



# On weighted norm inequalities for oscillatory integral operators

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## Abstract

We prove weighted norm inequalities with Muckenhoupt's  $A_p$ -weights, for a wide class of oscillatory integral operators. As a consequence, one also obtains the boundedness of commutators of the aforementioned operators with functions in BMO.

**Keywords** Oscillatory integral operators · Weighted norm inequalities · Muckenhoupt weights · Commutators with BMO

**Mathematics Subject Classification** 30H35 · 35S30 · 42B20 · 42B37

## 1 Introduction

In this paper we are concerned with global weighted norm inequalities for oscillatory integral operators, where the phase functions satisfy suitable growth and non-degeneracy conditions, the amplitudes have certain decay, and the weights are in the  $A_p$ -class of Muckenhoupt, with  $1 < p < \infty$ .

More precisely, we are interested in estimates of the form

$$\|T_a^\varphi f\|_{L_w^p} \leq C \|f\|_{L_w^p}, \quad (1)$$

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with  $w \in A_p$ ,  $1 < p < \infty$ , where

$$T_a^\varphi f(x) = \int_{\mathbb{R}^n} a(x, \xi) e^{i\varphi(x, \xi)} \widehat{f}(\xi) \, d\xi, \tag{2}$$

and  $d\xi := (2\pi)^{-n} d\xi$ . Moreover  $\|f\|_{L_w^p} := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p}$ .

The main assumption on the amplitude  $a$  is usually that it belongs to a Hörmander class  $S_{\rho, \delta}^m$  (see [7]), which are defined as follows:

**Definition 1.1** Let  $m \in \mathbb{R}$  and  $0 \leq \rho, \delta \leq 1$ . An amplitude  $a(x, \xi)$  in the class  $S_{\rho, \delta}^m$  is a function  $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  that verifies the estimate

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|},$$

for all multi-indices  $\alpha$  and  $\beta$  and  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ . One refers to  $m$  as the order of the amplitude, and to  $\rho, \delta$  as its type.

A wider class of amplitudes used are those which are merely bounded in the  $x$ -variable and were first introduced by Kenig and Staubach in [9].

**Definition 1.2** For  $\rho \in [0, 1]$ , an amplitude  $a(x, \xi)$  is in the class  $L^\infty S_\rho^m$  if it is essentially bounded in the  $x$  variable,  $C^\infty(\mathbb{R}^n)$  in the  $\xi$  variable and verifies the estimate

$$\left\| \partial_\xi^\alpha a(\cdot, \xi) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \langle \xi \rangle^{m - \rho|\alpha|},$$

for all multi-indices  $\alpha$  and all  $\xi \in \mathbb{R}^n$ . Clearly  $S_{\rho, \delta}^m \subset L^\infty S_\rho^m$ .

As far as the phase function  $\varphi$  is concerned, when it is linear, i. e.  $\varphi(x, \xi) = x \cdot \xi$  (which is the so-called pseudodifferential case), the problem of establishing (1) was completely solved by Miller in [12], for  $a(x, \xi) \in S_{1,0}^m$ . For amplitudes in more general Hörmander classes, Chanillo and Torchinsky [4] considered symbols in the class  $S_{\rho, \delta}^{\frac{n}{2}(\rho-1)}$  and showed estimate (1) for  $2 \leq p < \infty$  and  $w \in A_{\frac{p}{2}}$ . For  $a(x, \xi) \in S_{\rho, \delta}^{n(\rho-1)}$  with  $0 \leq \delta \leq \rho \leq \frac{1}{2}$ , Álvarez and Hounie [2] showed weighted boundedness of pseudodifferential operators for all weights  $w \in A_p$ ,  $1 < p < \infty$ . The result of Álvarez and Hounie was extended to the case of  $0 < \rho \leq 1$  and  $0 \leq \delta < 1$  by Michalowski, Rule and Staubach in [10] and an extension of Chanillo–Torchinsky’s result to the class of rough amplitudes (in  $x$ ) was obtained by Michalowski, Rule and Staubach in [11].

For non-linear phases, a suitable class of phase functions is given by the following:

**Definition 1.3** For  $s > 0$ , we say that a real-valued *phase function*  $\varphi$  is a phase function of order  $s$  if  $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ ,

$$abs \partial_\xi^\alpha (\varphi(x, \xi) - x \cdot \xi) \leq c_\alpha |\xi|^{s - |\alpha|}, \quad |\alpha| \geq 0 \tag{3}$$

for all  $x \in \mathbb{R}^n$  and  $\xi \neq 0$ , and there is a constant  $C > 0$  such that

$$|\nabla_{\xi} \varphi(x, \xi) - \nabla_{\xi} \varphi(y, \xi)| \geq C|x - y|, \tag{4}$$

for all  $x, y \in \mathbb{R}^n$  and all  $\xi \in \mathbb{R}^n \setminus \{0\}$ .

Having the phase functions and the amplitudes in suitable classes, any expression of the form (2) is referred to as an oscillatory integral operator (abbreviated as OIO).

When the phase functions  $\varphi(x, \xi)$  are positively homogeneous of degree one in  $\xi$  (i. e. the case of the so-called Fourier integral operators), Dos Santos Ferreira and Staubach proved in [5] the following result:

**Theorem 1.4** *Let  $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  be a phase function of order 1 that is positively homogeneous of degree one in  $\xi$  and satisfying the condition*

$$|\det_{n-1} \partial_{\xi'}^2 \varphi(x, \xi_1, \xi')| \geq C > 0,$$

for some constant  $C$  and for all  $x \in \mathbb{R}^n$  and all  $\xi = (\xi_1, \xi') \in (\mathbb{R} \times \mathbb{R}^{n-1}) \setminus \{0\}$ .

Then, given  $a(x, \xi) \in L^\infty S_1^{-(n+1)/2}$ , the Fourier integral operator  $T_a^\varphi$  satisfies the weighted norm inequality (1) for all  $1 < p < \infty$  and all  $w \in A_p$ . This result is sharp as far as the order of the amplitude is concerned.

One of the goals of our investigation here is to extend this theorem to the case of more general OIOs by suitably adapting the conditions on the phase function above. The main inspiration for this task came from the following striking result of Chanillo and Torchinsky [4].

**Theorem 1.5** *If  $\varphi(x, \xi) = x \cdot \xi + |\xi|^s$  with  $0 < s < 1$ , and  $a(x, \xi) \in S_{1,0}^{-ns/2}$  then (1) is valid for all  $1 < p < \infty$  and all  $w \in A_p$ . This result is sharp.*

The proof of this theorem was based on the sharp-function estimate

$$(T_a^\varphi f)^\sharp(x) \lesssim M_p f(x),$$

(which is stronger than (1)) established by Chanillo and Torchinsky for all  $1 < p < \infty$ , where the sharp-function  $f^\sharp$  of  $f$  is defined by

$$f^\sharp(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy,$$

with  $f_B = \frac{1}{|B|} \int_B f(y) dy$ , and the supremum is taken over balls  $B$  in  $\mathbb{R}^n$  containing  $x$ . Furthermore for  $f \in L^p_{loc}$ , the  $L^p$  maximal function  $M_p(u)$  is defined by

$$M_p(f)(x) = \sup_{B \ni x} \left( \frac{1}{|B|} \int_B |f(y)|^p \, dy \right)^{\frac{1}{p}}, \tag{5}$$

where the supremum is taken over balls  $B$  in  $\mathbb{R}^n$  containing  $x$ .

The main result of this paper is the following extension of Theorems 1.4 and 1.5.

**Theorem 1.6** *Given  $s > 0$ , let  $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  be a phase function of order  $s$  that satisfies the condition*

$$|\det \partial_{\xi\xi}^2 \varphi(x, \xi)| \geq C|\xi|^q$$

for some  $q \leq n(s - 2)$ , and some  $C = C(q, s) > 0$  and for all  $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ . Then, given  $a \in L^\infty S_1^{m(s,q)}$  with

$$m(s, q) = \frac{-n(3s - 4) + 2q}{2},$$

the oscillatory integral operator  $T_a^\varphi$  satisfies (1) for all  $1 < p < \infty$  and all  $w \in A_p$ ,

We note that for  $\varphi(x, \xi) = x \cdot \xi + |\xi|^s$ , with  $s > 0, s \neq 1$ , which is the phase function associated to the semigroup  $e^{i\Delta^s/2}$  and dispersive partial differential equations, all the conditions on the phase function in Theorem 1.6 are satisfied and the largest value of  $q$  is in fact  $n(s - 2)$ , and therefore  $m(s, q) = -\frac{ns}{2}$ . Furthermore since,  $S_{1,0}^{-ns/2} \subset L^\infty S_1^{-ns/2}$ , this recovers Theorem 1.5, and in fact extends it to the case of all  $s \neq 1$  and rough amplitudes. We also note that for  $s = 1$ , Theorem 1.6 is not applicable, since for the phase function of the wave equation,  $\varphi(x, \xi) = x \cdot \xi + |\xi|$ , the condition on the determinant of the Hessian in  $\xi$  variables is not satisfied. However in this case one has the result of Theorem 1.4 that saves the day.

Thus the main point of our investigation here is to extend weighted norm inequalities to a class of operators that go beyond the scope of Fourier integral operators.

To prove Theorem 1.6 we rely on a pointwise estimate for oscillatory integral operators involving the Hardy–Littlewood’s maximal function,  $L^2$  boundedness of oscillatory integral operators and extrapolation. As such, our proof is different from that of Chanillo–Torchinsky, and applies to a wider range of oscillatory integral operators.

As consequence of a general result of Álvarez, Bagby, Kurtz and C. Perez [1] and Theorem 1.6 we obtain

**Corollary 1.7** *Given  $b \in \text{BMO}$  and  $k$  a positive integer, the  $k$ -th commutator defined by*

$$T_{a,b,k}^\varphi u(x) := T_a^\varphi ((b(x) - b(\cdot))^k u)(x)$$

is bounded on  $L_w^p$  for all  $1 < p < \infty$  and all  $w \in A_p$ .

The paper is organised as follows. In Sect. 2 we recall some basic definitions and results regarding weights and weighted norm inequalities. In Sect. 3 we prove pointwise estimates for oscillatory integral operators via a refined frequency-space decomposition which is continuous and not discrete, as opposed to common practice. In Sect. 4 we prove Theorem 1.6, using extrapolation and some other geometric considerations.

## 2 Preliminaries

We shall briefly recall some definitions, facts and results regarding weighted norm inequalities that are needed in our investigation. Our main references for this material are [6] and [13].

The Hardy–Littlewood maximal function is given by

$$M(f) := M_1(f),$$

where  $M_p$  was defined in (5). An immediate consequence of Hölder’s inequality is that  $M(f)(x) \leq M_p(f)(x)$  for  $p \geq 1$ .

One can then define the class of Muckenhoupt  $A_p$  weights as follows.

**Definition 2.1** Let  $w \in L^1_{\text{loc}}$  be a positive function. One says that  $w \in A_1$  if,

$$[w]_{A_1} := \sup_{B \text{ balls in } \mathbb{R}^n} w_B \|w^{-1}\|_{L^\infty(B)} < +\infty,$$

and  $w \in A_p$  for  $1 < p < \infty$  if

$$[w]_{A_p} := \sup_{B \text{ balls in } \mathbb{R}^n} w_B \left( w^{-\frac{1}{p-1}} \right)_B^{p-1} < +\infty.$$

**Example 2.2** The function  $|x|^\alpha$  is in  $A_1$  if and only if  $-n < \alpha \leq 0$  and is in  $A_p$  with  $1 < p < \infty$  iff  $-n < \alpha < n(p-1)$ . Also  $u(x) = \log \frac{1}{|x|}$  when  $|x| < \frac{1}{e}$  and  $u(x) = 1$  otherwise, is an  $A_1$  weight.

The following four theorems are basic in proving weighted norm inequalities for linear operators on weighted  $L^p$  spaces with Muckenhoupt weights. All the proofs can be found in [6] or [13].

**Theorem 2.3** Suppose  $p > 1$  and  $w \in A_p$ . There exists an exponent  $q < p$ , which depends only on  $p$  and  $[w]_{A_p}$ , such that  $w \in A_q$ . There exists  $\varepsilon > 0$ , which depends only on  $p$  and  $[w]_{A_p}$ , such that  $w^{1+\varepsilon} \in A_p$ .

**Theorem 2.4** For  $1 < q < \infty$ , the Hardy–Littlewood maximal operator is bounded on  $L^q_w$  if and only if  $w \in A_q$ . Consequently, for  $1 \leq p < \infty$ ,  $M_p$  is bounded on  $L^p_w$  if and only if  $w \in A_{q/p}$

**Theorem 2.5** Suppose that  $K : \mathbb{R}^n \rightarrow \mathbb{R}$  is integrable non-increasing and radial. Then for one has for all  $x \in \mathbb{R}^n$ .

$$\left| \int K(x-y) f(y) \, dy \right| \leq \|K\|_{L^1} M(f)(x)$$

The following result of Rubio de Francia [6] is also basic in the context of weighted norm inequalities.

**Theorem 2.6** (Extrapolation theorem) *If  $\|Tu\|_{L_w^{p_0}} \leq C\|u\|_{L_w^{p_0}}$  for some fixed  $p_0 \in (1, \infty)$  and all  $w \in A_{p_0}$ , then one has in fact  $\|Tu\|_{L_w^p} \leq C\|u\|_{L_w^p}$  for all  $p \in (1, \infty)$  and  $w \in A_p$ .*

*Additional conventions* As is common practice, we will denote constants which can be determined by known parameters in a given situation, but whose value is not crucial to the problem at hand, by  $C$ . Such parameters in this paper would be, for example,  $m, p, n, [w]_{A_p}$ , and the constants  $C_\alpha$  in Definition 1.2. The value of  $C$  may differ from line to line, but in each instance could be estimated if necessary. We sometimes write  $a \lesssim b$  as shorthand for  $a \leq Cb$ . When  $C_1b \leq a \leq C_2b$  then we write  $a \sim b$ . Also we will use  $\langle D \rangle^s$  to denote the Fourier multiplier  $(1 - \Delta)^{\frac{s}{2}}$ .

### 3 Pointwise estimates for OIOs

Using suitable frequency-space decompositions we will demonstrate that the OIOs considered here can be pointwise bounded from above by operator involving the Hardy–Littlewood maximal function. First we consider the low frequency portion of the OIOs for which we have

**Lemma 3.1** *Let  $s > 0, a(x, \xi) \in L^\infty S_1^m$  be a symbol that is compactly supported in the  $\xi$ -variable and  $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  be a phase function satisfying condition (3) for all  $x$  and  $\xi$ . Then for all  $x \in \mathbb{R}^n$  one has that*

$$|T_a^\varphi f(x)| \lesssim M(f)(x). \tag{6}$$

**Proof** Assumption (3) implies that

$$\left| \partial_\xi^\alpha (\varphi(x, \xi) - x \cdot \xi) \right| \leq C_\alpha |\xi|^{s-|\alpha|}, \quad |\alpha| \geq 0$$

on the support of  $a$ . This in turn implies that the conditions of Lemma 4.3 in [3] are satisfied and therefore the modulus of the integral kernel

$$K(x, y) := \int_{\mathbb{R}^n} a(x, \xi) e^{i\varphi(x, \xi) - iy \cdot \xi} \, d\xi$$

is bounded by  $\langle x - y \rangle^{-n-\varepsilon s}$  for any  $0 \leq \varepsilon < 1$ . Hence

$$|T_a^\varphi f(x)| \leq \int |K(x, y)| |f(y)| \, dy \lesssim \int \langle x - y \rangle^{-n-\varepsilon s} |f(y)| \, dy,$$

and therefore Theorem 2.5 yields the pointwise estimate (6). □

The main device to show any sort of estimate for oscillatory integral operators is the Littlewood–Paley decomposition, whose definition we now recall.

**Definition 3.2** Let  $\psi_0 \in C_c^\infty(\mathbb{R}^n)$  be equal to 1 on  $B(0, 1)$  and have its support in  $B(0, 2)$ . Then let

$$\psi_j(\xi) := \psi_0\left(2^{-j}\xi\right) - \psi_0\left(2^{-(j-1)}\xi\right),$$

where  $j \geq 1$  is an integer and  $\psi(\xi) := \psi_1(\xi)$ . Then  $\psi_j(\xi) = \psi\left(2^{-(j-1)}\xi\right)$  and one has the following Littlewood–Paley partition of unity

$$\sum_{j=0}^\infty \psi_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{7}$$

Now we are ready to state and prove our main result concerning the pointwise estimate for OIOs.

**Theorem 3.3** Assume that  $s > 0$ ,  $a(x, \xi) \in L^\infty S_1^m$ ,  $\varphi(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  is a phase function of order  $s$  (strictly speaking only the estimate (3) is required) that satisfies the condition

$$|\det \partial_{\xi\xi}^2 \varphi(x, \xi)| \gtrsim |\xi|^q, \tag{8}$$

for all  $\xi \neq 0$  and some  $q \leq n(s - 2)$ , uniformly in  $x$ . Then if

$$m < \frac{-n(3s - 4) + 2q}{2}$$

one has the following pointwise estimate for  $T_a^\varphi$

$$|T_a^\varphi f(x)| \lesssim M(f)(x) + M(Mf)(x). \tag{9}$$

**Proof** We start by splitting the operator into low- and high frequency portions. Indeed, using the Littlewood-Paley decomposition (7), we can write

$$T_a^\varphi f(x) = T_L f(x) + T_H f(x) = T_L f(x) + \sum_{j=1}^\infty T_j f(x), \tag{10}$$

where the amplitude of  $T_L$  is  $a(x, \xi) \psi_0(\xi)$ , and is smooth compactly supported (in  $\xi$ ).

Using this decomposition, we can use Lemma 3.1 to reduce matters to the analysis of the high frequency portion of the operator, and we therefore need only to show that

$$|T_H f(x)| \lesssim M(Mf)(x). \tag{11}$$

To this end, given a Littlewood–Paley partition of unity as in (7), for  $j \geq 1$  we consider the operator

$$T_j f(x) = \int K_j(x, y) f(y) dy = \int \left( \int a(x, \xi) \psi_j(\xi) e^{-iy \cdot \xi + i\varphi(x, \xi)} d\xi \right) f(y) dy,$$

in the decomposition of  $T_H$  in (refdecompositionofT). The analysis then hinges on a continuous partition of unity. Let  $\lambda \geq -1$  to be fixed later, and take a radially symmetric  $\theta \in C_c^\infty(\mathbb{R}^n)$  supported in a ball of small radius and positive in the interior of the ball. Let  $O_j = \text{supp}(\psi_{j-1} + \psi_j + \psi_{j+1})$ . We then have for  $\xi \in \text{supp}\psi_j$  that

$$\begin{aligned} 1 &= \int_{O_j} \theta(2^{\lambda j}(\xi - \eta)) \, d\eta \bigg/ \int_{O_j} \theta(2^{\lambda j}(\xi - \omega)) \, d\omega \\ &= \int_{O_j} \left( \theta(2^{\lambda j}(\xi - \eta)) \bigg/ \int_{O_j} \theta(2^{\lambda j}(\xi - \omega)) \, d\omega \right) \, d\eta =: \int_{O_j} \rho_j(\xi, \eta) \, d\eta. \end{aligned}$$

For these cut-off functions one has that for all  $\xi \in \text{supp}\psi_j$

$$|\partial_\xi^\alpha \rho_j(\xi, \eta)| \leq C_\alpha 2^{n\lambda j + |\alpha|\lambda j}. \tag{12}$$

This is so because when  $\xi \in \text{supp}\psi_j$  and the support of  $\theta$  is sufficiently small, then the extra margins in  $O_j$  ensure that for all  $j \geq 0$ ,

$$\int_{O_j} \theta(2^{\lambda j}(\xi - \eta)) \, d\eta = \int_{\mathbb{R}^n} \theta(2^{\lambda j}(\xi - \eta)) \, d\eta = 2^{-\lambda n j} \int \theta(\omega) \, d\omega.$$

With the partition of unity introduced above, define the kernels

$$K_j(x, y, \eta) = \int a(x, \xi) \rho_j(\xi, \eta) \psi_j(\xi) e^{-iy \cdot \xi + i\varphi(x, \xi)} \, d\xi.$$

With these the kernel of  $T_j$  can be written as

$$K_j(x, y) = \int_{O_j} K_j(x, y, \eta) \, d\eta,$$

and therefore at least formally (subject to convergence issues that will be resolved momentarily)

$$|T_j f(x)| \leq \int_{O_j} \int |K_j(x, y, \eta) f(y)| \, dy \, d\eta. \tag{13}$$

Our goal now is to obtain a pointwise estimate for the operator  $T_j$ , by estimating its kernel  $K_j(x, y)$  in a certain way. To achieve this, we rewrite the phase function of  $K_j(x, y, \eta)$  as

$$-y \cdot \xi + \varphi(x, \xi) = (-y + \nabla_\xi \varphi(x, \eta)) \cdot \xi + h_j(x, \xi, \eta),$$

with  $\eta \in \mathbb{R}^n$  and  $h_j(x, \xi, \eta) := \varphi(x, \xi) - \nabla_\xi \varphi(x, \eta) \cdot \xi$ ,

which in turn yields

$$K_j(x, y, \eta) = \int b_j(x, \xi, \eta) e^{i(-y + \nabla_\xi \varphi(x, \eta)) \cdot \xi} \, d\xi,$$



where  $b_j(x, \xi, \eta) := a(x, \xi)\rho_j(\xi, \eta)\psi_j(\xi)e^{ih_j(x, \xi, \eta)}$ .

Now, the mean-value theorem yields that on the support of  $b_j$ ,  $\partial_{\xi_i} h_j(x, \xi, \eta) = (\nabla \partial_{\xi_i} \varphi)(x, \eta) \cdot (\xi - \eta)$  for some  $\eta$  on the line segment between  $\xi$  and  $v$ . Therefore on  $\text{supp } b_j$ , (3) yields that

$$|\partial_{\xi}^{\alpha} h_j(x, \xi, \eta)| \lesssim \begin{cases} 2^{(s-\lambda-2)j} & |\alpha| = 1 \\ 2^{(s-|\alpha|)j} & |\alpha| > 1. \end{cases}$$

Setting

$$\lambda = \frac{s}{2} - 1,$$

we have that for all  $\alpha$

$$|\partial_{\xi}^{\alpha} e^{ih_j(x, \xi, \eta)}| \lesssim 2^{\lambda|\alpha|j}. \tag{14}$$

Leibniz' formula, the assumption on the amplitude and estimates (12) and (14) yield that, on the support of  $b_j^v$ ,

$$\begin{aligned} |\partial_{\xi}^{\alpha} b_j(x, \xi, \eta)| &\leq C_{\alpha} \sum_{\sum \alpha_{\ell} = \alpha} \left| \partial_{\xi}^{\alpha_1} a(x, \xi) \partial_{\xi}^{\alpha_2} \rho_j(\xi, \eta) \partial_{\xi}^{\alpha_3} \psi_j(\xi) \partial_{\xi}^{\alpha_4} (e^{ih_j(x, \xi, \eta)}) \right| \\ &\leq C_{\alpha} \sum_{\sum \alpha_{\ell} = \alpha} 2^{(m+n\lambda-|\alpha_1+\alpha_3|+\lambda|\alpha_2+\alpha_4)j}. \end{aligned}$$

Thus we have that for all  $\alpha$

$$\left| \partial_{\xi}^{\alpha} b_j(x, \xi, \eta) \right| \leq C_{\alpha} 2^{mj+\lambda(n+|\alpha|)j} = C_{\alpha} 2^{mj+(s/2-1)(n+|\alpha|)j}. \tag{15}$$

Using integration by parts and estimate (15), we have for any nonnegative integer  $N$  that

$$\begin{aligned} |K_j(x, \nabla_{\xi} \varphi(x, \eta) + y, \eta)| &= 2^{-\lambda j n} \left| \int b_j(x, 2^{-\lambda j} \xi, \eta) e^{-i2^{-\lambda j} y \cdot \xi} \, d\xi \right| \\ &= 2^{-\lambda j n} \left| \int e^{-i2^{-\lambda j} y \cdot \xi} \left( \frac{1 - \Delta_{\xi}}{1 + 2^{-2\lambda j} |y|^2} \right)^N b_j(x, 2^{-\lambda j} \xi, \eta) \, d\xi \right| \\ &\lesssim \frac{2^{-\lambda j n} |\text{supp } b_j(x, 2^{-\lambda j}(\cdot), \eta)|}{(1 + 2^{-2\lambda j} |y|^2)^N} \sum_{|\alpha| \leq 2N} |\partial_{\xi}^{\alpha} b_j(x, 2^{-\lambda j} \xi, \eta)| \\ &\lesssim \frac{2^{-\lambda j n}}{(1 + 2^{-2\lambda j} |y|^2)^N} \sum_{|\alpha| \leq 2N} 2^{-\lambda j |\alpha|} 2^{mj+(s/2-1)(n+|\alpha|)j} \\ &\lesssim \frac{2^{mj}}{(1 + 2^{-2\lambda j} |y|^2)^N} \lesssim 2^{mj} (1 + 2^{-\lambda j} |y|)^{-2N}. \end{aligned}$$

Hence, returning to (13), the previous estimate yields that

$$\begin{aligned}
 \int |K_j(x, y, \eta) f(y)| \, dy &= \int |K_j(x, \nabla_\xi \varphi(x, \eta) + y, \eta)| |f(y + \nabla_\xi \varphi(x, \eta))| \, dy \\
 &\lesssim 2^{mj} \int \frac{|f(\nabla_\xi \varphi(x, \eta) + y)|}{(1 + 2^{-\lambda j} |y|)^{2N}} \, dy \\
 &\lesssim 2^{mj+n\lambda j} \int \frac{|f(2^{\lambda j} y + \nabla_\xi \varphi(x, \eta))|}{(1 + |y|)^{2N}} \, dy \\
 &\lesssim 2^{mj+n\lambda j} \sum_{k=0}^\infty 2^{-2Nk} \int_{|y| \leq 2^k} |f(2^{\lambda j} y + \nabla_\xi \varphi(x, \eta))| \, dy \\
 &\lesssim 2^{mj} \sum_{k=0}^\infty 2^{-2Nk} \int_{|y| \leq 2^{\lambda j} 2^k} |f(y + \nabla_\xi \varphi(x, \eta))| \, dy \\
 &\lesssim 2^{mj+n\lambda j} Mf(\nabla_\xi \varphi(x, \eta)) \\
 &= 2^{mj+n\lambda j} Mf(x + (\nabla_\xi \varphi(x, \eta) - x)).
 \end{aligned}$$

Now the change of variables  $\omega := \nabla_\xi \varphi(x, \eta) - x$ , condition (8), and estimate (3) imply that

$$\begin{aligned}
 |T_j f(x)| &\lesssim 2^{mj+n(s/2-1)j} \int_{|\eta| \sim 2^j} Mf(x + (\nabla_\xi \varphi(x, \eta) - x)) \, d\eta \\
 &= 2^{mj+n(s/2-1)j-qj} \int_{|\omega| \lesssim 2^{j(s-1)}} Mf(x + \omega) \, d\omega \\
 &\lesssim 2^{(m-q+n(s-1)+n(s/2-1))j} M Mf(x) \\
 &\lesssim 2^{(m-q+ns/2+n(s-2))j} M Mf(x).
 \end{aligned}$$

Therefore summing in  $j$  and taking  $m < q - ns/2 - n(s - 2) = (-n(3s - 4) + 2q)/2$  we obtain (11). □

**Remark 3.4** A calculation shows that if the phase function is of the form  $\varphi(x, \xi) = x \cdot \xi + \psi(|\xi|)$ , for a function  $\psi \in C^\infty(\mathbb{R})$ ,

$$|\det \partial_{\xi\xi}^2 \varphi(x, \xi)| = \left| \frac{\psi'(|\xi|)}{|\xi|} \right|^{n-1} |\psi''(|\xi|)|.$$

Now if we take  $\psi(r) = r^s$ , then this yields that

$$|\det \partial_{\xi\xi}^2 (x \cdot \xi + |\xi|^s)| = s(s - 1)s^{n-1} |\xi|^{n(s-2)}.$$

If we instead take  $\psi(r) = (1 + r^2)^{\frac{s}{2}}$  then

$$|\det \partial_{\xi\xi}^2 (x \cdot \xi + \langle \xi \rangle^s)| = \left( s \langle \xi \rangle^{s-2} \right)^{n-1} s \langle \xi \rangle^{s-4} (1 + (s - 1) |\xi|^2).$$

For  $s \neq 1$  and  $|\xi|$  large, we have (similar to the previous case) that

$$|\det \partial_{\xi\xi}^2(x \cdot \xi + \langle \xi \rangle^s)| \gtrsim |\xi|^{n(s-2)}.$$

This means that for both phase functions, as long as  $s \neq 1$  all the assumptions of the theorem are satisfied for the phase and one obtains a pointwise estimate for the corresponding OIOs when the order of amplitudes is strictly less than  $-\frac{ns}{2}$ .

However for  $s = 1$ , the phase function  $x \cdot \xi + |\xi|^s$  does not satisfy condition (8) and for the phase function  $x \cdot \xi + \langle \xi \rangle^s$  one has for  $|\xi|$  large enough

$$|\det \partial_{\xi\xi}^2(x \cdot \xi + \langle \xi \rangle)| \gtrsim |\xi|^{-(n+2)}.$$

This means that one can deduce a pointwise estimate for the Klein–Gordon operator  $e^{i\sqrt{1-\Delta}}$ , of the form

$$|\langle D \rangle^m e^{i\sqrt{1-\Delta}} f(x)| \lesssim M(f)(x) + M(Mf)(x), \tag{16}$$

provided that  $m < -\frac{(n+4)}{2}$ , but this estimate is not optimal. Indeed for both the wave  $e^{i\sqrt{-\Delta}}$  and the Klein–Gordon operators, it is possible to use Theorem 1.4 to improve the decay in the estimate above to  $m < -\frac{(n+1)}{2}$ .

### 4 Weighted estimates for OIOs

Note that, (9) jointly with the boundedness of the Hardy–Littlewood maximal operator  $M$  yields that, under the assumptions of Theorem 3.3, the operator  $T_a^\varphi$  is automatically bounded on any space where  $M^2$  is bounded. In particular, it satisfies (1) for all  $1 < p < \infty$  and  $w \in A_p$ . So the remaining difficulty in proving Theorem 1.6 is to obtain the result for amplitudes where the decay  $m$  is equal to the critical index  $m(s, q)$ .

The following theorem establishes the  $L^2$  boundedness of OIOs with phase functions that satisfy a certain weak non-degeneracy condition.

**Theorem 4.1** *Assume that  $a(x, \xi) \in L^\infty S_1^m$ , and  $\varphi(x, \xi)$  is a phase function of order  $s > 0$ . Then for  $m < 0$  the OIO  $T_a^\varphi$  is bounded on  $L^2(\mathbb{R}^n)$ .*

**Proof** We start by splitting the operator into low- and high frequency portions as in (10). In light of Lemma 3.1, we have that  $|T_L f(x)| \lesssim M(f)(x)$  and the  $L^2$ -boundedness of the Hardy–Littlewood maximal function (which is also a consequence of Theorem 2.4), we can confine ourselves to deal with the high frequency components  $T_j$  of  $T_a^\varphi$ , hence we can assume that  $\xi$  is large on the support of the amplitude  $a_j(x, \xi) := a(x, \xi) \psi(2^{-j}\xi)$ . Here we shall use a  $T_j T_j^*$  argument. The kernel of the operator  $S_j := T_j T_j^*$  reads

$$K_j(x, y) = 2^{jn} \int e^{i(\varphi(x, 2^j \xi) - \varphi(y, 2^j \xi))} \psi^2(\xi) a(x, 2^j \xi) \bar{a}(y, 2^j \xi) \, d\xi.$$

Now assumption (4) on the phase yields that there is a constant  $C > 0$  such that

$$|\nabla_{\xi}[\varphi(x, 2^j \xi) - \varphi(y, 2^j \xi)]| \geq C2^j |x - y|,$$

for  $j \geq 0, x, y \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}^n \setminus \{0\}$ . This enables us to use the non-stationary phase estimate (e.g. Theorem 7.7.1 in [8]), and the smoothness of the phase function  $\varphi(x, \xi)$  in the spatial variable, yield that for all integers  $N$

$$|K_j(x, y)| \leq C_N 2^{j(2m+n)} \langle 2^j(x - y) \rangle^{-N},$$

for some constant  $C_N > 0$ .

This implies

$$\sup_x \int |K_j(x, y)| \, dy \leq C_N 2^{2mj} \tag{17}$$

for all  $N > n$ , and by symmetry we also have that  $\sup_y \int |K_j(x, y)| \, dx \leq C_N 2^{2mj}$ . Therefore Schur’s lemma yields the  $L^2$ -boundedness of  $S_j$  and summing in  $j$  and using the fact that  $m < 0$  we have that  $S$  and therefore  $T$  are  $L^2$ -bounded.  $\square$

The following interpolation lemma is the main tool in proving the endpoint weighted boundedness of OIOs.

**Lemma 4.2** *Let  $1 < p < \infty$  and  $m_1 < m_2$ . Suppose that*

- (a) *the OIO  $T_a^\varphi$  with amplitude  $a \in L^\infty S_1^{m_1}$  and the phase  $\varphi$  are bounded on  $L_w^p$  for a fixed  $w \in A_p$ , and*
- (b) *the OIO  $T$  with amplitude  $a(x, \xi) \in L^\infty S_1^{m_2}$  and the same type of phases as in (a) are bounded on  $L^p$ ,*

*where the bounds depend only on a finite number of seminorms in Definition 1.2. Then, for each  $m \in (m_1, m_2)$ , operators with amplitudes in  $L^\infty S_1^m$  are bounded on  $L_w^{p_\nu}$ , where*

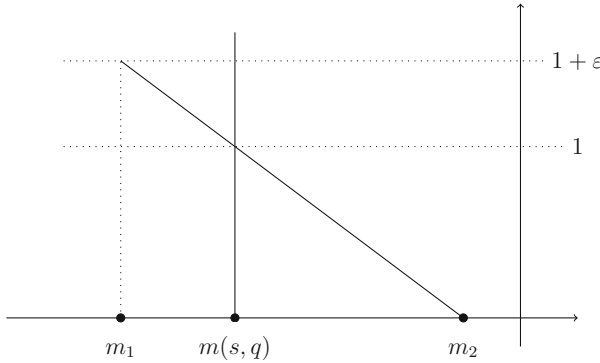
$$\nu = \frac{m_2 - m}{m_2 - m_1}.$$

**Proof** The proof is the same as Lemma 3.1 in [5].  $\square$

Now we are ready to prove our main result concerning weighted boundedness of OIOs. This is done by combining our previous results with a method based on the properties of the  $A_p$  weights and interpolation.

**Proof of Theorem 1.6**

First we note that the assumptions in the statement of the theorem and Theorem 3.3 guarantee the weighted boundedness for amplitudes of order  $m < m(s, q)$ . Next by the extrapolation Theorem 2.6, it is enough to show the boundedness of  $T_a^\varphi$  in  $L_w^2$  with  $w \in A_2$ .



**Fig. 1** We can choose  $m_1$  arbitrarily close to  $m(s, q)$ . This shows that  $(1 + \varepsilon)^{\frac{m_2 - m(s, q)}{m_2 - m_1}} = 1$

Let us fix  $m_2$  such that  $m(s, q) < m_2 < 0$ . By Theorem 2.3, given  $w \in A_2$  choose  $\varepsilon$  such that  $w^{1+\varepsilon} \in A_2$ . For this  $\varepsilon$  take  $m_1 < m(s, q)$  in such a way that the line that joins the points with coordinates  $(m_1, 1 + \varepsilon)$  and  $(m_2, 0)$  intersects the line  $x = m(s, q)$  in the  $(x, y)$  plane in a point with coordinates  $(m(s, q), 1)$ . Clearly this procedure could be carried out, since Theorem 3.3 ensures that we can choose the point  $m_1$  on the negative  $x$  axis as close as we like to the point  $m(s, q)$ . This is depicted in Fig. 1.

Using Theorem 3.3, given  $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$  satisfying the conditions of the theorem and  $a \in L^\infty S_1^{m_1}$ , the OIO  $T_a^\varphi$  is bounded on  $L^2_{w^{1+\varepsilon}}$

for  $w \in A_2$ , and by Theorem 4.1 the OIOs with amplitudes in  $L^\infty S_1^{m_2}$  and phases satisfying (4) are bounded on  $L^2$ . Therefore, we can use Lemma 4.2 on  $w^{1+\varepsilon}$  and conclude that the OIOs  $T_a^\varphi$  with phases and amplitudes as in the statement of the theorem are bounded operators on  $L^2_w$ . This concludes the proof of the theorem.

The result above also implies the extension of Theorem 1.5 in the introduction, namely.

**Corollary 4.3** For  $s > 0, s \neq 1$  one has the following two sharp weighted norm inequalities: For  $1 < p < \infty$  and all  $w \in A_p$  one has

$$\|\langle D \rangle^{-\frac{ns}{2}} e^{i\Delta^{s/2}} f\|_{L^p_w} \lesssim \|f\|_{L^p_w}, \tag{18}$$

and

$$\|\langle D \rangle^{-\frac{ns}{2}} e^{i\langle D \rangle^s} f\|_{L^p_w} \lesssim \|f\|_{L^p_w}. \tag{19}$$

**Proof** As was shown in Remark 3.4, the conditions on the phase functions  $x \cdot \xi + |\xi|^s$  and  $x \cdot \xi + \langle \xi \rangle^s$  are satisfied with  $q = n(s - 2)$ . Therefore Theorem 1.6 yields estimates (18) and (19).  $\square$

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## Declarations

**Conflict of interest** The authors declare no conflict of interest.

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