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Lattice sums for double periodic polyanalytic functions

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Abstract

In 1892, Lord Rayleigh estimated the effective conductivity of rectangular arrays of disks and proved, employing the Eisenstein summation, that the lattice sum S_2 is equal to π for the square array. Further, it became clear that such equality can be treated as a necessary condition of the macroscopic isotropy of composites governed by the Laplace equation. This yielded the description of two-dimensional conducting composites by the classic elliptic functions, including the conditionally convergent Eisenstein series. In 1935, Natanzon used a polyharmonic function to solve the plane elasticity problem. This paper is devoted to the extension of the classic lattice sums to the lattice sums for double periodic (pseudoperiodic) polyanalytic functions. The exact relations and computationally effective formulae between the polyanalytic and classic lattice sums are established. Polynomial representations of the lattice sums are obtained. They are a source of new exact formulae for the lattice sums.

Keywords Lattice sum · Double periodic polyanalytic functions · Eisenstein summation · Elliptic integrals

Mathematics Subject Classification 33E20 · 11M36 · 33C75 · 33E05

1 Introduction

Various lattice sums are widely used to study the mechanical properties of crystals and the optical properties of regular lattices of atoms [1]. The classical theory of

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elliptic functions [2] can be considered as the theory of doubly periodic analytic (meromorphic) functions or the theory of functions on torus [3]. The 2D lattice sums are an element of this theory. This paper extends the lattice sums to doubly periodic polyanalytic functions.

Let a doubly periodic lattice on the complex plane be determined by two fundamental translation vectors expressed by the complex numbers ω_1, ω_2. Without loss of generality we may assume that $\omega_1 > 0$ and $\mathrm{Im}\tau > 0$ where $\tau = \frac{\omega_2}{\omega_1}$. Let the area of each cell be normalized to unity, i.e., $\omega_1^2 \mathrm{Im}\tau = 1$. Though the assumptions on ω_1 and the area partly restrict the application of the elliptic modular group $SL(2, \mathbb{Z})$ [3, Sect. V.7] to the extension of the formulae derived in the present paper, they are natural in the theory of composites and explicitly demonstrate the role of the number π in the description of isotropic lattices.

The *classic lattice* sums are introduced as follows

$$S_q(\omega_1, \omega_2) = \sum_{m,n}{}' \frac{1}{(m\omega_1 + n\omega_2)^q} = (\mathrm{Im}\ \tau)^{\frac{q}{2}} \sum_{m,n}{}' \frac{1}{(m + n\tau)^q}. \tag{1}$$

The symbol $\sum'_{m,n}$ means that m and n run over all integer numbers, except the pair $m = n = 0$. The series $S_q(\omega_1, \omega_2)$ we denote by $S_q(\tau)$ or S_q for short. The series (1) are absolutely convergent for $q > 2$, and conditionally convergent for $q = 1, 2$ [4]. The classic Weierstrass functions are expanded into the Laurent series near zero

$$\zeta(z) = \frac{1}{z} - \sum_{l=2}^{\infty} S_{2l} z^{2l+1}, \quad \wp(z) = \frac{1}{z^2} + \sum_{l=2}^{\infty} (2l+1) S_{2l} z^{2l}. \tag{2}$$

The conditionally convergent sum S_2 can be defined by the Eisenstein summation method [4]

$$\sum_{m,n}{}^{(e)} := \lim_{N\to\infty} \sum_{n=-N}^{N} \left(\lim_{M\to\infty} \sum_{m=-M}^{M} \right). \tag{3}$$

The Eisenstein functions of first and second order are introduced by the summation (3) and can be expanded into the series [4]

$$E_1(z) = \frac{1}{z} - \sum_{l=1}^{\infty} S_{2l} z^{2l+1}, \quad E_2(z) = \frac{1}{z^2} + \sum_{l=1}^{\infty} (2l+1) S_{2l} z^{2l}. \tag{4}$$

Rayleigh [5] investigated doubly periodic problems for harmonic functions with one circular inclusion per periodicity cell and applied them to the calculation of the effective conductivity. Rayleigh [5] used the Eisenstein summation and proved that for rectangular lattices

$$S_2 = \pi^2 \mathrm{Im}\ \tau \left(\frac{1}{3} + 2 \sum_{m=1}^{\infty} \frac{1}{\sin^2 m\pi\tau} \right). \tag{5}$$

Rayleigh [5] demonstrated that $S_2 = \pi$ for the square array when $\tau = i$. The same formula $S_2 = \pi$ holds for the hexagonal (triangular) array when $\tau = \exp(\frac{\pi i}{3})$. It is worth noting that only the square and hexagonal arrays form macroscopically isotropic conducting regular composites.

Historical remarks and various applications of polyanalytic functions can be found in [6, 7]. A polyanalytic version of the Weierstrass sigma function and its application to signal analysis was discussed in [6] and works cited therein. A theory of polyanalytic doubly periodic functions having applications to 2D elastic problems was developed in [8–11]. The procedure of homogenization for double periodic elastic structures requires periodicity (pseudoperiodicity) of local elastic fields. This leads to the particular theory of periodic (pseudoperiodic) polyanalytic functions summarized in [10, Appendix 2]. One of the fundamental notions of this theory is the *lattice sums*

$$S_q^{(p)}(\omega_1, \omega_2) = \sum_{m,n}{}' \frac{\overline{(m\omega_1 + n\omega_2)}^p}{(m\omega_1 + n\omega_2)^q} = (\mathrm{Im}\ \tau)^{\frac{q-p}{2}} \sum_{m,n}{}' \frac{\overline{(m + n\tau)}^p}{(m + n\tau)^q}. \tag{6}$$

We use the following terminology: (1) is called the *classic lattice sum* and (6) the *p-analytic or polyanalytic lattice sums*. Similar to the expansions (4) for the Weierstrass functions, the main bianalytic doubly periodic functions can be expanded into series, including the lattice sums $S_q^{(1)}(\omega_1, \omega_2)$ following [8, 10]

$$\wp_{1,2}(z) = \sum_{m=2}^{\infty} (m+1) S_{m+2}^{(1)} z^m. \tag{7}$$

The function (7) used to provide the solution for the plane elasticity problem is polyharmonic with $p = 1$ [8, 10].

The sum $S_3^{(1)}(\omega_1, \omega_2)$ is conditionally convergent and can be defined by the Eisenstein summation, which yields [11, formula (2.10)]

$$S_3^{(1)}(\omega_1, \omega_2) = S_2(\omega_1, \omega_2) - 4\pi^3 i (\mathrm{Im}\ \tau)^2 \sum_{m=1}^{\infty} m \frac{\cos(m\pi\tau)}{\sin^3(m\pi\tau)}. \tag{8}$$

It was proved in [11] that $S_3^{(1)} = \frac{\pi}{2}$ for the hexagonal array, which is the unique regular periodic structure for 2D elastic composite. It is worth noting that the square array does not form an isotropic elastic structure, and $S_3^{(1)} = \frac{\pi}{2} + \frac{\Gamma(1/4)}{384\pi^3}$, in this case [11, formula (4.19)].

In the present paper, we study the p-analytic lattice sums (6) associated with the p-analytic functions. A computationally effective formula (18) for the p-analytic lattice sums (6) is derived. As the particular formulae (5) and (8), formula (18) is based on the trigonometric series, first arisen in the works due to Eisenstein [4] and further in [12], see for extended Refs. [1, 11]. Special attention is paid to the conditionally convergent sums $S_{p+2}^{(p)}$. A simple formula for some of $S_{p+2}^{(p)}$ based on numerical computations is suggested. In particular cases, it coincides with the known formulae $S_2^{(0)} = \pi$ and

$S_3^{(1)} = \frac{\pi}{2}$. The expression (22) of the p-analytic lattice sums $S_q^{(1)}$ through the classic lattice sums S_q is established.

2 Expression of *p*-analytic sums through the classic lattice sums

Consider the p-analytic lattice sums (6) for $q - p \geq 2$. If $q - p > 2$, the series absolutely converges. In the case $q = p + 2$ we arrive at the conditionally convergent series $S_{p+2}^{(p)}$. The Eisenstein summation method will be applied to $S_{p+2}^{(p)}$.

First, consider the function

$$\varepsilon_0(\tau) = \ln\ \sin(\pi\tau) \tag{9}$$

and the functions

$$\varepsilon_l(\tau) := \sum_m (j + \tau)^{-l}, \quad l = 1, 2, \ldots. \tag{10}$$

The n-th derivative of ε_0 satisfies equation [4]

$$\frac{d^n}{d\tau^n}\varepsilon_0(\tau) = (-1)^{n+1}(n-1)!\ \varepsilon_n(\tau). \tag{11}$$

The following formula was proved in [13]

$$S_q^{(1)} = S_{q-1} + 2i\frac{\text{Im}\ \omega_2}{\omega_1^q}\frac{(-1)^q}{(q-1)!}\sum_m m\left[\frac{d^q\varepsilon_0(x)}{dx^q}\right]_{x=m\tau}, \quad q > 2. \tag{12}$$

We now proceed to extend this formula to the series (6). Due to assumptions about translation vectors ω_1, ω_2 the expression $\overline{(m + n\tau)}^p$ is equal to $(m + n\overline{\tau})^p$. Using the binomial formula and adding and subtracting coefficient with nonconjugated τ, we have

$$\begin{aligned} S_q^{(p)} &= \sum_{m,n}\frac{1}{(m\omega_1 + n\omega_2)^{q-p}} \\ &\quad - 2i(\text{Im}\ \tau)^{\frac{q-p}{2}}\sum_{s=1}^{p}\binom{p}{s}\text{Im}\ (\tau^s)\sum_n n^s\sum_m\frac{m^{p-s}}{(m + n\tau)^q}, \end{aligned} \tag{13}$$

where $\binom{p}{s}$ denotes the binomial coefficient. Using the relation

$$\sum_m\frac{m^r}{(m + n\tau)^q} = \sum_{s=0}^{r}(-1)^s\binom{r}{s}(n\tau)^s\sum_m\frac{1}{(m + n\tau)^{q-r+s}} \tag{14}$$

we obtain

$$S_q^{(p)} = S_{q-p} - 2i(\text{Im }\tau)^{\frac{q-p}{2}} \sum_{s=1}^{p} \sum_{r=0}^{p-s} (-1)^r \binom{p}{s} \binom{p-s}{r} \tau^r \text{Im }(\tau^s) \sum_n n^{s+r} \sum_m \frac{1}{(m+n\tau)^{q-p+s+r}}. \quad (15)$$

Introduction of the summation index $t = s + r$ yields

$$S_q^{(p)} = S_{q-p} - 2i(\text{Im }\tau)^{\frac{q-p}{2}} \sum_{s=1}^{p} \sum_{t=s}^{p} (-1)^{t-s} \binom{p}{s} \binom{p-s}{t-s} \tau^{t-s} \text{Im }(\tau^s) \sum_n n^t \sum_m \frac{1}{(m+n\tau)^{q-p+t}} \quad (16)$$

Selecting the terms with the same values of $\sum_n n^t \sum_m \frac{1}{(m+n\tau)^{q-p+t}}$ we obtain

$$S_q^{(p)} = S_{q-p} - 2i(\text{Im }\tau)^{\frac{q-p}{2}} \sum_{t=1}^{p} \left(\sum_{s=1}^{t} (-1)^{t-s} \binom{p}{s} \binom{p-s}{t-s} \tau^{t-s} \text{Im }(\tau^s) \right) \sum_n n^t \sum_m \frac{1}{(m+n\tau)^{q-p+t}}. \quad (17)$$

Using (11) we arrive at the formula for $q - p \geq 2$

$$S_q^{(p)} = S_{q-p} + 2i(\text{Im }\tau)^{\frac{q-p}{2}} \times \sum_{t=1}^{p} \left(\sum_{s=1}^{t} \frac{(-1)^{q-p-s}}{(q-p+t-1)!} \binom{p}{s} \binom{p-s}{t-s} \tau^{t-s} \text{Im }(\tau^s) \right) \sum_n n^t \left[\frac{\mathrm{d}^{q-p+t}}{\mathrm{d}x^{q-p+t}} \varepsilon_0(x) \right]_{x=n\tau}. \quad (18)$$

One can see that formula (18) for $p = 1$ coincides with formula (12) from [13].

The formula (18) is effective in numerical computations. It contains the classic lattice sums for which fast computational formulae are known [14]. The sum $\sum_n n^t \frac{d^{q-p+t}}{d\tau^{q-p+t}} \varepsilon_0(n\tau)$ contains the derivatives of the elementary function (9), hence, it can be also also quickly computed. The results of numerical computations are presented in Table 1 for hexagonal lattice and in Table 2 for square lattice (Appendix B). The values in the tables without the decimal points are exact.

3 Recurrence formula for $S^{(1)}_{2l+1}$ hrough the classic lattice sums

The following well-known recurrence formula is useful to compute the classic lattice sums beginning from S_4 and S_6

$$(2l-1)(2l+1)(l-3)S_{2l} = 3\sum_{j=2}^{l-2}(2j-1)(2l-2j-1)S_{2j}S_{2l-2j} \tag{19}$$

Fast formulae for S_4 and S_6 are known in the theory of elliptic functions [2]. In the present section, we express the lattice sums $S^{(1)}_{2l+3}$ through the classic sums by an analogous formula. It is worth noting that the lattice sums S_{2l+1} and $S^{(1)}_{2l+2}$ vanish for $l = 0, 1, \ldots$ [4, 10].

We will use the expansion (7) of the Natanzon–Filshtinsky function and the following formula established in [10, 13]

$$\pi\wp_{1,2}(z) = \frac{1}{2}\wp'(z) + (\zeta(z) - (S_2-\pi)z)\wp(z) + (S_2-\pi)\zeta(z) + (5S_4+c)z + c_1. \tag{20}$$

It is assumed that the constants c and c_1 are undetermined. Substituting (2) and (7) into (20) we arrive at the formula

$$\begin{aligned}\pi\sum_{m=2}^{\infty}(m+1)S^{(1)}_{m+2}z^m &= c_1 + (10S_4+c)z + \sum_{n=1}^{\infty}((2n+3)(n+2)-1)\\ &\quad\times S_{2n+4}z^{2n+1} - (S_2-\pi)\sum_{n=1}^{\infty}2nS_{2n+2}z^{2n+1}\\ &\quad-\sum_{n=2}^{\infty}\sum_{l=1}^{n-1}(2l+1)S_{2l+2}S_{2(n-l)+2}z^{2n+1}.\end{aligned} \tag{21}$$

We arrive at the following assertions by selecting coefficients at the same powers of z. First, the linear part $c_1 + (10S_4 + c)z$ vanishes, hence, $c_1 = 0$ and $c = -10S_4$. Second, the sums $S^{(1)}_m$ are equal zero for even m. Third, the lattice sums $S^{(1)}_{2m+3}$ can be calculated through the classic sums S_m by means of the following algebraic relations

$$\begin{aligned}2\pi S^{(1)}_{2m+3} &= (2m+5)\,S_{2m+4} - \frac{2m}{m+1}(S_2-\pi)S_{2m+2}\\ &\quad-\sum_{l=1}^{m-1}\frac{2l+1}{m+1}S_{2l+2}S_{2(m-l)+2},\quad m=1,2,\ldots.\end{aligned} \tag{22}$$

Note that $S_2 = \pi$ for the hexagonal and square lattices. Hence, in these cases (22) is reduced to

$$2\pi S_{2m+3}^{(1)} = (2m+5)\, S_{2m+4} - \sum_{l=1}^{m-1} \frac{2l+1}{m+1} S_{2l+2} S_{2(m-l)+2}, \quad m = 1, 2, \ldots . \tag{23}$$

The relations (22) and (23) are useful for fast computations and can be applied to deduce new exact formulae. Consider, for instance, the square lattice for which $S_6 = 0$ and [15]

$$S_4 = \frac{\Gamma^8\left(\frac{1}{4}\right)}{2^6 \cdot 3 \cdot 5\pi^2}, \tag{24}$$

where $\Gamma(z)$ stands for Euler's gamma-function [16, vol. I]. It follows from (19) that $S_8 = \frac{3}{7} S_4^2$. Then, (23) yields the closed form expression for $S_7^{(1)} = \frac{10}{7\pi} S_4^2$ after the substitution (24), see the value in the Table 5.

4 Expression of lattice sums through elliptic integrals

4.1 An useful function *x* derived from elliptic integrals

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+$ the set of positive numbers. Following Akhiezer [2] and Erdélyi et al. [16, vol. II] introduce the complete elliptic integral of the first kind

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{25}$$

where the elliptic modulus $k \in (0, 1)$ and the complimentary modulus $k' = \sqrt{1-k^2}$ are used. The value

$$x \equiv x(k) = \frac{K(k')}{K(k)} \tag{26}$$

can be considered as a one-to-one monotonically decreasing continuously differentiable function $x : (0, 1) \to \mathbb{R}_+$. The previous papers implicitly used expressions for the derivative $\frac{dx}{dk}$. Below, such a formula is explicitly written and proved for the completeness of the presentation.

Proposition 1 *The derivative of the function $x(k)$ can be calculated by the formula*

$$\frac{dx}{dk} = -\frac{\pi}{2k(1-k^2)K^2(k)}. \tag{27}$$

Proof The complete elliptic integral $K(k)$ satisfies Legendre's relation

$$E(k)K(k') + E(k')K(k) - K(k')K(k) = \frac{\pi}{2}, \tag{28}$$

where $E(k)$ is the complete elliptic integral of the second kind

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}dt. \tag{29}$$

Its derivative can be calculated by the formula

$$\frac{dE}{dk} = \frac{E(k) - K(k)}{k}. \tag{30}$$

The pair of the functions $K(k)$ and $K(k')$ satisfy the differential equation [17]

$$\frac{d}{dk}\left(k(k')^2\frac{du}{dk}\right) = ku. \tag{31}$$

The functions $E(k)$ and $E(k') - K(k')$ satisfy the other differential equation [17]

$$(k')^2\frac{d}{dk}\left(k\frac{du}{dk}\right) + ku = 0. \tag{32}$$

Then, the derivative of $K(k)$ can be calculated by the formula

$$\frac{dK}{dk} = \frac{E(k) - (k')^2K(k)}{k(k')^2}. \tag{33}$$

It follows from Legendre's relation (28) that

$$K(k') = \frac{\frac{\pi}{2} - E(k')K(k)}{(E(k) - K(k))K(k)}. \tag{34}$$

Differentiation of (26) and using the expressions on the derivatives of the elliptic integrals yield

$$\frac{dx}{dk} = \frac{1}{k(1-k^2)K(k)}\left(K(k')\left(1 - \frac{E(k)}{K(k)}\right) - E(k')\right). \tag{35}$$

Application of Legendre's identity (28) reduces (35)–(27).

The proposition is proved. □

4.2 The basic modular forms

Let r be an arbitrary natural number $r \geq 2$. It will be convenient to consider the slightly modified lattice sums (1) called the modular form

$$V_r^{(0)}(\tau) := \sum_{m,n}{}' \frac{1}{(m+n\tau)^r}. \tag{36}$$

It follows from (1) that $V_r^{(0)}(\tau) = 0$ for r odd. Consider τ as a function $\tau = \tau(x(k))$ depending on k. Using (26) and (50)–(51) we arrive at the formula

$$\begin{aligned} V_2^{(0)}(\tau) &:= \sum_n \sum_m \frac{1}{(m+n\tau)^2} \\ &= \begin{cases} \frac{4}{3}(4k^2-5)K^2(k) + 8K(k)E(k), & \tau = \frac{1+ix}{2} \\ \frac{4}{3}(k^2-2)K^2(k) + 4K(k)E(k), & \tau = ix. \end{cases} \end{aligned} \tag{37}$$

The classic lattice sums S_4 and S_6 defined by (1) are expressed through the elliptic integrals [18, items 18.9.4 and 18.9.5]. Using (36) we can write these expressions in the form

$$V_4^{(0)}(\tau) = \begin{cases} \frac{16}{45}(16k^4 - 16k^2 + 1)K^4(k), & \tau = \frac{1+ix}{2} \\ \frac{16}{45}(k^4 - k^2 + 1)K^4(k), & \tau = ix. \end{cases} \tag{38}$$

$$V_6^{(0)}(\tau) = \begin{cases} \frac{128}{945}(2k^2-1)(32k^4 - 32k^2 - 1)K^6(k), & \tau = \frac{1+ix}{2} \\ \frac{64}{945}(k^2-2)(2k^2-1)(k^2+1)K^6(k), & \tau = ix. \end{cases} \tag{39}$$

Using (1) and (36) we rewrite relation (19) as the recursive formula to determine the values (36)

$$V_{2l}^{(0)}(\tau) = 3\sum_{j=2}^{l-2} \frac{(2j-1)(2l-2j-1)}{(2l-1)(2l+1)(l-3)} V_{2j}^{(0)}(\tau) V_{2l-2j}^{(0)}(\tau), \quad l = 4, 5, \ldots \tag{40}$$

The term $(\mathrm{Im}\tau)^{r/2}$ from (17) becomes $i^{r/2}$ in the first case $\tau = ix$ and $\left(\frac{i}{2}\right)^{r/2}$ in the second case $\tau = \frac{1+ix}{2}$. Consider the first case for definiteness. The expression $\tau^{r/2} = \left(i\frac{K'}{K}\right)^{r/2}$ contains K in the denominator with power $r/2$. But it is canceled with the multiplier K^r in $V_r^{(0)}$ for $r > 2$. Therefore, $S_r^{(0)}$ is a polynomial in k, K, K' and E for any even r. The theorem is proved for $p = 0$.

The proof for $p > 0$ can be obtained from Faá di Bruno's formula. But we shall follow a constructive recursive approach preferable in computations. Looking at τ as a function $\tau = \tau(x(k))$ depending on k, it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}k} V_r^{(0)}(\tau(x(k)))k = \frac{\mathrm{d}}{\mathrm{d}r} V_r^{(0)}(\tau) \frac{\mathrm{d}\tau}{\mathrm{d}x} \frac{\mathrm{d}x}{\mathrm{d}k}. \tag{41}$$

The derivatives of $V_r^{(0)}$ in τ are the series

$$\frac{\mathrm{d}^s}{\mathrm{d}\tau^s} V_r^{(0)}(\tau) = (-1)^s s! \binom{s+r-1}{r-1} \sum_n n^s \sum_m \frac{1}{(m+n\tau)^{s+r}}, \quad s = 1, 2, \ldots. \tag{42}$$

4.3 The extended modular forms

Using (41) introduce the functions

$$V_r^{(1)}(k) = \frac{\frac{\mathrm{d}}{\mathrm{d}k} V_r^{(0)}(\tau(x(k)))}{\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}k}},$$

$$V_r^{(s)}(k) = \frac{d^s}{d\tau^s} V_r^{(0)}(\tau) = \frac{\frac{\mathrm{d}}{\mathrm{d}k} V_r^{(s-1)}(k)}{\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}k}}, \; s = 1, 2, \ldots. \tag{43}$$

The derivative $\frac{\mathrm{d}\tau}{\mathrm{d}x}$ becomes the constant i in the first case $\tau = ix$ and $\frac{i}{2}$ in the second case $\tau = \frac{\pm 1 + ix}{2}$. As we have already assumed before, consider the first case for definiteness. It follows from (27) that

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}k}\right)^{-1} = \frac{2ki}{\pi}(1-k^2)K^2(k). \tag{44}$$

The second formula (43) can be written in the form

$$V_r^{(s)}(k) = \frac{2ki}{\pi}(1-k^2)K^2(k)\frac{d}{dk}V_r^{(s-1)}(k)k, \; s = 1, 2, \ldots \tag{45}$$

For example the first three functions $V_r^{(1)}(k)$ are calculated by (43)–(45) by using (30) and (33)

$$V_2^{(1)}(k) = \frac{4i}{3\pi}K^2(k)\left[-3E^2(k) - 2(k^2-2)E(k)K(k) + (k^2-1)K^2(k)\right]. \tag{46}$$

$$V_4^{(1)}(k) = \frac{16i}{45\pi}K^5(k)\left[(k^4-3k^2+2)K(k) - 2(k^4-k^2+1)E(k)\right]. \tag{47}$$

$$V_6^{(1)}(k) = \frac{128i}{945\pi}K^7(k)\left[(k^6+k^4-4k^2+2)K(k) - (2k^6-3k^4-3k^2+2)E(k)\right]. \tag{48}$$

Equation (42) implies that

$$\sum_n n^p \sum_m \frac{1}{(m+n\tau)^{p+r}} = \frac{(-1)^p}{p!\binom{p+r-1}{r-1}} V_r^{(p)}(k). \tag{49}$$

4.4 A recurrence relation providing polyanalytic lattice sums from modular forms

We will use the following Rayleigh formulae [5] written in the form [11]

$$S_2(ix) = \frac{4}{3}\, K(k')\left(3E(k) + (k^2-2)K(k)\right), \tag{50}$$

$$S_2\left(\frac{\pm 1 + ix}{2}\right) = 2\, K(k')\left(2\, E(k) + \frac{4k^2-5}{3}\, K(k)\right), \quad x > 0. \tag{51}$$

Consider now (17) with $q = p+2$

$$S^{(p)}_{p+2}(\tau) = S_2(\tau) - 2i\mathrm{Im}\,\tau \sum_{t=1}^{p}\left(\sum_{s=1}^{t}(-1)^{t-s}\binom{p}{s}\binom{p-s}{t-s}\tau^{t-s}\mathrm{Im}\,(\tau^s)\right)$$
$$\sum_n n^t \sum_m \frac{1}{(m+n\tau)^{t+2}}. \tag{52}$$

The formula (52) for $p=1$ becomes

$$S^{(1)}_3(\tau) = S_2(\tau) - 2i(\mathrm{Im}\,\tau)^2 \sum_n n \sum_m \frac{1}{(m+n\tau)^3}. \tag{53}$$

Using (53) we can write (52) for $p=2$ in the form

$$S^{(2)}_4(\tau) = 2S^{(1)}_3(\tau) - S_2(\tau)$$
$$- 2i\mathrm{Im}\,\tau(\mathrm{Im}\,(\tau^2) - 2\tau\mathrm{Im}\,\tau)\sum_n n^2 \sum_m \frac{1}{(m+n\tau)^4}. \tag{54}$$

This recurrence can be continued and the values $S^{(p)}_{p+2}(\tau)$ can be found by means of the previous ones $S^{(l)}_{l+2}(\tau)$ ($l = 0, 1, \ldots, p-1$) and by the absolutely convergent series $\sum_n n^p \sum_m \frac{1}{(m+n\tau)^{p+2}}$. This series can be differentiated term by term

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sum_n n^{t-1}\sum_m \frac{1}{(m+n\tau)^{t+1}}\tau = -\frac{1}{t+1}\sum_n n^t \sum_m \frac{1}{(m+n\tau)^{t+2}}$$

Theorem 2 *Let* $\tau = \frac{1+ix}{2}$ *or* $\tau = ix$, $(x > 0)$. *Then,* $S^{(p)}_{p+r}(\tau)$ *for any integer* $p \geq 0$, $r \geq 2$ *can be written as a polynomial in four variables* K, K', E *and* k.

Proof Using relation (49) formula (17) can be written in the form

$$S^{(p)}_{p+r}(\tau) = S_r(\tau) + 2i(\mathrm{Im}\,\tau)^{\frac{r}{2}} \times \sum_{t=1}^{p}\left(\sum_{s=1}^{t}(-1)^{t-s}\binom{p}{s}\binom{p-s}{t-s}\tau^{t-s}\mathrm{Im}\,(\tau^s)\right)$$
$$V^{(t)}_r(\tau) \tag{55}$$

We now proceed to prove by induction that the recurrence formulae (45) determine the function $V_r^{(p)}(k)$ for any integer $p \geq 0$ as a homogeneous polynomial in K and E of the total power in k, K at least $p+1$ and at most $2(p+1)$ for $r=2$ and at least $p+r$ and at most $2p+r$ for $r>2$.

Let us assume that the induction hypothesis is true for $p-1$, i.e., $V_r^{(p-1)}(k)$ is a polynomial of the variables K, E and k of the total power in K of the total power in K at least p and at most $2p$ for $r=2$ and at least $p+r-1$ and at most $2p+r-2$ for $r>2$. Consider the action of the operator determined by the right part of (45) up to a constant multiplier on the polynomial term

$$k(1-k^2)K^2\frac{\mathrm{d}}{\mathrm{d}k}[f(k)K^mE^l] = k(1-k^2)K^{m+2}E^l\frac{\mathrm{d}}{\mathrm{d}k}f(k) + f(k)K^{m+1}E^{l-1} \times [mE^2 + (1-k^2)(l-m)KE - l(1-k^2)K^2], \tag{56}$$

where $f(k)$ is a polynomial in k and $p \leq m \leq 2p$ for $r=2$ or $p+r-1 \leq m \leq 2p+r-2$ for $r>2$. Here, we omit the argument in the elliptic integrals for short. Hence, after application of the operator from (45) the polynomial $V_r^{(p-1)}(k)$ is transformed onto the homogeneous polynomial $V_r^{(p)}(k)$ in K, E and k. Moreover, $V_r^{(p)}$ is presented as a polynomial with the multiplier $K^{m'}$, where from (56) $p+1 \leq m' \leq 2(p+1)$ for $r=2$ and $p+r \leq m' \leq 2p+r$ for $r>2$.

The term $(\mathrm{Im}\tau)^{\frac{r}{2}}\tau^t\mathrm{Im}(\tau^s)$, $(0<s\leq p, 0\leq t\leq p-s)$ from (55) after substitution $\tau = i\frac{K'}{K}$ contains only K in the denominator with power less than $p+r$. But it is canceled with the multiplier K with power at least $p+1$ for $r=2$ and $p+r$ for $r>2$ in $V_r^{(p)}$. Substitution of (49) into (52) yields the required polynomial representation of $S_{p+r}^{(p)}(\tau)$. □

Remark 1 This yields a recursive method on $V_{q-p}^{(t)}$ $(p\geq 0, q\geq p+2)$ to compute the sums $S_q^{(p)}(\tau)$ as follows

$$S_q^{(p)}(\tau) = S_{q-p}(\tau) + 2i(\mathrm{Im}\,\tau)^{\frac{q-p}{2}} \times \sum_{t=1}^{p}\left(\sum_{s=1}^{t}(-1)^{t-s}\binom{p}{s}\binom{p-s}{t-s}\tau^{t-s}\mathrm{Im}\,(\tau^s)\right)V_{q-p}^{(t)}(\tau) \tag{57}$$

for $\tau = ix$ or $\tau = \frac{1+ix}{2}$.

4.5 Explicit expressions of polyanalytical lattice sums using elliptic integrals

Appendix contains the Mathematica code to compute the lattice sums $S_q^{(p)}(\tau)$ for $\tau = ix$ (Listing 1) and $\tau = \frac{1+ix}{2}$ (Listing 2) where x is given by (26). The computed values $S_2^{(0)}$ and $S_3^{(1)}$ coincide with the same values obtained in [11, Th.4.6].

Below, we write typical new formulae obtained by application of Theorem 2 and by (57)

$$S_4^{(2)}(ix) = \frac{4}{3\pi^2}\Big(4K'(16E(E-K)\Big(E+(k^2-1)K)K'^2 - 4(3E^2+2E(k^2-2)K-(k^2-1)K^2\Big)K'\pi + (3E+(k^2-2)K)\pi^2)\Big), \quad (58)$$

$$S_4^{(2)}\left(\frac{1+ix}{2}\right) = \frac{2}{3\pi^2}K'\Big(16(2E-K)(E^2+2E(k^2-1)K-(k^2-1)K^2)K'^2 - 8(3E^2+E(4k^2-5)K-2(k^2-1)K^2)K'\pi + (6E+(4k^2-5)K)\pi^2\Big) \quad (59)$$

$$S_5^{(3)}(ix) = \frac{4}{3\pi^3}K'\Big(48E(E-K)(E+(k^2-1)K)K'^2\pi - 16\Big(3E^4+4E^3(k^2-2)K-6E^2(k^2-1)K^2-(k^2-1)^2K^4\Big)K'^3 - 6(3E^2+2E(k^2-2)K-(k^2-1)K^2)K'\pi^2 + (3E+(k^2-2)K)\pi^3\Big), \quad (60)$$

$$S_5^{(3)}\left(\frac{1+ix}{2}\right) = \frac{2}{3\pi^3}K'\Big(48(2E-K)(E^2+K(2E-K)(k^2-1))K'^2\pi - 32\Big(3E^4+2E^3(4k^2-5)K+(k^2-1)K^2(6EK-12E^2-K^2)\Big)K'^3 - 12(3E^2+E(4k^2-5)K-2(k^2-1)K^2)K'\pi^2+(6E+(4k^2-5)K)\pi^3\Big) \quad (61)$$

$$S_6^{(4)}(ix) = \frac{4}{15\pi^4}K'\Big(256(3E^5+5E^4(k^2-2)K-10E^3(k^2-1)K^2 - 5E(k^2-1)^2K^4-(k^2-2)(k^2-1)^2K^5)K'^4 - 320(3E^4+4E^3(k^2-2)K-6E^2(k^2-1)K^2-(k^2-1)^2K^4)K'^3\pi - 40(3E^2+2E(k^2-2)K-(k^2-1)K^2)K'\pi^3+5(3E+(k^2-2)K)\pi^4 + 480E(E-K)(E+(k^2-1)K)K'^2\pi^2\Big) \quad (62)$$

$$S_6^{(4)}\left(\frac{1+ix}{2}\right) = \frac{2}{15\pi^4}K'\Big(256(6E^5+5E^4(4k^2-5)K-40E^3(k^2-1)K^2 + (k^2-1)(30E^2K^3-10EK^4+K^5))K'^4 - 640(3E^4+2E^3(4k^2-5)K+(k^2-1)(6EK^3-12E^2K^2-K^4))K'^3\pi - 80(3E^2+E(4k^2-5)K-2(k^2-1)K^2)K'\pi^3+5(6E+(4k^2-5)K)\pi^4 + 480(2E-K)(E^2+(k^2-1)(2EK-K^2))K'^2\pi^2\Big) \quad (63)$$

All the results given above are expressed by using the elliptic modulus k that is related to the geometric parameter x by (26). For any value of x, k is given by inversion of (26). However for many cases of interest, k is a simple function of x, as shown in the following section, avoiding the tedious inversion giving x.

5 Exact values for lattice sums

Theorem 2 yields exact startling formulae including the number π for the special lattice sums. Let k_r be such an elliptic modulus for which $x(k_r) = \sqrt{r}$, see (26). It follows from Borwein and Borwein [19] that

$$\begin{aligned} k_1 &= \frac{1}{\sqrt{2}}, \quad k_2 = \sqrt{2} - 1, \\ k_3 &= \frac{1}{4}\sqrt{2}(\sqrt{3} - 1), \ k_4 = 3 - 2\sqrt{2}, \end{aligned} \tag{64}$$

$$K(k_1) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}, \quad K(k_2) = \frac{(\sqrt{2}+1)^{1/2}\Gamma(1/8)\Gamma(3/8)}{2^{13/4}\sqrt{\pi}}, \tag{65}$$

$$K(k_3) = \frac{3^{1/4}\Gamma^3(1/3)}{2^{7/3}\pi}, \quad K(k_4) = \frac{(\sqrt{2}+1)\Gamma^2(1/4)}{2^{7/2}\sqrt{\pi}}, \tag{66}$$

and

$$k_{1/2} = \sqrt{2\sqrt{2} - 2}, \quad k_{1/3} = \frac{1}{4}\sqrt{2}(\sqrt{3} + 1), \quad k_{1/4} = 2^{1/4}(2\sqrt{2} - 2). \tag{67}$$

Here we use known fact that k_r' is the complementary modulus to k_r i.e $k_r' = k_{1/r}$. Using (26) we obtain

$$\begin{aligned} &K(k_{1/2}) = \sqrt{2}K(k_2), \quad K(k_{1/3}) = \sqrt{3}K(k_3), \\ &K(k_{1/4}) = 2K(k_4). \end{aligned} \tag{68}$$

Following Borwein and Borwein [19] consider the elliptic alpha function

$$\alpha(r) = \frac{E(k_r')}{K(k_r)} - \frac{\pi}{4(K(k_r))^2} = \frac{\pi}{4(K(k_r))^2} + \sqrt{r}\left(1 - \frac{E(k_r)}{K(k_r)}\right). \tag{69}$$

We have

$$E(k_r) = K(k_r) - \frac{\alpha(r)}{\sqrt{r}}K(k_r) + \frac{\pi}{4\sqrt{r}K(k_r)} \tag{70}$$

and

$$E(k_r') = \alpha(r)K(k_r) + \frac{\pi}{4K(k_r)}. \tag{71}$$

The first four values of $\alpha(r)$, $r = 1, 2, 3, 4$ are exactly written as follows

$$\alpha(1) = \frac{1}{2}, \quad \alpha(2) = \sqrt{2} - 1, \quad \alpha(3) = \frac{1}{2}(\sqrt{3} - 1), \quad \alpha(4) = 2(\sqrt{2} - 1)^2. \tag{72}$$

The value of $\alpha(r^{-1})$ is calculated by formula [19, p. 153]

$$\alpha\left(\frac{1}{r}\right) = \frac{\sqrt{r} - \alpha(r)}{r}. \tag{73}$$

This yields

$$\alpha\left(\frac{1}{2}\right) = \frac{1}{2}, \quad \alpha\left(\frac{1}{3}\right) = \frac{1}{6}(\sqrt{3}+1), \quad \alpha\left(\frac{1}{4}\right) = \sqrt{2} - 1. \tag{74}$$

Using the above and recursive formula (57) we calculate some $S_q^{(p)}$. For example, if $\tau = i$ we have $r = 1$ and $k_1 = k_1' = \frac{1}{\sqrt{2}}$. The corresponding elliptic integrals of the first kind are calculated by the formulae $K(k_1) = K(k_1') = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}$. The elliptic integrals of the second kind are calculated by (70) and (71)

$$E(k_1) = K(k_1) - \alpha(1)K(k_1) + \frac{\pi}{4K(k_1)} = \frac{8\pi^2 + \Gamma^4(1/4)}{8\sqrt{\pi} + \Gamma^2(1/4)},$$
$$E(k_1') = \alpha(1)K(k_1) + \frac{\pi}{4K(k_1)} = \frac{4\pi^2 + \Gamma^4(1/4)}{4\sqrt{\pi} + \Gamma^2(1/4)}.$$

Substituting the above values into (58) we have

$$S_4^{(2)}(i) = \frac{\pi}{3}. \tag{75}$$

Similarly, we can calculate some values of the $S_q^{(p)}$. The results are summarized in Tables 5, 6, 7, 8 and 9 (Appendix B).

6 Discussion and conclusion

The paper systematically describes the p-analytic lattice sums (6). In particular, the computationally effective formula (18) is derived. The expressions (22)–(23) of the p-analytic lattice sums $S_q^{(1)}$ through the classic lattice sums S_q are established. Theorem 2 about polynomial representations is constructive and is the source of exact formulae with the number π selected in the tables.

After preparing our paper for submission, the paper [20] was published. The formulae of Chen et al. [20] containing the p-analytic lattice sums overlap with our results. Though our formulae (18) and the values from Tables 5, 6, 7, 8 and 9 coincide with the formulae (20), (21) and partially with Table 1 from Chen et al. [20], we give our alternative proofs in the present paper to completely present polynomial representations of lattice sums in Theorem 1.

The obtained formulae have fundamental applications in the theory of fibrous composites since the effective properties of composites are expressed in terms of the lattice sums [14, 21]. For example, consider the hexagonal array of unidirectional circular fibers embedded in a host material. Let ϕ denote the concentration of fibers. The effective shear modulus G_e can be found by the following formula [21, formulas (3.83)–(3.84)]

$$G_e = G\frac{1+A(\phi)}{1-\kappa A(\phi)}, \tag{76}$$

where G and κ denote the shear modulus and Muskhelishvili's constant of the host. The value $A(\phi)$ can be considered as a function expanded in the powers of ϕ written here in the shortened form

$$A(\phi) = \varrho_3\phi + \frac{48(S_5^{(1)})^2}{\pi^4}\varrho_3^3\phi^5 - \frac{360S_5^{(1)}S_6}{\pi^5}\varrho_3^3\phi^6 + \frac{5S_6^2\varrho^3(\varrho_1\varrho_2+135\varrho_3^2)}{\pi^6}\varrho_3^3\phi^7 + \dots \tag{77}$$

Here, ϱ_j $(j = 1, 2, 3)$ are some combination of elastic constants of the components of the considered composite.

The theory of lattice sums for polyanalytic functions developed in the present paper can be helpful for further investigations of regular and random composites following the lines of Drygaś et al. [21].

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Code availability All used codes are available in this manuscript.

Declarations

Conflict of interest The authors declare that they have no known conflict of interest or competing interests that could have appeared to influence the work reported in this paper.

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Appendix A: Mathematica codes

Below, we give the Mathematica codes to compute $S_q^{(p)}(ix)$ and $S_q^{(p)}\left(\frac{1+ix}{2}\right)$.

Listing 1 Code for $S_q^{(p)}(ix)$

```
ReplaceKp = {K'[k] -> (E[k] - (1 - k^2) K[k]) /(k (1 - k^2)) };
ReplaceEp = {EE'[k] -> (E[k] - K[k]) /k };

xprim[k_] := -(\[Pi ]/( 2 k (1 - k^2) (K[k]) ^2)) ;

V[2, k_] := -(4/3) (2 - k^2) (K[k]) ^2 + 4 K[k] E[k];
V[4, k_] := 16/45 (1 - k^2 + k^4) (K[k]) ^4;
V[6, k_] := 64/945 ((k^2 - 2) (2 k^2 - 1) (k^2 + 1)) (K[k]) ^6;
V[l_, k_] := 3/(( l + 1) (l - 1) (l/2 - 3)) Sum[(2 n-1)(l-2n-1)V[2 n, k] V[l-2n, k],
{n ,2, l/2 -2}];
Vorth[st_, qminusp_, k_] :=
If [ st == 0, V[qminusp, k] // Simplify,
-1/(st +qminusp- 1) I D[Vorth[st - 1, qminusp, k], k]/xprim[k] /. ReplaceKp /. ReplaceEp //
    Simplify ];

Sorth[p_, q_, k_] := Simplify [(K[kp]/K[k]) ^((q - p)/2) (V[q - p, k]) \\
- 2 I (K[kp]/K[k]) ^((q - p)/2) (Sum[Sum[(-1)^(t-s) Binomial[p, s] Binomial[p - s, t - s] (I)^(
    t - s) Im[I^s] (K[kp]/K[k]) ^t, {s ,1, t }] Vorth[t, q-p, k]) , {t ,1, p }]];
```

Listing 2 Code for $S_q^{(p)}\left(\frac{1+ix}{2}\right)$

```
Vh[2, k_] := -(4/3) (5 - 4 k^2) (K[k]) ^2 + 8 K[k] E[k];
Vh[4, k_] := 16/45 (16 k^4 - 16 k^2 + 1) (K[k]) ^4;
Vh[6, k_] := 128/945 ((2 k^2 - 1) (32 k^4 - 32 k^2 - 1)) (K[k]) ^6;
Vh[l_, k_] := 3/(( l + 1) (l - 1) (l/2 - 3)) Sum[(2 n-1)(l-2n-1)Vh[2 n, k]
Vh[l-2n, k], {n ,2, l/2 -2}];
Vhex[st_, qminusp_, k_] :=
If [ st == 0, Vh[qminusp, k] // Simplify,
-1/(st + qminusp - 1) I/2 D[Vhhex[st - 1, qminusp, k], k]/xprim[k] /. ReplaceKp /. ReplaceEp
    //Simplify ];

Shex[p_, q_, k_] := Simplify [(K[kp]/(2 K[k]) ) ^((q - p)/2) (Vh[q - p, k]) - 2 I (K[kp]/(2 K[k])
    ) ^((q - p)/2) (Sum[Sum[(-1)^(t-s)Binomial[p, s] Binomial[p - s, t - s] ((1+I (K[kp]/ K[k
    ]) ) /2) ^(t - s)(2^(-s)) Sum[Binomial[s,j] Im[I^j ]( K[kp]/ K[k]) ^j, {j ,0, s }], {s ,1, t }]
    Vhex[t, q-p, k]) , {t ,1, p }]];
```

Appendix B: Numerical results

See Appendix Tables 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Table 1 The values of $S_q^{(p)}$ for the hexagonal lattice calculated by (18)

$q \setminus p$	1	2	3	4	5	6	7	8	9	10
3	1.5708	–	–	–	–	–	–	–	–	–
4	0	5.54874	–	–	–	–	–	–	–	–
5	4.2426	0	0.785398	–	–	–	–	–	–	–
6	0	0	0	0.628319	–	–	–	–	–	–
7	0	0	0	0	8.26383	–	–	–	–	–
8	0	0	0	5.21071	0	0.448799	–	–	–	–
9	0	0	4.09248	0	0	0	0.392699	–	–	–
10	0	3.42851	0	0	0	0	0	6.6867	–	–
11	2.93754	0	0	0	0	0	4.47981	0	0.314159	–
12	0	0	0	0	0	3.83829	0	0	0	0.285599
13	0	0	0	0	3.34954	0	0	0	0	0
14	0	0	0	2.91403	0	0	0	0	0	5.38661
15	0	0	2.52844	0	0	0	0	0	4.10976	0
16	0	2.19126	0	0	0	0	0	3.43033	0	0
17	1.89818	0	0	0	0	0	2.93774	0	0	0
18	0	0	0	0	0	2.53534	0	0	0	0
19	0	0	0	0	2.19326	0	0	0	0	0
20	0	0	0	1.89876	0	0	0	0	0	2.91342
21	0	0	1.64418	0	0	0	0	0	2.52836	0
22	0	1.42385	0	0	0	0	0	2.19125	0	0
23	1.23308	0	0	0	0	0	1.89818	0	0	0
24	0	0	0	0	0	1.64402	0	0	0	0
25	0	0	0	0	1.42381	0	0	0	0	0

Table 2 The values of $S_q^{(p)}$ for the square lattice calculated by (18)

$q \setminus p$	1	2	3	4	5	6	7	8	9	10
3	4.07845	–	–	–	–	–	–	–	–	–
4	0	1.0472	–	–	–	–	–	–	–	–
5	0	0	6.79031	–	–	–	–	–	–	–
6	0	5.03067	0	0.628319	–	–	–	–	–	–
7	4.51552	0	0	0	4.35815	–	–	–	–	–
8	0	0	0	3.44189	0	0.448799	–	–	–	–
9	0	0	3.60434	0	0	0	8.91917	–	–	–
10	0	3.77445	0	0	0	5.49471	0	0.349066	–	–
11	3.88073	0	0	0	4.60323	0	0	0	2.82712	–
12	0	0	0	4.27271	0	0	0	3.03028	0	0.285599
13	0	0	4.13019	0	0	0	3.50734	0	0	0
14	0	4.06372	0	0	0	3.7536	0	0	0	5.46435
15	4.03154	0	0	0	3.87642	0	0	0	4.58835	0
16	0	0	0	3.93797	0	0	0	4.26919	0	0
17	0	0	3.96889	0	0	0	4.12945	0	0	0
18	0	3.98442	0	0	0	4.06357	0	0	0	3.77621
19	3.9922	0	0	0	4.03151	0	0	0	3.88111	0
20	0	0	0	4.01569	0	0	0	3.93893	0	0
21	0	0	4.00783	0	0	0	3.96909	0	0	0
22	0	4.00391	0	0	0	3.98445	0	0	0	4.06317
23	4.00195	0	0	0	3.99221	0	0	0	4.03143	0
24	0	0	0	3.9961	0	0	0	4.01567	0	0
25	0	0	3.99805	0	0	0	4.00783	0	0	0

Table 3 The values of $\frac{1}{\pi} S_{p+2}^{(p)}$ for the hexagonal lattice

p	0	1	2	3	4	5	6	7	8	9	10
$\frac{S_{p+2}^{(p)}}{\pi}$	1	$\frac{1}{2}$	1.76622	$\frac{1}{4}$	$\frac{1}{5}$	2.63046	$\frac{1}{7}$	$\frac{1}{8}$	2.12844	$\frac{1}{10}$	$\frac{1}{11}$
p	11	12	13	14	15	16	17	18	19	20	21
$\frac{S_{p+2}^{(p)}}{\pi}$	3.15788	$\frac{1}{13}$	$\frac{1}{14}$	1.72012	$\frac{1}{16}$	$\frac{1}{17}$	3.58714	$\frac{1}{19}$	$\frac{1}{20}$	1.85034	$\frac{1}{22}$

Table 4 The values of $\frac{1}{\pi} S_{p+2}^{(p)}$ for the square lattice

p	0	1	2	3	4	5	6	7	8	9	10
$\frac{S_{p+2}^{(p)}}{\pi}$	1	1.29821	$\frac{1}{3}$	2.16142	$\frac{1}{5}$	1.38724	$\frac{1}{7}$	2.83906	$\frac{1}{9}$	0.899899	$\frac{1}{11}$
p	11	12	13	14	15	16	17	18	19	20	21
$\frac{S_{p+2}^{(p)}}{\pi}$	2.96621	$\frac{1}{13}$	1.45425	$\frac{1}{15}$	2.60551	$\frac{1}{17}$	0.79325	$\frac{1}{19}$	3.69472	$\frac{1}{21}$	1.13929

Table 5 The values of $S_{p+2}^{(p)}(ix)$ calculated by (57)

p	$S_{p+2}^{(p)}(i)$	$S_{p+2}^{(p)}(i\sqrt{2})$	$S_{p+2}^{(p)}(i\sqrt{3})$	$S_{p+2}^{(p)}(2i)$
0	π	$\pi+\frac{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{48\sqrt{2}\pi}$	$\pi+\frac{\sqrt{3}\Gamma^6\left(\frac{1}{3}\right)}{16\cdot 2^{2/3}\pi^2}$	$\pi+\frac{\Gamma^4\left(\frac{1}{4}\right)}{16\pi}$
1	$\frac{\pi}{2}+\frac{\Gamma^8\left(\frac{1}{4}\right)}{384\pi^3}$	$\frac{\pi}{2}+\frac{\Gamma^4\left(\frac{1}{8}\right)\Gamma^4\left(\frac{3}{8}\right)}{1024\pi^3}$	$\frac{\pi}{2}+\frac{3\Gamma^{12}\left(\frac{1}{3}\right)}{256\cdot 2^{1/3}\pi^5}$	$\frac{\pi}{2}+\frac{\Gamma^8\left(\frac{1}{4}\right)}{192\pi^3}$
2	$\frac{\pi}{3}$	$\frac{\pi}{3}+\frac{\Gamma^6\left(\frac{1}{8}\right)\Gamma^6\left(\frac{3}{8}\right)}{24576\sqrt{2}\pi^5}$	$\frac{\pi}{3}+\frac{\sqrt{3}\Gamma^{18}\left(\frac{1}{3}\right)}{2048\pi^8}$	$\frac{\pi}{3}+\frac{\Gamma^{12}\left(\frac{1}{4}\right)}{3072\pi^5}$
3	$\frac{\pi}{4}+\frac{\Gamma^{16}\left(\frac{1}{4}\right)}{49152\pi^7}$	$\frac{\pi}{4}+\frac{\Gamma^8\left(\frac{1}{8}\right)\Gamma^8\left(\frac{3}{8}\right)}{3145728\pi^7}$	$\frac{\pi}{4}+\frac{9\Gamma^{24}\left(\frac{1}{3}\right)}{65536\cdot 2^{2/3}\pi^{11}}$	$\frac{\pi}{4}+\frac{\Gamma^{16}\left(\frac{1}{4}\right)}{49152\pi^7}$
4	$\frac{\pi}{5}$	$\frac{\pi}{5}+\frac{\Gamma^{10}\left(\frac{1}{8}\right)\Gamma^{10}\left(\frac{3}{8}\right)}{62914560\sqrt{2}\pi^9}$	$\frac{\pi}{5}+\frac{9\sqrt{3}\Gamma^{30}\left(\frac{1}{3}\right)}{2621440\cdot 2^{1/3}\pi^{14}}$	$\frac{\pi}{5}+\frac{\Gamma^{20}\left(\frac{1}{4}\right)}{983040\pi^9}$
5	$\frac{\pi}{6}+\frac{\Gamma^{24}\left(\frac{1}{4}\right)}{23592960\pi^{11}}$	$\frac{\pi}{6}+\frac{23\Gamma^{12}\left(\frac{1}{8}\right)\Gamma^{12}\left(\frac{3}{8}\right)}{12079595520\pi^{11}}$	$\frac{\pi}{6}+\frac{9\Gamma^{36}\left(\frac{1}{3}\right)}{10485760\pi^{17}}$	$\frac{\pi}{6}+\frac{\Gamma^{24}\left(\frac{1}{4}\right)}{11796480\pi^{11}}$
6	$\frac{\pi}{7}$	$\frac{\pi}{7}+\frac{29\Gamma^{14}\left(\frac{1}{8}\right)\Gamma^{14}\left(\frac{3}{8}\right)}{676457349120\sqrt{2}\pi^{13}}$	$\frac{\pi}{7}+\frac{27\sqrt{3}\Gamma^{42}\left(\frac{1}{3}\right)}{1174405120\ 2^{2/3}\pi^{20}}$	$\frac{\pi}{7}+\frac{\Gamma^{28}\left(\frac{1}{4}\right)}{440401920\pi^{13}}$
7	$\frac{\pi}{8}+\frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{42278584320\pi^{15}}$	$\frac{\pi}{8}+\frac{181\Gamma^{16}\left(\frac{1}{8}\right)\Gamma^{16}\left(\frac{3}{8}\right)}{173173081374720\pi^{15}}$	$\frac{\pi}{8}+\frac{891\Gamma^{48}\left(\frac{1}{3}\right)}{150323855360\sqrt[3]{2}\pi^{23}}$	$\frac{\pi}{8}+\frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{42278584320\pi^{15}}$
8	$\frac{\pi}{9}$	$\frac{\pi}{9}+\frac{73\left(24+17\sqrt{2}\right)\Gamma^{18}\left(\frac{1}{8}\right)\Gamma^{18}\left(\frac{3}{8}\right)}{6234230929489920\left(1+\sqrt{2}\right)^4\pi^{17}}$	$\frac{\pi}{9}+\frac{9\sqrt{3}\Gamma^{54}\left(\frac{1}{3}\right)}{37580963840\pi^{26}}$	$\frac{\pi}{9}+\frac{\Gamma^{36}\left(\frac{1}{4}\right)}{84557168640\pi^{17}}$
9	$\frac{\pi}{10}+\frac{\Gamma^{40}\left(\frac{1}{4}\right)}{3382286745600\pi^{19}}$	$\frac{\pi}{10}+\frac{487\Gamma^{20}\left(\frac{1}{8}\right)\Gamma^{20}\left(\frac{3}{8}\right)}{199495389743677440\pi^{19}}$	$\frac{\pi}{10}+\frac{243\Gamma^{60}\left(\frac{1}{3}\right)}{4810363371520\ 2^{2/3}\pi^{29}}$	$\frac{\pi}{10}+\frac{\Gamma^{40}\left(\frac{1}{4}\right)}{1691143372800\pi^{19}}$
10	$\frac{\pi}{11}$	$\frac{\pi}{11}+\frac{211\left(58+41\sqrt{2}\right)\Gamma^{22}\left(\frac{1}{8}\right)\Gamma^{22}\left(\frac{3}{8}\right)}{35111188594887229440\left(1+\sqrt{2}\right)^5\pi^{21}}$	$\frac{\pi}{11}+\frac{243\sqrt{3}\Gamma^{66}\left(\frac{1}{3}\right)}{423311976693760\sqrt[3]{2}\pi^{32}}$	$\frac{\pi}{11}+\frac{13\Gamma^{44}\left(\frac{1}{4}\right)}{297641233612800\pi^{21}}$

Table 6 The values of $S_{p+2}^{(p)}(\frac{i}{x})$ calculated by (57)

p	$S_{p+2}^{(p)}(i)$	$S_{p+2}^{(p)}(i/\sqrt{2})$	$S_{p+2}^{(p)}(i/\sqrt{3})$	$S_{p+2}^{(p)}(i/2)$
0	π	$\pi - \frac{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{48\sqrt{2}\pi}$	$\pi - \frac{\sqrt{3}\Gamma^6\left(\frac{1}{3}\right)}{16\,2^{2/3}\pi^2}$	$\pi - \frac{\Gamma^4\left(\frac{1}{4}\right)}{16\pi}$
1	$\frac{\pi}{2} + \frac{\Gamma^8\left(\frac{1}{4}\right)}{384\pi^3}$	$\frac{\pi}{2} + \frac{\Gamma^4\left(\frac{1}{8}\right)\Gamma^4\left(\frac{3}{8}\right)}{1024\pi^3}$	$\frac{\pi}{2} + \frac{3\Gamma^{12}\left(\frac{1}{3}\right)}{256\sqrt[3]{2}\pi^5}$	$\frac{\pi}{2} + \frac{\Gamma^8\left(\frac{1}{4}\right)}{192\pi^3}$
2	$\frac{\pi}{3}$	$\frac{\pi}{3} - \frac{\Gamma^6\left(\frac{1}{8}\right)\Gamma^6\left(\frac{3}{8}\right)}{24576\sqrt{2}\pi^5}$	$\frac{\pi}{3} - \frac{\sqrt{3}\Gamma^{18}\left(\frac{1}{3}\right)}{2048\pi^8}$	$\frac{\pi}{3} - \frac{\Gamma^{12}\left(\frac{1}{4}\right)}{3072\pi^5}$
3	$\frac{\pi}{4} + \frac{\Gamma^{16}\left(\frac{1}{4}\right)}{49152\pi^7}$	$\frac{\pi}{4} + \frac{5\Gamma^8\left(\frac{1}{8}\right)\Gamma^8\left(\frac{3}{8}\right)}{3145728\pi^7}$	$\frac{\pi}{4} + \frac{9\Gamma^{24}\left(\frac{1}{3}\right)}{65536\,2^{2/3}\pi^{11}}$	$\frac{\pi}{4} + \frac{\Gamma^{16}\left(\frac{1}{4}\right)}{49152\pi^7}$
4	$\frac{\pi}{5}$	$\frac{\pi}{5} - \frac{\Gamma^{10}\left(\frac{1}{8}\right)\Gamma^{10}\left(\frac{3}{8}\right)}{62914560\sqrt{2}\pi^9}$	$\frac{\pi}{5} - \frac{9\sqrt{3}\Gamma^{30}\left(\frac{1}{3}\right)}{2621440\sqrt[3]{2}\pi^{14}}$	$\frac{\pi}{5} - \frac{\Gamma^{20}\left(\frac{1}{4}\right)}{983040\pi^9}$
5	$\frac{\pi}{6} + \frac{\Gamma^{24}\left(\frac{1}{4}\right)}{23592960\pi^{11}}$	$\frac{\pi}{6} + \frac{23\Gamma^{12}\left(\frac{1}{8}\right)\Gamma^{12}\left(\frac{3}{8}\right)}{12079595520\pi^{11}}$	$\frac{\pi}{6} + \frac{9\Gamma^{36}\left(\frac{1}{3}\right)}{10485760\pi^{17}}$	$\frac{\pi}{6} + \frac{\Gamma^{24}\left(\frac{1}{4}\right)}{11796480\pi^{11}}$
6	$\frac{\pi}{7}$	$\frac{\pi}{7} - \frac{29\Gamma^{14}\left(\frac{1}{8}\right)\Gamma^{14}\left(\frac{3}{8}\right)}{676457349120\sqrt{2}\pi^{13}}$	$\frac{\pi}{7} - \frac{27\sqrt{3}\Gamma^{42}\left(\frac{1}{3}\right)}{1174405120\,2^{2/3}\pi^{20}}$	$\frac{\pi}{7} - \frac{\Gamma^{28}\left(\frac{1}{4}\right)}{440401920\pi^{13}}$
7	$\frac{\pi}{8} + \frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{42278584320\pi^{15}}$	$\frac{\pi}{8} + \frac{181\Gamma^{16}\left(\frac{1}{8}\right)\Gamma^{16}\left(\frac{3}{8}\right)}{173173081374720\pi^{15}}$	$\frac{\pi}{8} + \frac{891\Gamma^{48}\left(\frac{1}{3}\right)}{150323855360\sqrt[3]{2}\pi^{23}}$	$\frac{\pi}{8} + \frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{42278584320\pi^{15}}$
8	$\frac{\pi}{9}$	$\frac{\pi}{9} - \frac{73\left(24+17\sqrt{2}\right)\Gamma^{18}\left(\frac{1}{8}\right)\Gamma^{18}\left(\frac{3}{8}\right)}{6234230929489920\left(1+\sqrt{2}\right)^4\pi^{17}}$	$\frac{\pi}{9} - \frac{9\sqrt{3}\Gamma^{54}\left(\frac{1}{3}\right)}{37580963840\pi^{26}}$	$\frac{\pi}{9} - \frac{\Gamma^{36}\left(\frac{1}{4}\right)}{84557168640\pi^{17}}$
9	$\frac{\pi}{10} + \frac{\Gamma^{40}\left(\frac{1}{4}\right)}{3382286745600\pi^{19}}$	$\frac{\pi}{10} + \frac{487\Gamma^{20}\left(\frac{1}{8}\right)\Gamma^{20}\left(\frac{3}{8}\right)}{199495389743677440\pi^{19}}$	$\frac{\pi}{10} + \frac{243\Gamma^{60}\left(\frac{1}{3}\right)}{4810363371520\,2^{2/3}\pi^{29}}$	$\frac{\pi}{10} + \frac{\Gamma^{40}\left(\frac{1}{4}\right)}{1691143372800\pi^{19}}$
10	$\frac{\pi}{11}$	$\frac{\pi}{11} - \frac{211\left(58+41\sqrt{2}\right)\Gamma^{22}\left(\frac{1}{8}\right)\Gamma^{22}\left(\frac{3}{8}\right)}{35111188594887229440\left(1+\sqrt{2}\right)^5\pi^{21}}$	$\frac{\pi}{11} - \frac{243\sqrt{3}\Gamma^{66}\left(\frac{1}{3}\right)}{423311976693760\sqrt[3]{2}\pi^{32}}$	$\frac{\pi}{11} - \frac{13\Gamma^{44}\left(\frac{1}{4}\right)}{297641233612800\pi^{21}}$

Table 7 The values of $S_{p+2}^{(p)}\left(\frac{1+ix}{2}\right)$ calculated by (57)

p	$S_{p+2}^{(p)}\left(\frac{1+i}{2}\right)$	$S_{p+2}^{(p)}\left(\frac{1+i\sqrt{2}}{2}\right)$	$S_{p+2}^{(p)}\left(\frac{1+i\sqrt{3}}{2}\right)$	$S_{p+2}^{(p)}\left(\frac{1+2i}{2}\right)$
0	π	$\pi+\frac{(2\sqrt{2}-3)\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{96\pi}$	π	$\pi+\frac{(3-2\sqrt{2})\Gamma^4\left(\frac{1}{4}\right)}{32\pi}$
1	$\frac{\pi}{2}-\frac{\Gamma^8\left(\frac{1}{4}\right)}{384\pi^3}$	$\frac{\pi}{2}-\frac{(\sqrt{2}-1)\Gamma^4\left(\frac{1}{8}\right)\Gamma^4\left(\frac{3}{8}\right)}{1024\pi^3}$	$\frac{\pi}{2}$	$\frac{\pi}{2}+\frac{(5-3\sqrt{2})\Gamma^8\left(\frac{1}{4}\right)}{768\pi^3}$
2	$\frac{\pi}{3}$	$\frac{\pi}{3}+\frac{(2\sqrt{2}-1)\Gamma^6\left(\frac{1}{8}\right)\Gamma^6\left(\frac{3}{8}\right)}{49152\pi^5}$	$\frac{\pi}{3}+\frac{\sqrt{3}\Gamma^{18}\left(\frac{1}{3}\right)}{2048\pi^8}$	$\frac{\pi}{3}-\frac{(\sqrt{2}-3)\Gamma^{12}\left(\frac{1}{4}\right)}{6144\pi^5}$
3	$\frac{\pi}{4}+\frac{\Gamma^{16}\left(\frac{1}{4}\right)}{49152\pi^7}$	$\frac{\pi}{4}+\frac{(5-2\sqrt{2})\Gamma^8\left(\frac{1}{8}\right)\Gamma^8\left(\frac{3}{8}\right)}{3145728\pi^7}$	$\frac{\pi}{4}$	$\frac{\pi}{4}+\frac{(4-3\sqrt{2})\Gamma^{16}\left(\frac{1}{4}\right)}{196608\pi^7}$
4	$\frac{\pi}{5}$	$\frac{\pi}{5}+\frac{(2\sqrt{2}-7)\Gamma^{10}\left(\frac{1}{8}\right)\Gamma^{10}\left(\frac{3}{8}\right)}{125829120\pi^9}$	$\frac{\pi}{5}$	$\frac{\pi}{5}+\frac{(12-5\sqrt{2})\Gamma^{20}\left(\frac{1}{4}\right)}{7864320\pi^9}$
5	$\frac{\pi}{6}-\frac{\Gamma^{24}\left(\frac{1}{4}\right)}{23592960\pi^{11}}$	$\frac{\pi}{6}+\frac{(23-5\sqrt{2})\Gamma^{12}\left(\frac{1}{8}\right)\Gamma^{12}\left(\frac{3}{8}\right)}{12079595520\pi^{11}}$	$\frac{\pi}{6}+\frac{9\Gamma^{36}\left(\frac{1}{3}\right)}{10485760\pi^{17}}$	$\frac{\pi}{6}+\frac{(40-9\sqrt{2})\Gamma^{24}\left(\frac{1}{4}\right)}{377487360\pi^{11}}$
6	$\frac{\pi}{7}$	$\frac{\pi}{7}+\frac{\left(-17+58\sqrt{2}\right)\Gamma^{14}\left(\frac{1}{8}\right)\Gamma^{14}\left(\frac{3}{8}\right)}{1352914698240\pi^{13}}$	$\frac{\pi}{7}$	$\frac{\pi}{7}+\frac{\left(24-23\sqrt{2}\right)\Gamma^{28}\left(\frac{1}{4}\right)}{7046430720\pi^{13}}$
7	$\frac{\pi}{8}+\frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{42278584320\pi^{15}}$	$\frac{\pi}{8}+\frac{\left(181-196\sqrt{2}\right)\Gamma^{16}\left(\frac{1}{8}\right)\Gamma^{16}\left(\frac{3}{8}\right)}{173173081374720\pi^{15}}$	$\frac{\pi}{8}$	$\frac{\pi}{8}+\frac{\left(416-147\sqrt{2}\right)\Gamma^{32}\left(\frac{1}{4}\right)}{1352914698240\pi^{15}}$
8	$\frac{\pi}{9}$	$\frac{\pi}{9}+\frac{\left(733+526\sqrt{2}\right)\Gamma^{18}\left(\frac{1}{8}\right)\Gamma^{18}\left(\frac{3}{8}\right)}{6234230929489920\left(1+\sqrt{2}\right)^4\pi^{17}}$	$\frac{\pi}{9}+\frac{9\sqrt{3}\Gamma^{54}\left(\frac{1}{3}\right)}{37580963840\pi^{26}}$	$\frac{\pi}{9}+\frac{\left(192-65\sqrt{2}\right)\Gamma^{36}\left(\frac{1}{4}\right)}{10823317585920\pi^{17}}$
9	$\frac{\pi}{10}-\frac{\Gamma^{40}\left(\frac{1}{4}\right)}{3382286745600\pi^{19}}$	$\frac{\pi}{10}+\frac{\left(6671+4705\sqrt{2}\right)\Gamma^{20}\left(\frac{1}{8}\right)\Gamma^{20}\left(\frac{3}{8}\right)}{199495389743677440\left(1+\sqrt{2}\right)^4\pi^{19}}$	$\frac{\pi}{10}$	$\frac{\pi}{10}+\frac{\left(640-489\sqrt{2}\right)\Gamma^{40}\left(\frac{1}{4}\right)}{865865406873600\pi^{19}}$
10	$\frac{\pi}{11}$	$\frac{\pi}{11}-\frac{\left(32391+22921\sqrt{2}\right)\Gamma^{22}\left(\frac{1}{8}\right)\Gamma^{22}\left(\frac{3}{8}\right)}{35111188594887229440\left(1+\sqrt{2}\right)^5\pi^{21}}$	$\frac{\pi}{11}$	$\frac{\pi}{11}+\frac{\left(4992-929\sqrt{2}\right)\Gamma^{44}\left(\frac{1}{4}\right)}{76196155804876800\pi^{21}}$

Table 8 The values of $S_q^{(p)}(i)$ calculated by (57)

$p \setminus q$	$p+2$	$p+4$	$p+6$	$p+8$
0	π	$\frac{\Gamma^8\left(\frac{1}{4}\right)}{960\pi^2}$	0	$\frac{\Gamma^{16}\left(\frac{1}{4}\right)}{2150400\pi^4}$
1	$\frac{\pi}{2} + \frac{\Gamma^8\left(\frac{1}{4}\right)}{384\pi^3}$	0	$\frac{\Gamma^{16}\left(\frac{1}{4}\right)}{645120\pi^5}$	0
2	$\frac{\pi}{3}$	$\frac{\Gamma^{16}\left(\frac{1}{4}\right)}{184320\pi^6}$	0	$\frac{\Gamma^{24}\left(\frac{1}{4}\right)}{743178240\pi^8}$
3	$\frac{\pi}{4} + \frac{\Gamma^{16}\left(\frac{1}{4}\right)}{49152\pi^7}$	0	$\frac{\Gamma^{24}\left(\frac{1}{4}\right)}{247726080\pi^9}$	0
4	$\frac{\pi}{5}$	$\frac{\Gamma^{24}\left(\frac{1}{4}\right)}{82575360\pi^{10}}$	0	$\frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{2615987404800\pi^{12}}$
5	$\frac{\pi}{6} + \frac{\Gamma^{24}\left(\frac{1}{4}\right)}{23592960\pi^{11}}$	0	$\frac{\Gamma^{32}\left(\frac{1}{4}\right)}{59454259200\pi^{13}}$	0
6	$\frac{\pi}{7}$	$\frac{\Gamma^{32}\left(\frac{1}{4}\right)}{15854469120\pi^{14}}$	0	$\frac{31\Gamma^{40}\left(\frac{1}{4}\right)}{2176501520793600\pi^{16}}$
7	$\frac{\pi}{8} + \frac{13\Gamma^{32}\left(\frac{1}{4}\right)}{42278584320\pi^{15}}$	0	$\frac{\Gamma^{40}\left(\frac{1}{4}\right)}{23917599129600\pi^{17}}$	0
8	$\frac{\pi}{9}$	$\frac{19\Gamma^{40}\left(\frac{1}{4}\right)}{167423193907200\pi^{18}}$	0	$\frac{773\Gamma^{48}\left(\frac{1}{4}\right)}{146260902197329920000\pi^{20}}$
9	$\frac{\pi}{10} + \frac{\Gamma^{40}\left(\frac{1}{4}\right)}{3382286745600\pi^{19}}$	0	$\frac{29\Gamma^{48}\left(\frac{1}{4}\right)}{16251211355258880\pi^{21}}$	0
10	$\frac{\pi}{11}$	$\frac{31\Gamma^{48}\left(\frac{1}{4}\right)}{4643203244359680\pi^{22}}$	0	$\frac{809\Gamma^{56}\left(\frac{1}{4}\right)}{530439538635649843200\pi^{24}}$

Table 9 The values of $S_q^{(p)}\left(\frac{1+i\sqrt{3}}{2}\right)$ calculated by (57)

$p \setminus q$	$p+2$	$p+4$	$p+6$	$p+8$
0	π	0	$\frac{3\sqrt{3}\Gamma^{18}\left(\frac{1}{3}\right)}{71680\pi^{6}}$	0
1	$\frac{\pi}{2}$	$\frac{3\sqrt{3}\Gamma^{18}\left(\frac{1}{3}\right)}{20480\pi^{7}}$	0	0
2	$\frac{\pi}{3}+\frac{\sqrt{3}\Gamma^{18}\left(\frac{1}{3}\right)}{2048\pi^{8}}$	0	0	$\frac{9\Gamma^{36}\left(\frac{1}{3}\right)}{734003200\pi^{14}}$
3	$\frac{\pi}{4}$	0	$\frac{27\Gamma^{36}\left(\frac{1}{3}\right)}{587202560\pi^{15}}$	0
4	$\frac{\pi}{5}$	$\frac{27\Gamma^{36}\left(\frac{1}{3}\right)}{146800640\pi^{16}}$	0	0
5	$\frac{\pi}{6}+\frac{9\Gamma^{36}\left(\frac{1}{3}\right)}{10485760\pi^{17}}$	0	0	$\frac{27\sqrt{3}\Gamma^{54}\left(\frac{1}{3}\right)}{6614249635840\pi^{23}}$
6	$\frac{\pi}{7}$	0	$\frac{243\sqrt{3}\Gamma^{54}\left(\frac{1}{3}\right)}{16535624089600\pi^{24}}$	0
7	$\frac{\pi}{8}$	$\frac{81\sqrt{3}\Gamma^{54}\left(\frac{1}{3}\right)}{1503238553600\pi^{25}}$	0	0
8	$\frac{\pi}{9}+\frac{9\sqrt{3}\Gamma\left(\frac{1}{3}\right)^{54}}{37580963840\pi^{26}}$	0	0	$\frac{65853\Gamma^{72}\left(\frac{1}{3}\right)}{15408555951652864000\pi^{32}}$
9	$\frac{\pi}{10}$	0	$\frac{12393\Gamma^{72}\left(\frac{1}{3}\right)}{770427797582643200\pi^{33}}$	0
10	$\frac{\pi}{11}$	$\frac{729\Gamma^{72}\left(\frac{1}{3}\right)}{110061113940377600\pi^{34}}$	0	0

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