$*, we assume for simplicity that* c *is smooth if not indicated otherwise. If* I *is not open, we assume that* c *can be extended to smooth curve defined on an open interval* $J \subseteq \mathbb{R}$ *containing* I*. Moreover, we implicitly assume that* 0 *is contained in* I *if we write* $0 \in I$*. Nevertheless, many results can be generalized by requiring less regularity.*

Notation 2.4 *If not indicated otherwise, we use Einstein summation convention.*

2.2 Lie groups

Copying and adapting [12, Sec. 3.1], we now introduce some notations and well-known facts concerning Lie groups and Lie algebras.

Let G be a Lie group and denote its Lie algebra by $\mathfrak{g}$. The identity of G is usually denoted by e. The left translation by an element $g \in G$ is denoted by

$$\ell_g\colon G \to G, \quad h \mapsto \ell_g(h) = gh \tag{2.1}$$

and we write

$$r_g\colon G \to G, \quad h \mapsto r_g(h) = hg \tag{2.2}$$

for the right translation by $g \in G$. The conjugation by an element $g \in G$ is given by

$$\mathrm{Conj}_g : G \to G, \quad h \mapsto \mathrm{Conj}_g(h) = (\ell_g \circ r_{g^{-1}})(h) = (r_{g^{-1}} \circ \ell_g)(h) = ghg^{-1} \tag{2.3}$$

and the adjoint representation of G is defined as

$$\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g = \big(\xi \mapsto \mathrm{Ad}_g(\xi) = T_e \mathrm{Conj}_g \xi\big). \tag{2.4}$$

Moreover, we denote the adjoint representation of $\mathfrak{g}$ by

$$\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \xi \mapsto \big(\eta \mapsto \mathrm{ad}_\xi(\eta) = [\xi, \eta]\big). \tag{2.5}$$

For $\xi \in \mathfrak{g}$, we denote by $\xi^L \in \Gamma^\infty(TG)$ and $\xi^R \in \Gamma^\infty(TG)$ the corresponding left and right-invariant vector fields, respectively, which are given by

$$\xi^L(g) = T_e \ell_g \xi \quad \text{and} \quad \xi^R(g) = T_e r_g \xi, \quad g \in G. \tag{2.6}$$

The exponential map of the Lie group G is denoted by $\exp: \mathfrak{g} \to G$. One has for $\xi \in \mathfrak{g}$ and $t \in \mathbb{R}$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi) = T_e \ell_{\exp(t\xi)} \xi = T_e r_{\exp(t\xi)} \xi \tag{2.7}$$

by the proof of [15, Prop. 19.5].

Next we recall that the tangent map of the group multiplication $m: G \times G \ni (g, h) \mapsto gh \in G$ is given by

$$T_{(g,h)} m(v_g, w_h) = T_g r_h v_g + T_h \ell_g w_h \tag{2.8}$$

for all $(g, h) \in G \times G$ and $(v_g, v_h) \in T_g G \times T_h G$, see e.g. [16, Lem. 4.2]. The tangent map of the inversion $\mathrm{inv}: G \ni g \mapsto \mathrm{inv}(g) = g^{-1} \in G$ reads

$$T_g \mathrm{inv} v_g = -(T_e \ell_{g^{-1}} \circ T_g r_{g^{-1}}) v_g \tag{2.9}$$

for all $g \in G$ and $v_g \in T_g G$, see e.g. [16, Cor. 4.3].

We now introduce the notation for some Lie groups that play a crucial role in this text.

Notation 2.5 *Let V be a finite dimensional $\mathbb{R}$-vector space. We write $\mathrm{GL}(V)$ for the general linear group of V. If V is a pseudo-Euclidean vector space, i.e. V is endowed with a scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, we denote the corresponding pseudo-orthogonal group by $\mathrm{O}(V, \langle \cdot, \cdot \rangle)$. Moreover, we often write $\mathrm{O}(V) = \mathrm{O}(V, \langle \cdot, \cdot \rangle)$ for short. Similarly, the special (pseudo-)orthogonal group is denoted by $\mathrm{SO}(V) = \mathrm{SO}(V, \langle \cdot, \cdot \rangle)$. More generally, a closed subgroup of $\mathrm{GL}(V)$, which is not further specified, is often denoted by $\mathrm{G}(V)$ and we write $\mathfrak{g}(V) \subseteq \mathfrak{gl}(V)$ for the corresponding Lie algebra. Sometimes, the exponential map of $\mathrm{G}(V)$ is denoted by*

$$\mathfrak{g}(V) \to \mathrm{G}(V), \quad \xi \mapsto \mathrm{e}^\xi = \sum_{k=0}^{\infty} \tfrac{1}{k!} \xi^k. \tag{2.10}$$

In the sequel, it is often convenient to denote the evaluation of $A \in \mathrm{GL}(V)$ *at* $v \in V$ *by* Av *instead of writing* $A(v)$.

2.3 Principal fiber bundles

Next we recall some well-known facts on principal fiber bundles and introduce some notations. For general facts on principal fiber bundles we refer to [16, Sec. 18–19] and [17, Sec. 1.1−1.3].

Notation 2.6 *Let* $P \to M$ *be an* H*-principal fiber bundle over* M *and let* $\mathfrak{h}$ *be the Lie algebra of* H. *The principal action is usually denoted by*

$$\triangleleft \colon P \times H \to P, \quad (p, h) \mapsto p \triangleleft h \tag{2.11}$$

and we denote for fixed $h \in H$ *by* $(\cdot \triangleleft h) \colon P \ni p \mapsto p \triangleleft h \in P$ *the induced diffeomorphism.*

Next, let $\eta \in \mathfrak{h}$. Then $\eta_P \in \Gamma^\infty(TP)$ denotes the fundamental vector field associated to the principal action. For $p \in P$, it is given by

$$\eta_P(p) = \tfrac{\mathrm{d}}{\mathrm{d}t}(p \triangleleft \exp(t\eta))\big|_{t=0}. \tag{2.12}$$

As a consequence of [17, Lem. 1.3.1], see also [16, Sec. 18.18], the vertical bundle $\mathrm{Ver}(P) = \ker(T\mathrm{pr}) \subseteq TP$ of $P \to M$ is fiber-wise given by

$$\mathrm{Ver}(P)_p = \big\{\eta_P(p) \mid \eta \in \mathfrak{h}\big\} = \big\{\tfrac{\mathrm{d}}{\mathrm{d}t}(p \triangleleft \exp(t\eta))\big|_{t=0} \mid \eta \in \mathfrak{h}\big\} \subseteq T_pP, \quad p \in P. \tag{2.13}$$

Recall that a complement of $\mathrm{Ver}(P)$, i.e. a subbundle $\mathrm{Hor}(P) \subseteq TP$ fulfilling $\mathrm{Hor}(P) \oplus \mathrm{Ver}(P) = TP$ is called horizontal bundle. It is well-known that such a complement defines a unique connection on P, i.e. an endomorphism $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TP)\big)$ such that $\mathcal{P}^2 = \mathcal{P}$ and $\mathrm{im}(\mathcal{P}) = \mathrm{Ver}(P)$ as well as $\ker(\mathcal{P}) = \mathrm{Hor}(P)$ holds. This fact can be regarded as a consequence of [16, Sec. 17.3]. Moreover, $\mathcal{P}$ corresponds to an $\mathfrak{h}$-valued one-form $\omega \in \Gamma^\infty(T^*P) \otimes \mathfrak{h}$ via

$$\omega\big|_p(v_p) = \big(T_e(p \triangleleft \cdot)\big)^{-1}\mathcal{P}\big|_p(v_p), \quad p \in P, \quad v_p \in T_pP, \tag{2.14}$$

see e.g. [16, Sec. 19.1, Eq. (1)]. A connection $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TP)\big)$ is called principal connection if

$$T_p(\cdot \triangleleft h)\big(\mathcal{P}\big|_p(v_p)\big) = \mathcal{P}\big|_{p \triangleleft h}\big(T_p(\cdot \triangleleft h)v_p\big), \quad p \in P, \quad v_p \in T_pP \tag{2.15}$$

holds for all $h \in H$, see e.g. [16, Sec. 19.1]. Next we recall how a principal connection $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TP)\big)$ is related to the corresponding connection one-form $\omega \in \Gamma^\infty(TP) \otimes \mathfrak{h}$ given by (2.14). This is the next lemma which is taken from [16, Sec. 19.1]

Lemma 2.7 *Let $\mathcal{P} \in \Gamma^\infty\big(\operatorname{End}(TP)\big)$ be a principal connection. Then the corresponding connection one-form $\omega \in \Gamma^\infty(T^*P) \otimes \mathfrak{h}$ satisfies:*

1. *For each $\eta \in \mathfrak{h}$, one has $\omega\big|_p\big(\eta_P(p)\big) = \eta$ for all $p \in P$.*
2. *For each $h \in H$ one has $\big((\cdot \triangleleft h)^*\omega\big)\big|_p(v_p) = \operatorname{Ad}_{h^{-1}}\big(\omega\big|_p(v_p)\big)$ for all $p \in P$ and $v_p \in T_pP$.*

Conversely, an $\mathfrak{h}$-valued one-form $\omega \in \Gamma^\infty(T^*P) \otimes \mathfrak{h}$ fulfilling Claim 1. and Claim 2. defines a principal connection on $P \to M$ via

$$\mathcal{P}\big|_p(v_p) = \big(T_e(p \triangleleft \cdot)\big)\omega\big|_p(v_p) = \big(\omega\big|_p(v_p)\big)_P(p) \tag{2.16}$$

for $p \in P$ and $v_p \in T_pP$ with the map $(p \triangleleft \cdot)\colon H \ni h \mapsto p \triangleleft h \in P$ for fixed $p \in P$.

Next we recall the notion of reductions of principal fiber bundles, see e.g. [16, Sec. 18.6]. Let $P \to M$ be an H-principal fiber bundle. Then an H_2-principal fiber bundle $P_2 \to M$ is called a reduction of P if there is a morphism of Lie groups $f\colon H_2 \to H$ and a morphism $\Psi\colon P_2 \to P$ of principal fiber bundles along f covering $\mathrm{id}_M\colon M \to M$. In particular, $\Psi(p_2 \triangleleft h_2) = \Psi(p_2) \triangleleft f(h_2)$ holds for all $h_2 \in H_2$ and $p_2 \in P_2$.

Furthermore, we need the notion of an associated bundle which we recall briefly from [16, Sec. 18.7]. Let F be some manifold and let $\triangleright\colon H \times F \to F$ be a smooth action of H on F from the left. Then the corresponding associated bundle is denoted by $\pi\colon P \times_H F \to M$, whose elements are given by

$$P \times_H F = \big\{[p, s] \mid (p, s) \in P \times F\big\}. \tag{2.17}$$

Here $[p, s]$ denotes the equivalence class of $(p, s) \in P \times F$ defined by the H-action

$$\overline{\triangleleft}\colon (P \times F) \times H \ni \big((p, f), h\big) \mapsto \big(p \triangleleft h, h^{-1} \triangleright f\big) \in P \times F, \tag{2.18}$$

i.e. $(p, s) \sim (p', s')$ iff there exists an $h \in H$ such that $(p', s') = (p \triangleleft h, h^{-1} \triangleright s)$ is fulfilled. The projection $\pi\colon P \times_H F \to M$, sometimes denoted by $\pi_{P \times_H F}\colon P \times_H F \to M$ to refer to $P \times_H F$ explicitly, is given by $\pi([p, s]) = \mathrm{pr}_P(p)$, where $\mathrm{pr}_P\colon P \to M$ denotes the projection of the principal fiber bundle. Furthermore we often write

$$\overline{\pi}\colon P \times F \to (P \times F)/H = P \times_H F, \quad (p, f) \mapsto [p, f] \tag{2.19}$$

for the H-principal fiber bundle over the associated bundle $P \times_H F$, where the principal action is given by (2.18). We also denote the projection in (2.19) by $\overline{\pi}_{P \times F}\colon P \times F \to P \times_H F$ to refer to $P \times F$ explicitly.

We will use the following identification of the tangent bundle of an associated bundle $P \times_H F \to M$ of an H-principal fiber bundle $P \to M$

$$T(P \times_H F) \cong TP \times_{TH} TF = \big\{[v_p, v_s] \mid (v_p, v_s) \in T_{(p,s)}(P \times F),\ (p, s) \in P \times F\big\}, \tag{2.20}$$

see e.g. [16, Sec. 18.18]. Here TP is considered as a TH-principal fiber bundle over TM with principal action $T\triangleleft\colon TP \times TH \to TP$, see e.g. [16, Sec. 18.18], and TH acts on TF via the tangent map of the H-action on F.

Finally, we introduce some notations concerning frame bundles of vector bundles. We refer to [16, Sec. 18.11] for general information on frame bundles.

Notation 2.8 *The frame bundle of a vector bundle $E \to M$ with typical fiber V is denoted by $\mathrm{GL}(V, E) \to M$. If E is equipped with a not necesarrily positive definite fiber metric, we denote the corresponding (pseudo-)orthogonal frame bundle by $\mathrm{O}(V, E) \to M$. More generally, let $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ be a closed subgroup of the general linear group $\mathrm{GL}(V)$. Then a $\mathrm{G}(V)$-reduction of $\mathrm{GL}(V, E)$ along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$ is often denoted by $\mathrm{G}(V, E)$ if it exists. We write $\mathrm{pr}_{\mathrm{G}(V,E)}\colon \mathrm{G}(V, E) \to M$ for the bundle projection.*

2.4 Reductive homogeneous spaces

In this subsection, we recall some well-known facts on reductive homogeneous spaces by adapting and copying some parts of [12, Sec. 3.2–3.3]. We refer to [15, Sec. 23.4] or [13, Chap. 11] for details.

Since reductive homogeneous spaces play a central role in this text, we recall their definition from [15, Def. 23.8], see also [10, Sec. 7] or [13, Chap. 11, Def. 21].

Definition 2.9 Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Moreover, let $H \subseteq G$ be a closed subgroup and denote its Lie algebra by $\mathfrak{h} \subseteq \mathfrak{g}$. Then the homogeneous space G/H is called reductive if there exists a subspace $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is fulfilled and

$$\mathrm{Ad}_h(\mathfrak{m}) \subseteq \mathfrak{m} \tag{2.21}$$

holds for all $h \in H$.

In the remainder part of this section, G/H always denotes a reductive homogeneous space with a fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ if not indicated otherwise.

The projection onto $\mathfrak{m}$ whose kernel is given by $\mathfrak{h}$ is denoted by $\mathrm{pr}_{\mathfrak{m}}\colon \mathfrak{g} \to \mathfrak{m}$. Analogously, we write $\mathrm{pr}_{\mathfrak{h}}\colon \mathfrak{g} \to \mathfrak{h}$ for the projection whose kernel is given by $\mathfrak{m}$. Moreover, we write for $\xi \in \mathfrak{g}$

$$\xi_{\mathfrak{m}} = \mathrm{pr}_{\mathfrak{m}}(\xi) \quad \text{and} \quad \xi_{\mathfrak{h}} = \mathrm{pr}_{\mathfrak{h}}(\xi). \tag{2.22}$$

The map

$$\tau\colon G \times G/H \to G/H, \quad (g, g' \cdot H) \mapsto (gg') \cdot H \tag{2.23}$$

is a smooth G-action on G/H from the left, where $g \cdot H \in G/H$ denotes the coset defined by $g \in G$. Borrowing the notation from [15, p. 676], for fixed $g \in G$, the associated diffeomorphism is denote by

$$\tau_g\colon G/H \to G/H, \quad g' \cdot H \mapsto \tau_g(g' \cdot H) = (gg') \cdot H. \tag{2.24}$$

In addition, we write

$$\mathrm{pr}\colon G \to G/H, \quad g \mapsto \mathrm{pr}(g) = g \cdot H \tag{2.25}$$

for the canonical projection. It is well-known that $\mathrm{pr}\colon G \to G/H$ carries the structure of an H-principle fiber bundle, see e.g. [16, Sec. 18.15]. In the sequel, we write

$$\triangleleft\colon G \times H \to G, \quad (g, h) \mapsto g \triangleleft h = r_h(g) = \ell_g(h) = gh \tag{2.26}$$

for the H-principal action on G if not indicated otherwise. The reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ can be used to define a principal connection on $\mathrm{pr}\colon G \to G/H$, see e.g. [18, Thm. 11.1]. Since this well-known fact will be used several times below, we state the next proposition which is copied from [12, Sec. 3.3].

Proposition 2.10 *Consider* $\mathrm{pr}\colon G \to G/H$ *as an H-principal fiber bundle, where G/H is a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and define* $\mathrm{Hor}(G) \subseteq TG$ *fiber-wise by*

$$\mathrm{Hor}(G)_g = (T_e\ell_g)\mathfrak{m}, \quad g \in G. \tag{2.27}$$

Then $\mathrm{Hor}(G)$ *is a subbundle of TG defining a horizontal bundle on TG, i.e. a complement of the vertical bundle* $\mathrm{Ver}(G) = \ker(T\mathrm{pr}) \subseteq TG$ *which yields a principal connection on* $\mathrm{pr}\colon G \to G/H$. *This principal connection $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$ corresponding to* $\mathrm{Hor}(G)$ *is given by*

$$\mathcal{P}\big|_g(v_g) = T_e\ell_g \circ \mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_g)^{-1} v_g, \quad g \in G, \quad v_g \in T_gG. \tag{2.28}$$

The corresponding connection one-form $\omega \in \Gamma^\infty(T^*G) \otimes \mathfrak{h}$ reads

$$\omega\big|_g(v_g) = \mathrm{pr}_{\mathfrak{h}} \circ (T_e\ell_g)^{-1} v_g \tag{2.29}$$

for $g \in G$ and $v_g \in T_gG$.

In the next lemma, following [15, Prop. 23.22], we recall a well-known property of the isotropy representation of a reductive homogeneous space.

Lemma 2.11 *The isotropy representation of a reductive homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ given by $H \ni h \mapsto T_{\mathrm{pr}(e)}\tau_h \in \mathrm{GL}\big(T_{\mathrm{pr}(e)}G/H\big)$ is equivalent to the representation $H \ni h \mapsto \mathrm{Ad}_h\big|_{\mathfrak{m}} = \big(X \mapsto \mathrm{Ad}_h(X)\big) \in \mathrm{GL}(\mathfrak{m})$, i.e.*

$$T_{\mathrm{pr}(e)}\tau_h \circ T_e\mathrm{pr}\big|_{\mathfrak{m}} = T_e\mathrm{pr} \circ \mathrm{Ad}_h\big|_{\mathfrak{m}} \tag{2.30}$$

is fulfilled for all $h \in H$.

Next we discuss invariant pseudo-Riemannian metrics on G/H briefly. A scalar product $\langle \cdot, \cdot \rangle\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ is called $\mathrm{Ad}(H)$-invariant if

$$\big\langle \mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)\big\rangle = \big\langle X, Y\big\rangle \tag{2.31}$$

holds for all $h \in H$ and $X, Y \in \mathfrak{m}$, see e.g [13, p. 301] or [15, Sec. 23.4] for the positive definite case. Reformulating and adapting [15, Def. 23.5], we call a pseudo-Riemannian metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle \in \Gamma^\infty\big(\mathrm{S}^2 T^*(G/H)\big)$ invariant if

$$\langle\!\langle v_p, w_p \rangle\!\rangle_p = \langle\!\langle T_p\tau_g v_p, T_p\tau_g w_p \rangle\!\rangle_{\tau_g(p)}, \quad p \in G/H, \quad v_p, w_p \in T_p(G/H) \quad (2.32)$$

holds for all $g \in G$. By requiring the linear isomorphism

$$T_e\mathrm{pr}\big|_{\mathfrak{m}} : \mathfrak{m} \to T_{\mathrm{pr}(e)}(G/H) \quad (2.33)$$

to be an isometry, there is a one-to-one correspondence between $\mathrm{Ad}(H)$-invariant scalar products on $\mathfrak{m}$ and invariant pseudo-Riemannian metrics on G/H, see e.g. [13, Chap. 11, Prop. 22] and also [15, Prop. 23.22] for the Riemannian case.

Naturally reductive homogeneous spaces are special reductive homogeneous spaces. We recall their definition from [13, Chap. 11, Def. 23].

Definition 2.12 Let G/H be a reductive homogeneous space equipped with an invariant pseudo-Riemannian metric and denote by $\langle \cdot, \cdot \rangle : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ the corresponding $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{m}$. Then G/H is called naturally reductive homogeneous space if

$$\big\langle [X, Y]_{\mathfrak{m}}, Z \big\rangle = \big\langle X, [Y, Z]_{\mathfrak{m}} \big\rangle \quad (2.34)$$

holds for all $X, Y, Z \in \mathfrak{m}$.

The following lemma can be considered as a generalization of [15, Prop. 23.29 (1)–(2)] to pseudo-Riemannian metrics and Lie groups which are not necessarily connected.

Lemma 2.13 *Let G be a Lie group and denote by $\mathfrak{g}$ its Lie algebra. Moreover, let G be equipped with a bi-invariant metric and let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the corresponding $\mathrm{Ad}(G)$-invariant scalar product. Moreover, let $H \subseteq G$ be a closed subgroup such that its Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is non-degenerated with respect to $\langle \cdot, \cdot \rangle$. Then G/H is a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$. Moreover, if G/H is equipped with the invariant metric corresponding to the scalar product on $\mathfrak{m}$ that is obtained by restricting $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}$, the reductive homogeneous space G/H is naturally reductive.*

Proof The claim can be proven analogously to the proof of [15, Prop. 23.29 (1)–(2)] by taking the assumption $\mathfrak{h} \oplus \mathfrak{h}^\perp = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{g}$ into account. □

Remark 2.14 Inspired by the terminology in [15, Sec. 23.6, p. 710], we refer to the naturally reductive spaces from Lemma 2.13 as normal naturally reductive spaces.

We now consider another special class of reductive homogeneous spaces. To this end, we state the following definition which can be found in [19, p. 209].

Definition 2.15 Let G be a connected Lie group and let H be a closed subgroup. Then (G, H) is called a symmetric pair if there exists a smooth involutive automorphism $\sigma : G \to G$, i.e. an automorphism of Lie groups fulfilling $\sigma^2 = \sigma$, such that $(H_\sigma)_0 \subseteq H \subseteq H_\sigma$ holds. Here H_σ denotes the set of fixed points of σ and $(H_\sigma)_0$ denotes the connected component of H_σ containing the identity $e \in G$.

Inspired by the terminology used in [15, Def. 23.13], we refer to the triple (G, H, σ) as symmetric pair, as well, where (G, H) is a symmetric pair with respect to the involutive automorphism $\sigma : G \to G$. These symmetric pairs lead to reductive homogeneous spaces which are called symmetric homogeneous spaces if a certain "canonical" reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is chosen, see e.g. [10, Sec. 14]. This decomposition is given by

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid T_e\sigma X = X\} \subseteq \mathfrak{g} \quad \text{and} \quad \mathfrak{m} = \{X \in \mathfrak{g} \mid T_e\sigma X = -X\} \subseteq \mathfrak{g}. \tag{2.35}$$

One can show that the decomposition from (2.35) turns G/H into a reductive homogeneous space and fulfills the inclusion $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, see e.g. [10, Sec. 14] and also [15, Prop. 23.33]. Note that the definition in [10, Sec. 14] does not require an invariant pseudo-Riemannian metric on G/H. Next we define symmetric homogeneous spaces and canonical reductive decompositions following [10, Sec. 14].

Definition 2.16 Let (G, H, σ) be a symmetric pair. Then the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ from (2.35) is called canonical reductive decomposition. Moreover, the reductive homogeneous space G/H with the canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called symmetric homogeneous space.

2.4.1 Invariant covariant derivatives

We discuss briefly invariant covariant derivatives on the reductive homogeneous space G/H corresponding to the well-known invariant affine connections from [10, Thm. 8.1]. In this context, we refer to [10, 12] for more details. We define invariant covariant derivatives and relate them to certain bilinear maps by adapting and copying some parts of [12, Sec. 4.1].

Definition 2.17 A covariant derivative $\nabla\colon \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ on G/H is called G-invariant, or invariant for short, if

$$\nabla_X Y = (\tau_{g^{-1}})_*\big(\nabla_{(\tau_g)_* X}(\tau_g)_* Y\big) \tag{2.36}$$

holds for all $g \in G$ and $X, Y \in \Gamma^\infty\big(T(G/H)\big)$. Here $(\tau_g)_* X$ denotes the push-forward of X by $\tau_g\colon G/H \to G/H$, i.e. $(\tau_g)_* X = T\tau_g \circ X \circ \tau_{g^{-1}}$.

Definition 2.18 A bilinear map $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is called $\mathrm{Ad}(H)$-invariant if

$$\mathrm{Ad}_h\big(\alpha(X, Y)\big) = \alpha\big(\mathrm{Ad}_h(X), \mathrm{Ad}_h(Y)\big) \tag{2.37}$$

holds for all $X, Y \in \mathfrak{m}$ and $h \in H$.

Let $X \in \mathfrak{g}$ and let $X_{G/H} \in \Gamma^\infty\big(T(G/H)\big)$ denote the fundamental vector field associated with the action $\tau\colon G \times G/H \to G/H$, i.e

$$X_{G/H}(p) = \tfrac{\mathrm{d}}{\mathrm{d}t}\tau_{\exp(tX)}(p)\big|_{t=0}, \quad p \in G/H. \tag{2.38}$$

We denote by $\nabla^\alpha \colon \Gamma^\infty\big(T(G/H)\big) \times \Gamma^\infty\big(T(G/H)\big) \to \Gamma^\infty\big(T(G/H)\big)$ the unique covariant derivative, see [12, Def. 4.16], corresponding to (or associated with) the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by requiring

$$\nabla^\alpha_{X_{G/H}} Y_{G/H}\big|_{\mathrm{pr}(e)} = T_e\mathrm{pr}\big(-[X, Y]_{\mathfrak{m}} + \alpha(X, Y)\big), \quad X, Y \in \mathfrak{m}. \tag{2.39}$$

A characterization of parallel vector fields along curves on G/H with respect to ∇^α is given in next proposition which is a reformulation of [12, Cor. 4.27].

Proposition 2.19 *Let $\gamma \colon I \to G/H$ be a curve and let $g \colon I \to G$ be a horizontal lift of γ with respect to the principal connection from Proposition 2.10 defined by the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Moreover, let $\widehat{Z} \colon I \to T(G/H)$ be a vector field along γ with horizontal lift $\overline{Z} \colon I \ni t \mapsto \big(T_{g(t)}\mathrm{pr}|_{\mathrm{Hor}(G)_{g(t)}}\big)^{-1}\widehat{Z}(t) \in \mathrm{Hor}(G)$ along $g \colon I \to G$. Define the curves $x, z \colon I \to \mathfrak{m}$ by*

$$x(t) = (T_e\ell_{g(t)})^{-1}\dot{g}(t) \quad \textit{and} \quad z(t) = (T_e\ell_{g(t)})^{-1}\overline{Z}(t) \tag{2.40}$$

for $t \in I$. Then $\widehat{Z} \colon I \to T(G/H)$ is parallel along $\gamma \colon I \to G/H$ with respect to ∇^α defined by the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ iff the ODE

$$\dot{z}(t) = -\alpha\big(x(t), z(t)\big) \tag{2.41}$$

is satisfied for all $t \in I$.

The next Proposition which is copied from [12, Prop. 4.22] characterizes metric invariant covariant derivatives.

Proposition 2.20 *Let $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map defining the invariant covariant derivative ∇^α on G/H. Then ∇^α is metric with respect to the invariant pseudo-Riemannian metric on G/H defined by the $\mathrm{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle \colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ iff for each $X \in \mathfrak{m}$ the linear map*

$$\alpha(X, \cdot) \colon \mathfrak{m} \to \mathfrak{m}, \quad Y \mapsto \alpha(X, Y) \tag{2.42}$$

is skew-adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e.

$$\big\langle \alpha(X, Y), Z \big\rangle = -\big\langle Y, \alpha(X, Z) \big\rangle \tag{2.43}$$

holds for all $X, Y, Z \in \mathfrak{m}$.

Following [12, Sec. 4.6], we introduce the canonical invariant covariant derivatives. They correspond to the canonical affine connections from [10, Sec. 10].

Definition 2.21 1. The canonical invariant covariant derivative of first kind ∇^{can1} corresponds to the $\mathrm{Ad}(H)$-invariant bilinear map $\mathfrak{m} \times \mathfrak{m} \ni (X, Y) \mapsto \frac{1}{2}[X, Y]_{\mathfrak{m}} \in \mathfrak{m}$.

2. The canonical invariant covariant derivative of second kind $\nabla^{\mathrm{can}2}$ corresponds to the $\mathrm{Ad}(H)$-invariant bilinear map $\mathfrak{m} \times \mathfrak{m} \ni (X, Y) \mapsto 0 \in \mathfrak{m}$.

The canonical invariant covariant derivatives correspond to the Levi-Civita covariant derivatives on certain pseudo-Riemannian homogeneous spaces. Following [12, Re. 4.36], we state the next remark.

Remark 2.22 Assume that G/H is a naturally reductive homogeneous space. Then the Levi-Civita covariant derivative coincides with the canonical covariant derivative of first kind, i.e $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can}1}$. This has already been proven in [10, Thm. 13.1 and Eq. (13.2)].

Concerning the canonical covariant derivatives on symmetric homogeneous spaces, we state the next remark following [10, Thm. 15.1], see also [12, Sec. 4.6].

Remark 2.23 Let (G, H, σ) be a symmetric pair and let G/H be the corresponding symmetric homogeneous space. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ denote the canonical reductive decomposition. Then $\frac{1}{2}[X, Y]_{\mathfrak{m}} = 0$ holds for all $X, Y \in \mathfrak{m}$ due to $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. Hence $\nabla^{\mathrm{can}1} = \nabla^{\mathrm{can}2}$ is fulfilled. Moreover, if G/H is a pseudo-Riemannian symmetric space, i.e. G/H is equipped with an invariant pseudo-Riemannian metric corresponding to an $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{m}$, then $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can}1} = \nabla^{\mathrm{can}2}$ holds.

3 Intrinsic rolling

In this section, a notion of rolling intrinsically a manifold M over another manifold $\widehat{M}$ of equal dimension $\dim(M) = n = \dim(\widehat{M})$ is recalled from the literature and slightly generalized. As preparation to define the configuration space, we state the following lemma which can be regarded as a slight generalization of the definition of the configuration space in [6, Sec. 3.1]. In particular, the definition of the map Ψ in Lemma 3.1, Claim 2., below, is very similar to [6, Eq. (4)].

Lemma 3.1 *Let $E \to M$ and $\widehat{E} \to \widehat{M}$ be two vector bundles both having typical fiber V and let $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ be a closed subgroup. Assume that the frame bundles of E and $\widehat{E}$ admit both a $\mathrm{G}(V)$-reduction along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$ which we denote by $\mathrm{G}(V, E) \to M$ and $\mathrm{G}(V, \widehat{E}) \to \widehat{M}$, respectively. Let*

$$Q = \big(\mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})\big)/\mathrm{G}(V) \tag{3.1}$$

be defined as the quotient of $\mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})$ by the diagonal action of $\mathrm{G}(V)$, where the action on each component is given by the $\mathrm{G}(V)$-principal action. Moreover, define

$$\pi : Q \to M \times \widehat{M}, \quad [f, \widehat{f}] \mapsto \big(\mathrm{pr}_{\mathrm{G}(V,E)}(f), \mathrm{pr}_{\mathrm{G}(V,\widehat{E})}(\widehat{f})\big), \tag{3.2}$$

where $[f, \widehat{f}] \in Q$ denotes the equivalence class defined by $(f, \widehat{f}) \in \mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})$. Then the following assertions are fulfilled:

1. *$\pi : Q = \big(\mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})\big)/\mathrm{G}(V) \to M \times \widehat{M}$ is a $\mathrm{G}(V)$-fiber bundle over $M \times \widehat{M}$.*

2. *Let* $(x, \widehat{x}) \in M \times \widehat{M}$ *and define*

$$\widetilde{Q}_{(x,\widehat{x})} = \{\widetilde{q} \colon E_x \to \widehat{E}_{\widehat{x}} \mid \widehat{f}^{-1} \circ \widetilde{q} \circ f \in \mathrm{G}(V) \textit{ for all } (f, \widehat{f}) \in \mathrm{G}(V, E)_x \times \mathrm{G}(V, \widehat{E}_{\widehat{x}})\}. \tag{3.3}$$

Then the map

$$\Psi \colon Q = \big(\big(\mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})\big)/\mathrm{G}(V)\big)_{(x,\widehat{x})} \ni [f, \widehat{f}] \mapsto \widehat{f} \circ f^{-1} \in \widetilde{Q}_{(x,\widehat{x})} \tag{3.4}$$

is bijective.

Proof The action

$$\begin{aligned}\big(\mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})\big) \times \mathrm{G}(V) \to \mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E}),\\ ((f, \widehat{f}), A) \mapsto (f \triangleleft A, \widehat{f} \triangleleft A)\end{aligned}$$

is free and proper since the action on each component is free and proper. Thus $Q = \big(\mathrm{G}(V, E) \times \mathrm{G}(V, \widehat{E})\big)/\mathrm{G}(V)$ is a smooth manifold. Moreover, $(f, \widehat{f}) \sim (f', \widehat{f}')$ holds iff there is an $A \in \mathrm{G}(V)$ such that $(f', \widehat{f}') = (f \triangleleft A, \widehat{f} \triangleleft A)$ is fulfilled. Let $(x, \widehat{x}) \in M \times \widehat{M}$ and let $U \subseteq M$ and $\widehat{U} \subseteq \widehat{M}$ be open neighbourhoods of x and $\widehat{x}$, respectively, such that

$$\varphi \colon \mathrm{G}(V, E)\big|_U \to U \times \mathrm{G}(V) \quad \text{and} \quad \widehat{\varphi} \colon \mathrm{G}(V, \widehat{E})\big|_{\widehat{U}} \to \widehat{U} \times \mathrm{G}(V)$$

are local trivializations of $\mathrm{G}(V, E) \to M$ and $\mathrm{G}(V, \widehat{E}) \to \widehat{M}$ as $\mathrm{G}(V)$-principal fiber bundles, respectively. Locally, one obtains for the principal action for $A \in \mathrm{G}(V)$

$$\varphi(f \triangleleft A) = (\mathrm{pr}_1(\varphi(f)), \mathrm{pr}_2(\varphi(f)) \circ A), \quad \widehat{\varphi}(\widehat{f} \triangleleft A) = (\mathrm{pr}_1(\widehat{\varphi}(\widehat{f})), \mathrm{pr}_2(\widehat{\varphi}(\widehat{f})) \circ A), \tag{3.5}$$

see e.g. [16, Sec. 18, p. 211]. We now define the local trivialization $\phi \colon Q\big|_{U \times \widehat{U}} \to U \times \widehat{U} \times \mathrm{G}(V)$ of $\pi : Q \to M \times \widehat{M}$ by

$$\phi\big([f, \widehat{f}]\big) = \big(\mathrm{pr}_1(\varphi(f)), \mathrm{pr}_1(\widehat{\varphi}(\widehat{f})), \big(\mathrm{pr}_2(\varphi(f))\big) \circ \big(\mathrm{pr}_2(\widehat{\varphi}(\widehat{f}))\big)^{-1}\big), \quad [f, \widehat{f}] \in Q\big|_{U \times \widehat{U}}.$$

Using (3.5), one shows that ϕ is well-defined. Moreover, it is straightforward to verify that ϕ is a local trivialization of $\pi : Q \to M \times \widehat{M}$. This shows Claim 1..

It remains to prove Claim 2.. Let $(x, \widehat{x}) \in M \times \widehat{M}$ and $f \in \mathrm{G}(V, E)_x$ as well as $\widehat{f} \in \mathrm{G}(V, \widehat{E})_{\widehat{x}}$. In particular, $f \colon V \to E_x$ and $\widehat{f} \colon V \to \widehat{E}_{\widehat{x}}$ are invertible linear maps. Hence $\widehat{f} \circ f^{-1} \colon E_x \to \widehat{E}_{\widehat{x}}$ is a linear isomorphism.

Moreover, $\Psi\big([f, \widehat{f}]\big)$ is independent of the representative of $[f, \widehat{f}] \in Q$ due to

$$\Psi\big([f \circ A, \widehat{f} \circ A]\big) = (\widehat{f} \circ A) \circ (f \circ A)^{-1} = \widehat{f} \circ f^{-1} = \Psi\big([f, \widehat{f}]\big)$$

for all $A \in \mathrm{G}(V)$.

Next we show that $\Psi\big([f, \widehat{f}]\big) \in \widetilde{Q}_{(x,\widehat{x})}$ holds for all $[f, \widehat{f}] \in Q_{(x,\widehat{x})}$. Let $[f, \widehat{f}] \in Q_{(x,\widehat{x})}$. By the fiber-wise transitivity of the principal $\mathrm{G}(V)$-actions on $\mathrm{G}(V, E)$ and

$\mathrm{G}(V, \widehat{E})$, respectively, we obtain for $A, B \in \mathrm{G}(V)$

$$(\widehat{f} \circ B)^{-1} \circ \Psi\big([f, \widehat{f}]\big) \circ (f \circ A) = B^{-1} \circ \widehat{f}^{-1} \circ \big(\widehat{f} \circ f^{-1}\big) \circ f \circ A = B^{-1} \circ A \in \mathrm{G}(V)$$

showing $\Psi\big([f, \widehat{f}]\big) \in \widetilde{Q}_{(x,\widehat{x})}$ for all $[f, \widehat{f}] \in Q_{(x,\widehat{x})}$, i.e. $\Psi\colon Q_{(x,\widehat{x})} \to \widetilde{Q}_{(x,\widehat{x})}$ is well-defined. Moreover, Ψ is injective. Let $[f, \widehat{f}], [f', \widehat{f}'] \in Q_{(x,\widehat{x})}$ with $\Psi\big([f, \widehat{f}]\big) = \Psi\big([f', \widehat{f}']\big)$. Since the $\mathrm{G}(V)$-principal actions on $\mathrm{G}(V, E)$ and $\mathrm{G}(V, \widehat{E})$ are free and fiber-wise transitive, we can write $f' = f \circ A$ and $\widehat{f}' = \widehat{f} \circ B$ with some uniquely determined $A, B \in \mathrm{G}(V)$. By this notation, we obtain

$$\widehat{f} \circ f^{-1} = \Psi\big([f, \widehat{f}]\big) = \Psi\big([f \circ A, \widehat{f} \circ B]\big) = (\widehat{f} \circ B) \circ (f \circ A)^{-1} = \widehat{f} \circ (B \circ A^{-1}) \circ f^{-1},$$

implying $B \circ A^{-1} = \mathrm{id}_V \iff A = B$ because $f\colon V \to E_x$ and $\widehat{f}\colon V \to \widehat{E}_{\widehat{x}}$ are both linear isomorphisms. Thus $[f', \widehat{f}'] = [f \circ A, \widehat{f} \circ A] = [f, \widehat{f}]$ is shown. It remains to show that Ψ is surjective. To this end, let $\widetilde{q} \in \widetilde{Q}_{(x,\widehat{x})}$ and chose some $f \in \mathrm{G}(V, E)_{(x,\widehat{x})}$ and $\widehat{f} \in \mathrm{G}(V, \widehat{E})_{(x,\widehat{x})}$. Then $\widehat{f}^{-1} \circ \widetilde{q} \circ f \in \mathrm{G}(V)$ holds. We now compute

$$\Psi\big([f, \widehat{f} \circ (\widehat{f}^{-1} \circ \widetilde{q} \circ f)]\big) = \big(\widehat{f} \circ (\widehat{f}^{-1} \circ \widetilde{q} \circ f)\big) \circ f^{-1} = (\widehat{f} \circ \widehat{f}^{-1}) \circ \widetilde{q} \circ (f \circ f^{-1}) = \widetilde{q},$$

i.e. Ψ is surjective. This yields the desired result. □

After this preparation, we consider intrinsic rollings. Let M and $\widehat{M}$ be two manifolds with $\dim(M) = n = \dim(\widehat{M})$. Moreover, let $\mathrm{G}(\mathbb{R}^n) \subseteq \mathrm{GL}(\mathbb{R}^n)$ be a closed subgroup and assume that the frame bundles $\mathrm{GL}(\mathbb{R}^n, TM) \to M$ and $\mathrm{GL}(\mathbb{R}^n, T\widehat{M}) \to \widehat{M}$ admit both a $\mathrm{G}(\mathbb{R}^n)$-reduction along the canonical inclusion $\mathrm{G}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$. These reductions are denoted by

$$\mathrm{G}(\mathbb{R}^n, TM) \to \mathrm{GL}(\mathbb{R}^n, TM) \quad \text{and} \quad \mathrm{G}(\mathbb{R}^n, T\widehat{M}) \to \mathrm{GL}(\mathbb{R}^n, T\widehat{M}), \tag{3.6}$$

respectively. In this section, we denote by

$$\pi\colon Q = \big(\mathrm{G}(\mathbb{R}^n, TM) \times \mathrm{G}(\mathbb{R}^n, T\widehat{M})\big)/\mathrm{G}(\mathbb{R}^n) \to M \times \widehat{M} \tag{3.7}$$

the $\mathrm{G}(\mathbb{R}^n)$-fiber bundle over $M \times \widehat{M}$ obtained by applying Lemma 3.1 to the frame bundles from (3.6).

We now define a notion of rolling of M over $\widehat{M}$ intrinsically, where M and $\widehat{M}$ are both equipped with a covariant derivative ∇ and $\widehat{\nabla}$, respectively.

Definition 3.2 An intrinsic ($\mathrm{G}(\mathbb{R}^n)$-reduced) rolling of (M, ∇) over $(\widehat{M}, \widehat{\nabla})$ is a curve

$$q\colon I \to Q = \big(\mathrm{G}(\mathbb{R}^n, TM) \times \mathrm{G}(\mathbb{R}^n, T\widehat{M})\big)/\mathrm{G}(\mathbb{R}^n) \tag{3.8}$$

with projection $(x, \widehat{x}) = \pi \circ q\colon I \to M \times \widehat{M}$ such that the following conditions are fulfilled:

1. No slip condition: $\dot{\widehat{x}}(t) = q(t)\dot{x}(t)$ for all $t \in I$.
2. No twist condition: $Z: I \to TM$ is a parallel vector field along x iff $\widehat{Z}: I \to T\widehat{M}$ defined by $\widehat{Z}(t) = q(t)Z(t)$ for $t \in I$ is parallel along $\widehat{x}$.

Here Lemma 3.1, Claim 2. is used to identify $q(t)$ with the linear isomorphism $q(t): T_{x(t)}M \to T_{\widehat{x}(t)}\widehat{M}$ which is denoted by $q(t)$, as well. We call the curve $x: I \to M$ rolling curve. The curve $\widehat{x}: I \to \widehat{M}$ is called development curve. The curve $q: I \to Q$ is often called rolling for short.

The next remark yields an other perspective on the intrinsic rollings from Definition 3.2.

Remark 3.3 Let $q: I \to Q$ be a ($\mathrm{G}(\mathbb{R}^n)$-reduced) intrinsic rolling of M over $\widehat{M}$ in the sense of Definition 3.2 and write $(x, \widehat{x}) = \pi \circ q: I \to M \times \widehat{M}$. Then we can view this rolling as a triple $(x(t), \widehat{x}(t), A(t))$, where $A(t) = q(t): T_{x(t)}M \to T_{\widehat{x}(t)}\widehat{M}$ is the linear isomorphism defined by $q(t)$ as in Lemma 3.1, Claim 2.. This point of view allows for relating a rolling $q: I \to Q$ from Definition 3.2 to [11, Def. 1], where a rolling is defined as a triple $(x(t), \widehat{x}(t), A(t))$ satisfying certain properties.

Definition 3.2 of an intrinsic rolling of M over $\widehat{M}$ generalizes several notions of intrinsic rolling from the literature.

Remark 3.4 Assume that M and $\widehat{M}$ are both orientible and both equipped with a Riemannian metric. Let $\mathrm{SO}(\mathbb{R}^n, TM)$ and $\mathrm{SO}(\mathbb{R}^n, T\widehat{M})$ be the corresponding reductions of their frame bundles. Moreover, let M and $\widehat{M}$ be endowed with the Levi-Civita covariant derivatives ∇^{LC} and $\widehat{\nabla^{\mathrm{LC}}}$ corresponding to the Riemannian metrics on M and $\widehat{M}$, respectively. Then Definition 3.2 specializes to [6, Def. 3]. Here the no twist condition is rewritten as in [11, Prop. 2]. More generally, if M and $\widehat{M}$ are oriented and equipped with a pseudo-Riemannian metric, Definition 3.2 specializes to [7, Def. 4]. If M and $\widehat{M}$ are both equipped with an arbitrary covariant derivative ∇ and $\widehat{\nabla}$, respectively, Definition 3.2 yields the definition proposed in [9, p. 35] and [8, Sec. 7] by setting $\mathrm{G}(\mathbb{R}^n, TM) = \mathrm{GL}(\mathbb{R}^n, TM)$ and $\mathrm{G}(\mathbb{R}^n, T\widehat{M}) = \mathrm{GL}(\mathbb{R}^n, T\widehat{M})$.

Studying properties of rollings in the sense of Definition 3.2 for general manifolds is out of the scope of this text. However, in Sect. 5 below, we discuss intrinsic rollings in the context of reductive homogeneous spaces in detail.

4 Frame bundles of associated vector bundles

In this section, we identify (certain reductions of) the frame bundle of a reductive homogeneous space G/H with certain principal fiber bundles obtained as associated bundles of the H-principal fiber bundle $\mathrm{pr}: G \to G/H$. We point out that the results of this section might be well-known since the statement of Corollary 4.11 can be found as an exercise in the German book [20, Ex. 2.7]. However, we were not able to find a reference including a proof. Hence we provide one in this section in order to keep this text as self-contained as possible. Here we first start with a more general situation that is applied to reductive homogeneous spaces later. We first determine (certain reductions of) the frame bundles of vector bundles given as associated bundles of some principal fiber bundle.

4.1 Frame bundles of associated vector bundles

Let $P \to M$ be an H-principal fiber bundle. We describe (reductions of) the frame bundle of a vector bundle associated to P in terms of another fiber bundle associated to P. To this end, we state the following lemma as preparation.

Lemma 4.1 *Let $P \to M$ be an H-principal fiber bundle and let $\rho\colon H \to \mathrm{GL}(V)$ be a smooth representation of H on a finite dimensional $\mathbb{R}$-vector space V. Moreover, let $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ be a closed subgroup such that $\rho_h \in \mathrm{G}(V)$ is fulfilled for all $h \in H$. Then the following assertions are fulfilled:*

1. *The Lie group H acts on $\mathrm{G}(V)$ via*

$$H \times \mathrm{G}(V) \to \mathrm{G}(V), \quad (h, A) \mapsto \rho_h \circ A \tag{4.1}$$

smoothly from the left.

2. *The map*

$$\triangleleft\colon (P \times_H \mathrm{G}(V)) \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V) \quad ([g, A], B) \to [g, A \circ B], \tag{4.2}$$

denoted by the same symbol as the principal action $\triangleleft\colon P \times H \to P$, yields a well-defined, smooth, free and proper $\mathrm{G}(V)$-right action on the associated bundle

$$\pi : P \times_H \mathrm{G}(V) \to M \tag{4.3}$$

turning

$$\widetilde{\pi} : P \times_H \mathrm{G}(V) \to \big(P \times_H \mathrm{G}(V)\big)/\mathrm{G}(V) \tag{4.4}$$

into a $\mathrm{G}(V)$-principal fiber bundle, where $\widetilde{\pi}$ denotes the canonical projection.

3. *The map*

$$\phi\colon (P \times_H \mathrm{G}(V))/\mathrm{G}(V) \ni \widetilde{\pi}([p, S]) \mapsto \mathrm{pr}(p) \in M \tag{4.5}$$

is a diffeomorphism such that $\phi \circ \widetilde{\pi} = \pi$ holds. Moreover,

$$\mathrm{id}_{G \times_H \mathrm{G}(\mathfrak{m})} : G \times_H \mathrm{G}(\mathfrak{m}) \to G \times_H \mathrm{G}(\mathfrak{m}) \tag{4.6}$$

is an isomorphism of $\mathrm{G}(V)$-principal fiber bundles covering ϕ.

Proof Claim 1. is obvious.

We now show Claim 2.. The $\mathrm{G}(V)$-right action $\triangleleft$ on $P \times_H \mathrm{G}(V)$ is well-defined due to

$$[p \triangleleft h, \rho_{h^{-1}} \circ A] \triangleleft B = [p \triangleleft h, \rho_{h^{-1}} \circ A \circ B] = [g, A \circ B] = [g, A] \triangleleft B$$

for all $p \in P$, $h \in H$ and $A, B \in \mathrm{G}(V)$. Next we show that $\triangleleft$ is smooth. To this end, we consider the diagram

$$
\begin{array}{ccc}
(P \times \mathrm{G}(V)) \times \mathrm{G}(V) & & \\
\downarrow \scriptstyle{\overline{\pi} \times \mathrm{id}_{\mathrm{G}(V)}} & \searrow^{(\overline{\pi} \times \mathrm{id}_{\mathrm{G}(V)}) \circ \widetilde{\triangleleft}} & \\
(P \times_H \mathrm{G}(V)) \times \mathrm{G}(V) & \xrightarrow{\quad \triangleleft \quad} & P \times_H \mathrm{G}(V)
\end{array}
\tag{4.7}
$$

where $\overline{\pi} \colon P \times \mathrm{G}(V) \to (P \times \mathrm{G}(V))/H = P \times_H \mathrm{G}(V)$ denotes the canonical projection and $\widetilde{\triangleleft}$ is given by

$$
\widetilde{\triangleleft} \colon (P \times \mathrm{G}(V)) \times \mathrm{G}(V) \to P \times \mathrm{G}(V), \quad ((p, A), B) \mapsto (p, A \circ B)
$$

which is clearly a smooth and free $\mathrm{G}(V)$-right action on $P \times \mathrm{G}(V)$. Moreover, the action $\widetilde{\triangleleft}$ is proper since the $\mathrm{G}(V)$-action on $\mathrm{G}(V)$ by right translations is proper, see e.g. [21, Prop. 9.29].

The map $\overline{\pi} \times \mathrm{id}_{\mathrm{G}(V)}$ is a surjective submersion and $(\overline{\pi} \times \mathrm{id}_{\mathrm{G}(V)}) \circ \widetilde{\triangleleft}$ is smooth as the composition of smooth maps. Thus the action $\triangleleft$ is smooth since the diagram (4.7) commutes.

Next let $[p, A] \in P \times_H \mathrm{G}(V)$ and $B \in \mathrm{G}(V)$. Then

$$
[p, A] \triangleleft B = [p, A \circ B] = [p, A] \implies B = \mathrm{id}_V
$$

holds proving that $\triangleleft$ is free.

We now show that $\triangleleft$ is proper. To this end, we use the characterization of a proper Lie group action in terms of sequences, see e.g. [16, Sec. 6.20].

Let $([p_i, A_i])_{i \in \mathbb{N}}$ be a convergent sequence in $P \times_H \mathrm{G}(V)$ with limit $[p, A] \in P \times_H \mathrm{G}(V)$. Next let $(B_i)_{i \in \mathbb{N}}$ be a sequence in $\mathrm{G}(V)$ such that the sequence defined by $[p_i, A_i] \triangleleft B_i = [p_i, A_i \circ B_i]$ converges. Then the action $\triangleleft$ is proper iff $(B_i)_{i \in \mathbb{N}}$ has a convergent subsequence. Let $s \colon U \to P \times \mathrm{G}(V)$ be a local section of the H-principal fiber bundle $\overline{\pi} \colon P \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V)$ defined on some open $U \subseteq P \times_H \mathrm{G}(V)$ such that $[p, A] \in U$ holds. Then $[p_i, A_i] \in U$ is fulfilled for all $i \geq N$ with $N \in \mathbb{N}$ sufficiently large. We define the sequence $(\widehat{p}_i, \widehat{A}_i)_{i \in \mathbb{N}}$ in $P \times \mathrm{G}(V)$ by setting

$$
(\widehat{p}_i, \widehat{A}_i) = s([p_i, A_i]), \quad i \geq N
$$

and choosing $(\widehat{p}_i, \widehat{A}_i) \in \overline{\pi}^{-1}([p_i, A_i])$ for $i < N$ arbitrarily. By construction, we have

$$
[p_i, A_i] = (\overline{\pi} \circ s)([p_i, A_i]) = \overline{\pi}(\widehat{p}_i, \widehat{A}_i) = [\widehat{p}_i, \widehat{A}_i] \tag{4.8}
$$

for all $i \in \mathbb{N}$. Moreover, the sequence $(\widehat{p}_i, \widehat{A}_i)_{i \in \mathbb{N}}$ converges to

$$
(\widehat{p}, \widehat{A}) = \lim_{i \to \infty} s([p_i, A_i]) = s([p, A]) \in P \times \mathrm{G}(V)
$$

by the continuity of the local section $s\colon U \to P \times \mathrm{G}(V)$ and the convergence of $[p_i, A_i]$. Moreover, let $(B_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{G}(V)$ such that the sequence defined by

$$[p_i, A_i] \triangleleft B_i = [p_i, A_i \circ B_i], \quad i \in \mathbb{N}$$

is convergent in $P \times_H \mathrm{G}(V)$. We denote its limit by $[p, C] = \lim_{i\to\infty}[p_i, A_i \circ B_i] \in P \times_H \mathrm{G}(V)$. Clearly,

$$[p_i, A_i] \triangleleft B_i = [\widehat{p}_i, \widehat{A}_i] \triangleleft B_i = [\widehat{p}_i, \widehat{A}_i \circ B_i], \quad i \in \mathbb{N}$$

holds by (4.8). Next we choose a local section $s_2\colon U_2 \to P \times \mathrm{G}(V)$ of $\overline{\pi}\colon P \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V)$ such that $[p, C] \in U_2 \subseteq P \times_H \mathrm{G}(V)$ is fulfilled. Then there exists an $N_2 \in \mathbb{N}$ with $[p_i, A_i \circ B_i] \in U_2$ for all $i \geq N_2$. We define the sequence

$$(\widetilde{p}_i, \widetilde{C}_i) = s_2([\widehat{p}_i, \widehat{A}_i \circ B_i]), \quad i \geq N_2 \tag{4.9}$$

and select $(\widetilde{p}_i, \widetilde{C}_i) \in \overline{\pi}^{-1}([p_i, \widehat{A}_i \circ B_i])$ for $i < N_2$ arbitrarily. Recall from [16, Sec. 18, p. 211] that the map

$$\sigma\colon P \oplus P \to H, \quad (p, p') \mapsto \sigma(p, p')$$

is smooth, where $\sigma(p, p') \in H$ is defined by $p \triangleleft \sigma(p, p') = p'$ for $(p, p') \in P \oplus P$. Next we define the map

$$\Theta\colon (P \oplus P) \times \mathrm{G}(V) \to P \times \mathrm{G}(V), \quad \big((p, p'), A\big) \mapsto \big(p, \rho_{\sigma(p,p')^{-1}} \circ A\big)$$

which is a smooth map as the composition of smooth maps. The definition of $(\widetilde{p}_i, \widetilde{C}_i)_{i\in\mathbb{N}}$ in (4.9) implies

$$\begin{aligned}(\widetilde{p}_i, \widetilde{C}_i) &= s_2([\widehat{p}_i, \widehat{A}_i \circ B_i]) \\ &= \big(\widehat{p}_i \triangleleft \sigma(\widehat{p}_i, \widetilde{p}_i), \rho_{(\sigma(\widehat{p}_i,\widetilde{p}_i))^{-1}} \circ \widehat{A}_i \circ B_i\big) \\ &= \big(\widetilde{p}_i, \rho_{(\sigma(\widehat{p}_i,\widetilde{p}_i))^{-1}} \circ \widehat{A}_i \circ B_i\big)\end{aligned} \tag{4.10}$$

since $[\widetilde{p}_i, \widetilde{C}_i] = [\widehat{p}_i, \widehat{A}_i \circ B_i]$ holds iff there exists a $h_i \in H$ with $\widetilde{p}_i = \widehat{p}_i \triangleleft h_i$ and $\widetilde{C}_i = \rho_{h_i^{-1}} \circ \widehat{A}_i \circ B_i$. Next consider the sequence defined for $i \in \mathbb{N}$ by

$$\Theta\big((\widetilde{p}_i, \widehat{p}_i), \widehat{A}_i\big) = (\widehat{p}_i \triangleleft \sigma(\widehat{p}_i, \widetilde{p}_i), \rho_{(\sigma(\widehat{p}_i,\widetilde{p}_i))^{-1}} \circ \widehat{A}_i) = (\widetilde{p}_i, \rho_{(\sigma(\widehat{p}_i,\widetilde{p}_i))^{-1}} \circ \widehat{A}_i), \tag{4.11}$$

which converges by the continuity of Θ as well as the convergence of the sequences $(\widetilde{p}_i, \widetilde{C}_i)_{i\in\mathbb{N}}$ and $(\widehat{p}_i, \widehat{A}_i)_{i\in\mathbb{N}}$ in $P \times \mathrm{G}(V)$. By (4.11), we obtain

$$\begin{aligned}\Theta\big((\widetilde{p}_i, \widehat{p}_i), \widehat{A}_i\big)\widetilde{\triangleleft}B_i &= \big(\widetilde{p}_i, \rho_{(\sigma(\widehat{p}_i,\widetilde{p}_i))^{-1}} \circ \widehat{A}_i\big)\widetilde{\triangleleft}B_i \\ &= \big(\widetilde{p}_i, \rho_{(\sigma(\widehat{p}_i,\widetilde{p}_i))^{-1}} \circ \widehat{A}_i \circ B_i\big) \\ &= (\widetilde{p}_i, \widetilde{C}_i) \to (\widetilde{p}, \widetilde{C}), \quad i \to \infty,\end{aligned}$$

where we used (4.10) to obtain last equality. Since the action $\widetilde{\triangleleft}\colon (P \times \mathrm{G}(V)) \times \mathrm{G}(V) \to P \times \mathrm{G}(V)$ is proper and the sequence $\big(\Theta((\widetilde{p}_i, \widehat{p}_i), \widehat{A}_i)\big)_{i\in\mathbb{N}}$ defined in (4.11) is convergent in $P \times \mathrm{G}(V)$, the sequence $(B_i)_{i\in\mathbb{N}}$ has a convergent subsequence. Thus the right action $\triangleleft\colon (P \times_H \mathrm{G}(V)) \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V)$ is indeed proper. Therefore $P \times_H \mathrm{G}(V) \to (P \times_H \mathrm{G}(V))/\mathrm{G}(V)$ is a principal fiber bundle by [17, Re. 1.1.2].

It remains to prove Caim 3.. We first show that ϕ is a diffeomorphism. The equivalence classes $\widetilde{\pi}([p, A])) = \widetilde{\pi}([p, A \circ B]) \in \big(P \times_H \mathrm{G}(V)\big)/\mathrm{G}(V)$ represented by $[p, A], [p, A \circ B] \in P \times_H \mathrm{G}(V)$ are equal, where $p \in P$, $A, B \in \mathrm{G}(V)$. Thus we have

$$\phi\big(\widetilde{\pi}([p, A \circ B])\big) = \mathrm{pr}(p) = \phi\big(\widetilde{\pi}([p, A])\big)$$

showing that ϕ is well-defined. We now consider the diagrams

$$\begin{array}{ccc} P \times_H \mathrm{G}(V) & \xrightarrow{\mathrm{id}_{(P\times_H \mathrm{G}(V))}} & P \times_H \mathrm{G}(V) \\ \Big\downarrow{\scriptstyle \widetilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ \big(P \times_H \mathrm{G}(V)\big)/\mathrm{G}(V) & \xrightarrow{\phi} & M \end{array} \tag{4.12}$$

and

$$\begin{array}{ccc} P \times_H \mathrm{G}(V) & \xleftarrow{\mathrm{id}_{(P\times_H \mathrm{G}(V))}} & P \times_H \mathrm{G}(V) \\ \Big\downarrow{\scriptstyle \widetilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ \big(P \times_H \mathrm{G}(V)\big)/\mathrm{G}(V) & \xleftarrow{\phi^{-1}} & M \end{array}, \tag{4.13}$$

where ϕ^{-1} is given by

$$\phi^{-1}\colon M \ni \mathrm{pr}(p) \mapsto \widetilde{\pi}([p, \mathrm{id}_V]) \in \big(P \times_H \mathrm{G}(V)\big)/\mathrm{G}(V).$$

Clearly $\phi^{-1} \circ \phi = \mathrm{id}_M$ and $\phi \circ \phi^{-1} = \mathrm{id}_{(P\times_H \mathrm{G}(V))/\mathrm{G}(V)}$ holds showing that ϕ is bijective. In addition, ϕ and ϕ^{-1} are smooth since (4.12) and (4.13) commute and $\widetilde{\pi}$ as well as π are both surjective submersions. Hence the commutativity of (4.12) implies that $\mathrm{id}_{P\times_H \mathrm{G}(V)}$ is indeed an isomorphism of $\mathrm{G}(V)$-principal fiber bundles over ϕ as desired. □

Remark 4.2 By Lemma 4.1, Claim 3., we view $P \times_H \mathrm{G}(V) \to \big(P \times_H \mathrm{G}(V)\big)/\mathrm{G}(V) \cong M$ as an $\mathrm{G}(V)$-principal fiber bundle over M which is denoted by the same symbol as the associated bundle, i.e. from now on, we write

$$\pi\colon P \times_H \mathrm{G}(V) \to M, \tag{4.14}$$

if we view $P \times_H \mathrm{G}(V)$ as an $\mathrm{G}(V)$-principal fiber bundle over M.

Under certain conditions on the representation $\rho\colon H \to \mathrm{GL}(V)$, one can determine a reduction of the $\mathrm{GL}(V)$-principal fiber bundle $P \times_H \mathrm{GL}(V) \to M$ obtained by setting $\mathrm{G}(V) = \mathrm{GL}(V)$ in Lemma 4.1.

Corollary 4.3 *Let $P \to M$ be an H-principal fiber bundle and let $\rho\colon H \to \mathrm{GL}(V)$ be a representation of H on V. Moreover, let $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ be a closed subgroup such that $\rho_h \in \mathrm{G}(V)$ holds for all $h \in H$. Then*

$$\iota_{P\times_H \mathrm{G}(V)}\colon P \times_H \mathrm{G}(V) \to P \times_H \mathrm{GL}(V), \quad [p, A] \mapsto [p, A] \tag{4.15}$$

is a reduction of the $\mathrm{GL}(V)$-principal fiber bundle $\pi_{P\times \mathrm{GL}(V)}\colon P \times_H \mathrm{GL}(V) \to M$ along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$.

Proof Let $\overline{\iota_{P\times_H \mathrm{G}(V)}}\colon P \times \mathrm{G}(V) \to P \times \mathrm{GL}(V)$ denote the canonical inclusion. Consider the diagram

$$\begin{array}{ccc} P \times \mathrm{G}(V) & \xrightarrow{\overline{\iota_{P\times_H \mathrm{G}(V)}}} & P \times \mathrm{GL}(V) \\ {\scriptstyle \overline{\pi}_{P\times \mathrm{G}(V)}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\pi}_{P\times \mathrm{GL}(V)}} \\ P \times_H \mathrm{G}(V) & \xrightarrow{\iota_{P\times_H \mathrm{G}(V)}} & P \times_H \mathrm{GL}(V) \end{array} \tag{4.16}$$

which clearly commutes. Since the map $\overline{\pi}_{P\times \mathrm{G}(V)}$ is a surjective submersion and $\overline{\pi}_{\mathrm{GL}(V)} \circ \overline{\iota_{P\times \mathrm{G}(V)}}$ is smooth as the composition of smooth maps, the map $\iota_{P\times_H \mathrm{G}(V)}$ is smooth, as well, because (4.16) commutes. Clearly, the map $\iota_{P\times_H \mathrm{G}(V)}$ covers the map $\mathrm{id}_M\colon M \to M$. We now compute for $[p, A] \in P \times_H \mathrm{G}(V)$ and $B \in \mathrm{G}(V)$

$$\iota_{P\times_H \mathrm{G}(V)}([p, A] \triangleleft B) = [p, A] \triangleleft B = \big(\iota_{P\times_H \mathrm{G}(V)}([p, A])\big) \triangleleft B$$

showing that $\iota_{P\times_H \mathrm{G}(V)}$ is a morphism of principal fiber bundles along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$ covering id_M, i.e. $\iota_{P\times_H \mathrm{G}(V)}$ is a reduction of $P \times_H \mathrm{GL}(V) \to M$. ◻

The next proposition shows that $\pi : P \times_H \mathrm{GL}(V) \to M$ can be identified with the frame bundle of the associated vector bundle $P \times_H V \to M$, where H acts on V via the representation viewed as the left action

$$\rho\colon H \times V \to V, \quad (h, v) \mapsto \rho_h(v). \tag{4.17}$$

Proposition 4.4 *Let $\mathrm{pr}\colon P \to M$ be an H-principal fiber bundle and let $\rho\colon H \to \mathrm{GL}(V)$ be a representation of H. The frame bundle of the associated vector bundle $P \times_H V \to M$ is isomorphic to $P \times_H \mathrm{GL}(V) \to M$ as $\mathrm{GL}(V)$-principal fiber bundle via the isomorphism*

$$\Psi\colon P \times_H \mathrm{GL}(V) \to \mathrm{GL}(V, P \times_H V), \quad [p, A] \mapsto \Psi([p, A]) \tag{4.18}$$

covering id_M*, where, for fixed* $[p, A] \in P \times \mathrm{GL}(V)$*, the linear isomorphism* $\Psi([p, A]) \colon \{\mathrm{pr}(p)\} \times V \cong V \to (P \times_H V)_{\mathrm{pr}(p)}$ *is given by*

$$\big(\Psi([p, A])\big)(v) = [p, Av] \tag{4.19}$$

for all $v \in V$*. Here we view* $\mathrm{GL}(V, P \times_H V)$ *as an open subset of the morphism bundle* $\mathrm{Hom}(M \times V, P \times_H V) \to M$ *as in [16, Sec. 18.11]. Moreover, we write* $\big(\Psi([p, A])\big)(v) = \big(\Psi([p, A])\big)(\mathrm{pr}(p), v)$ *for short, i.e. we suppress the first component* $\mathrm{pr}(p) \in M$ *of* $(\mathrm{pr}(p), v) \in M \times V$ *in the notation.*

Proof We start with showing that Ψ is well-defined. Let $h \in H$. Indeed, Ψ is independent of the chosen representative of $[p, A] \in P \times_H \mathrm{GL}(V)$ due to

$$\big(\Psi([p \triangleleft h, \rho_{h^{-1}} \circ A])\big)(v) = [p \triangleleft h, (\rho_{h^{-1}} \circ A)(v)] = [p, Av]$$

for all $v \in V$. Moreover, for fixed $[p, A] \in P \times_H \mathrm{GL}(V)$, the map

$$V \ni v \mapsto \Psi([p, A])(v) \in (P \times_H V)_{\mathrm{pr}(p)}$$

is clearly linear. In addition, this map is invertible and its inverse is given by

$$\big(\Psi([p, A])\big)^{-1} \colon (P \times_H V)_{\mathrm{pr}(p)} \ni ([p, v]) \mapsto \big(\Psi([p, A]\big)^{-1}([p, v]) = A^{-1}v \in V.$$

Indeed, $\big(\Psi([p, A])\big)^{-1}$ is well-defined. Let $h, h' \in H$. Then one has $p \triangleleft h' = \big(p \triangleleft (hh^{-1})\big) \triangleleft h' = (p \triangleleft h) \triangleleft (h^{-1}h')$. Thus we obtain

$$\begin{aligned}
\big(\Psi(p \triangleleft h, \rho_{h^{-1}} \circ A)\big)^{-1}([p \triangleleft h', \rho_{h'^{-1}}(v)]) &= (\rho_{(h^{-1}h')^{-1}} \circ A)^{-1}(\rho_{(h^{-1}h')^{-1}}(v)) \\
&= A^{-1}v \\
&= \big(\Psi([p, A])\big)^{-1}([p, v])
\end{aligned}$$

for all $v \in V$ showing that $\big(\Psi([p, A])\big)^{-1}$ is well-defined. Moreover, one has

$$\big(\Psi([p, A]) \circ \big(\Psi([p, A])\big)^{-1}\big)([p, v]) = \big(\Psi([p, A])\big)(A^{-1}v) = [p, AA^{-1}v] = [p, v]$$

as well as

$$\big((\Psi([p, A]))^{-1} \circ \Psi([p, A])\big)(v) = \big(\Psi([p, A])\big)^{-1}([p, Av]) = A^{-1}(Av) = v$$

showing that $\Psi([p, A]) \colon V \to (P \times_H V)_{\mathrm{pr}(p)}$ is a linear isomorphism for all $[p, A] \in P \times_H \mathrm{GL}(V)$. Thus $\Psi \colon P \times_H \mathrm{GL}(V) \to \mathrm{GL}(V, P \times_H V)$ is well-defined.

Next we show that Ψ is a morphism of principal fiber bundles over id_M. Clearly, $\mathrm{id}_M \circ \pi_{P \times_H \mathrm{GL}(V)} = \mathrm{pr}_{\mathrm{GL}(V, P \times_H V)}$ holds, i.e. Ψ covers id_M.

We now show that Ψ is smooth. To this end, let $P \times_H \operatorname{End}(V) \to M$ denote the vector bundle associated to the H-principal fiber bundle $\mathrm{pr}\colon P \to M$ with typical fiber $\operatorname{End}(V)$, where H acts on $\operatorname{End}(V)$ via

$$H \times \operatorname{End}(V) \ni (h, A) \mapsto \rho_h \circ A \in \operatorname{End}(V)$$

from the left. We now define the map

$$\widetilde{\Psi}\colon P \times_H \operatorname{End}(V) \to \operatorname{Hom}(M \times V, P \times_H V),$$
$$[p, A] \mapsto \widetilde{\Psi}([p, A]) = \big((x, v) \mapsto \big(\widetilde{\Psi}([p, A])\big)(x, v) = [p, Av]\big).$$

An argument analogously to the one at the beginning of this proof, showing that Ψ is well-defined, proves that the map $\widetilde{\Psi}$ is well-defined, i.e. $\widetilde{\Psi}$ is independent of the representative $(p, A) \in P \times \operatorname{End}(V)$ of $[p, A] \in P \times_H \operatorname{End}(V)$ and that $\widetilde{\Psi}$ takes values in $\operatorname{Hom}(M \times V, P \times_H V) \to M$. Next we show that $\widetilde{\Psi}$ is a smooth morphism of vector bundles. To this end, we prove that

$$\overline{\Psi}\colon \Gamma^\infty\big(P \times_H \operatorname{End}(V)\big) \to \Gamma^\infty\big(\operatorname{Hom}(M \times V, P \times_H V)\big), \quad s \mapsto \widetilde{\Psi} \circ s \tag{4.20}$$

is $\mathscr{C}^\infty(M)$-linear. Then the desired properties of $\widetilde{\Psi}$ follow by [14, Lem. 10.29].

We first show that $\overline{\Psi}$ is well-defined, i.e. that $\overline{\Psi}(s) \in \Gamma^\infty\big(\operatorname{Hom}(M \times V, P \times_H V)\big)$ is a smooth section of $\operatorname{Hom}(M \times V, P \times_H V) \to M$ for all $s \in \Gamma^\infty\big(P \times_H \operatorname{End}(V)\big)$. In other words, we have to show that for fixed $s \in \Gamma^\infty\big(P \times_H \operatorname{End}(V)\big)$ the map $\overline{\Psi}(s)$ is a smooth vector bundle morphism $\overline{\Psi}(s)\colon M \times V \to P \times_H V$ over id_M. Obviously, $\overline{\Psi}(s)$ is fiber-wise linear and covers id_M. It remains to prove the smoothness of $\overline{\Psi}(s)$. To this end, we proceed locally. Let $x_0 \in M$ and let $U \subseteq M$ be open with $x_0 \in U$. Moreover, after shrinking U if necessary, let $\widetilde{U} \subseteq P \times_H \operatorname{End}(V)$ be open with $s(x) \in \widetilde{U}$ for all $x \in U$ such that there is a smooth local section $\overline{s}\colon \widetilde{U} \to P \times \operatorname{End}(V)$ of the H-principal fiber bundle $\overline{\pi}_{P \times \operatorname{End}(V)}\colon P \times \operatorname{End}(V) \to P \times_H \operatorname{End}(V)$. Then $\overline{s} \circ s\colon U \to P \times \operatorname{End}(V)$ is smooth and $(\overline{s} \circ s)(x) = (p(x), A(x))$ holds for all $x \in U$ with some smooth maps $U \ni x \mapsto p(x) \in P$ and $U \ni x \mapsto A(x) \in \operatorname{End}(V)$. Thus $s(x) = (\overline{\pi}_{P \times \operatorname{End}(V)} \circ \overline{s} \circ s)(x) = [p(x), A(x)]$ is fulfilled for all $x \in U$. By this notation, we obtain for $(x, v) \in U \times V$

$$\big(\overline{\Psi}(s)\big)(x, v) = [p(x), A(x)v] = \overline{\pi}_{P \times V} \circ (\mathrm{id}_P \times e) \circ ((\overline{s} \circ s) \times \mathrm{id}_V)(x, v) \tag{4.21}$$

with $e\colon \operatorname{End}(V) \times V \ni (A, v) \mapsto Av \in V$. Hence the map $\overline{\Psi}(s)\big|_{U \times V}$ is smooth as the composition of smooth maps by (4.21). Thus $\overline{\Psi}(s)$ is smooth since $x_0 \in M$ is arbitrary.

Next we prove the $\mathscr{C}^\infty(M)$-linearity of $\overline{\Psi}$. Let $s_1, s_2 \in \Gamma^\infty\big(P \times_H \operatorname{End}(V)\big)$ be two sections point-wise given by

$$s_1(x) = [p(x), A_1(x)] \quad \text{and} \quad s_2(x) = [p(x), A_2(x)], \quad x \in M.$$

Here we assume without loss of generality that their first component is represented by the same element $p(x) \in P$ for all $x \in M$. Moreover, let $f, g \in \mathscr{C}^\infty(M)$. By the vector bundle structure on associated vector bundles, see e.g. [17, Re. 1.2.9], we obtain for $(x, v) \in M \times V$

$$\begin{aligned}\big(\widetilde{\Psi} \circ (fs_1 + gs_1)\big)(x, v) &= \big(\widetilde{\Psi}\big(f(x)[p(x), A_1(x)] + g(x)[p(x), A_2(x)]\big)\big)(x, v)\\ &= \big[p(x), \big(f(x)A_1(x) + g(x)A_2(x)\big)v\big]\\ &= f(x)[p(x), A_1(x)v] + g(x)[p(x), A_2(x)v]\\ &= \big(f(\widetilde{\Psi} \circ s_1)\big)(x, v) + \big(g(\widetilde{\Psi} \circ s_2)\big)(x, v)\end{aligned}$$

showing the $\mathscr{C}^\infty(M)$-linearity of $\overline{\Psi}$ by its definition in (4.20). Hence $\widetilde{\Psi}$ is indeed a smooth morphism of vector bundles by [14, Lem. 10.29].

In order to prove the smoothness of Ψ, we consider the map

$$i\colon P \times_H \mathrm{GL}(V) \to P \times_H \mathrm{End}(V), \quad [p, A] \mapsto [p, A]$$

whose smoothness can be proven analogously to the proof of Corollary 4.3 by exploiting the smoothness of the canonical inclusion $P \times \mathrm{GL}(V) \to P \times \mathrm{End}(V)$. We now obtain for $[p, A] \in P \times_H \mathrm{GL}(V)$ and $(x, v) \in M \times V$

$$\big((\widetilde{\Psi} \circ i)([p, A])\big)(x, v) = \big(\widetilde{\Psi}([p, A])\big)(x, v) = [p, Av] = \big(\Psi([p, A])\big)(x, v).$$

Thus $\Psi = \widetilde{\Psi} \circ i$ is smooth as the composition of smooth maps.

It remains to show that Ψ is an isomorphism of $\mathrm{GL}(V)$-principal fiber bundles. To this end, we recall that the $\mathrm{GL}(V)$-action on $\mathrm{GL}(V, P \times_H V)$ is given by composition from the right, see e.g. [16, Sec. 18.11]. Thus we have for $[p, A] \in P \times_H \mathrm{GL}(V)$ and $B \in \mathrm{GL}(V)$ as well as $v \in V$

$$\big(\Psi([p, A] \triangleleft B)\big)(v) = \big(\Psi([p, A \circ B])\big)(v) = [p, (A \circ B)v] = \Psi([p, A])(Bv) = \big(\Psi[p, A] \circ B\big)(v).$$

proving that Ψ is a morphism of $\mathrm{GL}(V)$-principal fiber bundles over $\mathrm{id}_M\colon M \to M$. Therefore it is an isomorphism of principal fiber bundles by [21, Prop. 9.23]. □

Assuming that $P \times_H \mathrm{GL}(V)$ admits a reduction as in Corollary 4.3, we obtain a reduction of $\mathrm{GL}(V, P \times_H V)$.

Corollary 4.5 *Let $P \to M$ be an H-principal fiber bundle and let $\rho\colon H \to \mathrm{GL}(V)$ be a representation of H on V such that $\rho_h \in \mathrm{G}(V)$ holds for all $h \in H$, where $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ is a closed subgroup. Moreover, let $\Psi\colon P \times_H \mathrm{GL}(V) \to \mathrm{GL}(V, P \times_H V)$ be the isomorphism of principal fiber bundles from Proposition 4.4 and let $\iota_{P \times_H \mathrm{G}(V)}\colon P \times_H \mathrm{G}(V) \to P \times_H \mathrm{GL}(V)$ be the reduction of principal fiber bundles from Corollary 4.3. Then*

$$P \times_H \mathrm{G}(V) \to \mathrm{GL}(V, P \times_H V), \quad [p, A] \mapsto \big(\Psi \circ \iota_{P \times_H \mathrm{G}(V)}\big)([p, A]) \tag{4.22}$$

is a $\mathrm{G}(V)$-reduction of the frame bundle $\mathrm{GL}(V, P \times_H V)$ along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$.

Proof Obviously, the map $\Psi \circ \iota_{P \times_H \mathrm{G}(V)}$ is smooth as the composition of smooth maps. Moreover, since Ψ is an isomorphism of principal fiber bundles covering id_M by Proposition 4.4 and $\iota_{P \times_H \mathrm{G}(V)}$ is a reduction of principal fiber bundles along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$ by Corollary 4.3, one verifies by a straightforward computation that (4.22) is a reduction of principal fiber bundles along the canonical inclusion $\mathrm{G}(V) \to \mathrm{GL}(V)$. □

Corollary 4.6 *Let $P \times_H V \to M$ be a vector bundle associated to $P \to M$, where $\rho\colon H \to \mathrm{GL}(V)$ is a representation. Moreover, let $E \to N$ be another vector bundle and let $\Phi\colon P \times_H V \to E$ be an isomorphism of vector bundles covering the diffeomorphism $\phi\colon M \to N$. Then*

$$\chi\colon P \times_H \mathrm{GL}(V) \to \mathrm{GL}(V, E), \quad [p, A] \mapsto \chi([p, A]) = (\Phi \circ \Psi)([p, A]) \quad (4.23)$$

is an isomorphism of $\mathrm{GL}(V)$-principal fiber bundles over the diffeomorphism $\phi\colon M \to N$, where $\Psi\colon P \times_H \mathrm{GL}(V) \to \mathrm{GL}(V, P \times_H V)$ denotes the isomorphism from Proposition 4.4.

Proof Obviously, for fixed $[p, A] \in P \times_H \mathrm{GL}(V)$, the map $\chi([p, A]) = (\Phi \circ \Psi)([p, A])\colon V \to E_{\phi(\mathrm{pr}(p))}$ is linear and invertible since Φ is an isomorphism of vector bundles. Hence χ is well-defined. Moreover, the map χ is smooth as the composition of the smooth maps Ψ and Φ. Its inverse is given by the composition of the smooth maps $\chi^{-1} = \Psi^{-1} \circ \Phi^{-1}\colon \mathrm{GL}(V, E) \to P \times_H \mathrm{GL}(V)$, i.e. χ^{-1} is clearly smooth, as well. Let $B \in \mathrm{GL}(V)$ and $[p, A] \in P \times_H \mathrm{GL}(V)$. Then

$$\chi([p, A] \triangleleft B)(v) = (\Phi \circ \Psi)([p, A \circ B])(v) = (\Phi \circ \Psi)([p, A])(Bv) = (\chi([p, A]) \circ B)(v)$$

holds for all $v \in V$ by the definition of Ψ. Hence χ is an isomorphism of $\mathrm{GL}(V)$-principal fiber bundles which covers the diffeomorphism $\phi\colon M \to N$. □

4.2 Principal fiber bundles over frame bundles and principal connections

Since the $\mathrm{G}(V)$-principal fiber bundle $\pi\colon P \times_H \mathrm{G}(V) \to M$ is obtained as a fiber bundle associated to the H-principal fiber bundle $P \to M$, we have the H-principal fiber bundle $\overline{\pi}\colon P \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V)$ over $P \times_H \mathrm{G}(V)$. Given a principal connection on $P \to M$, we construct a principal connection on $\overline{\pi}\colon P \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V)$. This construction will be applied to the configuration space of an intrinsic rolling of a reductive homogeneous space in Proposition 5.4 below.

Proposition 4.7 *Let $\mathrm{pr}\colon P \to M$ be an H-principal fiber bundle and let $\rho\colon H \to \mathrm{GL}(V)$ be a representation of H on the finite dimensional $\mathbb{R}$-vector space V. Assume that there exists a closed subgroup $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ with Lie algebra $\mathfrak{g}(V) \subseteq \mathfrak{gl}(V)$ such that $\rho_h \in \mathrm{G}(V)$ holds for all $h \in H$. Moreover, let*

$$\rho'\colon \mathfrak{h} \to \mathfrak{g}(V) \subseteq \mathfrak{gl}(V), \quad \eta \mapsto (T_e\rho)\eta = \rho'_\eta \in \mathfrak{g}(V) \subseteq \mathfrak{gl}(V) \quad (4.24)$$

denote the induced morphism of Lie algebras. Consider the H-principal fiber bundle

$$\overline{\pi}\colon P \times \mathrm{G}(V) \to P \times_H \mathrm{G}(V) \tag{4.25}$$

over the associated bundle $\pi\colon P \times_H \mathrm{G}(V) \to M$, where H acts on $\mathrm{G}(V)$ via

$$H \times \mathrm{G}(V) \ni (h, A) \mapsto \rho_h \circ A \in \mathrm{G}(V). \tag{4.26}$$

Moreover, let $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TP)\big)$ be a principal connection on $\mathrm{pr}\colon P \to M$ with corresponding connection one-form $\omega \in \Gamma^\infty(T^*M)\otimes\mathfrak{h}$. Then the following assertions are fulfilled:

1. *The vertical bundle $\mathrm{Ver}(P\times\mathrm{G}(V)) \subseteq T(P\times\mathrm{G}(V)) \cong TP\times T\mathrm{G}(V)$ is fiber-wise given by*

$$\mathrm{Ver}(P \times \mathrm{G}(V))_{(p,A)} = \big\{\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(p \triangleleft \exp(t\eta)\big)\big|_{t=0}, -\rho'_\eta \circ A\big) \mid \eta \in \mathfrak{h}\big\} \tag{4.27}$$

   where $(p, A) \in P \times \mathrm{G}(V)$.

2. *Defining $\overline{\mathcal{P}} \in \Gamma^\infty\big(\mathrm{End}(T(P\times\mathrm{GL}(V)))\big)$ for $(p, A) \in P\times\mathrm{G}(V)$ and $(v_p, v_A) \in T_{(p,A)}(P \times \mathrm{G}(V))$ by*

$$\overline{\mathcal{P}}\big|_{(p,A)}(v_p, v_A) = (\mathcal{P}\big|_p(v_p), -\rho'_{\omega|_p(v_p)} \circ A) \tag{4.28}$$

   yields a principal connection on $\overline{\pi}\colon P\times\mathrm{G}(V) \to P\times_H\mathrm{G}(V)$ with corresponding connection one-form $\overline{\omega} \in \Gamma^\infty\big(T^*(P \times \mathrm{G}(V))\big) \otimes \mathfrak{h}$ given by

$$\overline{\omega}\big|_{(p,A)}(v_p, v_A) = \omega\big|_p(v_p) \tag{4.29}$$

   for all $(p, A) \in P \times \mathrm{G}(V)$ and $(v_p, v_A) \in T_{(p,A)}(P \times \mathrm{G}(V))$.

3. *Let $\overline{q}\colon I \ni t \mapsto \overline{q}(t) = (p(t), A(t)) \in P \times \mathrm{G}(V)$ be a curve which is horizontal with respect to the principal connection $\overline{\mathcal{P}}$. Then the curve $p\colon I \to P$ given by the first component of $\overline{q}$ is horizontal with respect to the principal connection $\mathcal{P}$ on $P \to M$.*

Proof First we recall that $\rho'\colon \mathfrak{h} \to \mathfrak{g}(V)$ is indeed a morphism of Lie algebras, see e.g. [16, Lem. 4.13]. Next we prove Claim 1.. To this end, we compute for $(p, A) \in P \times \mathrm{G}(V)$

$$\begin{aligned}
\mathrm{Ver}(P \times \mathrm{G}(V))_{(p,A)} &= \big\{\tfrac{\mathrm{d}}{\mathrm{d}t}\big((p, A)\overline{\triangleleft}\exp(t\eta)\big)\big|_{t=0} \mid \eta \in \mathfrak{h}\big\} \\
&= \big\{\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(p \triangleleft (\exp(t\eta))\big)\big|_{t=0}, \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\rho_{\exp(-t\eta)} \circ A\big)\big|_{t=0}\big) \mid \eta \in \mathfrak{h}\big\} \\
&= \big\{\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(p \triangleleft (\exp(t\eta))\big)\big|_{t=0}, -\rho'_\eta \circ A\big) \mid \eta \in \mathfrak{h}\big\}
\end{aligned}$$

showing Claim 1., where $\overline{\triangleleft}$ denotes the H-principal action on $P \times \mathrm{G}(V)$ similar to (2.18).

We now prove Claim 2.. Obviously, $\overline{\mathcal{P}} \in \Gamma^\infty\big(\mathrm{End}(T(P \times \mathrm{G}(V)))\big)$ holds. Next we show that $\overline{\mathcal{P}}$ is a projection, i.e. $\overline{\mathcal{P}}^2 = \overline{\mathcal{P}}$ is fulfilled. By using the correspondence of $\mathcal{P}$ and ω from (2.14) as well as $\mathcal{P}^2 = \mathcal{P}$, we calculate for $p \in P$ and $v_p \in T_p P$

$$\begin{aligned} \omega\big|_p(\mathcal{P}\big|_p(v_p)) &= \big(T_e(p \triangleleft \cdot)\big)^{-1}\mathcal{P}\big|_p\big(\mathcal{P}\big|_p(v_p)\big) \\ &= \big(T_e(p \triangleleft \cdot)\big)^{-1}\mathcal{P}\big|_p(v_p) \\ &= \omega\big|_p(v_p). \end{aligned} \tag{4.30}$$

Using (4.30) and $\mathcal{P}^2 = \mathcal{P}$, we have for $(p, A) \in P \times \mathrm{G}(V)$ and $(v_p, v_A) \in T_p P \times T_A \mathrm{G}(V)$

$$\begin{aligned} \overline{\mathcal{P}}\big|_{(p,A)}\big(\overline{\mathcal{P}}\big|_{p,A}(v_p, v_A)\big) &= \overline{\mathcal{P}}\big|_{(p,A)}\big(\mathcal{P}\big|_p(v_p), -\rho'_{\omega|_p(v_p)} \circ A\big) \\ &= \big(\mathcal{P}\big|_p\big(\mathcal{P}\big|_p(v_p)\big), -\rho'_{\omega|_p(\mathcal{P}|_p(v_p))} \circ A\big) \\ &= \big(\mathcal{P}\big|_p(v_p), -\rho'_{\omega|_p(v_p)} \circ A\big) \\ &= \overline{\mathcal{P}}\big|_{(p,A)}(v_p, A), \end{aligned}$$

proving that $\overline{\mathcal{P}}^2 = \overline{\mathcal{P}} \in \Gamma^\infty\big(\mathrm{End}(T(P \times \mathrm{G}(V)))\big)$ is a projection.

Moreover, $\mathrm{im}(\overline{\mathcal{P}}) = \mathrm{Ver}(P \times \mathrm{G}(V))$ holds by $\mathrm{im}(\mathcal{P}) = \mathrm{Ver}(P)$ and the characterization of the vertical bundle in (4.27).

We now show that $\overline{\mathcal{P}}$ corresponds to $\overline{\omega}$. To this end, let $\eta \in \mathfrak{h}$ and denote by $\eta_{P\times \mathrm{G}(V)} \in \Gamma^\infty\big(T(P \times \mathrm{G}(V))\big)$ the corresponding fundamental vector field associated to the H-principal action given by

$$\eta_{P\times\mathrm{G}(V)}(p, A) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big((p, A)\overline{\triangleleft}\exp(t\eta)\big)\big|_{t=0}, \quad (p, A) \in P \times \mathrm{G}(V).$$

By this notation and the definition of $\overline{\omega}$ in (4.29), we obtain

$$\begin{aligned} &\big(\overline{\omega}\big|_{(p,A)}(v_p, v_A)\big)_{P\times\mathrm{G}(V)}(p, A) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big((p, A)\overline{\triangleleft}\exp\big(t\overline{\omega}\big|_{(p,A)}(v_p, v_A)\big)\big)\big|_{t=0} \\ &= \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(p \triangleleft \exp\big(t\omega\big|_p(v_p)\big)\big)\big|_{t=0}, \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\rho_{\exp(-t\omega|_p(v_p))} \circ A\big)\big|_{t=0}\big) \\ &= \big(\mathcal{P}\big|_g(v_g), -\rho'_{\omega|_p(v_p)} \circ A\big) \\ &= \overline{\mathcal{P}}\big|_{(u,A)}(v_p, v_A). \end{aligned} \tag{4.31}$$

Moreover, denoting by $\eta_P \in \Gamma^\infty(TP)$ the fundamental vector field on P defined by $\eta \in \mathfrak{h}$, as usual, we compute

$$\begin{aligned} \overline{\omega}\big|_{(p,A)}\big(\eta_{P\times \mathrm{G}(V)}(p,A)\big) &= \overline{\omega}\big|_{(p,A)}\Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(p \triangleleft \exp(t\eta)\big)\big|_{t=0}, \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\rho_{\exp(-t\eta)} \circ A\big)\big|_{t=0}\Big) \\ &= \omega\big|_p\big(\eta_P(p)\big) \\ &= \eta \end{aligned} \tag{4.32}$$

since ω, being the connection one-form associated to $\mathcal{P}$, fulfills $\omega(\eta_P) = \eta$ for all $\eta \in \mathfrak{h}$. Thus $\overline{\omega}$ is the connection one-form corresponding to the connection $\overline{\mathcal{P}}$ due to (4.31) and (4.32). In order to show that $\overline{\mathcal{P}}$ is a principal connection, we show that $\overline{\omega}$ has the desired equivarience-property. By exploiting that $\omega \in \Gamma^\infty(T^*G) \otimes \mathfrak{h}$ is a principal connection one-form, we compute for $h \in H$

$$\begin{aligned} \big((\cdot\overline{\triangleleft}h)^*\overline{\omega}\big)\big|_{(p,A)}(v_p, v_A) &= \overline{\omega}\big|_{(p,A)\overline{\triangleleft}h}\big(T_{(p,A)}(\cdot\overline{\triangleleft}h)(v_p, v_A)\big) \\ &= \overline{\omega}\big|_{(p\triangleleft h, \rho_{h^{-1}}\circ A)}\big(T_p(\cdot \triangleleft h)v_p, T_A(\rho_{h^{-1}} \circ (\cdot))v_A\big) \\ &= \omega_{p\triangleleft h}\big(T_p(\cdot \triangleleft h)v_p\big) \\ &= \mathrm{Ad}_{h^{-1}}\big(\omega\big|_p(v_p)\big) \\ &= \mathrm{Ad}_{h^{-1}}\big(\overline{\omega}\big|_{(p,A)}(v_p, v_A)\big) \end{aligned}$$

as desired.

It remains to show Claim 3.. Let $\overline{q}\colon I \ni t \mapsto \overline{q}(t) = (p(t), A(t)) \in P \times \mathrm{G}(V)$ be horizontal with respect to $\overline{\mathcal{P}}$. Then

$$0 = \overline{\mathcal{P}}\big|_{\overline{q}(t)}(\dot{\overline{q}}(t)) = \big(\mathcal{P}\big|_{p(t)}(\dot{p}(t)), -\mathrm{ad}_{\omega|_{p(t)}(\dot{p}(t))} \circ A(t)\big)$$

holds. In particular, this implies $\mathcal{P}\big|_{p(t)}(\dot{p}(t)) = 0$. Hence $p\colon I \to P$ is horizontal with respect to the principal connection $\mathcal{P}$ on $P \to M$. ◻

4.3 Frame bundles of reductive homogeneous spaces

We now consider (certain reductions) of the frame bundle of a reductive homogeneous space by applying Proposition 4.4 to the H-principal fiber bundle $\mathrm{pr}\colon G \to G/H$. To this end, we recall that the tangent bundle of a reductive homogeneous space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is isomorphic to the vector bundle $G \times_H \mathfrak{m} \to G/H$, where H acts on $\mathfrak{m}$ via

$$H \times \mathfrak{m} \ni (h, X) \mapsto \mathrm{Ad}_h(X) \in \mathfrak{m}. \tag{4.33}$$

This statement as well as the statement of Corollary 4.11 below seem to be well-known since they can be found in [20, Ex. 2.7]. Moreover, exploiting that the isotropy representation $H \ni h \mapsto T_{\mathrm{pr}(e)}\tau_h \in \mathrm{GL}(T_{\mathrm{pr}(e)}(G/H))$ is equivalent to the representation

$H \mapsto \mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{GL}(\mathfrak{m})$, see Lemma 2.11, one obtains that

$$G \times_H \mathfrak{m} \to T(G/H), \quad [g, X] \mapsto (T_g \mathrm{pr} \circ T_e \ell_g)X \tag{4.34}$$

is an isomorphism of vector bundles over $\mathrm{id}_{G/H}$ by adapting the proof in [16, Sec. 18.16].

Corollary 4.8 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Moreover, assume that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$, where $\mathrm{G}(\mathfrak{m})$ is some closed subgroup of $\mathrm{GL}(\mathfrak{m})$. Then*

$$G \times_H \mathrm{G}(\mathfrak{m}) \ni [g, A] \mapsto (X \mapsto [g, AX]) \in \mathrm{GL}(\mathfrak{m}, G \times_H \mathfrak{m}) \tag{4.35}$$

is a reduction of the frame bundle of $G \times_H \mathfrak{m} \to G/H$ along the canonical inclusion $\mathrm{G}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m})$. Moreover, the map

$$G \times_H \mathrm{G}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m}, T(G/H)), \quad [g, A] \mapsto \big(X \mapsto (T_g \mathrm{pr} \circ T_e \ell_g \circ A)X\big) \tag{4.36}$$

is a reduction of $\mathrm{GL}(\mathfrak{m}, T(G/H))$ along the canonical inclusion $\mathrm{G}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m})$.

Proof The map defined in (4.35) is a reduction of the frame bundle of $G \times_H \mathfrak{m} \to G/H$ by Proposition 4.4.

It remains to show that (4.36) is a reduction of principal fiber bundles. In fact,

$$G \times_H \mathrm{GL}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m}, T(G/H)), \quad [g, A] \mapsto \big(X \mapsto (T_g \mathrm{pr} \circ T_e \ell_g \circ A)X\big) \tag{4.37}$$

is an isomorphism of principal fiber bundles covering $\mathrm{id}_{G/H}$ by Corollary 4.6 since (4.34) is an isomorphism of vector bundles covering $\mathrm{id}_{G/H}$. The desired result follows by exploiting that (4.36) is the composition of the isomorphism (4.37) and the reduction (4.35). □

Remark 4.9 In the sequel, under the assumption of Corollary 4.8, we often identify $G \times_H \mathrm{G}(\mathfrak{m})$ with the image of the reduction (4.36) from Corollary 4.8 as in [17, Re. 1.1.8]. This is indicated by the notation $\mathrm{G}(\mathfrak{m}, T(G/H)) \subseteq \mathrm{GL}(\mathfrak{m}, T(G/H))$.

Corollary 4.10 *Let G/H be a pseudo-Riemannian reductive homogeneous space whose invariant metric corresponds to the $\mathrm{Ad}(H)$-invariant scalar product $\langle \cdot, \cdot \rangle \colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$. Moreover, denote by $\mathrm{O}(\mathfrak{m})$ the pseudo-orthogonal group of $\mathfrak{m}$ with respect to $\langle \cdot, \cdot \rangle$. Then*

$$G \times_H \mathrm{O}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m}, T(G/H)), \quad [g, S] \mapsto \big(X \mapsto (T_g \mathrm{pr} \circ T_e \ell_g \circ S)X\big) \tag{4.38}$$

is a reduction of the frame bundle of $T(G/H)$ along the canonical inclusion $\mathrm{O}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m})$.

Proof This is a consequence of Corollary 4.8. □

If $G \times_H \mathrm{O}(\mathfrak{m})$ is identified with the image of (4.38), it is often denoted by $\mathrm{O}(\mathfrak{m}, T(G/H))$.

Corollary 4.11 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the frame bundle of $T(G/H)$ is isomorphic to $G \times_H \mathrm{GL}(\mathfrak{m}) \to G/H$ as $\mathrm{GL}(\mathfrak{m})$-principal fiber bundle via the isomorphism*

$$G \times_H \mathrm{GL}(\mathfrak{m}) \to \mathrm{GL}(\mathfrak{m}, T(G/H)), \quad [g, A] \mapsto \big(X \mapsto (T_g \mathrm{pr} \circ T_e \ell_g \circ A)X\big) \tag{4.39}$$

of $\mathrm{GL}(\mathfrak{m})$-principal fiber bundles.

Proof This follows by setting $\mathrm{G}(\mathfrak{m}) = \mathrm{GL}(\mathfrak{m})$ in Corollary 4.8. □

Remark 4.12 Corollary 4.11 seems to be well-known since the statement that $G \times_H \mathrm{GL}(\mathfrak{m}) \to G/H$ is isomorphic to the frame bundle of G/H can be found in [20, Ex. 2.7].

5 Intrinsic rollings of reductive homogeneous spaces

Let G/H be a reductive homogeneous space with fixed reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In the sequel, we always endow $\mathfrak{m}$ with the covariant derivative $\nabla^{\mathfrak{m}}$ which is defined in (5.1) below. Let $V : \mathfrak{m} \ni v \mapsto (v, V_2(v)) \in \mathfrak{m} \times \mathfrak{m} \cong T\mathfrak{m}$ and $W : \mathfrak{m} \ni v \mapsto (v, W_2(v)) \in \mathfrak{m} \times \mathfrak{m} \cong T\mathfrak{m}$ by vector fields on $\mathfrak{m}$, where $V_2, W_2 : \mathfrak{m} \to \mathfrak{m}$ are smooth maps. Then $\nabla^{\mathfrak{m}} : \Gamma^\infty(T\mathfrak{m}) \times \Gamma^\infty(T\mathfrak{m}) \to \Gamma^\infty(T\mathfrak{m})$ is defined by

$$\nabla^{\mathfrak{m}}_V W\big|_v = \big(v, (T_v W_2) V_2(v)\big), \quad v \in \mathfrak{m}. \tag{5.1}$$

Clearly, for $\mathfrak{m} = \mathbb{R}^n$ the covariant derivative $\nabla^{\mathfrak{m}}$ coincides with the covariant derivative from [13, Chap. 3, Def. 8].

In this section, we consider intrinsic ($\mathrm{G}(\mathfrak{m})$-reduced) rollings of $(\mathfrak{m}, \nabla^{\mathfrak{m}})$ over G/H equipped with an invariant covariant derivative ∇^α. Such intrinsic rollings are called rollings of $\mathfrak{m}$ over G/H with respect to ∇^α, rollings of G/H with respect to ∇^α, or simply rollings of G/H, for short.

Notation 5.1 *In the sequel, we do not explicitly refer to the $\mathrm{G}(\mathfrak{m})$-reduction if this reduction is clear by the context, for instance by denoting the configuration space by $Q = \mathfrak{m} \times (G \times \mathrm{G}(\mathfrak{m}))$ as in Lemma 5.2, below.*

5.1 Configuration space

The goal of this subsection is to derive an explicit description of the configuration space for rollings of $\mathfrak{m}$ over G/H with respect to an invariant covariant derivative ∇^α. Moreover, we consider an H-principal fiber bundle over the configuration space equipped with a suitable principal connection. This allows for lifting rollings, i.e. certain curves on the configuration space, horizontally to curves on that principal fiber bundle. We start with investigating the configuration space.

Lemma 5.2 *Let G/H be a reductive homogeneous space and let $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ be a closed subgroup such that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$. Then the following assertions are fulfilled:*

1. *Let $Q = \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m}))$ and define*

$$\pi : Q \to \mathfrak{m} \times G/H, \quad (v, [g, S]) \mapsto \pi(v, [g, S]) = (v, \mathrm{pr}(g)). \tag{5.2}$$

Then $\pi : Q \to \mathfrak{m} \times G/H$ is isomorphic to the configuration space of the intrinsic rolling of $\mathfrak{m}$ over G/H, i.e. to the $\mathrm{G}(\mathfrak{m})$-fiber bundle

$$\begin{aligned} &\big(\mathrm{G}(\mathfrak{m}, T\mathfrak{m}) \times \mathrm{G}(\mathfrak{m}, T(G/H))\big)/\mathrm{G}(\mathfrak{m}) \\ &\cong \big((\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m}))\big)/\mathrm{G}(\mathfrak{m}) \to \mathfrak{m} \times G/H \end{aligned} \tag{5.3}$$

via the isomorphism of $\mathrm{G}(\mathfrak{m})$-fiber bundles

$$\begin{aligned} \Psi \colon \big((\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m}))\big)/\mathrm{G}(\mathfrak{m}) &\to \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m})), \\ \big[(v, S_1), [g, S_2]\big] &\mapsto \big(v, [g, S_2 \circ S_1^{-1}]\big) \end{aligned} \tag{5.4}$$

covering the identity $\mathrm{id}_{\mathfrak{m} \times G/H} \colon \mathfrak{m} \times G/H \to \mathfrak{m} \times G/H$ whose inverse is given by

$$\begin{aligned} \Psi^{-1} \colon \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m})) &\to \big((\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m}))\big)/\mathrm{G}(\mathfrak{m}), \\ (v, [g, S]) &\mapsto \big[(v, \mathrm{id}_{\mathfrak{m}}), [g, S]\big]. \end{aligned} \tag{5.5}$$

2. *Let $q = (v, [g, S]) \in Q$ with $\pi(q) = (v, \mathrm{pr}(g))$. Then q defines the linear isomorphism*

$$T_v\mathfrak{m} \cong \mathfrak{m} \ni Z \mapsto \big(T_g\mathrm{pr} \circ T_e\ell_g \circ S\big)Z \in T_{\mathrm{pr}(g)}G/H \tag{5.6}$$

via Lemma 3.1, Claim 2., where q is identified with $\Psi^{-1}(q) \in \big((\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m}))\big)/\mathrm{G}(\mathfrak{m})$. In the sequel, we often denote this isomorphism by q, as well, i.e. we write $q(Z) = qZ = \big(T_g\mathrm{pr} \circ T_e\ell_g \circ S\big)Z$.

Proof By Corollary 4.8 we have $\mathrm{G}(\mathfrak{m}, T(G/H)) \cong G \times_H \mathrm{G}(\mathfrak{m})$. Moreover, $\mathrm{G}(\mathfrak{m}, T\mathfrak{m}) \cong \mathfrak{m} \times \mathrm{G}(\mathfrak{m})$ is clearly fulfilled. We first show that Ψ is smooth. Consider

$$\begin{array}{ccc} (\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m})) & & \\ \Big\downarrow \overline{\mathrm{pr}} & \searrow^{\overline{\Psi}} & \\ \big((\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m}))\big)/\mathrm{G}(\mathfrak{m}) & \xrightarrow{\Psi} & \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m})) \end{array}, \tag{5.7}$$

where $\overline{\mathrm{pr}}$ is the canonical projection and $\overline{\Psi}$ is given by

$$\overline{\Psi}\big((v, S_1), [g, S_2]\big) = (v, [g, S_2 \circ S_1^{-1}]) \tag{5.8}$$

for $((v, S_1), [g, S_2]) \in (\mathfrak{m} \times \mathrm{G}(\mathfrak{m})) \times (G \times_H \mathrm{G}(\mathfrak{m}))$. Clearly, since (5.7) commutes and the canonical projection $\overline{\mathrm{pr}}$ is a surjective submersion, the map Ψ defined by (5.4) is smooth by the smoothness of $\overline{\Psi}$. In addition, Ψ maps fibers into fibers, i.e. it is a morphism of $\mathrm{G}(\mathfrak{m})$-fiber bundles covering the identity of $\mathfrak{m} \times G/H$. Therefore Ψ is an isomorphism of fiber bundles, see e.g. [21, Prop. 9.3]. The formula (5.5) for Ψ^{-1} is verified by a straightforward calculation.

It remains to show Claim 2.. Let $q = (v, [g, S]) \in Q$ and let $Z \in T_v\mathfrak{m}$. Then $\Psi^{-1}((v, [g, S])) = \big[(v, \mathrm{id}_{\mathfrak{m}}), [g, S]\big]$ holds. Using the bijection from Lemma 3.1, Claim 2., this element is identified with a linear isomorphism which we denote by the same symbol. Evaluated at $Z \in T_v\mathfrak{m} \cong \mathfrak{m}$, it is given by

$$\begin{aligned}(\Psi^{-1}(v, [g, S]))\big)(Z) &= \Big(\big(\big[(v, \mathrm{id}_{\mathfrak{m}}), [g, S]\big)\Big)(Z) \\ &= \Big(\big(T_g\mathrm{pr} \circ T_e\ell_g \circ S\big) \circ (\mathrm{id}_{\mathfrak{m}})^{-1}\Big)(Z) \\ &= \big(T_g\mathrm{pr} \circ T_e\ell_g \circ S\big)Z,\end{aligned}$$

where the second equality follows by Lemma 3.1, Claim 2. and Corollary 4.8. □

Remark 5.3 The configuration space $\pi : Q \to \mathfrak{m} \times G/H$ can be viewed as a $\mathrm{G}(\mathfrak{m})$-principal fiber bundle. Indeed, as a consequence of Lemma 4.1, the $\mathrm{G}(\mathfrak{m})$-right action

$$Q \times \mathrm{G}(\mathfrak{m}) \to Q, \quad \big((v, [g, S]), S_2\big) \mapsto (v, [g, S \circ S_2]) \tag{5.9}$$

is a principal action.

Moreover, since the configuration space $Q = \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m}))$ is the product of $\mathfrak{m}$ and the associated bundle $G \times_H \mathrm{G}(\mathfrak{m})$, we obtain an H-principal fiber bundle over Q.

Proposition 5.4 *Let G/H be a reductive homogeneous space and let $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ be a closed subgroup such that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$. Then the following assertions are fulfilled:*

1. *Define $\overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$. Then*

$$\overline{\pi}\colon \overline{Q} \ni (v, g, S) \mapsto (v, [g, S]) \in Q \tag{5.10}$$

becomes an H-principal fiber bundle over $Q = \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m}))$ with H-principal action given by

$$\begin{aligned}&\triangleleft_{\overline{Q}}\colon \overline{Q} \times H \to \overline{Q}, \\ &\quad (v, g, S) \mapsto (v, g, S) \triangleleft_{\overline{Q}} h = (v, gh, \mathrm{Ad}_{h^{-1}} \circ S) = (v, g \triangleleft h, \mathrm{Ad}_{h^{-1}} \circ S),\end{aligned} \tag{5.11}$$

where $\triangleleft\colon G \times H \ni (g, h) \mapsto g \triangleleft h = gh \in G$ denotes the H-principal action from (2.26) on $\mathrm{pr}\colon G \to G/H$.

2. *For* $(v, g, S) \in \overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$ *the vertical bundle* $\mathrm{Ver}(\overline{Q}) \subseteq T\overline{Q}$ *is given by*

$$\mathrm{Ver}(\overline{Q})_{(v,g,S)} = \big\{(0, T_e\ell_g \eta, -\mathrm{ad}_\eta \circ S) \mid \eta \in \mathfrak{h}\big\} \subseteq T_{(v,g,S)}(\mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})). \tag{5.12}$$

3. *Let* $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$ *and let* $\omega \in \Gamma^\infty(T^*G) \otimes \mathfrak{h}$ *denote the principal connection and connection one-form from Proposition 2.10 on* $\mathrm{pr}\colon G \to G/H$*, respectively. Defining for* $(v, g, S) \in \overline{Q}$ *and* $(u, v_g, v_S) \in T_{(v,g,S)}\overline{Q}$

$$\overline{\mathcal{P}}\big|_{(v,g,S)}(u, v_g, v_S) = (0, \mathcal{P}\big|_g(v_g), -\mathrm{ad}_{\omega|_g(v_g)} \circ S), \tag{5.13}$$

yields a principal connection on $\overline{\pi}\colon \overline{Q} \to Q$ *with corresponding connection one-form* $\overline{\omega} \in \Gamma^\infty(T^*\overline{Q}) \otimes \mathfrak{h}$ *given by*

$$\overline{\omega}\big|_{(v,g,S)}(u, v_g, v_S) = \omega\big|_g(v_g), \quad (v, g, S) \in \overline{Q},\ (u, v_g, v_S) \in T_{(v,g,S)}\overline{Q}. \tag{5.14}$$

4. *Let* $\overline{q}\colon I \ni t \mapsto \overline{q}(t) = (v(t), g(t), S(t)) \in \overline{Q}$ *be a horizontal curve with respect to the principal connection* $\overline{\mathcal{P}}$*. Then the curve* $g\colon I \to G$ *defined by the second component of* $\overline{q}$ *is horizontal with respect to* $\mathrm{Hor}(G)$ *from Proposition 2.10.*

Proof We consider $\mathrm{pr}\colon G \to G/H$ as an H-principal fiber bundle. Then $\mathfrak{m} \times G \ni (v, g) \mapsto (v, \mathrm{pr}(g)) \in \mathfrak{m} \times G/H$ becomes clearly an H-principal fiber bundle with principal action

$$(\mathfrak{m} \times G) \times H \ni ((v, g), h) \mapsto (v, g \triangleleft h) = (v, gh) \in \mathfrak{m} \times G/H$$

and Q can be viewed as a $\mathrm{G}(\mathfrak{m})$-fiber bundle associated to the H-principal fiber bundle $\mathfrak{m} \times G/H$, where H acts on $\mathrm{G}(\mathfrak{m})$ via

$$H \times \mathrm{G}(V) \to \mathrm{G}(V), \quad (h, S) \mapsto \mathrm{Ad}_h\big|_{\mathfrak{m}} \circ S.$$

Thus, by the definition of an associated bundle, $\overline{\pi}\colon \overline{Q} \to Q$ becomes an H-principal fiber bundle over Q with principal action given by (5.11), i.e. Claim 1. is shown.

Next, let $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TP)\big)$ be the principal connection on G from Proposition 2.10. It is straightforward to verify that $\widetilde{\mathcal{P}} \in \Gamma^\infty\big(\mathrm{End}(TP)\big)$ defined by

$$\widetilde{\mathcal{P}}\big|_{(v,g)}(u, v_g) = \big(0, \mathcal{P}(v_g)\big), \quad (v, g) \in \mathfrak{m} \times G, \quad (u, v_g) \in \mathfrak{m} \times T_gG \cong T_v\mathfrak{m} \times T_gG$$

yields a principal connection on $\mathfrak{m} \times G \to \mathfrak{m} \times G/H$ with corresponding connection one-form given by $\widetilde{\omega}\big|_{(v,g)}(u, v_g) = \omega\big|_g(v_g)$. Thus Proposition 4.7 applied to $\mathfrak{m} \times G \to \mathfrak{m} \times G/H$ equipped with the principal connection $\widetilde{\mathcal{P}}$ yields Claim 2., Claim 3., and Claim 4.. ◻

5.2 The distribution characterizing intrinsic rollings

Motivated by [6, Sec. 4], we determine a distribution on Q characterizing intrinsic rollings of $\mathfrak{m}$ over G/H. More precisely, a curve $q\colon I \to Q$ is horizontal with respect to this distributions iff it is a rolling of G/H with respect to ∇^α.

Applying the description of the tangent bundle of an associated bundle from (2.20) to the configuration space Q, we obtain for its tangent bundle

$$TQ = T(\mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m}))) \cong T\mathfrak{m} \times T(G \times_H \mathrm{G}(\mathfrak{m})) \cong T\mathfrak{m} \times (TG \times_{TH} T\mathrm{G}(\mathfrak{m})). \tag{5.15}$$

Before we proceed, we state a simple lemma concerning this identification. We start with considering a situation which is slightly more general than (5.15).

Lemma 5.5 *Let V be a finite dimensional $\mathbb{R}$-vector space and let $\rho\colon H \to \mathrm{GL}(V)$ be a representation. Moreover, let $\mathrm{G}(V) \subseteq \mathrm{GL}(V)$ be a closed subgroup such that $\rho_h \in \mathrm{G}(V)$ holds for all $h \in H$ and consider the associated bundle $G \times_H \mathrm{G}(V) \to G/H$, where H acts on $\mathrm{G}(V)$ via $H \times \mathrm{G}(V) \ni (h, S) \mapsto \rho_h \circ S \in \mathrm{G}(V)$. Let $(v_g, v_S) \in TG \times T\mathrm{G}(V)$ and $h \in H$. Then*

$$[v_g, v_S] = [T_g r_h v_g, \rho_{h^{-1}} \circ v_S] \in TG \times_{TH} \mathrm{G}(V) \tag{5.16}$$

holds.

Proof We denote by $\triangleleft\colon G \times H \ni (g, h) \mapsto g \triangleleft h = gh \in G$ the principal action on $G \to G/H$. Its tangent map is given by

$$T_{(g,h)}(\cdot \triangleleft \cdot)(v_g, v_h) = T_h \ell_g v_h + T_g r_h v_g, \tag{5.17}$$

due to (2.8). Moreover, the tangent map of

$$\phi\colon H \times \mathrm{G}(V) \to \mathrm{G}(V), \quad (h, S) \mapsto \rho_{h^{-1}} \circ S$$

reads

$$T_{(h,S)}\phi(v_h, v_S) = T_{(h,S)}\phi(0, v_S) + T_{(h,S)}\phi(v_h, 0), \tag{5.18}$$

where we identify $T(H \times \mathrm{G}(V)) = TH \times T\mathrm{G}(V)$. By setting $v_h = 0$ in (5.17) and (5.18), respectively, we obtain $T_{(g,h)}(\cdot \triangleleft \cdot)(v_g, 0) = T_g r_h v_g$ and

$$T_{(h,S)}\phi(v_h, v_S) = T_{(h,S)}\phi(0, v_S) = T_S\phi(h, \cdot)v_S = T_S\big(\rho_{h^{-1}} \circ (\cdot)\big)v_S = \rho_{h^{-1}} \circ v_S.$$

Thus the desired result follows by the definition of the equivalence relation in $TG \times_{TH} T\mathrm{G}(V)$, i.e. $(v_g, v_S) \sim (v'_g, v'_S) \in TG \times T\mathrm{G}(V)$ iff there exists an $h \in H$ and $v_h \in T_h H$ such that

$$v'_g = T_{(g,h)}(\cdot \triangleleft \cdot)(v_g, v_h) \quad \text{and} \quad v'_S = T_{(h,S)}\phi(v_h, v_S)$$

holds. □

Corollary 5.6 *Let G/H be a reductive homogeneous space and let $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ be a closed subgroup such that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$. Let $(v_g, v_S) \in TG \times T\mathrm{G}(\mathfrak{m})$ and $h \in H$. Then*

$$[v_g, v_S] = [T_g r_h v_g, \mathrm{Ad}_{h^{-1}} \circ v_S] \in TG \times_{TH} \mathrm{G}(\mathfrak{m}) \tag{5.19}$$

is fulfilled.

Proof Applying Lemma 5.5 to the representation $H \ni h \mapsto \mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{GL}(\mathfrak{m})$ yields the desired result because of $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ for all $h \in H$. □

In order to determine the distribution on Q which characterizes rollings of $\mathfrak{m}$ over G/H with respect to ∇^α, we first define a distribution on $\overline{Q}$. Afterwards, this distribution is used to obtain the desired distribution on the configuration space Q.

Lemma 5.7 *Let G/H be a reductive homogenoues space. Moreover, let $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ be a closed subgroup and let $\mathfrak{g}(\mathfrak{m}) \subseteq \mathfrak{gl}(\mathfrak{m})$ denote its Lie algebra. Assume that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$ and let $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map such that for each $X \in \mathfrak{m}$ the linear map*

$$\alpha(X, \cdot) \colon \mathfrak{m} \to \mathfrak{m}, \quad Y \mapsto \alpha(X, \cdot)(Y) = \alpha(X, Y) \tag{5.20}$$

is an element in $\mathfrak{g}(\mathfrak{m})$, i.e. $\alpha(X, \cdot) \in \mathfrak{g}(\mathfrak{m})$. Moreover, let $\overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$ as in Proposition 5.4 and define

$$\overline{\Psi^\alpha} \colon \overline{Q} \times \mathfrak{m} \to T\overline{Q}, \quad (\overline{q}, u) = ((v, g, S), u) \mapsto \big(u, (T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S\big). \tag{5.21}$$

Then $\overline{\Psi^\alpha}$ is a morphism of vector bundles covering $\mathrm{id}_{\overline{Q}} \colon \overline{Q} \to \overline{Q}$ and $\overline{D^\alpha} = \mathrm{im}(\overline{\Psi^\alpha}) \subseteq T\overline{Q}$ is a regular distribution on $\overline{Q}$ given fiber-wise by

$$\overline{D^\alpha}_{(v,g,S)} = \big\{(u, (T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S) \mid u \in T_v\mathfrak{m} \cong \mathfrak{m}\big\} \subseteq T_{(v,g,S)}\overline{Q} \tag{5.22}$$

for all $(v, g, S) \in \overline{Q}$. Moreover, $\overline{D^\alpha}$ is contained in the the horizontal bundle defined by the principal connection $\overline{\mathcal{P}}$ from Proposition 5.4, i.e.

$$\overline{\mathcal{P}}\big|_{(v,g,S)}(u, v_g, v_S) = 0 \quad \textit{for all} \quad (v, g, S) \in Q, \quad (u, v_g, v_S) \in \overline{D^\alpha}_{(v,g,S)} \tag{5.23}$$

is fulfilled.

Proof The image of $\overline{\Psi^\alpha}$ defined by (5.21) is contained in $T\overline{Q}$. Indeed, by the assumption on $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, we have $\alpha(Su, \cdot) \in \mathfrak{g}(\mathfrak{m})$ for $S \in \mathrm{G}(\mathfrak{m})$ and $u \in \mathfrak{m}$. Hence we obtain

$$\alpha(Su, \cdot) \circ S = (T_{\mathrm{id}_{\mathfrak{m}}} r_S)(\alpha(Su, \cdot)) \in T_S\mathrm{G}(\mathfrak{m})$$

proving $\overline{\Psi^\alpha}((v, g, S), u) \in T_v\mathfrak{m} \times T_gG \times T_S\mathrm{G}(\mathfrak{m}) \cong T_{(v,g,S)}\overline{Q}$ for all $((v, g, S), u) \in \overline{Q} \times \mathfrak{m}$. Thus $\overline{\Psi^\alpha}$ is clearly a smooth vector bundle morphism covering the identity. Furthermore, the rank of $\overline{\Psi^\alpha}$ is obviously constant. Hence its image $\overline{D^\alpha} = \mathrm{im}(\overline{\Psi^\alpha})$

is a vector subbundle of $T\overline{Q}$ by [14, Thm. 10.34]. The fiber-wise description of $\overline{D^\alpha}$ in (5.22) holds by the definition of $\overline{\Psi^\alpha}$ due to $\overline{D^\alpha} = \mathrm{im}(\overline{\Psi^\alpha})$.

We now show that $\overline{D^\alpha}$ is contained in the horizontal bundle. Obviously, this is equivalent to $\overline{\mathcal{P}}\big|_{(v,g,S)}(u, v_g, v_S) = 0$ for all $(v, g, S) \in \overline{Q}$ and $(u, v_g, v_S) \in \overline{D^\alpha}_{(v,g,S)}$. Using the definition of $\overline{\mathcal{P}} \in \Gamma^\infty\big(\mathrm{End}(T\overline{Q})\big)$ from Proposition 5.4 and writing $(u, v_g, v_S) \in \overline{D^\alpha}_{(v,g,S)}$ as

$$(u, v_g, v_S) = (u, (T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S)$$

for some $u \in \mathfrak{m}$, we obtain

$$\begin{aligned}\overline{\mathcal{P}}\big|_{(v,g,S)}(u, v_g, v_S) &= \overline{\mathcal{P}}\big|_{(v,g,S)}(u, (T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S) \\ &= (0, \mathcal{P}\big|_g(v_g), -\mathrm{ad}_{\omega|_g(v_g)} \circ S) \\ &= (0, 0, 0)\end{aligned}$$

due to $v_g = (T_e\ell_g \circ S)u \in \mathrm{Hor}(G)_g$ because of $Su \in \mathfrak{m}$, where we used $\mathcal{P}\big|_g(v_g) = 0$ as well as $\omega\big|_g(v_g) = 0$ by the definitions of $\mathcal{P} \in \Gamma^\infty\big(\mathrm{End}(TG)\big)$ and $\omega \in \Gamma^\infty(T^*G) \otimes \mathfrak{h}$ in Proposition 2.10. □

Next we use the distribution $\overline{D^\alpha}$ on $\overline{Q}$ to construct the desired distribution on Q.

Lemma 5.8 *Using the notations and assumptions of Lemma 5.7, we define $D^\alpha \subseteq TQ$ by*

$$D^\alpha = (T\overline{\pi})(\overline{D^\alpha}) \subseteq TQ. \tag{5.24}$$

Then the following assertions are fulfilled:

1. *Let $(v, [g, S]) \in Q$. Then D^α is fiber-wise given by*

$$D^\alpha_{(v,[g,S])} = \big\{(u, [(T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S]) \mid u \in T_v\mathfrak{m} \cong \mathfrak{m}\big\} \subseteq T_{(v,[g,S])}Q \tag{5.25}$$

   using the identification (5.15) implicitly.
2. *Let $(v, g, S) \in \overline{Q}$. Then the map*

$$T_{(v,g,S)}\overline{\pi}\big|_{\overline{D^\alpha}_{(v,g,S)}} : \overline{D^\alpha}_{(v,g,S)} \to D^\alpha_{(v,[g,S])} \tag{5.26}$$

   is a linear isomorphism.
3. *Let $q : I \to Q$ be a curve and let $\overline{q} : I \to \overline{Q}$ denote a horizontal lift of q with respect to the principal connection from Proposition 5.4. Then q is horizontal with respect to D^α, i.e. $\dot{q}(t) \in D^\alpha_{q(t)}$ iff $\overline{q}$ is horizontal with respect to $\overline{D^\alpha}$, i.e. $\dot{\overline{q}}(t) \in \overline{D^\alpha}_{\overline{q}(t)}$.*
4. *D^α is the image of the morphism of vector bundles*

$$\Psi^\alpha : Q \times \mathfrak{m} \to TQ, \quad ((v, [g, S]), u) \mapsto (u, [(T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S]) \tag{5.27}$$

over $\mathrm{id}_Q \colon Q \to Q$ *of constant rank. In particular,* D^α *is a regular distribution on* Q.

Proof We start with determining D^α point-wise. Let $(v, g, S) \in \overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$ and $(u, v_g, v_S) \in T\overline{Q} \cong T\mathfrak{m} \times TG \times T\mathrm{G}(\mathfrak{m})$. Then

$$\overline{\pi}(v, g, S) = (v, [g, S]) \quad \text{and} \quad T\overline{\pi}(u, v_g, v_S) = (u, [v_g, v_S]) \tag{5.28}$$

holds by the identification (5.15). Evaluating (5.28) at $(u, v_g, v_S) \in \overline{D^\alpha}_{(v,g,S)}$, i.e.

$$(u, v_g, v_S) = (u, (T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S)$$

for some $u \in \mathfrak{m}$, yields Claim 1. because of $D^\alpha_{\overline{\pi}(v,g,S)} = (T\overline{\pi})\big(\overline{D^\alpha}_{(v,g,S)}\big)$.

Next we show Claim 2., i.e. that the restriction

$$T_{(v,g,S)}\overline{\pi}\big|_{\overline{D^\alpha}_{(v,g,S)}} \colon \overline{D^\alpha}_{(v,g,S)} \to D^\alpha_{(v,[g,S])} \tag{5.29}$$

is bijective. Clearly, the linear map in (5.29) is injective since $\overline{D^\alpha}_{(v,g,S)} \subseteq \mathrm{Hor}(\overline{Q})_{(v,g,S)}$ holds according to Lemma 5.7. We now show that (5.29) is surjective. Let $h \in H$. Moreover, let $(v, g, S) \in \overline{Q}$ and $(v, gh, \mathrm{Ad}_{h^{-1}} \circ S) \in \overline{Q}$ be two representatives of

$$\overline{\pi}\big(v, g, S\big) = (v, [g, S]) = (v, [gh, \mathrm{Ad}_{h^{-1}} \circ S]) = \overline{\pi}\big(v, gh, \mathrm{Ad}_{h^{-1}} \circ S\big) \in Q$$

and let $(u, v_g, v_S) \in \overline{D^\alpha}_{(v,g,S)}$. We show that there exists a $(u, v'_g, v'_S) \in \overline{D^\alpha}_{(v,gh,\mathrm{Ad}_{h^{-1}}\circ S)}$ such that

$$T_{(v,g,S)}\overline{\pi}(u, v_g, v_S) = T_{(v,gh,\mathrm{Ad}_{h^{-1}}\circ S)}\overline{\pi}(u, v'_g, v'_S)$$

holds. To this end, we define

$$v'_g = T_g(\cdot \triangleleft h)v_g = T_g r_h v_g \quad \text{and} \quad v'_S = T_S\big(\mathrm{Ad}_{h^{-1}}(\cdot)\big) \circ v_S = \mathrm{Ad}_{h^{-1}} \circ v_S. \tag{5.30}$$

By using $(u, v_g, v_S) \in \overline{D^\alpha}_{(v,g,S)}$, i.e.

$$v_g = (T_e\ell_g \circ S)u \quad \text{and} \quad v_S = -\alpha(Su, \cdot) \circ S \tag{5.31}$$

for some $u \in \mathfrak{m}$, we show that $(u, v'_g, v'_S) \in \overline{D^\alpha}_{(v,gh,\mathrm{Ad}_{h^{-1}}\circ S)}$ holds. To this end, we calculate

$$\begin{aligned} v'_g &= T_g r_h v_g \\ &= T_g r_h\big((T_e\ell_g \circ S)u\big) \\ &= T_e(r_h \circ \ell_g)Su \\ &= T_e(\ell_g \circ r_h)Su \\ &= T_e(\ell_{gh} \circ \ell_{h^{-1}} \circ r_h)Su \\ &= T_e\ell_{gh} \circ T_e(\ell_{h^{-1}} \circ r_h)Su \\ &= (T_e\ell_{gh} \circ \mathrm{Ad}_{h^{-1}} \circ S)u. \end{aligned} \tag{5.32}$$

Moreover, using the definition of v'_S in (5.30) and v_S in (5.31), we have by the $\mathrm{Ad}(H)$-invariance of $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$

$$\begin{aligned} v'_S &= \mathrm{Ad}_{h^{-1}} \circ v_S \\ &= \mathrm{Ad}_{h^{-1}} \circ \big(-\alpha(Su, \cdot)\big) \circ S \\ &= -\alpha(\mathrm{Ad}_{h^{-1}}(Su), \cdot) \circ \mathrm{Ad}_{h^{-1}} \circ S. \end{aligned} \tag{5.33}$$

By (5.32) and (5.33), we obtain

$$(u, v'_g, v'_S) = (u, (T_e\ell_{gh} \circ \mathrm{Ad}_{h^{-1}} \circ S)u, -\alpha(\mathrm{Ad}_{h^{-1}}(Su), \cdot) \circ \mathrm{Ad}_{h^{-1}} \circ S) \tag{5.34}$$

showing $(u, v'_g, v'_S) \in \overline{D^\alpha}_{(v,gh,\mathrm{Ad}_{h^{-1}}\circ S)}$ as desired. Equation (5.34) implies

$$\begin{aligned} T_{(v,g,S)}\overline{\pi}(u, v_g, v_S) &= (u, [v_g, v_S]) \\ &= (u, [T_g r_h v_g, \mathrm{Ad}_{h^{-1}} \circ v_S]) \\ &= T_{(v,gh,\mathrm{Ad}_{h^{-1}}\circ S)}\overline{\pi}(u, v'_g, v'_S) \end{aligned}$$

due to Corollary 5.6. Thus the linear map (5.29) is surjective. Hence Claim 2. is proven.

Next let $q\colon I \to Q$ be a curve and let $\overline{q}\colon I \to \overline{Q}$ be a horizontal lift with respect to the principal connection $\overline{\mathcal{P}}$ from Proposition 5.4. In particular $\overline{\pi}(\overline{q}(t)) = q(t)$ holds. Assume that $\dot{\overline{q}}(t) \in \overline{D^\alpha}_{\overline{q}(t)}$ holds. This assumption yields

$$\dot{q}(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}(\overline{\pi} \circ \overline{q})(t) = (T_{\overline{q}(t)}\overline{\pi})\dot{\overline{q}}(t) \in (T_{\overline{q}(t)}\overline{\pi})\big(\overline{D^\alpha}_{\overline{q}(t)}\big) = D^\alpha_{q(t)}$$

by the definition of D^α since $\overline{q}(t)$ is horizontal with respect to $\overline{D^\alpha}$. Conversely, assume that $\dot{q}(t) \in D^\alpha_{q(t)}$ holds. Then $\dot{\overline{q}}(t) \in T_{q(t)}\overline{Q}$ is the unique horizontal tangent vector which fulfills $(T_{\overline{q}(t)}\overline{\pi})\dot{\overline{q}}(t) = \dot{q}(t)$ or equivalently

$$\dot{\overline{q}}(t) = \big(T_{\overline{q}(t)}\overline{\pi}\big|_{\mathrm{Hor}(\overline{Q})_{\overline{q}(t)}}\big)^{-1}\dot{q}(t).$$

Since (5.29) is a linear isomorphism, we obtain $\big(T_{\overline{q}(t)}\overline{\pi}\big|_{\overline{D^\alpha}_{\overline{q}(t)}}\big)^{-1}(D^\alpha_{q(t)}) = \overline{D^\alpha}_{\overline{q}(t)}$. This yields Claim 3. because of $\dot{q}(t) \in D^\alpha_{q(t)}$ and $\overline{D^\alpha} \subseteq \mathrm{Hor}(\overline{Q})$.

It remains to proof Claim 4.. To this end, using the identification (5.15), we consider the diagram

$$
\begin{array}{ccc}
\overline{Q} \times \mathfrak{m} & \xrightarrow{\quad \overline{\Psi^\alpha} \quad} & T\overline{Q} \\
{\scriptstyle \overline{\pi} \times \mathrm{id}_{\mathfrak{m}}} \big\downarrow & & \big\downarrow {\scriptstyle T\overline{\pi}} \\
Q \times \mathfrak{m} & \xrightarrow{\quad \Psi^\alpha \quad} & TQ
\end{array}
\tag{5.35}
$$

which clearly commutes. Thus Ψ^α is smooth since $T\overline{\pi} \circ \overline{\Psi^\alpha}$ is smooth and $\overline{\pi} \times \mathrm{id}_{\mathfrak{m}}$ is a surjective submersion. In addition, Ψ^α is fiber-wise linear, i.e. Ψ^α is a vector bundle morphism covering $\mathrm{id}_Q \colon Q \to Q$. Moreover, Claim 2. implies that the rank of Ψ is constant. Hence the image of Ψ^α is a subbundle of TQ according to [14, Thm. 10.34]. In addition, we obtain $D^\alpha = \mathrm{im}(\Psi^\alpha)$ due to

$$
D^\alpha = (T\overline{\pi})(\overline{D^\alpha}) = (T\overline{\pi} \circ \overline{\Psi^\alpha})(\overline{Q} \times \mathfrak{m}) = (\Psi^\alpha \circ (\overline{\pi} \times \mathrm{id}_{\mathfrak{m}}))(\overline{Q} \times \mathfrak{m}) = \Psi^\alpha(Q \times \mathfrak{m})
\tag{5.36}
$$

since (5.35) commutes. This yields the desired result. □

Theorem 5.9 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and let $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ be a closed subgroup such that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$. Moreover, let $\alpha \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map defining the invariant covariant derivative ∇^α such that for each $X \in \mathfrak{m}$ the linear map*

$$
\alpha(X, \cdot) \colon \mathfrak{m} \to \mathfrak{m}, \quad Y \mapsto \alpha(X, Y)
\tag{5.37}
$$

belongs to $\mathfrak{g}(\mathfrak{m})$, i.e. to the Lie algebra of $\mathrm{G}(\mathfrak{m})$. Let $\overline{D^\alpha}$ denote the distribution on $\overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$ from Lemma 5.7 associated to α and let $D^\alpha = T\overline{\pi}(\overline{D^\alpha})$ be the distribution defined in Lemma 5.8. Then the following assertions are fulfilled:

1. *Let $q \colon I \to Q$ and let*

$$
(v, \gamma) \colon I \to \mathfrak{m} \times G/H, \quad t \mapsto (\pi \circ q)(t) = (v(t), \gamma(t)).
\tag{5.38}
$$

   Let $\overline{q} \colon I \ni t \mapsto (v(t), g(t), S(t)) \in \overline{Q}$ be a horizontal lift of q with respect to the principal connection $\overline{\mathcal{P}}$ from Proposition 5.4. Then q is horizontal with respect to D^α iff the ODE

$$
\begin{aligned}
\dot{S}(t) &= -\alpha\big(S(t)\dot{v}(t), \cdot\big) \circ S(t), \\
\dot{g}(t) &= \big(T_e \ell_{g(t)} \circ S(t)\big)\dot{v}(t)
\end{aligned}
\tag{5.39}
$$

   is fulfilled. Moreover, the development curve is given by $\gamma = \mathrm{pr} \circ g \colon I \to G/H$.
2. *Let $q \colon I \to Q$ be a curve and let $(v, \gamma) = (\pi \circ q) \colon I \to \mathfrak{m} \times G/H$. Then q is horizontal with respect to D^α, i.e. $\dot{q}(t) \in D^\alpha_{q(t)}$, iff q defines a ($\mathrm{G}(\mathfrak{m})$-reduced) intrinsic rolling of $\mathfrak{m}$ over G/H with respect to ∇^α with rolling curve v and development curve γ.*

Proof We first show Claim 1.. Let $q\colon I \to Q$ be some curve and let

$$\overline{q}\colon I \to \overline{Q}, \quad t \mapsto (v(t), g(t), S(t)) \tag{5.40}$$

be a horizontal lift of q with respect to the principal connection $\overline{\mathcal{P}}$ from Proposition 5.4. Clearly, the development curve $\gamma\colon I \to G/H$ defined by $q\colon I \to Q$ is given by $\gamma = \mathrm{pr} \circ g$. Moreover, by Lemma 5.8, Claim 3., q is horizontal with respect to D^α iff $\overline{q}$ is horizontal with respect to $\overline{D^\alpha}$. Hence it is sufficient to show that $\overline{q}$ from (5.40) fulfills the ODE (5.39) iff $\overline{q}$ is horizontal with respect to $\overline{D^\alpha}$. First we assume $\dot{\overline{q}}(t) \in \overline{D^\alpha}_{\overline{q}(t)}$ for all $t \in I$. Writing $\dot{\overline{q}}(t) = (\dot{v}(t), \dot{g}(t), \dot{S}(t))$ and using the definition of $\overline{D^\alpha}$, one obtains

$$\dot{\overline{q}}(t) \in \overline{D^\alpha}_{\overline{q}(t)} = \big\{(u, \big(T_e\ell_{g(t)} \circ S(t)\big)u, -\alpha(S(t)u, \cdot) \circ S(t)) \mid u \in T_{v(t)}\mathfrak{m} \cong \mathfrak{m}\big\}. \tag{5.41}$$

Thus $\dot{g}(t)$ and $\dot{S}(t)$ are uniquely determined by

$$\dot{g}(t) = \big(T_e\ell_{g(t)} \circ S(t)\big)\dot{v}(t) \quad \text{and} \quad \dot{S}(t) = -\alpha\big(S(t)\dot{v}(t), \cdot\big) \circ S(t)$$

due to (5.41). Hence the curve $\overline{q}\colon I \to \overline{Q}$ which is horizontal with respect to $\overline{D^\alpha}$ fulfills the ODE (5.39). Conversely, assume that $\overline{q}\colon I \to \overline{Q}$ given by $\overline{q}(t) = (v(t), g(t), S(t))$ fulfills (5.39). Then $\overline{q}(t)$ is clearly horizontal with respect to $\overline{D^\alpha}$ by the definition of $\overline{D^\alpha}$. Thus Claim 1. is proven.

Next we show Claim 2.. To this end, let $q\colon I \to Q$ be horizontal with respect to D^α. Then a horizontal lift $\overline{q}\colon I \ni t \mapsto \overline{q}(t) = (v(t), g(t), S(t)) \in \overline{Q}$ of q fulfills the ODE (5.39) by Claim 1.. Moreover, $q\colon I \to Q$ can be represented by

$$q(t) = (\overline{\pi} \circ \overline{q})(t) = (v(t), [g(t), S(t)]), \quad t \in I.$$

Hence the linear isomorphism associated with $q(t)$ is given by

$$T_{v(t)}\mathfrak{m} \cong \mathfrak{m} \ni Z \mapsto q(t)Z = \big(T_{g(t)}\mathrm{pr} \circ T_e\ell_{g(t)} \circ S(t)\big)Z \in T_{\mathrm{pr}(g(t))}(G/H) \tag{5.42}$$

according to Lemma 5.2, Claim 2.. Using(5.39) and (5.42) we obtain

$$\dot{\gamma}(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{pr} \circ g)(t) = T_{g(t)}\mathrm{pr}\dot{g}(t) = \big(T_{g(t)}\mathrm{pr}\big)\Big((T_e\ell_{g(t)} \circ S(t))\dot{v}(t)\Big) = q(t)\dot{v}(t)$$

showing the no-slip condition. Next we prove the no-twist condition. Let

$$Z\colon I \to T\mathfrak{m}, \quad t \mapsto (v(t), Z_2(t)) \in T\mathfrak{m}$$

be a vector field along $v\colon I \to \mathfrak{m}$ which we identify with the map $I \ni t \mapsto Z_2(t) \in \mathfrak{m}$ defined by its second component. Then Z is parallel along v iff $\dot{Z}_2(t) = 0$ holds, i.e. $Z_2(t) = Z_0$ for all $t \in I$ and some $Z_0 \in \mathfrak{m}$. We need to show that the vector field

$$\widehat{Z}\colon I \to T(G/H), \quad t \mapsto q(t)Z(t) = q(t)Z_0 = \big(T_{g(t)}\mathrm{pr} \circ T_e\ell_{g(t)} \circ S(t)\big)Z_0$$

is parallel along $\gamma\colon I \to G/H$ with respect to ∇^α. By Proposition 5.4, Claim 4., the curve $g\colon I \to G$ is a horizontal lift of the curve $\gamma\colon I \ni t \mapsto \mathrm{pr}(g(t)) \in G/H$. In addition, we have

$$S(t)\dot{v}(t) = \big(T_e\ell_{g(t)}\big)^{-1}\dot{g}(t)$$

by (5.39). Moreover, the horizontal lift of $\widehat{Z}$ along $g\colon I \to G$ is given by

$$\overline{Z}\colon I \to \mathrm{Hor}(G), \quad t \mapsto \bar{Z}(t) = \big(T_{g(t)}\mathrm{pr}\big|_{\mathrm{Hor}(G)}\big)^{-1}\widehat{Z}(t) = \big(T_e\ell_{g(t)} \circ S(t)\big)Z_0$$

and the curve $z\colon I \ni t \mapsto z(t) = \big(T_e\ell_{g(t)}\big)^{-1}\overline{Z}(t) \in \mathfrak{m}$ fulfills

$$z(t) = \big(T_e\ell_{g(t)}\big)^{-1}\big(T_e\ell_{g(t)} \circ S(t)\big)Z_0 = S(t)Z_0.$$

Thus we obtain by exploiting (5.39)

$$\dot{z}(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}(S(t)Z_0) = \dot{S}(t)Z_0 = -\big(\alpha\big(S(t)\dot{v}(t), \cdot\big) \circ S(t)\big)Z_0 = -\alpha\big(S(t)\dot{v}(t), z(t)\big).$$

Hence $\widehat{Z}$ is parallel along γ by Proposition 2.19.

Conversely, assume that $\widehat{Z}(t) = q(t)Z(t)$ is parallel along $\gamma\colon I \to G/H$, where $Z\colon I \ni t \mapsto (v(t), Z_2(t)) \in T\mathfrak{m}$ is some vector field along along $v\colon I \to \mathfrak{m}$ which we identify with the map $I \ni t \mapsto Z_2(t) \in \mathfrak{m}$. Let $\{A_1, \ldots, A_N\} \subseteq \mathfrak{m}$ be some basis of $\mathfrak{m}$. We define a parallel frame along $\gamma\colon I \to G/H$ by $A_i(t) = q(t)A_i$ for $i \in \{1, \ldots, N\}$ and $t \in I$. Then $\widehat{Z}\colon I \to T(G/H)$ is parallel along γ iff its coefficient functions $z^i\colon I \to \mathbb{R}$ defined by $\widehat{Z}(t) = z^i(t)A_i(t)$ are constant, i.e. $z^i(t) = z_0^i$ for all $t \in I$ with some $z_0^i \in \mathbb{R}$, see e.g. [22, Chap. 4, p. 109]. By the linearity of $q(t)\colon T_{v(t)}\mathfrak{m} \cong \mathfrak{m} \to T_{\gamma(t)}(G/H)$, we obtain

$$\widehat{Z}(t) = z_0^i A_i(t) = z_0^i\big(q(t)A_i\big) = q(t)\big(z_0^i A_i\big) = q(t)Z_0 = q(t)Z_2(t)$$

showing $Z_2(t) = z_0^i A_i = Z_0$ for $t \in I$, where $Z_0 = z_0^i A_i \in \mathfrak{m}$ is constant. Hence $Z\colon I \ni t \mapsto (v(t), Z_2(t)) = (v(t), Z_0) \in \mathfrak{m} \times \mathfrak{m} \cong T\mathfrak{m}$ is a parallel vector field along the curve $v\colon I \to \mathfrak{m}$. Thus the curve $q\colon I \to Q$ which is horizontal with respect to D^α is a rolling.

It remains to prove the converse. Let $q\colon I \to Q$ be a curve defining a rolling. We show that q is horizontal with respect to D^α.

Let $\overline{q}\colon I \to Q$ be a horizontal lift of q with respect to the principal connection from Proposition 5.4. By Lemma 5.8, Claim 3. q is horizontal with respect to D^α iff $\overline{q}$ is horizontal with respect to $\overline{D^\alpha}$. Writing $\overline{q}(t) = (v(t), g(t), S(t))$, the linear isomorphism $T_{v(t)}\mathfrak{m} \cong \mathfrak{m} \to T_{\mathrm{pr}(g(t))}(G/H)$ defined by $q(t) = \overline{\pi}(\overline{q}(t))$ is given by

$$q(t)Z = \big(T_{g(t)}\mathrm{pr} \circ T_e\ell_{g(t)} \circ S(t)\big)Z, \quad Z \in T_{v(t)}\mathfrak{m} \cong \mathfrak{m}$$

according to Lemma 5.2, Claim 2.. Hence the no slip condition yields

$$\dot{\gamma}(t) = \big(T_{g(t)}\mathrm{pr} \circ T_e\ell_{g(t)} \circ S(t)\big)\dot{v}(t). \tag{5.43}$$

By Proposition 5.4, Claim 3. the curve $g\colon I \to G$ is horizontal with respect to $\mathrm{Hor}(G)$ from Proposition 2.10. In addition, $\gamma = \mathrm{pr} \circ g$ holds, i.e. $g\colon I \to G$ is a horizontal lift of $\gamma\colon I \to G/H$. Thus $g\colon I \to G$ fulfills the ODE $\dot{g}(t) = \big(T_e\ell_{g(t)} \circ S(t)\big)\dot{v}(t)$ by (5.43). Moreover, since $g\colon I \to G$ is a horizontal lift of $\gamma\colon I \to G/H$, the no twist condition yields

$$S(t)Z_0 = -\alpha\big(S(t)\dot{v}(t), S(t)Z_0\big) \tag{5.44}$$

for all $Z_0 \in \mathfrak{m}$ by Proposition 2.19. Clearly, (5.44) is equivalent to the ODE

$$\dot{S}(t) = -\alpha\big(S(t)\dot{v}(t), \cdot\big) \circ S(t)$$

for $S\colon I \to \mathrm{G}(\mathfrak{m})$. Therefore $q\colon I \to Q$ is horizontal with respect to D^α by Claim 1. □

In particular, Theorem 5.9 applies to (pseudo-)Riemannian reductive homogeneous spaces. We comment on this particular situation in the next remark.

Remark 5.10 Let G/H be a reductive homogeneous space equipped with an invariant pseudo-Riemannian metric and let $\langle\cdot,\cdot\rangle\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ denote the corresponding $\mathrm{Ad}(H)$-invariant scalar product. Moreover, let ∇^α be a *metric* invariant covariant derivative on G/H. Then Proposition 2.20 yields $\alpha(X, \cdot) \in \mathfrak{so}(\mathfrak{m})$ for all $X \in \mathfrak{m}$. Thus Theorem 5.9 can be applied to $\mathrm{G}(\mathfrak{m}) = \mathrm{O}(\mathfrak{m})$, i.e. the configuration space can be reduced to $Q = \mathfrak{m} \times (G \times_H \mathrm{O}(\mathfrak{m}))$ since $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{O}(\mathfrak{m})$ holds for all $h \in H$.

Remark 5.10 can be specialized further to naturally reductive homogeneous space equipped with the Levi-Civita covariant derivative.

Remark 5.11 Let G/H be a naturally reductive homogeneous space. Then Theorem 5.9 can be applied to G/H equipped with ∇^{LC}, where the configuration space can be reduced to $Q = \mathfrak{m} \times (G \times_H \mathrm{O}(\mathfrak{m}))$ and $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is given by $\alpha(X, Y) = \frac{1}{2}\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X(Y)$ for $X, Y \in \mathfrak{m}$ since $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can1}}$ holds by Remark 2.22.

5.3 Kinematic equations and control theoretic perspective

Throughout this section we denote by G/H a reductive homogeneous space and we assume that $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ is a closed subgroup such that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds. Moreover, we assume that the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ fulfills $\alpha(X, \cdot) \in \mathfrak{g}(\mathfrak{m})$ for all $X \in \mathfrak{m}$. If not indicated otherwise, we consider the "reduced" configuration space $Q = \mathfrak{m} \times (G \times_H \mathrm{G}(\mathfrak{m}))$.

We start with relating rollings of $\mathfrak{m}$ over G/H to a control system.

Remark 5.12 Let G/H be a reductive homogeneous space equipped with an invariant covariant derivative ∇^α defined by the $\mathrm{Ad}(H)$-invariant bilinear map $\alpha\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. Recall the definition of the morphism of vector bundles Ψ^α in Lemma 5.8, Claim 4., i.e.

$$\begin{aligned} &\Psi^\alpha\colon Q \times \mathfrak{m} \to TQ, \\ &((v, [g, S]), u) \mapsto \Psi^\alpha((v, [g, S]), u) = (u, [(T_e\ell_g \circ S)u, -\alpha(Su, \cdot) \circ S]). \end{aligned} \tag{5.45}$$

Then Ψ^α defines a control system in the sense of [23, p. 21] with state space Q and control set $\mathfrak{m}$. Obviously, for each $u \in \mathfrak{m}$, the map $\Psi^\alpha(\cdot, u)\colon Q \to TQ$ is a section of D^α, where $D^\alpha \subseteq TQ$ is the distribution characterizing the rolling of $\mathfrak{m}$ over G/H with respect to ∇^α.

Moreover, if G/H is equipped with an invariant pseudo-Riemannian metric and an invariant metric covariant derivative ∇^α, we can endow Q with an additional structure which is similar to a sub-Riemannian structure. We refer to [24, Def. 3.2] for a definition of sub-Riemannian structures.

Remark 5.13 Let G/H be a reductive homogeneous space equipped with a G-invariant pseudo-Riemannian metric corresponding to the scalar product $\langle\cdot,\cdot\rangle\colon \mathfrak{m}\times\mathfrak{m}\to\mathbb{R}$. Moreover, let $\alpha\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant bilinear map defining the *metric* invariant covariant derivative ∇^α. As in Remark 5.10, we set $Q = \mathfrak{m}\times(G\times_H \mathrm{O}(\mathfrak{m}))$. Moreover, motivated by [8, Eq. (3)], we equip the trivial vector bundle $Q\times\mathfrak{m}\to Q$ with the fiber metric $h\in\Gamma^\infty\big(\mathrm{S}^2(Q\times\mathfrak{m})^*\big)$ defined by $h_q(X,Y)=\langle X,Y\rangle$ for $q\in Q$ and $X, Y\in\mathfrak{m}$. Then the pair $(\Psi^\alpha, Q\times\mathfrak{m})$ is formally similar to a sub-Riemannian structure on Q except for the following facts:

1. The fiber metric on $Q\times\mathfrak{m}\to\mathfrak{m}$ is allowed to be indefinite.
2. In general, the manifold Q is *not* connected.
3. The distribution $D^\alpha=\mathrm{im}(\Psi)$ might be *not* bracket generating.

However, by imposing further restrictions on G/H, Q and $\langle\cdot,\cdot\rangle$ it might be possible to obtain a sub-Riemannian structure on Q. In particular, if we assume that G/H is a Riemannian reductive homogeneous space, the fiber metric h on Q is positive definite. Moreover, if we assume that G is connected and $\mathrm{Ad}_h\big|_{\mathfrak{m}}\colon\mathfrak{m}\to\mathfrak{m}$ is an orientation preserving isometry, i.e. $\mathrm{Ad}_h\big|_{\mathfrak{m}}\in\mathrm{SO}(\mathfrak{m})$ for all $h\in H$, the configuration space can be reduced to $Q=\mathfrak{m}\times(G\times_H\mathrm{SO}(\mathfrak{m}))$, which is obviously connected. Under these assumptions, the pair $(\Psi^\alpha, Q\times\mathfrak{m})$ defines a structure on Q which fulfills the requirements of a sub-Riemannian structure on Q in the sense of [24, Def. 3.2] except for the fact that D^α might be not bracket generating. Investigating conditions on G/H and α such that D^α is bracket generating is out of the scope of this text. Nevertheless, in this context, we refer to [8], where the controllability of rollings of oriented Riemannian manifolds are considered. Moreover, we mention [25], where optimal control problems associated to rollings of certain manifolds are considered.

Using terminologies of control theory, we call a curve $u\colon I\to\mathfrak{m}$ a control curve. Such a curve can be used to determine a rolling of $\mathfrak{m}$ over G/H, where the rolling curve $v\colon I\to\mathfrak{m}$ satisfies the ODE $\dot v(t)=u(t)$. Inspired by the terminology used in [2], we introduce a notion of a kinematic equation for rollings of $\mathfrak{m}$ over G/H with respect to ∇^α. To this end, we first state the following proposition.

Proposition 5.14 *Let* $u\colon I\to\mathfrak{m}$ *be a control curve and let* $\alpha\colon\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ *be an* $\mathrm{Ad}(H)$*-invariant bilinear map satisfying* $\alpha(X,\cdot)\in\mathfrak{g}(\mathfrak{m})$*. Moreover, let* $\overline{q}\colon I\ni t\mapsto$

$\overline{q}(t) = (v(t), g(t), S(t)) \in \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m}) = \overline{Q}$ *be a curve satisfying the ODE*

$$\begin{aligned} \dot{v}(t) &= u(t), \\ \dot{S}(t) &= -\alpha\big(S(t)u(t), \cdot\big) \circ S(t), \\ \dot{g}(t) &= \big(T_e\ell_{g(t)} \circ S(t)\big)u(t). \end{aligned} \tag{5.46}$$

Then $q\colon I \ni t \mapsto (\overline{\pi} \circ \overline{q})(t) \in Q$ *is a rolling of* $\mathfrak{m}$ *over* G/H *with respect to* ∇^α.

Proof The curve $q\colon I \ni t \mapsto (\overline{\pi} \circ \overline{q})(t) = (v(t), [g(t), S(t)]) \in Q$ is horizontal with respect to $D^\alpha \subseteq TQ$ by Theorem 5.9, Claim 1. because of $u(t) = \dot{v}(t)$ for all $t \in I$. Hence $q\colon I \to Q$ is a rolling of $\mathfrak{m}$ over G/H by Theorem 5.9, Claim 2.. □

Definition 5.15 The ODE (5.46) in Proposition 5.14 is called the kinematic equation for ($\mathrm{G}(\mathfrak{m})$-reduced) rollings of $\mathfrak{m}$ over G/H with respect to ∇^α. An initial value problem associated with the ODE (5.46) with some initial condition $(v(t_0), g(t_0), S(t_0)) \in \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$ for some $t_0 \in I$ is called kinematic equation, as well.

In the sequel, we are mainly interested in the initial value problem associated with (5.46) defined by the initial condition $(v(0), g(0), S(0)) = (0, e, \mathrm{id}_\mathfrak{m}) \in \overline{Q} = \mathfrak{m} \times G \times \mathrm{G}(\mathfrak{m})$.

Remark 5.16 By specializing ∇^α in Definition 5.15, one obtains:

1. The kinematic equation with respect to $\nabla^{\mathrm{can}1}$ reads

$$\begin{aligned} \dot{v}(t) &= u(t), \\ \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \\ \dot{g}(t) &= \big(T_e\ell_{g(t)} \circ S(t)\big)u(t). \end{aligned} \tag{5.47}$$

2. For $\nabla^{\mathrm{can}2}$ one obtains the kinematic equation

$$\begin{aligned} \dot{v}(t) &= u(t), \\ \dot{S}(t) &= 0, \\ \dot{g}(t) &= \big(T_e\ell_{g(t)} \circ S(0)\big)u(t) \end{aligned} \tag{5.48}$$

since $\dot{S}(t) = 0$ implies $S(t) = S(t_0)$ for all $t \in I$ and some $t_0 \in I$. We point out that setting $S(t) = \mathrm{id}_\mathfrak{m}$ for all $t \in I$ yields an expression which is similar to the ODE describing rollings of a symmetric space over a flat space obtained [11, Sec. 4.2].

Next, we state the kinematic equation for a naturally reductive homogeneous space.

Corollary 5.17 *Let* G/H *be a naturally reductive homogeneous space and let* $\langle \cdot, \cdot \rangle\colon \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ *be the* $\mathrm{Ad}(H)$*-invariant scalar product corresponding to the pseudo-Riemannian metric. Then the kinematic equation for* ($\mathrm{O}(\mathfrak{m})$*-reduced) rollings of* $\mathfrak{m}$ *over* G/H *with respect to* ∇^{LC} *is given by (5.47) from Remark 5.16, Claim 1.. In particular, the curve* $S\colon I \to \mathrm{O}(\mathfrak{m})$ *takes values in the pseudo-Euclidean group of* $\mathfrak{m}$ *with respect to* $\langle \cdot, \cdot \rangle$ *provided that the initial condition lies in* $\mathrm{O}(\mathfrak{m})$.

Proof Since G/H has an invariant pseudo-Riemannian metric corresponding to the $\mathrm{Ad}(H)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, one obtains $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{O}(\mathfrak{m})$ for all $h \in H$. Moreover, Remark 5.16, Claim 1. yields the desired result since ∇^{LC} is metric and $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can1}}$ holds for naturally reductive homogeneous spaces by Remark 2.22. □

Remark 5.18 In general, it is not clear to us whether the maximal solution of the initial value problem

$$\dot{S}(t) = -\alpha\big(S(t)u(t),\cdot\big) \circ S(t), \quad S(0) = \mathrm{id}_{\mathfrak{m}} \tag{5.49}$$

associated with the kinematic equation in Proposition 5.14 is defined on the whole interval I. In principal, it could only be defined on a proper subinterval $I_1 \subsetneq I$.

However, if we assume that G/H is equipped with an invariant Riemannian metric, i.e. an invariant positive definite metric, ∇^α is metric and the control curve $u\colon \mathbb{R} \to \mathfrak{m}$ is bounded, we can prove that the time-independent vector field on $\mathbb{R}\times\mathrm{O}(\mathfrak{m})$ associated to (5.49), see e.g. [16, Sec. 3.30], given by

$$X(t,S) = \big(1, -\alpha\big(Su(t),\cdot\big) \circ S\big), \quad (t,S) \in \mathbb{R} \times \mathrm{O}(\mathfrak{m}) \tag{5.50}$$

is complete. To this end, we show that this vector field is bounded in a complete Riemannian metric on $\mathbb{R} \times \mathrm{O}(\mathfrak{m})$. Then the completeness of $X \in \Gamma^\infty\big(T(\mathbb{R} \times \mathrm{O}(\mathfrak{m}))\big)$ follows by [16, Prop. 23.9]. We view $\mathrm{O}(\mathfrak{m})$ as subset of $\mathrm{End}(\mathfrak{m})$ and denote by $\langle\cdot,\cdot\rangle\colon \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ the $\mathrm{Ad}(H)$-invariant inner product corresponding to the Riemannian metric on G/H. The norm on $\mathfrak{m}$ induced by $\langle\cdot,\cdot\rangle$ is denoted by $\|\cdot\|$. We denote an extension of these maps to $\mathfrak{g}$ by the same symbols. We now endow $\mathrm{End}(\mathfrak{m})$ with the Frobenius scalar product given by $\langle S, T\rangle_F = \mathrm{tr}(S^\top T)$, where $S^\top$ is the adjoint of S with respect to $\langle\cdot,\cdot\rangle$. Then $\langle\cdot,\cdot\rangle_F$ induces a bi-invariant and hence a complete Riemannian metric on $\mathrm{O}(\mathfrak{m})$. Moreover, the norm $\|\cdot\|_F$ defined by the Frobenius scalar product is equivalent to the operator norm $\|\cdot\|_2$. In particular, there is a $C > 0$ such that $\|S\|_F \le C\|S\|_2$ holds for all $S \in \mathrm{End}(\mathfrak{m})$. We now endow $\mathbb{R}\times\mathrm{O}(\mathfrak{m})$ with the Riemannian metric defined for $(s,S) \in \mathbb{R}\times\mathrm{O}(\mathfrak{m})$ and $(v,V),(w,W) \in T_{(s,S)}(\mathbb{R}\times\mathrm{O}(\mathfrak{m}))$ by

$$\big\langle (v,V),(w,W)\big\rangle_{(s,S)} = vw + \mathrm{tr}\big(V^\top W\big), \tag{5.51}$$

which is clearly complete. Moreover, $\alpha\colon \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$ is bounded since $\mathfrak{m}$ is finite dimensional. Hence there exists a $C' \ge 0$ with $\|\alpha(X,Y)\| \le C'\|X\|\|Y\|$ for $X, Y \in \mathfrak{m}$. Thus, for fixed $X \in \mathfrak{m}$, the operator norm of the linear map $\alpha(X,\cdot)\colon \mathfrak{m} \to \mathfrak{m}$ is bounded by

$$\|\alpha(X,\cdot)\|_2 \le C'\|X\|. \tag{5.52}$$

By this notation, we obtain

$$\begin{aligned}
\|X(t,S)\|^2 &= 1 + \big\|\alpha\big(Su(t),\cdot\big)\circ S\big\|_F^2 \\
&\le 1 + C^2\big\|\alpha\big(Su(t),\cdot\big)\circ S\big\|_2^2 \\
&\le 1 + C^2\big\|\alpha\big(Su(t),\cdot\big)\big\|_2^2\|S\|_2^2 \\
&\le 1 + (CC')^2\|S\|_2^2\|u(t)\|^2 \\
&\le 1 + (CC')^2\|u\|_\infty^2 < \infty,
\end{aligned} \tag{5.53}$$

where we exploited $\|S\|_2 = 1$ for all $S \in \mathrm{O}(\mathfrak{m})$ and $\|u\|_\infty$ denotes the supremum norm of $u\colon I \to \mathfrak{m}$. Equation (5.53) shows that $X \in \Gamma^\infty\big(T(\mathbb{R} \times \mathrm{O}(\mathfrak{m}))\big)$ is bounded in a complete Riemannian metric on $\mathbb{R} \times \mathrm{O}(\mathfrak{m})$ as desired. Thus the maximal solution of the initial value problem

$$\dot{S}(t) = -\alpha(S(t)u(t), \cdot) \circ S(t), \quad S(0) = S_0 \in \mathrm{O}(\mathfrak{m}) \tag{5.54}$$

is defined on $\mathbb{R}$.

5.4 Rolling along special curves

Next we consider a rolling of $\mathfrak{m}$ over G/H along a curve such that the development curve $\gamma\colon I \to G/H$ is the projection of a not necessarily horizontal one-parameter subgroup of G, i.e.

$$\gamma(t) = \mathrm{pr}(\exp(t\xi)), \quad t \in I \tag{5.55}$$

for some $\xi \in \mathfrak{g}$. In this subsection, we focus on the invariant covariant derivatives $\nabla^{\mathrm{can}1}$ and $\nabla^{\mathrm{can}2}$ on G/H. This discussion is motivated by the rolling and unwrapping technique for solving interpolation problems, see e.g. [2, 5], for which an explicit expression for a rolling along a curve connecting two points is desirable. A natural choice for such a curve would be a projection of a horizontal one-parameter subgroup in G, i.e. a geodesic with respect to $\nabla^{\mathrm{can}1}$ or $\nabla^{\mathrm{can}2}$. However, even if such a curve connecting two given points exists, as far as we know, in general, no closed-formula for such curves are known. In this context, we refer to [26], where the problem of connecting two points $X, X_1 \in \mathrm{St}_{n,k} \subseteq \mathbb{R}^{n\times k}$ on the Stiefel manifold $\mathrm{St}_{n,k}$ by a curve of the form $t \mapsto \mathrm{e}^{t\xi_1} X \mathrm{e}^{t\xi_2}$ with some suitable $(\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$ is addressed.

As a preparation for deriving the desired rollings, we state the following lemma.

Lemma 5.19 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, let $\xi \in \mathfrak{g}$ and let $\gamma\colon I \ni t \mapsto \mathrm{pr}(\exp(t\xi)) \in G/H$. Then the curve*

$$g\colon I \to G, \quad t \mapsto g(t) = \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) \tag{5.56}$$

is the horizontal lift of γ through $g(0) = e$ with respect to the principal connection from Proposition 2.10. Moreover, g is the solution of the initial value problem

$$\dot{g}(t) = T_e\ell_{g(t)}\big(\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}})\big), \quad g(0) = e. \tag{5.57}$$

Proof Obviously, $\gamma(t) = \mathrm{pr}(\exp(t\xi)) = \mathrm{pr}(g(t))$ holds for all $t \in I$ due to $\exp(-t\xi_{\mathfrak{h}}) \in H$ for all $t \in I$. Moreover, $g(0) = e$ is fulfilled. It remains to prove that

$g\colon I \to G$ is horizontal. To this end, we compute by exploiting (2.8) and (2.7)

$$
\begin{aligned}
\dot{g}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\exp(t\xi)\exp(-t\xi_{\mathfrak{h}})\big) \\
&= T_{\exp(t\xi)} r_{\exp(-t\xi_{\mathfrak{h}})} \tfrac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi) + T_{\exp(-t\xi_{\mathfrak{h}})} \ell_{\exp(t\xi)} \tfrac{\mathrm{d}}{\mathrm{d}t}\exp(-t\xi_{\mathfrak{h}}) \\
&= \big(T_{\exp(t\xi)} r_{\exp(-t\xi_{\mathfrak{h}})} \circ T_e \ell_{\exp(t\xi)}\big)\xi + \big(T_{\exp(-t\xi_{\mathfrak{h}})} \ell_{\exp(t\xi)} \circ T_e r_{\exp(-t\xi_{\mathfrak{h}})}\big)(-\xi_{\mathfrak{h}}) \\
&= T_e\big(r_{\exp(-t\xi_{\mathfrak{h}})} \circ \ell_{\exp(t\xi)}\big)\xi - T_e\big(\ell_{\exp(t\xi)} \circ r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{h}} \\
&= T_e\big(\ell_{\exp(t\xi)} \circ r_{\exp(-t\xi_{\mathfrak{h}})}\big)(\xi - \xi_{\mathfrak{h}}) \\
&= T_e\big(\ell_{\exp(t\xi)} \circ r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{m}}.
\end{aligned}
$$

Consequently, we obtain by the chain rule

$$
\begin{aligned}
(T_e\ell_{g(t)})^{-1}\dot{g}(t) &= \big(T_e\ell_{\exp(t\xi)\exp(-t\xi_{\mathfrak{h}})}\big)^{-1}\dot{g}(t) \\
&= \big(T_{\exp(-t\xi_{\mathfrak{h}})}\ell_{\exp(t\xi)} \circ T_e\ell_{\exp(-t\xi_{\mathfrak{h}})}\big)^{-1}\dot{g}(t) \\
&= \big(T_e\ell_{\exp(-t\xi_{\mathfrak{h}})}\big)^{-1} \circ \big(T_{\exp(-t\xi_{\mathfrak{h}})}\ell_{\exp(t\xi)}\big)^{-1}\dot{g}(t) \\
&= \big(T_e\ell_{\exp(-t\xi_{\mathfrak{h}})}\big)^{-1}\big(T_{\exp(t\xi)\exp(-t\xi_{\mathfrak{h}})}\ell_{\exp(-t\xi)}\big)\big(T_e\big(\ell_{\exp(t\xi)} \circ r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{m}}\big) \\
&= \big(T_e\ell_{\exp(-t\xi_{\mathfrak{h}})}\big)^{-1} T_e\big(\ell_{\exp(-t\xi)} \circ \ell_{\exp(t\xi)} \circ r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{m}} \\
&= \big(T_e\ell_{\exp(-t\xi_{\mathfrak{h}})}\big)^{-1} T_e\big(r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{m}} \\
&= \big(T_{\exp(-t\xi_{\mathfrak{h}})}\ell_{\exp(t\xi_{\mathfrak{h}})} \circ T_e r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{m}} \\
&= T_e\big(\ell_{\exp(t\xi_{\mathfrak{h}})} \circ r_{\exp(-t\xi_{\mathfrak{h}})}\big)\xi_{\mathfrak{m}} \\
&= \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}}).
\end{aligned}
$$

Since G/H is reductive, this implies

$$
(T_e\ell_{g(t)})^{-1}\dot{g}(t) = \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}}) \in \mathfrak{m} \tag{5.58}
$$

due to $\exp(t\xi_{\mathfrak{h}}) \in H$. Thus g is horizontal. Moreover, the curve $g\colon I \to G$ is a solution of the initial value problem (5.57) by (5.58). □

Remark 5.20 Let G be equipped with a bi-invariant metric which induces a positive definite fiber metric on $\mathrm{Hor}(G)$, i.e. a sub-Riemannian structure on G, and let $H \subseteq G$ be a closed subgroup such that its Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ is non-degenerated with respect to the scalar product corresponding to the metric. Then the curve given by $g(t) = \exp(t\xi)\exp(-t\xi_{\mathfrak{h}})$ from Lemma 5.19 is a sub-Riemannian geodesic on G according to [27, Sec. 11.3.7].

5.4.1 Rolling along special curves with respect to $\nabla^{\mathrm{can}1}$

We now derive an expression for a rolling of $\mathfrak{m}$ over G/H with respect to $\nabla^{\mathrm{can}1}$ such that the development curve is given by $\gamma\colon I \ni t \mapsto \mathrm{pr}(\exp(t\xi)) \in G/H$ for some $\xi \in \mathfrak{g}$. To this end, we determine a curve

$$
\overline{q}\colon I \to \overline{Q} = \mathfrak{m} \times G \times \mathrm{GL}(\mathfrak{m}), \quad t \mapsto (v(t), g(t), S(t)) \tag{5.59}
$$

which is horizontal with respect to the principal connection $\overline{\mathcal{P}}$ from Proposition 5.4, Claim 3. such that $q = \overline{\pi} \circ \overline{q}\colon I \to Q$ is the desired rolling. In particular,

$$\mathrm{pr}(g(t)) = \gamma(t) = \mathrm{pr}(\exp(t\xi)) \tag{5.60}$$

has to be fulfilled and q has to be tangent to the distribution D^{α} by Theorem 5.9, Claim 2.. Thus $\overline{q}\colon I \to \overline{Q}$ has to be tangent to $\overline{D^{\alpha}}$ by Lemma 5.8, Claim 3.. Furthermore, by Proposition 5.4, Claim 4., the curve $g\colon I \to G$ fulfilling $\gamma = \mathrm{pr} \circ g$ is tangent to $\mathrm{Hor}(G) \subseteq TG$ from Proposition 2.10, i.e. g is a horizontal lift of γ. Hence Lemma 5.19 yields

$$g(t) = \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) \tag{5.61}$$

for all $t \in I$ which fulfills the initial value problem

$$\dot{g}(t) = T_e\ell_{g(t)}\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}}), \quad g(0) = e. \tag{5.62}$$

Next we recall the kinematic equation from Remark 5.16, Claim 1. for convenience

$$\begin{aligned} \dot{v}(t) &= u(t), \\ \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \\ \dot{g}(t) &= \big(T_e\ell_{g(t)} \circ S(t)\big)u(t). \end{aligned} \tag{5.63}$$

By comparing (5.63) with (5.62), we obtain

$$S(t)u(t) = \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}}). \tag{5.64}$$

Thus the ODE for $S\colon I \to \mathrm{GL}(\mathfrak{m})$ in (5.63) becomes

$$\dot{S}(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}})} \circ S(t). \tag{5.65}$$

In order to obtain the desired rolling, we need to solve the initial value problem associated with (5.65) explicitly. This is the next lemma.

Lemma 5.21 *The solution of the initial value problem*

$$\dot{S}(t) = -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}})} \circ S(t), \quad S(0) = S_0 \in \mathrm{GL}(\mathfrak{m}) \tag{5.66}$$

is given by

$$S\colon I \to \mathrm{GL}(\mathfrak{m}), \quad t \mapsto \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\Big(-t\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{h}} + \frac{1}{2}\xi_{\mathfrak{m}}}\Big) \circ S_0 \tag{5.67}$$

Proof We make the following Ansatz. We set $S(t) = \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \widetilde{S}(t)$ for all $t \in I$, where $\widetilde{S}\colon I \to \mathrm{GL}(\mathfrak{m})$ is given by

$$\widetilde{S}(t) = \exp\big(-t(\mathrm{ad}_{\xi_{\mathfrak{h}}} + \tfrac{1}{2}(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{m}}}))\big) \circ S_0$$

for $t \in I$ and some $S_0 \in \mathrm{GL}(\mathfrak{m})$. Obviously,

$$\dot{\widetilde{S}}(t) = -\big(\mathrm{ad}_{\xi_\mathfrak{h}} + \tfrac{1}{2}(\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{m}})\big) \circ \widetilde{S}(t)$$

holds for all $t \in I$. Using the well-known identity $\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} = \mathrm{e}^{t\,\mathrm{ad}_{\xi_\mathfrak{h}}}$, we compute

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t\,\mathrm{ad}_{\xi_\mathfrak{h}}} = \mathrm{e}^{t\,\mathrm{ad}_{\xi_\mathfrak{h}}} \circ \mathrm{ad}_{\xi_\mathfrak{h}} = \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \mathrm{ad}_{\xi_\mathfrak{h}}.$$

Thus we obtain

$$\begin{aligned}
\dot{S}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \widetilde{S}(t)\big) \\
&= \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})}\big) \circ \widetilde{S}(t) + \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \dot{\widetilde{S}}(t) \\
&= \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \mathrm{ad}_{\xi_\mathfrak{h}} \circ \widetilde{S}(t) - \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \big(\mathrm{ad}_{\xi_\mathfrak{h}} + \tfrac{1}{2}(\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{m}})\big) \circ \widetilde{S}(t) \\
&= -\tfrac{1}{2}\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{m}} \circ \widetilde{S}(t) \\
&= -\tfrac{1}{2}\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})}(\xi_\mathfrak{m})} \circ \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \widetilde{S}(t) \\
&= -\tfrac{1}{2}\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})}(\xi_\mathfrak{m})} \circ S(t),
\end{aligned}$$

where we exploited that $\mathrm{pr}_\mathfrak{m}$ and Ad_h commutes and that Ad_h is a morphism of Lie algebras for all $h \in H$. Moreover, $S(0) = \widetilde{S}(0) = S_0$ is clearly fulfilled. Hence

$$S\colon I \to \mathrm{GL}(\mathfrak{m}), \quad t \mapsto \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \exp\big(-t(\mathrm{ad}_{\xi_\mathfrak{h}} + \tfrac{1}{2}(\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{m}}))\big) \circ S_0$$

is the unique solution of the initial value problem fulfilling $S(0) = S_0$. The desired result follows by

$$\mathrm{ad}_{\xi_\mathfrak{h}} + \tfrac{1}{2}\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{m}}\Big|_\mathfrak{m} = \mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{h} + \frac{1}{2}\xi_\mathfrak{m}}\Big|_\mathfrak{m},$$

where $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ is exploited. □

We now choose $S(0) = \mathrm{id}_\mathfrak{m}$ in the expression for $S(t)$ from Lemma 5.21 and obtain

$$S(t) = \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})} \circ \exp\Big(-t\,\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{h} + \frac{1}{2}\xi_\mathfrak{m}}\Big), \quad t \in I. \tag{5.68}$$

Clearly, the inverse of $S(t)$ is given by

$$S(t)^{-1} = \exp\big(t\,\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{h} + \frac{1}{2}\xi_\mathfrak{m}}\big) \circ \mathrm{Ad}_{\exp(-t\xi_\mathfrak{h})}. \tag{5.69}$$

By (5.64), i.e. $S(t)u(t) = \mathrm{Ad}_{\exp(t\xi_\mathfrak{h})}(\xi_\mathfrak{m})$, we have

$$u(t) = S(t)^{-1}\big(\mathrm{Ad}_{\exp(t\xi_\mathfrak{h})}(\xi_\mathfrak{m})\big) = \exp\Big(t\,\mathrm{pr}_\mathfrak{m} \circ \mathrm{ad}_{\xi_\mathfrak{h} + \frac{1}{2}\xi_\mathfrak{m}}\Big)(\xi_\mathfrak{m}). \tag{5.70}$$

According to (5.63), the rolling curve $v\colon I \to \mathfrak{m}$ is defined by $\dot{v}(t) = u(t)$. Choosing $v(0) = 0$ as initial condition yields

$$v(t) = \int_0^t \exp\left(s\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{h}}+\frac{1}{2}\xi_{\mathfrak{m}}}\right)(\xi_{\mathfrak{m}})\mathrm{d}s, \quad t \in I. \tag{5.71}$$

Thus the desired rolling is determined. The discussion above is summarized in the next proposition.

Proposition 5.22 *Let G/H be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and let $\xi \in \mathfrak{g}$ be arbitrary. Moreover, let $\overline{q}\colon I \ni t \mapsto \overline{q}(t) = (v(t), g(t), S(t)) \in \mathfrak{m} \times G \times \mathrm{GL}(\mathfrak{m}) = \overline{Q}$ be defined by*

$$\begin{aligned} v(t) &= \int_0^t \exp\left(s\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{h}}+\frac{1}{2}\xi_{\mathfrak{m}}}\right)(\xi_{\mathfrak{m}})\mathrm{d}s \\ g(t) &= \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) \\ S(t) &= \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\left(-t\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{h}}+\frac{1}{2}\xi_{\mathfrak{m}}}\right) \end{aligned} \tag{5.72}$$

for $t \in I$. Then $q\colon I \ni t \mapsto (\overline{\pi} \circ \overline{q})(t) = (v(t), [g(t), S(t)]) \in Q$ defines a rolling of $\mathfrak{m}$ over G/H with rolling curve $v\colon I \to \mathfrak{m}$ and development curve

$$\gamma\colon I \to G/H, \quad t \mapsto \mathrm{pr}(g(t)) = \mathrm{pr}(\exp(t\xi)). \tag{5.73}$$

Furthermore, this intrinsic rolling viewed as a triple as in Remark 3.3 is given by $(v(t), \gamma(t), A(t))$, where $A(t)$ reads

$$A(t)\colon T_{v(t)}\mathfrak{m} \cong \mathfrak{m} \to T_{\gamma(t)}(G/H), \quad Z \mapsto \big(T_{g(t)}\mathrm{pr} \circ T_e\ell_{g(t)} \circ S(t)\big)Z. \tag{5.74}$$

Proof This is a consequent of the above discussion. □

Remark 5.23 Assume that $\mathrm{G}(\mathfrak{m}) \subseteq \mathrm{GL}(\mathfrak{m})$ is a closed subgroup such that $\mathrm{Ad}_h\big|_{\mathfrak{m}} \in \mathrm{G}(\mathfrak{m})$ holds for all $h \in H$ and that $\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_X\big|_{\mathfrak{m}} \in \mathfrak{g}(\mathfrak{m})$ holds for all $X \in \mathfrak{m}$. Then the curve $S\colon I \to \mathrm{GL}(\mathfrak{m})$ from Proposition 5.22 is actually contained in $\mathrm{G}(\mathfrak{m})$, i.e. $S(t) \in \mathrm{G}(\mathfrak{m})$ for all $t \in I$. In particular, if G/H is a naturally reductive homogeneous space, one has $S(t) \in \mathrm{O}(\mathfrak{m})$ for all $t \in I$.

Corollary 5.24 *Let $\xi_{\mathfrak{m}} \in \mathfrak{m}$ and define the curves*

$$\begin{aligned} &v\colon I \to \mathfrak{m}, \quad t \mapsto v(t) = t\xi_{\mathfrak{m}} \\ &S\colon I \to \mathrm{GL}(\mathfrak{m}), \quad t \mapsto S(t) = \exp(-\tfrac{1}{2}t\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{m}}}) \\ &g\colon I \to G, \quad t \mapsto g(t) = \exp(t\xi_{\mathfrak{m}}) \end{aligned} \tag{5.75}$$

Then $q\colon I \ni t \mapsto q(t) = (v(t), [g(t), S(t)]) \in Q$ is an intrinsic rolling with respect to $\nabla^{\mathrm{can}1}$ whose development curve is a geodesic with respect to $\nabla^{\mathrm{can}1}$.

Proof This is an immediate consequence of Proposition 5.22 due to $\xi_{\mathfrak{h}} = 0$. □

5.4.2 Rolling along special curves with respect to ∇^{can2}

We now consider a rolling of a reductive homogeneous space with respect to the covariant derivative ∇^{can2} such that the development curve is given by $\gamma\colon I \ni t \mapsto \operatorname{pr}(\exp(t\xi))$ for some $\xi \in \mathfrak{g}$. This is the next proposition.

Proposition 5.25 *Let $\xi \in \mathfrak{g}$ be arbitrary and define $\overline{q}\colon I \ni t \mapsto (v(t), g(t), S(t)) \in \mathfrak{m} \times G \times \mathrm{GL}(\mathfrak{m}) = \overline{Q}$ by*

$$\begin{aligned} v(t) &= \int_0^t \mathrm{Ad}_{\exp(s\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}})\mathrm{d}s \\ S(t) &= \mathrm{id}_{\mathfrak{m}} \\ g(t) &= \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}). \end{aligned} \tag{5.76}$$

Then $q = \overline{\pi} \circ \overline{q}\colon I \to Q$ is an intrinsic rolling of $\mathfrak{m}$ over G/H with respect to ∇^{can2} whose development curve is given by $\gamma\colon I \ni t \mapsto \operatorname{pr}(\exp(t\xi)) \in G/H$. This intrinsic rolling viewed as a triple as in Remark 3.3 is given by $(v(t), \gamma(t), A(t))$, where $A(t)$ reads

$$A(t)\colon T_{v(t)}\mathfrak{m} \cong \mathfrak{m} \to T_{\gamma(t)}(G/H), \quad Z \mapsto \big(T_{g(t)}\operatorname{pr} \circ T_e\ell_{g(t)}\big)Z. \tag{5.77}$$

Proof The curve $g\colon I \ni t \mapsto \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}) \in G$ is the horizontal lift of γ through $g(0) = e$ by Lemma 5.19. We now show that $\overline{q}\colon I \to \overline{Q}$ defined by (5.76) fulfills the kinematic equation from Remark 5.16, Claim 2.. Indeed, we have

$$u(t) = \dot{v}(t) = \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}})$$

and $g(t) = \exp(t\xi)\exp(-t\xi_{\mathfrak{h}})$ is the solution of the initial value problem

$$\dot{g}(t) = T_e\ell_{g(t)}\mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})}(\xi_{\mathfrak{m}}) = T_e\ell_{g(t)}u(t) = \big(T_e\ell_{g(t)} \circ \mathrm{id}_{\mathfrak{m}}\big)u(t), \quad g(0) = e$$

by Lemma 5.19 as desired. Therefore $q = \overline{\pi} \circ \overline{q}\colon I \to Q$ is indeed a rolling of $\mathfrak{m}$ over G/H with respect to ∇^{can2} whose developement curve is given by $\gamma(t) = \operatorname{pr}(\exp(t\xi))$. □

6 Applications and examples

In this section, we consider some examples. Before we study Lie groups and Stiefel manifolds in detail, we briefly comment on symmetric homogeneous spaces. By recalling $\nabla^{\text{can1}} = \nabla^{\text{can2}}$ for symmetric homogeneous space and $\nabla^{\text{LC}} = \nabla^{\text{can1}} = \nabla^{\text{can2}}$ for pseudo-Riemannian symmetric homogeneous spaces from Remark 2.23, the kinematic equation from Remark 5.16, Claim 2. yields the next lemma.

Lemma 6.1 *Let (G, H, σ) be a symmetric pair and let G/H be the corresponding symmetric homogeneous space with canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.*

Moreover, let $u\colon I \to \mathfrak{m}$ *be a curve. Define the curve* $\overline{q}\colon I \ni t \mapsto (v(t), g(t), \mathrm{id}_{\mathfrak{m}}) \in \overline{Q} = \mathfrak{m} \times G \times \mathrm{GL}(\mathfrak{m})$ *by the initial value problem*

$$\begin{aligned} \dot{v}(t) &= u(t), & v(0) &= 0 \\ \dot{g}(t) &= T_e \ell_{g(t)} u(t), & g(0) &= e. \end{aligned} \tag{6.1}$$

Then $q = \overline{\pi} \circ \overline{q}\colon I \ni t \mapsto (v(t), [g(t), \mathrm{id}_{\mathfrak{m}}]) \in Q = \mathfrak{m} \times (G \times_H \mathrm{GL}(\mathfrak{m}))$ *is a rolling of* $\mathfrak{m}$ *over* G/H *with respect* $\nabla^{\mathrm{can}1} = \nabla^{\mathrm{can}2}$.

If G/H is a pseudo-Riemannian symmetric space, we can consider an $\mathrm{O}(\mathfrak{m})$-reduced rolling, i.e. we can take $Q = \mathfrak{m} \times (G \times \mathrm{O}(\mathfrak{m}))$ in Lemma 6.1 and $q\colon I \to Q$ is a rolling of $\mathfrak{m}$ over G/H with respect to the covariant derivative $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can}1} = \nabla^{\mathrm{can}2}$. In this case, Lemma 6.1 is very similar the result obtained in [11, Sec. 4.2].

6.1 Rolling Lie groups

In this subsection, we discuss intrinsic rollings of Lie groups. First we discuss rollings of $\mathfrak{g}$ over G, where we view G as the reductive homogeneous space $G/\{e\}$ equipped with the covariant derivative $\nabla^{\mathrm{can}1}$. Afterwards, we discuss rollings of a connected Lie group G viewed as the symmetric homogeneous space $(G \times G)/\Delta G$ equipped with $\nabla^{\mathrm{can}1} = \nabla^{\mathrm{can}2}$. It turns out that both points of view are closely related.

6.1.1 Rollings of Lie groups as reductive homogeneous spaces

We first consider the rolling of a Lie-group G viewed as a reductive homogeneous space $G/\{e\}$ equipped with the covariant derivative $\nabla^{\mathrm{can}1}$. Obviously, the reductive decomposition is given by $\mathfrak{h} = \{0\}$ and $\mathfrak{m} = \mathfrak{g}$. Clearly, this implies $\mathrm{pr}_{\mathfrak{m}} = \mathrm{id}_{\mathfrak{m}} = \mathrm{id}_{\mathfrak{g}}$. Moreover, the configuration space becomes

$$Q = \mathfrak{g} \times (G \times_{\{e\}} \mathrm{GL}(\mathfrak{g})) = \mathfrak{g} \times G \times \mathrm{GL}(\mathfrak{g}). \tag{6.2}$$

We now determine a rolling $q\colon I \ni t \mapsto (v(t), g(t), S(t)) \in Q = \mathfrak{g} \times G \times \mathrm{GL}(\mathfrak{g})$ of $\mathfrak{g}$ over G with respect to $\nabla^{\mathrm{can}1} = \nabla^{\alpha}$, where

$$\alpha\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad (X, Y) \mapsto \alpha(X, Y) = \tfrac{1}{2}[X, Y]. \tag{6.3}$$

To this end, we solve the following initial value problem associated with the kinematic equation from Remark 5.16, Claim 1.

$$\begin{aligned} \dot{v}(t) &= u(t), & v(0) &= 0, \\ \dot{S}(t) &= -\tfrac{1}{2}\mathrm{ad}_{S(t)u(t)} \circ S(t), & S(0) &= \mathrm{id}_{\mathfrak{g}}, \\ \dot{g}(t) &= \big(T_e \ell_{g(t)} \circ S(t)\big) u(t), & g(0) &= g_0, \end{aligned} \tag{6.4}$$

where $u\colon I \to \mathfrak{g}$ denotes a prescribed control curve. Motivated by [2, Sec. 3.2], where rollings of $\mathrm{SO}(n)$ over one if its affine tangent spaces are determined by using an extrinsic point of view, we make the following Ansatz.

We define the curves $k\colon I \to G$ and $W\colon I \to G$ by the initial value problems

$$\dot{k}(t) = \tfrac{1}{2} T_e \ell_{k(t)} u(t), \quad k(0) = g_0 \quad \text{and} \quad \dot{W}(t) = -\tfrac{1}{2} T_e \ell_{W(t)} u(t), \quad W(0) = e. \tag{6.5}$$

Moreover, we set

$$S\colon I \to \mathrm{GL}(\mathfrak{g}), \quad t \mapsto S(t) = \mathrm{Ad}_{W(t)} \tag{6.6}$$

as well as

$$g\colon I \to G, \quad t \mapsto g(t) = k(t)W(t)^{-1}. \tag{6.7}$$

Clearly, $S(0) = \mathrm{Ad}_e = \mathrm{id}_{\mathfrak{g}}$ and $g(0) = g_0 e^{-1} = g_0$ holds. Next we show that $S\colon I \to \mathrm{GL}(\mathfrak{g})$ defined by (6.6) is a solution of (6.4). To this end, we calculate

$$\dot{W}(t) = -\tfrac{1}{2} T_e \ell_{W(t)} u(t) = \tfrac{\mathrm{d}}{\mathrm{d}s}\Big(\ell_{W(t)}\big(\exp(-\tfrac{1}{2} s u(t))\big)\Big)\Big|_{s=0}, \tag{6.8}$$

where we used the chain-rule and exploited the definition of W in (6.5). In other words, the smooth curve

$$\gamma\colon \mathbb{R} \to G, \quad s \mapsto \ell_{W(t)}\big(\exp(-\tfrac{1}{2} s u(t))\big) = W(t) \exp(-\tfrac{1}{2} s u(t)) \tag{6.9}$$

fulfills $\gamma(0) = W(t)$ and $\tfrac{\mathrm{d}}{\mathrm{d}s}\gamma(s)\big|_{s=0} = \dot{W}(t)$ according to 6.8. Thus we calculate for $Z \in \mathfrak{g}$ by using the definition of $S\colon I \to \mathrm{GL}(\mathfrak{g})$ from (6.6) and the chain rule

$$\begin{aligned}
\dot{S}(t)Z &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{Ad}_{W(t)}(Z)\big) \\
&= T_{W(t)} \mathrm{Ad}_{(\cdot)}(Z) \dot{W}(t) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{Ad}_{\gamma(s)}(Z)\big|_{s=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{Ad}_{W(t)\exp(-\frac{1}{2} s u(t))}(Z)\big|_{s=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{Ad}_{W(t)}\Big(\mathrm{Ad}_{\exp(-\frac{1}{2} s u(t))}(Z)\Big)\Big|_{s=0} \\
&= \mathrm{Ad}_{W(t)}\Big(\tfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{Ad}_{\exp(-\frac{1}{2} s u(t))}(Z)\big|_{s=0}\Big) \\
&= -\tfrac{1}{2}\big(\mathrm{Ad}_{W(t)} \circ \mathrm{ad}_{u(t)}\big)(Z) \\
&= -\tfrac{1}{2} \mathrm{ad}_{\mathrm{Ad}_{W(t)}(u(t))} \circ \mathrm{Ad}_{W(t)}(Z) \\
&= -\tfrac{1}{2} \mathrm{ad}_{S(t)u(t)} \circ S(t)Z
\end{aligned} \tag{6.10}$$

as desired, where we exploited that $\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}$ is a morphism of Lie algebras for all $g \in G$ and $\gamma\colon \mathbb{R} \to G$ is defined by (6.9).

It remains to show that $g\colon I \to G$ defined in (6.7) fulfills (6.4). To this end, using the chain-rule several times, we obtain by $g(t) = k(t)W(t)$ and (2.8) as well as (2.9) and the definition of $k\colon I \to G$ and $W\colon I \to G$ in (6.5)

$$
\begin{aligned}
\dot{g}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(k(t)W(t)^{-1}\big) \\
&= T_{k(t)} r_{W(t)^{-1}} \dot{k}(t) + T_{W(t)^{-1}} \ell_{k(t)} \tfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{inv}(W(t)) \\
&= T_{k(t)} r_{W(t)^{-1}} \dot{k}(t) + T_{W(t)^{-1}} \ell_{k(t)} \big(- T_e \ell_{W(t)^{-1}} \circ T_{W(t)} r_{W(t)^{-1}} \big) \dot{W}(t) \\
&= \tfrac{1}{2} \big(T_{k(t)} r_{W(t)^{-1}} \circ T_e \ell_{k(t)} \big) u(t) \\
&\quad + \tfrac{1}{2} \big(T_{W(t)^{-1}} \ell_{k(t)} \circ T_e \ell_{W(t)^{-1}} \circ T_{W(t)} r_{W(t)^{-1}} \circ T_e \ell_{W(t)} \big) u(t) \\
&= \tfrac{1}{2} \big(T_e (r_{W(t)^{-1}} \circ \ell_{k(t)}) \big) u(t) \\
&\quad + \tfrac{1}{2} \big(T_{W(t)^{-1}} \ell_{k(t)} \circ T_e (\ell_{W(t)^{-1}} \circ r_{W(t)^{-1}} \circ \ell_{W(t)}) \big) u(t) \\
&= \tfrac{1}{2} T_e (r_{W(t)^{-1}} \circ \ell_{k(t)}) u(t) + \tfrac{1}{2} \big(T_{W(t)^{-1}} \ell_{k(t)} \circ T_e r_{W(t)^{-1}} \big) u(t) \\
&= \tfrac{1}{2} T_e \big(r_{W(t)^{-1}} \circ \ell_{g(t)W(t)} \big) u(t) + \tfrac{1}{2} \big(T_{W(t)^{-1}} \ell_{g(t)W(t)} \circ T_e r_{W(t)^{-1}} \big) u(t) \\
&= \tfrac{1}{2} T_e \big(r_{W(t)^{-1}} \circ \ell_{g(t)} \circ \ell_{W(t)} \big) u(t) + \tfrac{1}{2} T_e \big(\ell_{g(t)} \circ \ell_{W(t)} \circ r_{W(t)^{-1}} \big) u(t) \\
&= \tfrac{1}{2} T_e \big(\ell_{g(t)} \circ \ell_{W(t)} \circ r_{W(t)^{-1}} \big) u(t) + \tfrac{1}{2} T_e \big(\ell_{g(t)} \circ \ell_{W(t)} \circ r_{W(t)^{-1}} \big) u(t) \\
&= \big(T_e \ell_{g(t)} \circ T_e \mathrm{Conj}_{W(t)} \big) u(t) \\
&= T_e \ell_{g(t)} \circ \mathrm{Ad}_{W(t)} (u(t)) \\
&= \big(T_e \ell_{g(t)} \circ S(t) \big) u(t)
\end{aligned}
\tag{6.11}
$$

as desired. Hence

$$
q\colon I \ni t \mapsto (v(t), g(t), S(t)) \in Q \tag{6.12}
$$

is an intrinsic rolling of $\mathfrak{g}$ over G/H with respect to $\nabla^{\mathrm{can}1}$, where S and g are defined in (6.6) and (6.7), respectively. Moreover, $v\colon I \to \mathfrak{g}$ is determined by $\dot{v}(t) = u(t)$ and the initial value $v(0) = 0$. We summarize the above discussion in the next proposition.

Proposition 6.2 *Let G be a Lie group viewed as reductive homogeneous space $G/\{e\}$ equipped with $\nabla^{\mathrm{can}1}$. Let $u\colon I \to \mathfrak{g}$ be some control curve and define $k\colon I \to G$ as well as $W\colon I \to G$ by the initial value problems*

$$
\dot{k}(t) = \tfrac{1}{2} T_e \ell_{k(t)} u(t), \quad k(0) = g_0 \quad \textit{and} \quad \dot{W}(t) = -\tfrac{1}{2} T_e \ell_{W(t)} u(t), \quad W(0) = e. \tag{6.13}
$$

Then

$$
q\colon I \ni t \mapsto (v(t), g(t), S(t)) = \big(v(t), k(t)W(t)^{-1}, \mathrm{Ad}_{W(t)} \big) \in \mathfrak{g} \times G \times \mathrm{GL}(\mathfrak{g}) = Q \tag{6.14}
$$

is an intrinsic rolling of $\mathfrak{g}$ over G, where the development curve $v\colon I \to \mathfrak{g}$ is defined by

$$
v(t) = \int_0^t u(s) \mathrm{d}s \tag{6.15}
$$

and the rolling curve is given by $g\colon I \ni t \mapsto k(t)W(t)^{-1} \in G$. This rolling can be viewed as a triple $(v(t), g(t), A(t))$ as in Remark 3.3, where the linear isomorphism

$A(t)\colon T_{v(t)}\mathfrak{g} \cong \mathfrak{g} \to T_{g(t)}G$ is given by

$$A(t)Z = \big(T_e\ell_{g(t)} \circ \mathrm{Ad}_{W(t)}\big)Z = \big(T_e\ell_{k(t)W(t)^{-1}} \circ \mathrm{Ad}_{W(t)}\big)Z \tag{6.16}$$

for all $Z \in \mathfrak{g}$.

Remark 6.3 Let $u\colon I \to \mathfrak{g}$ be a control curve. Then the intrinsic rolling $q\colon I \to \mathfrak{m} \times G \times \mathrm{GL}(\mathfrak{g})$ of $\mathfrak{g}$ over G with respect to $\nabla^{\mathrm{can}1}$ is defined on the whole interval I by the form of the initial value problem in (6.13).

Corollary 6.4 *Let G be a Lie group equipped with a bi-invariant pseudo-Riemannian metric. Then rollings of $\mathfrak{g}$ over G with respect to ∇^{LC} with a prescribed control curve $u\colon I \to \mathfrak{g}$ are given by Proposition 6.2.*

Proof Let G be equipped with a pseudo-Riemannian bi-invariant metric. Then the corresponding scalar product $\langle\cdot,\cdot\rangle\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is $\mathrm{Ad}(G)$-invariant, see e.g. [13, Chap. 11, Prop. 9], i.e. $G/\{e\}$ is a naturally reductive homogeneous space. Thus we have $\nabla^{\mathrm{LC}} = \nabla^{\mathrm{can}1}$ by Remark 2.22. This yields the desired result. □

Remark 6.5 For the special case $G = \mathrm{SO}(n) \subseteq \mathbb{R}^{n\times n}$, equipped with the bi-invariant metric induced by the Frobenius scalar product on $\mathbb{R}^{n\times n}$, expressions for extrinsic rollings considered as curves in the Euclidean group are derived in [2, Thm. 3.2]. Rollings of pseudo-orthogonal groups have been studied in [4]. The tangential part of these rollings is very similar to the result of Proposition 6.2. Indeed, the linear isomorphism defined by the rolling from Proposition 6.2 in (6.16) simplifies for a matrix Lie group to

$$A(t)Z = \big(T_e\ell_{k(t)W(t)^{-1}} \circ \mathrm{Ad}_{W(t)}\big)Z = k(t)ZW(t)^{-1} \tag{6.17}$$

for all $Z \in \mathfrak{g}$.

6.1.2 Rollings of Lie groups as symmetric homogeneous spaces

We now identify G with the symmetric homogeneous spaces $(G \times G)/\Delta G$ and study the rolling of $\mathfrak{m}$ over $(G \times G)/\Delta G$ with respect to $\nabla^{\mathrm{can}1} = \nabla^{\mathrm{can}2}$. To this end, we state the next lemma as preparation which is an adaption of [15, Sec. 23.9.5], see also [19, Chap. IV, 6], where it is stated for the Riemannian case.

Lemma 6.6 *Let G be a connected Lie group and define*

$$\sigma\colon G \times G \to G \times G, \quad (g_1, g_2) \mapsto (g_2, g_1). \tag{6.18}$$

Then σ is an involutive automorphism of $G \times G$ and $\Delta G = \{(g, g) \mid g \in G\} \subseteq G \times G$ is the set of fixed points of σ. Moreover, $(G \times G)/\Delta G$ is a symmetric homogeneous space and the corresponding canonical reductive decomposition $\mathfrak{g} \times \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is given by

$$\mathfrak{h} = \{(X, X) \mid X \in \mathfrak{g}\} \quad \textit{and} \quad \mathfrak{m} = \{(X, -X) \mid X \in \mathfrak{g}\}. \tag{6.19}$$

In addition, the map

$$\phi\colon (G\times G)/\Delta G \to G, \quad (g_1, g_2)\cdot \Delta G \mapsto g_1 g_2^{-1} \tag{6.20}$$

is a diffeomorphism and the map

$$\overline{\phi}\colon G\times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2^{-1} \tag{6.21}$$

is a surjective submersion which fulfills $\phi \circ \mathrm{pr} = \overline{\phi}$.

Next we determine the tangent map of $\overline{\phi}$ evaluated at elements in $\mathrm{Hor}(G\times G) \subseteq TG \times TG$. We point out that the identity $\big(T_{(e,e)}\overline{\phi}\big)(X, X) = 2X$ for all $X \in \mathfrak{g}$ is well-known, see e.g. [15, Sec. 23.9.5] or [19, Chap. IV, 6].

Lemma 6.7 *Let G be a connected Lie group and let* $X \in \mathfrak{g}$. *Then* $(T_e\ell_{g_1}X, -T_e\ell_{g_2}X) \in \mathrm{Hor}(G\times G)_{(g_1,g_2)}$ *holds. Moreover, the tangent map of* $\overline{\phi}\colon G\times G \ni (g_1, g_2) \mapsto g_1 g_2^{-1} \in G$ *fulfills*

$$\big(T_{(g_1,g_2)}\overline{\phi}\big)(T_e\ell_{g_1}X, -T_e\ell_{g_2}X) = \big(T_e\ell_{g_1 g_2^{-1}} \circ \mathrm{Ad}_{g_2}\big)(2X). \tag{6.22}$$

In particular $T_{(e,e)}\overline{\phi}(X, -X) = 2X$ *holds and*

$$\big(T_{(e,e)}\overline{\phi}\big|_{\mathfrak{m}}\big)^{-1}(X) = \big(\tfrac{1}{2}X, -\tfrac{1}{2}X\big) \tag{6.23}$$

is satisfied for all $X \in \mathfrak{g}$.

Proof Obviously, $(T_e\ell_{g_1}X, -T_e\ell_{g_2}X) \in \mathrm{Hor}(G\times G)_{(g_1,g_2)}$ is satisfied by $\mathrm{Hor}(G\times G)_{(g_1,g_2)} = T_{(e,e)}\ell_{(g_1,g_2)}\mathfrak{m}$ and the definition of $\mathfrak{m} \subseteq \mathfrak{g}\times\mathfrak{g}$ in Lemma 6.6. Next we prove (6.22). To this end, we consider the curve

$$\gamma\colon \mathbb{R} \to G\times G, \quad t \mapsto \big(g_1 \exp(tX), g_2\exp(-tX)\big)$$

which fulfills $\gamma(0) = (g_1, g_2)$ and $\dot{\gamma}(0) = (T_e\ell_{g_1}X, -T_e\ell_{g_2}X)$. Next we calculate

$$\begin{aligned}
\big(T_{(g_1,g_2)}\overline{\phi}\big)(T\ell_{g_1}X, -T_e\ell_{g_2}X) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\overline{\phi}(\gamma(t))\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\overline{\phi}\big(g_1\exp(tX), g_2(\exp(-tX))\big)\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}g_1\exp(tX)\big(g_2(\exp(-tX))\big)^{-1}\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}g_1\exp(tX)\exp(tX)g_2^{-1}\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}g_1\exp(2tX)g_2^{-1}\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}g_1g_2^{-1}g_2\exp(2tX)g_2^{-1}\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}g_1g_2^{-1}\mathrm{Conj}_{g_2}\big(\exp(2tX)\big)\big|_{t=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}g_1g_2^{-1}\exp\big(\mathrm{Ad}_{g_2}(2tX)\big)\big|_{t=0} \\
&= T_e\ell_{g_1g_2^{-1}}\tfrac{\mathrm{d}}{\mathrm{d}t}\exp\big(t\mathrm{Ad}_{g_2}(2X)\big)\big|_{t=0} \\
&= T_e\ell_{g_1g_2^{-1}}\mathrm{Ad}_{g_2}(2X)
\end{aligned}$$

proving (6.22) as desired. Evaluating (6.22) at $(g_1, g_2) = (e, e) \in G \times G$ yields

$$T_{(e,e)}\overline{\phi}(X, -X) = (T_e\ell_e \circ \mathrm{Ad}_e)(2X) = 2X$$

for all $X \in \mathfrak{g}$. Now (6.23) is verified by a straightforward calculation. □

Next we consider intrinsic rollings of $\mathfrak{m}$ over $(G \times G)/\Delta G$ with respect to $\nabla^{\mathrm{can}2}$ and relate them to the intrinsic rollings of $\mathfrak{g}$ over G with respect to $\nabla^{\mathrm{can}1}$. This is the next proposition.

Proposition 6.8 *Let G be a connected Lie group and let $u\colon I \to \mathfrak{g}$ be a control curve. Consider the initial value problem*

$$\begin{aligned}
\dot{v}(t) &= u(t), & v(0) &= 0, \\
\dot{g}_1(t) &= \tfrac{1}{2}T_e\ell_{g_1(t)}u(t), & g_1(0) &= g_0, \\
\dot{g}_2(t) &= -\tfrac{1}{2}T_e\ell_{g_1(t)}u(t), & g_2(0) &= e.
\end{aligned} \tag{6.24}$$

Then the following assertions are fulfilled:

1. *The curve $\widetilde{q}\colon I \to \mathfrak{m} \times ((G \times G) \times_{\Delta G} \mathrm{GL}(\mathfrak{m}))$ defined for $t \in I$ by*

$$\widetilde{q}(t) = \big((\tfrac{1}{2}v(t), -\tfrac{1}{2}v(t)), [(g_1(t), g_2(t)), \mathrm{id}_{\mathfrak{m}}]\big) \tag{6.25}$$

is a rolling of $\mathfrak{m}$ over $(G \times G)/\Delta G$ with respect to $\nabla^{\mathrm{can}2}$ with rolling curve $\widetilde{v}\colon I \ni t \mapsto \widetilde{v} = (\tfrac{1}{2}v(t), -\tfrac{1}{2}v(t)) \in \mathfrak{m}$ and development curve $\widetilde{\gamma}\colon I \ni t \mapsto \widetilde{\gamma}(t) = \mathrm{pr}(g_1(t), g_2(t)) \in (G \times G)/\Delta G$.

2. *The curve*

$$q\colon I \ni t \mapsto \big(v(t), g_1(t)g_2(t)^{-1}, \mathrm{Ad}_{g_2(t)}\big) \in \mathfrak{g} \times G \times \mathrm{GL}(\mathfrak{g}) \tag{6.26}$$

is a rolling of $\mathfrak{g}$ *over* G *with respect to* $\nabla^{\mathrm{can}1}$ *with rolling curve* $v\colon I \to \mathfrak{g}$ *and development curve* $g\colon I \ni t \mapsto g(t) = g_1(t)g_2(t)^{-1} \in G$.

3. *Let* $\phi\colon (G \times G)/\Delta G \to G$ *be the diffeomorphism from Lemma 6.6. Then one has for all* $Z \in \mathfrak{g} \cong T_{v(t)}\mathfrak{g}$ *and* $t \in I$

$$q(t)Z = T\phi \circ \widetilde{q}(t) \circ \big(T_{(e,e)}(\phi \circ \mathrm{pr})\big|_{\mathfrak{m}}\big)^{-1} Z, \tag{6.27}$$

where $q(t)$ *as well as* $\widetilde{q}(t)$ *are identified with the linear isomorphisms given by*

$$q(t)\colon T_{v(t)}\mathfrak{g} \cong \mathfrak{g} \to T_{g(t)}G, \quad Z \mapsto \big(T_e\ell_{g(t)} \circ \mathrm{Ad}_{g_2(t)}\big)Z, \tag{6.28}$$

where $g(t) = g_1(t)g_2(t)^{-1}$, *and*

$$\begin{aligned} \widetilde{q}(t)\colon T_{\widetilde{v}(t)}\mathfrak{m} \cong \mathfrak{m} &\to T_{\mathrm{pr}(g_1(t),g_2(t))}(G \times G)/\Delta G, \\ (Z,-Z) &\mapsto \big(T_{(g_1(t),g_2(t))}\mathrm{pr} \circ (T_e\ell_{g_1(t)}, T_e\ell_{g_2(t)})\big)(Z,-Z), \end{aligned} \tag{6.29}$$

respectively.

Proof Claim 1. follows by Remark 5.16, Claim 2.. Moreover, Claim 2. is a consequence of Proposition 6.2.

It remains to show Claim 3.. Let $Z \in \mathfrak{g}$. Then one has

$$(T_{(e,e)}(\phi \circ \mathrm{pr})\big|_{\mathfrak{m}})^{-1} Z = (T_{(e,e)}\overline{\phi}\big|_{\mathfrak{m}})^{-1} Z = \big(\tfrac{1}{2}Z, -\tfrac{1}{2}Z\big)$$

according to Lemma 6.7. Moreover $\overline{\phi} = \phi \circ \mathrm{pr}$ holds by Lemma 6.6 implying $T\overline{\phi} = T\phi \circ T\mathrm{pr}$. Therefore we obtain by Lemma 6.7

$$\begin{aligned} T\phi \circ \widetilde{q}(t) \circ (T_{(e,e)}(\phi \circ \mathrm{pr})\big|_{\mathfrak{m}})^{-1} Z &= T\phi \circ \widetilde{q}(t)\big(\tfrac{1}{2}Z, -\tfrac{1}{2}Z\big) \\ &= T\phi \circ \big(T_{(g_1(t),g_2(t))}\mathrm{pr} \circ (T_e\ell_{g_1(t)}, T_e\ell_{g_2(t)})\big)\big(\tfrac{1}{2}Z, -\tfrac{1}{2}Z\big) \\ &= T_{(g_1(t),g_2(t))}(\phi \circ \mathrm{pr})\big(\tfrac{1}{2}T_e\ell_{g_1(t)}Z, -\tfrac{1}{2}T_e\ell_{g_2(t)}Z\big) \\ &= \big(T_{(g_1(t),g_2(t))}\overline{\phi}\big)\big(\tfrac{1}{2}T_e\ell_{g_1(t)}Z, -\tfrac{1}{2}T_e\ell_{g_2(t)}Z\big) \\ &= \big(T_e\ell_{(g_1(t)g_2(t)^{-1})} \circ \mathrm{Ad}_{g_2(t)}\big)Z \\ &= q(t)Z \end{aligned}$$

for all $Z \in \mathfrak{g}$ as desired. □

6.2 Rolling Stiefel manifolds

Rollings of Stiefel manifolds have been already considered in the literature in [3] and [11], however not from an intrinsic point of view. In this section we apply the general theory developed in Sect. 5 to the Stiefel manifold $\mathrm{St}_{n,k}$ endowed with the Levi-Civita covariant derivative defined by a so-called α-metric. These metrics have been recently introduced in [28].

Remark 6.9 We point out that in contrast to the previous sections, where α denotes a bilinear map $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ defining an invariant covariant derivative, in this section α denotes an element in $\mathbb{R} \setminus \{0\}$. There is no danger of confusion because in this section, we consider rollings of Stiefel manifolds exclusively with respect to the Levi-Civita covariant derivative ∇^{LC} defined by an α-metric. Since the Stiefel manifold $\mathrm{St}_{n,k}$ equipped with an α-metric is a naturally reductive homogeneous space, see Lemma 6.11 below, the Levi-Civita covariant derivative ∇^{LC} corresponds to the invariant covariant derivative defined by the bilinear map $\mathfrak{m}\times\mathfrak{m} \ni (X, Y) \mapsto \frac{1}{2}[X, Y]_{\mathfrak{m}} \in \mathfrak{m}$ according to Remark 2.22.

6.2.1 Stiefel manifolds equipped with α-metrics

We start with recalling some results from [28], in particular [28, Sec. 2-3]. The Stiefel manifold $\mathrm{St}_{n,k}$ can be considered as the embedded submanifold of $\mathbb{R}^{n\times k}$ given by

$$\mathrm{St}_{n,k} = \{X \in \mathbb{R}^{n\times k} \mid X^\top X = I_k\}, \quad 1 \leq k \leq n. \tag{6.30}$$

In the sequel, we write $\mathrm{O}(n) = \{R \in \mathbb{R}^{n\times n} \mid R^\top R = I_n\}$ for the orthogonal group. We now identify $\mathrm{St}_{n,k}$ with a normal naturally reductive space G/H, where $G = \mathrm{O}(n) \times \mathrm{O}(k)$ is equipped with a suitable bi-invariant pseudo-Riemannian metric. To this end, we consider the action of the Lie group $G = \mathrm{O}(n) \times \mathrm{O}(k)$ on $\mathbb{R}^{n\times k}$ from the left via

$$\Phi\colon (\mathrm{O}(n) \times \mathrm{O}(k)) \times \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}, \quad ((R, \theta), X) \mapsto \Phi\big((R, \theta), X\big) = RX\theta^\top. \tag{6.31}$$

For fixed $(R, \theta) \in \mathrm{O}(n) \times \mathrm{O}(k)$, the induced diffeomorphism

$$\Phi_{(R,\theta)}\colon \mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}, \quad X \mapsto RX\theta^\top \tag{6.32}$$

is clearly linear. Restricting the second argument of Φ to $\mathrm{St}_{n,k}$ yields the action

$$(\mathrm{O}(n) \times \mathrm{O}(k)) \times \mathrm{St}_{n,k} \to \mathrm{St}_{n,k}, \quad ((R, \theta), X) \mapsto \Phi\big((R, \theta), X\big) = RX\theta^\top, \tag{6.33}$$

which is known to be transitive. This action is denoted by Φ, as well.

Let $X \in \mathrm{St}_{n,k}$ be fixed and denote by $H = \mathrm{Stab}(X)$ the stabilizer subgroup of X under the action Φ. We identify $\mathrm{St}_{n,k} \cong (\mathrm{O}(n) \times \mathrm{O}(k))/H$ via the $(\mathrm{O}(n) \times \mathrm{O}(k))$-equivariant diffeomorphism

$$\iota_X\colon G/H \to \mathrm{St}_{n,k}, \quad (R, \theta) \cdot H \mapsto \Phi\big((R, \theta), X\big) = RX\theta^\top, \tag{6.34}$$

where $(R, \theta) \cdot H \in (\mathrm{O}(n) \times \mathrm{O}(k))/H$ denotes the coset defined by $(R, \theta) \in \mathrm{O}(n) \times \mathrm{O}(k)$. Moreover, the map

$$\mathrm{pr}_X\colon \mathrm{O}(n) \times \mathrm{O}(k) \to \mathrm{St}_{n,k}, \quad (R, \theta) \mapsto RX\theta^\top \tag{6.35}$$

is a surjective submersion. Note that $\iota_X\colon G/H \to \mathrm{St}_{n,k} \subseteq \mathbb{R}^{n\times k}$ becomes a $(\mathrm{O}(n) \times \mathrm{O}(k))$-equivariant embedding and

$$\mathrm{pr}_X = \iota_X \circ \mathrm{pr} \tag{6.36}$$

holds, where $\mathrm{pr}\colon \mathrm{O}(n)\times\mathrm{O}(k) \to \big(\mathrm{O}(n)\times\mathrm{O}(k)\big)/H$ denotes the canonical projection. The Lie algebra of H is given by

$$\mathfrak{h} = \ker\big(T_{(I_n,I_k)}\mathrm{pr}_X\big) \subseteq \mathfrak{g} = \mathfrak{so}(n) \times \mathfrak{so}(k). \tag{6.37}$$

By [28, Eq. (14)], the stabilizer subgroup $H \subseteq \mathrm{O}(n) \times \mathrm{O}(k)$ is isomorphic to the Lie group $\mathrm{O}(n-k) \times \mathrm{O}(k)$.

Next we recall the definition of the so-called α-metrics from [28]. To this end, a bi-invariant metric on $\mathfrak{so}(n) \times \mathfrak{so}(k)$ is introduced following [28, Def. 3.1]. Define for $0 \neq \alpha \in \mathbb{R}$

$$\begin{aligned}\langle\cdot,\cdot\rangle^\alpha\colon (\mathfrak{so}(n)\times\mathfrak{so}(k)) \times (\mathfrak{so}(n)\times\mathfrak{so}(k)) &\to \mathbb{R},\\ \big((\Omega_1,\Omega_2),(\eta_1,\eta_2)\big) &\mapsto -\mathrm{tr}(\Omega_1\Omega_2) - \tfrac{1}{\alpha}\mathrm{tr}(\eta_1\eta_2).\end{aligned} \tag{6.38}$$

By [28, Prop. 2], the subspace $\mathfrak{h} \subseteq \mathfrak{so}(n)\times\mathfrak{so}(k)$ defined in (6.37) is non-degenerated iff $\alpha \neq -1$ holds. In this case, we write $\mathfrak{m} = \mathfrak{h}^\perp$ and

$$\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g} = \mathfrak{so}(n) \times \mathfrak{so}(k) \tag{6.39}$$

is fulfilled. Next we reformulate [28, Def. 3.3].

Definition 6.10 Let $\alpha \in \mathbb{R} \setminus \{0,-1\}$ and let $\mathrm{O}(n) \times \mathrm{O}(k)$ be equipped with the bi-invariant metric defined by the scalar product $\langle\cdot,\cdot\rangle^\alpha$ from (6.38). The metric on $\mathrm{St}_{n,k}$ defined by requiring that the map $\mathrm{pr}_X\colon \mathrm{O}(n)\times\mathrm{O}(k) \to \mathrm{St}_{n,k}$ from (6.35) is a pseudo-Riemannian submersion is called α-metric.

The Stiefel manifold equipped with an α-metric is a naturally reductive homogeneous space.

Lemma 6.11 *Let* $\alpha \in \mathbb{R} \setminus \{-1,0\}$. *Then* $(\mathrm{O}(n)\times\mathrm{O}(k))/H \cong \mathrm{St}_{n,k}$ *equipped with an* α*-metric from Definition 6.10 is a naturally reductive homogeneous space.*

Proof Obviously, the scalar product $\langle\cdot,\cdot\rangle^\alpha$ on $\mathfrak{so}(n)\times\mathfrak{so}(k)$ from Definition 6.10 is $\mathrm{Ad}(\mathrm{O}(n)\times\mathrm{O}(k))$-invariant for $\alpha \in \mathbb{R}\setminus\{0\}$. In addition, the subspace $\mathfrak{h} = \ker(T_{(I_n,I_k)}\mathrm{pr}_X) \subseteq \mathfrak{so}(n)\times\mathfrak{so}(k)$ is non-degenerated for $\alpha \in \mathbb{R}\setminus\{0,-1\}$ by [28, Prop. 2]. Thus Lemma 2.13 yields the desired result. □

In the sequel, an explicit expression for the orthogonal projection $\mathrm{pr}_\mathfrak{m}\colon \mathfrak{so}(n)\times\mathfrak{so}(k) \to \mathfrak{m}$ with respect to the scalar product $\langle\cdot,\cdot\rangle^\alpha$ is needed. Therefore we state the next lemma which is taken from [28, Lem. 3.2].

Lemma 6.12 *Let* $\alpha \in \mathbb{R}\setminus\{-1,0\}$ *and let* $X \in \mathrm{St}_{n,k}$*. The orthogonal projection*

$$\mathrm{pr}_{\mathfrak{m}} : \mathfrak{so}(n) \times \mathfrak{so}(k) \ni (\Omega, \eta) \mapsto (\Omega^{\perp_X}, \eta^{\perp_X}) \in \mathfrak{m} \subseteq \mathfrak{so}(n) \times \mathfrak{so}(k) \tag{6.40}$$

is given by

$$\begin{aligned} \Omega^{\perp_X} &= XX^\top\Omega + \Omega XX^\top - \tfrac{2\alpha+1}{\alpha+1} XX^\top \Omega XX^\top - \tfrac{1}{\alpha+1} X\eta X^\top, \\ \eta^{\perp_X} &= \tfrac{\alpha}{\alpha+1}\big(\eta - X^\top \Omega X\big). \end{aligned} \tag{6.41}$$

Proof This is just a reformulation of [28, Lem. 3.2]. □

Furthermore, the following lemma is a trivial reformulation of [28, Prop. 3].

Lemma 6.13 *Let* $\alpha \in \mathbb{R} \setminus \{-1,0\}$ *and let* $X \in \mathrm{St}_{n,k}$*. The map*

$$\big(T_{(I_n,I_k)}(\iota_X \circ \mathrm{pr})\big|_{\mathfrak{m}}\big)^{-1} : T_X\mathrm{St}_{n,k} \ni V \mapsto (\Omega(V)^{\perp_X}, \eta(V)^{\perp_X}) \in \mathfrak{m} \subseteq \mathfrak{so}(n) \times \mathfrak{so}(k) \tag{6.42}$$

is given by

$$\begin{aligned} \Omega(V)^{\perp_X} &= VX^\top - XV^\top + \tfrac{2\alpha+1}{\alpha+1} XV^\top XX^\top, \\ \eta(V)^{\perp_X} &= -\tfrac{\alpha}{\alpha+1} X^\top V \end{aligned} \tag{6.43}$$

for all $V \in T_X\mathrm{St}_{n,k}$*.*

Proof This is a consequent of [28, Prop. 3]. □

6.2.2 Intrinsic rolling

We now determine intrinsic rollings of the Stiefel manifold equipped with an α-metric over one of its tangent spaces.

By Lemma 5.2, the configuration space for rolling $T_X\mathrm{St}_{n,k} \cong \mathfrak{m}$ over $\mathrm{St}_{n,k}$ intrinsically is given by the fiber bundle

$$\pi : Q = \mathfrak{m} \times \big((\mathrm{O}(n) \times \mathrm{O}(k)) \times_H \mathrm{O}(\mathfrak{m})\big) \to \mathfrak{m} \times (\mathrm{O}(n) \times \mathrm{O}(k))/H, \tag{6.44}$$

where $H = \mathrm{Stab}(X) \subseteq \mathrm{O}(n) \times \mathrm{O}(k) = G$. By identifying $T_X\mathrm{St}_{n,k} \cong \mathfrak{m}$ via the linear isometry $T_{(I_n,I_k)}\mathfrak{m} \to T_X\mathrm{St}_{n,k}$ from Lemma 6.13 and $\mathrm{St}_{n,k} \cong (\mathrm{O}(n) \times \mathrm{O}(k))/H$ via the $(\mathrm{O}(n) \times \mathrm{O}(k))$-equivariant isometry $\iota_X : (\mathrm{O}(n) \times \mathrm{O}(k))/H \to \mathrm{St}_{n,k}$, we obtain the following proposition describing intrinsic rollings of $T_X\mathrm{St}_{n,k}$ over $\mathrm{St}_{n,k}$.

Proposition 6.14 *Let* $\mathrm{St}_{n,k}$ *be equipped with an* α*-metric for* $\alpha \in \mathbb{R} \setminus \{-1,0\}$ *and let*

$$V : I \to T_X\mathrm{St}_{n,k}, \quad t \mapsto V(t) \tag{6.45}$$

be a given rolling curve. Denote by $v : I \to \mathfrak{m}$ *the corresponding curve in* $\mathfrak{m}$ *given by*

$$\begin{aligned} v(t) &= \big(T_{I_{n,k}}(\iota_X \circ \mathrm{pr}\big|_{\mathfrak{m}})\big)^{-1} V(t) \\ &= \big(V(t)X^\top - XV(t)^\top + \tfrac{2\alpha+1}{\alpha+1} XV(t)^\top XX^\top, -\tfrac{\alpha}{\alpha+1} X^\top V(t)\big) \end{aligned} \tag{6.46}$$

for $t \in I$. Then the kinematic equation for the intrinsic rolling of $\mathrm{St}_{n,k}$ *over* $\mathfrak{m} \cong T_X \mathrm{St}_{n,k}$ *with respect to* ∇^{LC} *defined by the α-metric along* $v \colon I \to \mathfrak{m}$ *is given by*

$$\begin{aligned} \dot{v}(t) &= u(t), \\ \dot{S}(t) &= -\tfrac{1}{2}\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{S(t)u(t)} \circ S(t), \\ \dot{g}(t) &= \big(T_e \ell_{g(t)} \circ S(t)\big)u(t), \end{aligned} \tag{6.47}$$

where $\mathrm{pr}_{\mathfrak{m}} \colon \mathfrak{so}(n) \times \mathfrak{so}(k) \to \mathfrak{m}$ *is explicitly given by Lemma 6.12. Let* $\overline{q} \colon I \ni t \mapsto (v(t), g(t), S(t)) \in \overline{Q} = \mathfrak{m} \times (\mathrm{O}(n) \times \mathrm{O}(k)) \times \mathrm{O}(\mathfrak{m})$ *be a curve satisfying (6.47). Then*

$$q \colon I \to Q, \quad t \mapsto q(t) = (\overline{\pi} \circ \overline{q})(t) = (v(t), [g(t), S(t)]) \tag{6.48}$$

is an intrinsic rolling of $T_X \mathrm{St}_{n,k} \cong \mathfrak{m}$ *over* $\mathrm{St}_{n,k}$ *with respect to the given α-metric along the rolling curve v. The development curve* $I \ni t \mapsto \mathrm{pr}(g(t)) = (R(t), \theta(t)) \in \mathrm{O}(n) \times \mathrm{O}(k)$ *is mapped by the embedding* $\iota_X \colon (\mathrm{O}(n) \times \mathrm{O}(k))/H \to \mathbb{R}^{n \times k}$ *to the curve*

$$\gamma \colon I \to \mathrm{St}_{n,k}, \quad t \mapsto \gamma(t) = (\iota_X \circ \mathrm{pr})(g(t)) = \mathrm{pr}_X(g(t)) = R(t) X \theta(t)^\top. \tag{6.49}$$

Proof Since $\mathrm{St}_{n,k}$ equipped with an α-metric is a naturally reductive homogeneous space by Lemma 6.11, this is a direct consequence of Corollary 5.17 combined with Lemma 6.12 and Lemma 6.13. □

Next we consider the intrinsic rolling of the Stiefel manifolds along curves of a special form by using Sect. 5.4.1. This yields the next remark.

Remark 6.15 Let $\xi = (\xi_1, \xi_2) \in \mathfrak{so}(n) \times \mathfrak{so}(k)$. Then

$$q \colon I \to \mathfrak{m} \times (G \times_H \mathrm{O}(\mathfrak{m})), \quad t \mapsto (v(t), [g(t), S(t)]), \tag{6.50}$$

where

$$\begin{aligned} g(t) &= \exp(t\xi)\exp(-t\xi_{\mathfrak{h}}), \\ S(t) &= \mathrm{Ad}_{\exp(t\xi_{\mathfrak{h}})} \circ \exp\Big(-t\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{h}} + \frac{1}{2}\xi_{\mathfrak{m}}}\Big), \\ v(t) &= \int_0^t \exp\Big(s\Big(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}_{\xi_{\mathfrak{h}} + \frac{1}{2}\xi_{\mathfrak{m}}}\Big)\Big)(\xi_{\mathfrak{m}})\mathrm{d}s \end{aligned} \tag{6.51}$$

is an intrinsic rolling of $\mathfrak{m}$ along the rolling curve $v \colon I \to \mathfrak{m}$ with development curve $\gamma(t) = \mathrm{pr}(g(t)) = \mathrm{pr}(\exp(t\xi))$. Identifying $\mathrm{St}_{n,k} \cong (\mathrm{O}(n) \times \mathrm{O}(k))/H$ with the embedded submanifold $\mathrm{St}_{n,k} \subseteq \mathbb{R}^{n \times k}$ via $\iota_X \colon (\mathrm{O}(n) \times \mathrm{O}(k))/H \to \mathrm{St}_{n,k} \subseteq \mathbb{R}^{n \times k}$, the development curve is given by

$$\gamma(t) = \mathrm{e}^{t\xi_1} X e^{-t\xi_2} \tag{6.52}$$

and the rolling curve reads

$$V(t) = T_X(\iota_X \circ \mathrm{pr})v(t) = v_1(t)X - Xv_2(t), \tag{6.53}$$

where we write $v(t) = (v_1(t), v_2(t)) \in \mathfrak{m}$ with $v_1(t) \in \mathfrak{so}(n)$ and $v_2(t) \in \mathfrak{so}(k)$ for all $t \in I$.

7 Conclusion

In this text, we investigated intrinsic rollings of reductive homogeneous spaces equipped with invariant covariant derivatives. As preparation, we considered frame bundles of vector bundles associated to principal fiber bundles in detail. Afterwards, using an abstract definition of intrinsic rolling as starting point, we investigated rollings of $\mathfrak{m}$ over the reductive homogeneous spaces G/H with respect to an invariant covariant derivative ∇^{α}. For a given control curve, we obtained the so-called kinematic equation which is a time-variant explicit ODE on a Lie group, whose solutions describe rollings of $\mathfrak{m}$ over G/H. Moreover, for the case, where the development curve is the projection of a one-parameter subgroup, we provided explicit solutions of the kinematic equation describing intrinsic rollings of $\mathfrak{m}$ over G/H with respect to the canonical covariant derivative of first kind and second kind, respectively. As examples, we discussed intrinsic rollings of Lie groups and Stiefel manifolds.

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Declarations

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