Analysis and Mathematical Physics (2022) 12:61
https://doi.org/10.1007/s13324-022-00669-7

Ricci solitons on four-dimensional Lorentzian Lie groups

M. Ferreiro-Subrido[1] · E. García-Río[1] · R. Vázquez-Lorenzo[2]

Received: 3 November 2021 / Accepted: 19 February 2022 / Published online: 26 March 2022

Abstract
We determine all non-Einstein Ricci solitons on four-dimensional Lorentzian Lie groups whose soliton vector field is left-invariant. In addition to pp-wave and plane wave Lie groups, there are four families of Lorentzian metrics on semi-direct extensions $\mathbb{R}^3 \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$. We show that some of these Ricci solitons are conformally Einstein and they may be expanding, steady or shrinking.

Keywords Ricci soliton · Left-invariant Lorentz metric · pp-wave

Mathematics Subject Classification Primary 53C50; Secondary 53C25 · 53C30

1 Introduction

A *Ricci soliton* is a triple (M, g, X) consisting of a vector field X on a pseudo-Riemannian manifold (M, g) satisfying the differential equation

$$\mathcal{L}_X g + \rho = \mu g \tag{1}$$

where $\mathcal{L}$ denotes the Lie derivative, ρ is the Ricci tensor and $\mu \in \mathbb{R}$. Ricci solitons not only generalize Einstein metrics but also are self-similar solutions of the Ricci flow and

Supported by projects PID2019-105138GB-C21(AEI/FEDER, Spain), and ED431C 2019/10, ED431F 2020/04 (Xunta de Galicia, Spain).

✉ E. García-Río
eduardo.garcia.rio@usc.es

M. Ferreiro-Subrido
mariaferreiro.subrido@usc.es

R. Vázquez-Lorenzo
ravazlor@edu.xunta.gal

1 CITMAga Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain

2 I.E.S. de Ribadeo Dionisio Gamallo, Ribadeo, Spain

conversely, thus corresponding to geometric fixed points of the flow (modulo scaling and diffeomorphisms). A Ricci soliton is said to be *expanding*, *steady*, or *shrinking* if the soliton constant $\mu < 0$, $\mu = 0$ or $\mu > 0$, respectively. Furthermore, if the soliton vector field X is the gradient of some potential function, then the soliton is said to be a *gradient Ricci soliton*. We refer to [11] for more information.

A Ricci soliton is said to be *trivial* if the pseudo-Riemannian metric is Einstein, in which case one may solve Equation (1) setting $X = 0$. It immediately follows from (1) that two Ricci soliton vector fields X_1 and X_2 on a given manifold (M, g) differ on a homothetic vector field $\xi = X_1 - X_2$. While the existence of homothetic vector fields is a very rigid condition in the positive definite case, Lorentzian manifolds may admit homothetic vector fields without being flat. Moreover, the Ricci soliton equation (1) is invariant by homotheties in the sense that (M, g, X) is a Ricci soliton with soliton constant μ if and only if $(M, \kappa g, \frac{1}{\kappa} X)$ is a Ricci soliton with soliton constant $\frac{\mu}{\kappa}$ for any $\kappa > 0$. Hence we work modulo homotheties in what follows.

A metric Lie group $(G, \langle, \rangle)$ is an *algebraic Ricci soliton* if the Ricci operator satisfies $\mathrm{Ric} = \mu \,\mathrm{Id} + D$ for some derivation of the corresponding Lie algebra [24]. Algebraic Ricci solitons are critical points of the scalar curvature for an appropriately restricted family of metrics [24] and, moreover, they are critical for a quadratic curvature functional with zero energy in dimensions three and four [6]. Algebraic Ricci solitons give rise to Ricci solitons whose soliton vector field is generically not left-invariant and there is a relation between Riemannian and Lorentzian algebraic Ricci solitons in the nilpotent case (see [30]). In contrast, Ricci solitons on Lie groups with left-invariant soliton vector field are not necessarily critical for any quadratic curvature functional, thus being of a different nature.

Non-trivial homogeneous Ricci solitons are necessarily expanding in the Riemannian setting and they are algebraic in dimension four [1]. Left-invariant Ricci solitons do not exist on Riemannian unimodular Lie groups, and there are no three-dimensional non-trivial left-invariant Ricci solitons on Riemannian Lie groups [14]. In sharp contrast, the Lorentzian signature supports such solitons (see [4]).

The purpose of this work is to classify left-invariant Ricci solitons on four-dimensional Lorentzian Lie groups. After reviewing left-invariant Einstein metrics and plane waves, we recall the situation in dimension three, which is much simpler than the four-dimensional one. Our main result (Theorem 1.2) gives a complete description modulo homotheties of non-trivial left-invariant Ricci solitons which are neither symmetric nor pp-waves. The symmetric case is treated in Remark 1.5 and the pp-wave Lie groups are considered in Sect. 5.

1.1 Einstein metrics on Lorentzian four-dimensional Lie groups

While four-dimensional homogeneous Einstein metrics are locally symmetric in the Riemannian setting [19], the Lorentzian signature allows other possibilities. Left-invariant Einstein metrics on four-dimensional Lorentzian Lie groups were studied in [9] and a different approach shows that left-invariant Einstein metrics split into three categories: symmetric spaces, plane waves and left-invariant metrics which do not correspond to any of these.

Indecomposable locally symmetric Lorentzian spaces either are irreducible (and hence of constant sectional curvature), or they correspond to Cahen-Wallach symmetric spaces [7], which are a special class of plane waves (see Sect. 1.2). Four-dimensional products $\mathbb{R} \times N^3$ are Einstein if and only if they are flat and so the only decomposable four-dimensional Einstein Lorentzian symmetric spaces of non-constant sectional curvature are products $M_1(c) \times M_2(c)$ of two surfaces with the same constant sectional curvature. The other possibilities are covered by the following (see [28]).

Theorem 1.1 *Let $(G, \langle , \rangle)$ be a four-dimensional Lie group with a left-invariant Einstein Lorentzian metric which is neither locally symmetric nor a plane wave. Then, it is locally homothetic to the Lie group determined by one of the following:*

(i) The Ricci-flat semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[e_1, e_4] = -2e_1, \quad [e_2, e_4] = e_2 + \sqrt{3}e_3, \quad [e_3, e_4] = -\sqrt{3}e_2 + e_3, \quad \textit{or}$$

(ii) the semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[u_1, u_4] = -u_1 + \delta u_2, \quad [u_2, u_4] = 5u_2, \quad [u_3, u_4] = 2u_3, \quad \delta \neq 0, \quad \textit{or}$$

(iii) the semi-direct product $\mathbb{R}^3 \rtimes \mathbb{R}$ with Lie algebra given by

$$[u_1, u_4] = 4u_1, \quad [u_2, u_4] = -2u_2 + \delta u_3, \quad [u_3, u_4] = \delta u_1 + u_3, \quad \delta \neq 0,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis with e_3 timelike, and $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$.

The curvature operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$ of metrics corresponding to Assertion (i) has real and complex eigenvalues, and moreover $\|\nabla R\|^2 \neq 0$. Metrics corresponding to Assertion (ii) have scalar curvature $\tau = -48$ and their Weyl curvature operator is two-step nilpotent. Moreover, they are locally isometric to the only non-reductive homogeneous space which is Einstein but not of constant sectional curvature [10, 15]. Metrics corresponding to Assertion (iii) have scalar curvature $\tau = -12$ and their Weyl curvature operator is three-step nilpotent.

1.2 Homogeneous pp-waves and plane waves

Let $(M, g, \mathcal{U})$ be a Brinkmann wave, i.e., a Lorentzian manifold admitting a parallel degenerate line field $\mathcal{U}$. $(M, g, \mathcal{U})$ is said to be a *pp-wave* if the parallel line field is locally generated by a parallel null vector field and (M, g) is transversally flat, i.e., its curvature tensor satisfies $R(X, Y) = 0$ for all $X, Y \in \mathcal{U}^\perp$. In such case there exist local coordinates (u, v, x^1, x^2) so that

$$g = du \circ dv + H(v, x^1, x^2) dv \circ dv + dx^1 \circ dx^1 + dx^2 \circ dx^2 .$$

Leistner showed in [25] that a Brinkmann wave $(M, g, \mathcal{U})$ is a pp-wave if and only if it is transversally flat and Ricci isotropic, i.e., $g(\mathrm{Ric}\, X, \mathrm{Ric}\, X) = 0$ for any vector field X on M.

A pp-wave is said to be a *plane wave* if the covariant derivative of the curvature tensor satisfies $\nabla_X R = 0$ for all $X \in \mathcal{U}^\perp$. In this case the local coordinates above can be specialized so that $H(v, x^1, x^2) = a_{ij}(v)x^i x^j$. The Ricci operator of any pp-wave is two-step nilpotent and the metric is Ricci-flat if $\Delta_x H = 0$, being $\Delta_x = \partial_{x^1x^1} + \partial_{x^2x^2}$ the spacelike Laplacian. It was shown in [17] that locally homogeneous Ricci-flat pp-waves are plane waves in the four-dimensional case. Homogeneous steady Ricci solitons on pp-waves which are not plane waves are given in Sect. 5, thus showing that the result in [17] does not extend to Ricci solitons.

Homogeneous plane waves in dimension four are described in terms of a 2×2 skew-symmetric matrix F and a 2×2 symmetric matrix A_0 so that the defining function $H(v, x^1, x^2)$ takes the form $H = \mathbf{x}^T A(v)\mathbf{x}$, where the matrix $A(v)$ is given by (see [2])

$$A(v) = e^{vF} A_0 e^{-vF}, \quad \text{or} \quad A(v) = \frac{1}{(v+b)^2} e^{\log(v+b)F} A_0 e^{-\log(v+b)F} .$$

Furthermore, the plane wave metric is Ricci-flat if and only if A_0 is trace-free.

The existence of Ricci solitons on plane waves was investigated in [5] where it is shown that any plane wave is a steady gradient Ricci soliton. Due to the existence of homothetic vector fields, one also has the existence of expanding and shrinking Ricci solitons on some special classes of plane waves. In any case, the soliton vector field needs not be left-invariant for a plane wave Lie group, and hence the existence of left-invariant Ricci solitons on plane wave Lie groups will be considered in Sect. 5.

1.3 Left-invariant Ricci solitons on 3-dimensional Lorentzian Lie groups

Non-trivial three-dimensional left-invariant Ricci solitons are either non-symmetric pp-waves or locally isometric to a left-invariant metric on $G = O(1, 2)$, the universal cover of $SL(2, \mathbb{R})$ or the non-unimodular semi-direct extension $\mathbb{R}^2 \rtimes \mathbb{R}$ given by the Lorentzian Lie algebras

$$\begin{array}{llll} (i) \ [u_1, u_2] = \lambda u_3, & [u_1, u_3] = -\lambda u_1 \mp u_2, & [u_2, u_3] = \lambda u_2, & \lambda \neq 0, \\ (ii) \ [u_1, u_2] = u_1 + \lambda u_3, & [u_1, u_3] = -\lambda u_1, & [u_2, u_3] = \lambda u_2 + u_3, & \lambda \neq 0, \\ (iii) \ [e_1, e_3] = e_1 - e_2, & [e_2, e_3] = e_1 + e_2 & & \end{array}$$

where $\{u_1, u_2, u_3\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$, and $\{e_1, e_2, e_3\}$ is an orthonormal basis with timelike e_1.

It was shown in [4] that three-dimensional Lorentzian Lie groups corresponding to cases (i) and (ii) have a single Ricci curvature which is a double or triple root of the corresponding minimal polynomial. Moreover, the Lie group corresponding to (iii), which was omitted in [4], has complex Ricci curvatures $-2 \pm 2i$.

There are two different possibilities for three-dimensional left-invariant pp-waves which are Ricci solitons: a locally conformally flat plane wave (thus locally isometric to a $\mathcal{P}_c$–space), or a pp-wave locally isometric to a $\mathcal{N}_b$–space. We refer to [16] for a classification of homogeneous pp-waves in dimension three, definitions of $\mathcal{P}_c$ and $\mathcal{N}_b$–spaces and more details.

1.4 Left-invariant Ricci solitons on 4-dimensional Lorentzian Lie groups

The four-dimensional situation is more complicated than the corresponding three-dimensional one, as in the Einstein case. We consider separately the case of left-invariant Ricci solitons on pp-wave Lie groups, which is treated in Sect. 5. The remaining possibilities are given as follows, which is the main result of this paper.

Theorem 1.2 *A non-symmetric four-dimensional Lorentzian Lie group which is not a pp-wave is a non-trivial left-invariant Ricci soliton if and only if it is homothetic to one of the following:*

(i) $G_\alpha = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_1, e_4] = \alpha e_1, \quad [e_2, e_4] = \varepsilon \left(1 - \tfrac{\alpha^2}{2}\right)^{\frac{1}{2}} e_2 - e_3, \quad [e_3, e_4] = e_2 + \varepsilon \left(1 - \tfrac{\alpha^2}{2}\right)^{\frac{1}{2}} e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ *is an orthonormal basis with* e_3 *timelike, and the parameter* $0 \le \alpha \le \sqrt{2}$*. If* $\alpha = 0$ *then* $\varepsilon = 1$*, while if* $0 < \alpha < \sqrt{2}$ *then* $\varepsilon^2 = 1$*; in this latter case,* $\alpha \neq \frac{2}{\sqrt{3}}$ *whenever* $\varepsilon = -1$.

(ii) $G_\alpha = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_4] = \alpha u_1, \quad [u_2, u_4] = -\alpha u_2 + u_3, \quad [u_3, u_4] = u_1, \qquad \alpha > 0,$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis with* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$.

(iii) $G = E(1, 1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[e_2, e_4] = -[e_1, e_2] = e_2, \quad [e_1, e_3] = [e_3, e_4] = \tfrac{1}{2}[e_1, e_4] = e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ *is an orthonormal basis with* e_3 *timelike.*

(iv) $G_{\alpha\beta} = E(1, 1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_2] = u_1, \qquad [u_1, u_4] = -2\alpha(\alpha\beta + 1)u_1, \quad [u_2, u_3] = u_3,$$
$$[u_2, u_4] = \beta u_1, \quad [u_3, u_4] = \alpha u_3,$$

where $\{u_1, u_2, u_3, u_4\}$ *is a pseudo-orthonormal basis with* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$*, and the parameters* $\alpha > 0$ *and* $\alpha\beta \notin \left\{-2, -1, -\frac{1}{2}\right\}$.

Remark 1.3 Left-invariant Ricci solitons corresponding to G_α in Assertion (i) are steady and the left-invariant soliton vector field is defined by $X = X_1 e_1 + e_4$ if the parameter $\alpha = 0$, and by $X = \frac{1}{2}\left(\alpha + \varepsilon\sqrt{4 - 2\alpha^2}\right) e_4$ otherwise. Moreover, the Ricci operator has eigenvalues

$$\xi_1 = 0, \qquad \xi_2 = -\alpha\left(\alpha + \varepsilon\left(4 - 2\alpha^2\right)^{\frac{1}{2}}\right),$$
$$\xi_3 = \alpha^2 - 2 - \varepsilon\alpha\left(1 - \tfrac{\alpha^2}{2}\right)^{\frac{1}{2}} + \left(\alpha^2 - 4 - 2\varepsilon\alpha\left(4 - 2\alpha^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},$$
$$\xi_4 = \alpha^2 - 2 - \varepsilon\alpha\left(1 - \tfrac{\alpha^2}{2}\right)^{\frac{1}{2}} - \left(\alpha^2 - 4 - 2\varepsilon\alpha\left(4 - 2\alpha^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Hence the Ricci curvatures are $\{0, 0, -2 \pm 2i\}$ if $\alpha = 0$, $\{0, \lambda, \alpha \pm \beta i\}$ with $\lambda\alpha\beta \neq 0$ if $0 < \alpha < \sqrt{2}$, and $\{0, -2, \pm\sqrt{2}\, i\}$ if $\alpha = \sqrt{2}$.

Left-invariant Ricci solitons corresponding to G_α in Assertion (ii) are steady and their left-invariant soliton vector field is defined by $X = X_1u_1 - X_1\alpha u_3 - \frac{1}{2}\alpha u_4$. Moreover, their Ricci operator is three-step nilpotent.

Left-invariant Ricci solitons corresponding to Assertion (iii) are steady and their left-invariant soliton vector field is defined by $X = -\frac{1}{2}e_1 + \frac{3}{2}e_4$. Moreover, their Ricci operator has eigenvalues $\{0, -2, -2 \pm \sqrt{6}\,i\}$.

Left-invariant Ricci solitons corresponding to $G_{\alpha\beta}$ in Assertion (iv) are expanding with $\mu = -(2(\alpha\beta+1)^2+1)\alpha^2$ and their left-invariant soliton vector field is defined by $X = X_1u_1 + X_2u_2 + X_4u_4$, where

$$\begin{aligned} X_1 &= \tfrac{1}{2(2\alpha\beta+1)}(\alpha\beta+2)(2(\alpha\beta+1)\alpha\beta-1),\\ X_2 &= \tfrac{1}{2\alpha\beta+1}(\alpha\beta+2)(2(\alpha\beta+2)\alpha\beta+3)\alpha^2,\\ X_4 &= \tfrac{1}{2\alpha\beta+1}(\alpha\beta+2)^2\,\alpha\,. \end{aligned}$$

Moreover, the Ricci operator is diagonalizable with non-zero real eigenvalues

$$\begin{aligned} &\xi_1 = \xi_2 = -(2\alpha\beta+1)(\alpha\beta+1)\alpha^2,\\ &\xi_3 = (2\alpha\beta+1)\alpha^2, \qquad \xi_4 = -(2(\alpha\beta+2)\alpha\beta+3)\alpha^2\,. \end{aligned}$$

Remark 1.4 Let $(G_1, \langle,\rangle_1)$ and $(G_2, \langle,\rangle_2)$ be two Lorentzian Lie groups with non-zero scalar curvatures. If $(G_1, \langle,\rangle_1)$ and $(G_2, \langle,\rangle_2)$ are homothetic, then one has that $\tau_1^{-2}\|R_1\|^2 = \tau_2^{-2}\|R_2\|^2$ and $\tau_1^{-2}\|W_1\|^2 = \tau_2^{-2}\|W_2\|^2$, where R_i and W_i denote the curvature tensor and the Weyl conformal curvature tensor for $i = 1, 2$, respectively. We use the quadratic scalar curvature invariants to show that left-invariant metrics in different assertions in Theorem 1.2 correspond to distinct homothetic classes. It also follows that different values of the parameter in Assertion (i) determine distinct homothetic classes. Metrics in Assertion (iv) with different $\alpha\beta$ correspond to distinct homothetic classes.

Remark 1.5 Locally symmetric Lorentzian spaces which are neither of constant sectional curvature nor a Cahen-Wallach symmetric space split as a product [7]. Left-invariant symmetric Ricci solitons which are neither Einstein nor a plane wave are locally isometric to $\mathbb{L}^2 \times N(c)$, where $N(c)$ is a surface of constant curvature, and correspond to one of the following Lie groups:

- $G_{\alpha\beta}$ in Assertion (iv) of Theorem 1.2 for $\alpha\beta = -1$, as discussed in Sect. 2.4.1.
- The Lie group $H^3 \rtimes \mathbb{R}$ determined by the Lie algebra

  $$[u_1, u_2] = \lambda_1 u_1,\ \ [u_1, u_4] = -\tfrac{\gamma_3\lambda_1^2}{\gamma_4}u_1,\ \ [u_2, u_4] = \gamma_3\lambda_1^2 u_3,\ \ [u_3, u_4] = \gamma_4\lambda_1 u_3,$$

  with $\lambda_1\gamma_4 \neq 0$, where $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$. It is a expanding Ricci soliton with $\mu = -\lambda_1^2$ and left-invariant soliton vector field $X = -\frac{\gamma_3\lambda_1^2}{\gamma_4^2}u_2 + \frac{\gamma_3^2\lambda_1^3}{2\gamma_4^3}u_3 - \frac{\lambda_1}{\gamma_4}u_4$, as discussed in Sect. 4.2.2.3.

Remark 1.6 The Bach tensor of a four-dimensional manifold is defined by $\mathfrak{B} = \operatorname{div}_1 \operatorname{div}_4 W + \frac{1}{2}W[\rho]$ (see [23]). Four-dimensional Bach-flat metrics are conformally invariant and Bach-flatness is a necessary condition to be conformally Einstein. Left-invariant metrics in Theorem 1.2 are Bach-flat if and only if they correspond to Assertion (iv) with $\alpha\beta = -\frac{5}{4}$. Furthermore, in this case the vector field $X = \frac{3}{2}u_1 - \frac{3\alpha}{2}u_4$ is locally a gradient and satisfies $\operatorname{div}_4 W + \frac{1}{2}W(\cdot,\cdot,\cdot,X) = 0$. A straightforward calculation shows that the Weyl operator acting on the space of two-forms has non-zero eigenvalues and thus the metric is weakly-generic. Hence it is conformally Einstein (see [20] for more information).

1.5 Left-invariant metrics and Gröbner basis

Connected and simply connected four-dimensional Lie groups are either products $SU(2) \times \mathbb{R}$, $\widetilde{SL}(2,\mathbb{R}) \times \mathbb{R}$, or one of the solvable semi-direct extensions of three-dimensional unimodular Lie groups $\widetilde{E}(2) \rtimes \mathbb{R}$, $E(1,1) \rtimes \mathbb{R}$, $H^3 \rtimes \mathbb{R}$ or $\mathbb{R}^3 \rtimes \mathbb{R}$, where $\widetilde{E}(2)$, $E(1,1)$, H^3 and $\mathbb{R}^3$ denote the Euclidean, the Poincaré, the Heisenberg and the Abelian three-dimensional Lie algebras, respectively. Since our purpose is to investigate left-invariant Ricci solitons, we work at the purely algebraic level, and therefore we restrict to the corresponding Lie algebras. Left-invariant Riemannian metrics are described, using the work of Milnor [26], in terms of the corresponding derivations on the three-dimensional unimodular Lie subalgebras. The Lorentzian situation is more subtle due to the fact that the restriction of the metric to the three-dimensional subalgebras $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{e}(2)$, $\mathfrak{e}(1,1)$, $\mathfrak{h}$ or $\mathfrak{r}^3$ may be a positive definite, Lorentzian or degenerate inner product. We follow [8] and consider separately the three possibilities above.

Let $(G, \langle,\rangle)$ be a four-dimensional Lie group and let X be a left-invariant vector field on G. Then $(G, \langle,\rangle, X, \mu)$ is a left-invariant Ricci soliton if and only if the symmetric tensor field $\frac{1}{2}\mathfrak{P} = \mathcal{L}_X\langle,\rangle + \rho - \mu\langle,\rangle$ vanishes identically. It is now immediate, since the vector field X is left-invariant, that the condition $\mathfrak{P} = 0$ equals to a system of polynomial equations on the structure constants which one has to solve in order to obtain a complete classification. When the system under consideration is simple, it is an elementary problem to find all common roots, but if the number of equations, unknowns and their degrees increase, it may become a quite unmanageable task. Given a set $\mathcal{S}$ of polynomials $\mathfrak{P}_{ij} \in \mathbb{R}[x_1,\ldots,x_n]$, an n-tuple of real numbers $\mathbf{a} = (a_1,\ldots,a_n)$ is a solution of $\mathcal{S}$ if and only if $\mathfrak{P}_{ij}(\mathbf{a}) = 0$ for all i, j. It is immediate to recognize that $\mathbf{a}$ is a solution of $\mathcal{S}$ if and only if it is a solution of $\mathcal{I} = \langle\mathfrak{P}_{ij}\rangle$, the ideal generated by the $\mathfrak{P}_{ij}$: if two sets of polynomials generate the same ideal, the corresponding zero sets must be identical. The theory of Gröbner basis provides a well-known strategy to solve rather large polynomial systems obtaining "better" polynomials that belong to the ideal generated by the initial polynomial system. We make use of Gröbner basis to show non-existence results in some cases (see [12, 13] for mor information on Gröbner basis).

2 Extensions of Lorentzian Lie groups

Let $(G, \langle , \rangle)$ be a four-dimensional Lorentzian Lie group $G_3 \rtimes \mathbb{R}$ so that the restriction of the metric to the three-dimensional subalgebra $\mathfrak{g}_3$ is Lorentzian. Three-dimensional unimodular Lie algebras are completely described by using a Milnor type frame associated to the self-dual structure tensor L given by $L(X \times Y) = [X, Y]$, where "$\times$" denotes the vector-cross product $\langle X \times Y, Z\rangle = \det(X, Y, Z)$. Self-duality of L ensures the existence of an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}_3$ diagonalizing the structure tensor in the positive definite case [26]. If the inner product is of Lorentzian signature, then L may have non-trivial Jordan normal form as follows (see, for example [27]).

Ia. L is real diagonalizable. Hence there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where we assume e_3 to be timelike, so that $L(e_i) = \lambda_i e_i$.

Ib. L has complex eigenvalues. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$, where we assume e_3 to be timelike, so that

$$L = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}, \quad \beta \neq 0\,.$$

II. L has a double root of its minimal polynomial. Then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ so that

$$L = \begin{pmatrix} \lambda_1 & 0 & 0 \\ \varepsilon & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \varepsilon = \pm 1, \quad \text{where} \quad \langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1\,.$$

III. L has a triple root of its minimal polynomial. Then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ so that

$$L = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \quad \text{where} \quad \langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1\,.$$

In what follows, we set $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ and L denotes the structure operator of the unimodular subalgebra $\mathfrak{g}_3$. We follow the work of Rahmani [29] to describe Lorentzian left-invariant metrics on $\mathfrak{g}_3$, and to analyse the existence of left-invariant Ricci solitons on each one of the possibilities above. It follows that all left-invariant metrics in Theorem 1.2 are realized as extensions of unimodular Lorentzian Lie groups.

2.1 The structure operator L is diagonalizable

There exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with e_3 timelike, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, so that

$$[e_1, e_2] = -\lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad \underset{(i=1,2,3)}{[e_i, e_4]} = \sum_{j=1}^{3} \alpha_i^j e_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ depending on the eigenvalues λ_i. The Jacobi identity leads to the following different possibilities.

2.1.1 Structure operator with non-zero eigenvalues: metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$

Assume $\lambda_1\lambda_2\lambda_3 \neq 0$. Then left-invariant metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ or $SU(2) \times \mathbb{R}$ are described by the corresponding Lie algebra structure

$$\begin{aligned}
&[e_1, e_2] = -\lambda_3 e_3, \; [e_1, e_3] = -\lambda_2 e_2, && [e_1, e_4] = \gamma_1\lambda_2 e_2 + \gamma_2\lambda_3 e_3,\\
&[e_2, e_3] = \lambda_1 e_1, \quad [e_2, e_4] = -\gamma_1\lambda_1 e_1 + \gamma_3\lambda_3 e_3, && [e_3, e_4] = \gamma_2\lambda_1 e_1 + \gamma_3\lambda_2 e_2,
\end{aligned}$$

where $\{e_1, \dots, e_4\}$ is an orthonormal basis. A straightforward calculation shows that a left-invariant vector field $X = \sum_\ell X_\ell e_\ell$ is a Ricci soliton if and only if the tensor field $\frac{1}{2}\mathfrak{P} = \mathcal{L}_X\langle , \rangle + \rho - \mu\langle , \rangle$ vanishes identically. Equivalently $\{\mathfrak{P}_{ij} = 0\}$, where the polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
\mathfrak{P}_{11} &= (\gamma_1^2 - \gamma_2^2 - 1)\lambda_1^2 - (\gamma_1^2 - 1)\lambda_2^2 + (\gamma_2^2 + 1)\lambda_3^2 - 2\lambda_2\lambda_3 - 2\mu,\\
\mathfrak{P}_{12} &= \gamma_2\gamma_3(\lambda_3^2 - \lambda_1\lambda_2) - 2(X_4\gamma_1 - X_3)(\lambda_1 - \lambda_2),\\
\mathfrak{P}_{13} &= -\gamma_1\gamma_3(\lambda_2^2 - \lambda_1\lambda_3) + 2(X_4\gamma_2 - X_2)(\lambda_1 - \lambda_3),\\
\mathfrak{P}_{14} &= \gamma_3(\lambda_2 - \lambda_3)^2 + 2(X_2\gamma_1 - X_3\gamma_2)\lambda_1,\\
\mathfrak{P}_{22} &= -(\gamma_1^2 - 1)\lambda_1^2 + (\gamma_1^2 - \gamma_3^2 - 1)\lambda_2^2 + (\gamma_3^2 + 1)\lambda_3^2 - 2\lambda_1\lambda_3 - 2\mu,\\
\mathfrak{P}_{23} &= \gamma_1\gamma_2(\lambda_1^2 - \lambda_2\lambda_3) + 2(X_4\gamma_3 + X_1)(\lambda_2 - \lambda_3),\\
\mathfrak{P}_{24} &= -\gamma_2(\lambda_1 - \lambda_3)^2 - 2(X_1\gamma_1 + X_3\gamma_3)\lambda_2,\\
\mathfrak{P}_{33} &= -(\gamma_2^2 + 1)\lambda_1^2 - (\gamma_3^2 + 1)\lambda_2^2 + (\gamma_2^2 + \gamma_3^2 + 1)\lambda_3^2 + 2\lambda_1\lambda_2 + 2\mu,\\
\mathfrak{P}_{34} &= \gamma_1(\lambda_1 - \lambda_2)^2 + 2(X_1\gamma_2 + X_2\gamma_3)\lambda_3,\\
\mathfrak{P}_{44} &= -\gamma_1^2(\lambda_1 - \lambda_2)^2 + \gamma_2^2(\lambda_1 - \lambda_3)^2 + \gamma_3^2(\lambda_2 - \lambda_3)^2 - 2\mu\,.
\end{aligned}$$

Since $\lambda_1\lambda_2\lambda_3 \neq 0$, we may assume $\lambda_1 = 1$ just working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_1}e_i$. Let $\mathcal{I} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \lambda_2, \lambda_3, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded reverse lexicographical order and we get that the polynomials

$$\mathbf{g}_1 = \mu^2 \quad \text{and} \quad \mathbf{g}_2 = 4\lambda_2\lambda_3 + 3(\lambda_2 + \lambda_3 + 1)\mu$$

belong to $\mathcal{G}$. Since $\lambda_2\lambda_3 \neq 0$, there are no left-invariant Ricci solitons in this case.

2.1.2 Structure operator with a zero eigenvalue: metrics on $\tilde{E}(2) \rtimes \mathbb{R}$ or $E(1,1) \rtimes \mathbb{R}$

We distinguish two possibilities depending on the causality of $\ker L$. If $\ker L$ is spacelike then either $\lambda_1 = 0$ or $\lambda_2 = 0$, while if $\ker L$ is timelike then $\lambda_3 = 0$. Next we show that left-invariant Ricci solitons exist only in the flat case.

2.1.2.1. *Structure operator L with spacelike kernel.* Without loss of generality, we assume $\lambda_1 = 0$ and $\lambda_2\lambda_3 \neq 0$. Left-invariant metrics are described by

$$[e_1, e_2] = -\lambda_3 e_3, \qquad [e_1, e_3] = -\lambda_2 e_2, \qquad [e_1, e_4] = \gamma_1 e_2 + \gamma_2 e_3,$$
$$[e_2, e_4] = \gamma_3 e_2 + \gamma_4 \lambda_3 e_3, \; [e_3, e_4] = \gamma_4 \lambda_2 e_2 + \gamma_3 e_3,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis. We focus on the following components of the tensor field $\mathfrak{P}$:

$$\mathfrak{P}_{11} = (\lambda_3 - \lambda_2)^2 - \gamma_1^2 + \gamma_2^2 - 2\mu, \qquad \mathfrak{P}_{14} = \gamma_4(\lambda_3 - \lambda_2)^2,$$
$$\mathfrak{P}_{22} = -(\gamma_4^2 + 1)(\lambda_2^2 - \lambda_3^2) + \gamma_1^2 - 4(\gamma_3 - X_4)\gamma_3 - 2\mu,$$
$$\mathfrak{P}_{33} = -(\gamma_4^2 + 1)(\lambda_2^2 - \lambda_3^2) + \gamma_2^2 + 4(\gamma_3 - X_4)\gamma_3 + 2\mu,$$
$$\mathfrak{P}_{44} = \gamma_4^2(\lambda_3 - \lambda_2)^2 - \gamma_1^2 + \gamma_2^2 - 4\gamma_3^2 - 2\mu\,.$$

One easily checks that $\mathfrak{P}_{11} + \gamma_4\mathfrak{P}_{14} - \mathfrak{P}_{44} = (\lambda_2 - \lambda_3)^2 + 4\gamma_3^2$ and therefore $\lambda_3 = \lambda_2$ and $\gamma_3 = 0$. Now, we have $\mathfrak{P}_{22} + \mathfrak{P}_{33} = \gamma_1^2 + \gamma_2^2$ which implies $\gamma_1 = \gamma_2 = 0$ and the metric is flat.

2.1.2.2. *Structure operator L with timelike kernel.*
If $\lambda_3 = 0$ and $\lambda_1\lambda_2 \neq 0$ then left-invariant metrics are described by

$$[e_1, e_3] = -\lambda_2 e_2, \qquad [e_1, e_4] = \gamma_1 e_1 + \gamma_2\lambda_2 e_2, \; [e_2, e_3] = \lambda_1 e_1,$$
$$[e_2, e_4] = -\gamma_2\lambda_1 e_1 + \gamma_1 e_2, \; [e_3, e_4] = \gamma_3 e_1 + \gamma_4 e_2,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis. We get the following components of the tensor field $\mathfrak{P}$:

$$\mathfrak{P}_{11} = (\gamma_2^2 - 1)(\lambda_1^2 - \lambda_2^2) - \gamma_3^2 - 4(\gamma_1 - X_4)\gamma_1 - 2\mu, \qquad \mathfrak{P}_{34} = \gamma_2(\lambda_1 - \lambda_2)^2,$$
$$\mathfrak{P}_{33} = -(\lambda_1 - \lambda_2)^2 - \gamma_3^2 - \gamma_4^2 + 2\mu, \qquad \mathfrak{P}_{44} = -\gamma_2^2(\lambda_1 - \lambda_2)^2 - 4\gamma_1^2 + \gamma_3^2 + \gamma_4^2 - 2\mu\,.$$

It now follows that $\mathfrak{P}_{33} + \gamma_2\mathfrak{P}_{34} + \mathfrak{P}_{44} = -(\lambda_1 - \lambda_2)^2 - 4\gamma_1^2$ and thus $\lambda_2 = \lambda_1$ and $\gamma_1 = 0$. Now, $\mathfrak{P}_{11} + \mathfrak{P}_{33} = -2\gamma_3^2 - \gamma_4^2$ which implies $\gamma_3 = \gamma_4 = 0$ and the metric is flat as in the previous case.

2.1.3 Structure operator of rank one: metrics on $H^3 \rtimes \mathbb{R}$

We consider separately the cases when the restriction of the metric to $\ker L$ is positive definite ($\lambda_3 \neq 0$) or Lorentzian ($\lambda_3 = 0$). We make use of Gröbner basis to show non-existence of left-invariant Ricci solitons in both cases.

2.1.3.1. *Structure operator L with positive definite kernel.*
Setting $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$ left-invariant metrics are described by

$$[e_1, e_2] = -\lambda_3 e_3, \qquad [e_1, e_4] = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,$$
$$[e_2, e_4] = \gamma_4 e_1 + \gamma_5 e_2 + \gamma_6 e_3, \; [e_3, e_4] = (\gamma_1 + \gamma_5) e_3,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis. Now, $X \in \mathfrak{h} \rtimes \mathbb{R}$ determines a left-invariant Ricci soliton if and only if the system of polynomial equations $\{\mathfrak{P}_{ij} = 0\}$ is satisfied, where the polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
\mathfrak{P}_{11} &= \lambda_3^2 - 4\gamma_1^2 - \gamma_2^2 + \gamma_3^2 + \gamma_4^2 - 4\gamma_1\gamma_5 + 4X_4\gamma_1 - 2\mu,\\
\mathfrak{P}_{12} &= -\gamma_1\gamma_2 - 3\gamma_1\gamma_4 - 3\gamma_2\gamma_5 + \gamma_3\gamma_6 - \gamma_4\gamma_5 + 2X_4(\gamma_2 + \gamma_4),\\
\mathfrak{P}_{13} &= 2X_2\lambda_3 + 2\gamma_1\gamma_3 + 3\gamma_3\gamma_5 - \gamma_4\gamma_6 - 2X_4\gamma_3,\\
\mathfrak{P}_{14} &= \gamma_6\lambda_3 - 2X_1\gamma_1 - 2X_2\gamma_4,\\
\mathfrak{P}_{22} &= \lambda_3^2 + \gamma_2^2 - \gamma_4^2 - 4\gamma_5^2 + \gamma_6^2 - 4\gamma_1\gamma_5 + 4X_4\gamma_5 - 2\mu,\\
\mathfrak{P}_{23} &= -2X_1\lambda_3 + 3\gamma_1\gamma_6 - \gamma_2\gamma_3 + 2\gamma_5\gamma_6 - 2X_4\gamma_6,\\
\mathfrak{P}_{24} &= -\gamma_3\lambda_3 - 2X_1\gamma_2 - 2X_2\gamma_5, \qquad \mathfrak{P}_{34} = 2\{X_3(\gamma_1 + \gamma_5) + X_1\gamma_3 + X_2\gamma_6\},\\
\mathfrak{P}_{33} &= \lambda_3^2 + 4\gamma_1^2 + \gamma_3^2 + 4\gamma_5^2 + \gamma_6^2 + 8\gamma_1\gamma_5 - 4X_4(\gamma_1 + \gamma_5) + 2\mu,\\
\mathfrak{P}_{44} &= -4\gamma_1^2 - \gamma_2^2 + \gamma_3^2 - \gamma_4^2 - 4\gamma_5^2 + \gamma_6^2 - 4\gamma_1\gamma_5 - 2\gamma_2\gamma_4 - 2\mu\,.
\end{aligned}$$

Since $\lambda_3 \neq 0$, we may assume $\lambda_3 = 1$ just working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_3} e_i$. Let $\mathcal{I}_1 \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order and get that the polynomials

$$\begin{aligned}
\mathbf{g}_{11} &= X_3\Big(2617344X_4^8 + 13139712X_4^6 + 18557248X_4^4 + 7213356X_4^2 + 61803\Big),\\
\mathbf{g}_{12} &= X_4\Big(83755008X_4^{14} + 776429568X_4^{12} + 2689679360X_4^{10} + 4517104000X_4^8\\
&\quad +4237066048X_4^6 + 2362718304X_4^4 + 591574590X_4^2 + 5006043\Big)
\end{aligned}$$

belong to $\mathcal{G}_1$. Thus, $X_3 = X_4 = 0$. Next, we compute a second Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1 \cup \{X_3, X_4\} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ with respect to the lexicographical order, obtaining that the polynomial $\mathbf{g}_{21} = X_1^2 + X_2^2$ belongs to $\mathcal{G}_2$, which shows that $X = 0$ and Ricci solitons reduce to Einstein metrics, which do not exist in this case.

2.1.3.2. *Structure operator L with Lorentzian kernel.* In this case $\lambda_3 = 0$ and we may assume without loss of generality that $\lambda_1 = 0$ and $\lambda_2 \neq 0$ so that left-invariant metrics are described by

$$[e_1, e_3] = -\lambda_2 e_2, \; [e_1, e_4] = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3,$$
$$[e_2, e_4] = \gamma_4 e_2, \quad [e_3, e_4] = \gamma_5 e_1 + \gamma_6 e_2 - (\gamma_1 - \gamma_4) e_3,$$

where $\{e_1, \ldots, e_4\}$ is an orthonormal basis. Proceeding as in the previous case, one has that the polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
\mathfrak{P}_{11} &= \lambda_2^2 - \gamma_2^2 + \gamma_3^2 - \gamma_5^2 - 4\gamma_1\gamma_4 + 4\gamma_1 X_4 - 2\mu,\\
\mathfrak{P}_{12} &= -2X_3\lambda_2 + \gamma_1\gamma_2 - 3\gamma_2\gamma_4 - \gamma_5\gamma_6 + 2X_4\gamma_2,\\
\mathfrak{P}_{13} &= -2\gamma_1(\gamma_3 + \gamma_5) - \gamma_2\gamma_6 + 3\gamma_3\gamma_4 - \gamma_4\gamma_5 - 2X_4(\gamma_3 - \gamma_5),\\
\mathfrak{P}_{14} &= \gamma_6\lambda_2 - 2(X_1\gamma_1 + X_3\gamma_5), \qquad \mathfrak{P}_{34} = \gamma_2\lambda_2 - 2X_3(\gamma_1 - \gamma_4) + 2X_1\gamma_3,\\
\mathfrak{P}_{22} &= -\lambda_2^2 + \gamma_2^2 - 4\gamma_4^2 - \gamma_6^2 + 4X_4\gamma_4 - 2\mu, \quad \mathfrak{P}_{24} = -2\{X_1\gamma_2 + X_2\gamma_4 + X_3\gamma_6\},\\
\mathfrak{P}_{23} &= 2X_1\lambda_2 - \gamma_1\gamma_6 - \gamma_2\gamma_3 - 2\gamma_4\gamma_6 + 2X_4\gamma_6,\\
\mathfrak{P}_{33} &= -\lambda_2^2 + \gamma_3^2 + 4\gamma_4^2 - \gamma_5^2 - \gamma_6^2 - 4\gamma_1\gamma_4 + 4X_4(\gamma_1 - \gamma_4) + 2\mu,\\
\mathfrak{P}_{44} &= -4\gamma_1^2 - \gamma_2^2 + \gamma_3^2 - 4\gamma_4^2 + \gamma_5^2 + \gamma_6^2 + 4\gamma_1\gamma_4 - 2\gamma_3\gamma_5 - 2\mu\,.
\end{aligned}$$

Since $\lambda_2 \neq 0$, we assume $\lambda_2 = 1$ just working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\lambda_2}e_i$. Let $\mathcal{I}_1 \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}_1$ of $\mathcal{I}_1$ with respect to the lexicographical order so that that the polynomials

$$\begin{aligned}
\mathbf{g}_{11} &= X_2\Big(2617344X_4^8 + 13139712X_4^6 + 18557248X_4^4 + 7213356X_4^2 + 61803\Big),\\
\mathbf{g}_{12} &= X_4\Big(83755008X_4^{14} + 776429568X_4^{12} + 2689679360X_4^{10} + 4517104000X_4^8\\
&\quad +4237066048X_4^6 + 2362718304X_4^4 + 591574590X_4^2 + 5006043\Big)
\end{aligned}$$

belong to $\mathcal{G}_1$. Thus, $X_2 = X_4 = 0$. We compute a second Gröbner basis $\mathcal{G}_2$ of the ideal generated by the polynomials $\mathcal{G}_1 \cup \{X_2, X_4\} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \mu, X_1, X_2, X_3, X_4]$ with respect to the lexicographical order and we get that the polynomial $\mathbf{g}_{21} = \gamma_4^2 + 1$ belongs to $\mathcal{G}_2$, which shows that there are no left-invariant Ricci solitons in this case.

2.1.4 Structure operator with zero eigenvalues: metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$

Since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, any linear map $D : \mathfrak{r}^3 \to \mathfrak{r}^3$ is a derivation. In order to simplify the structure constants, we proceed as follows. Let $\Phi(x, y) = \langle Dx, y\rangle$ be the bilinear form associated to $D(\cdot) = [\cdot, e_4]$, and let $\Phi_s = \frac{1}{2}(\Phi + {}^t\Phi)$ and $\Phi_a = \frac{1}{2}(\Phi - {}^t\Phi)$ be the symmetric and skew-symmetric parts of Φ, respectively. Moreover, let D_{sad} and D_{asad} defined by $\Phi_s(x, y) = \langle D_{sad}x, y\rangle$ and $\Phi_a(x, y) = \langle D_{asad}x, y\rangle$ be the corresponding self-adjoint and anti-self-adjoint endomorphisms. We analyse separately the different Jordan normal forms of D_{sad}.

2.1.4.1. *The self-adjoint part of the derivation* D_{sad} *is diagonalizable.*

In this case, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}^3$, with e_3 timelike, so that

$$D_{sad} = \begin{pmatrix} \eta_1 & 0 & 0\\ 0 & \eta_2 & 0\\ 0 & 0 & \eta_3 \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} 0 & \gamma_1 & \gamma_2\\ -\gamma_1 & 0 & \gamma_3\\ \gamma_2 & \gamma_3 & 0 \end{pmatrix}$$

and therefore left-invariant metrics are described by

$$[e_1, e_4] = \eta_1 e_1 - \gamma_1 e_2 + \gamma_2 e_3,\ [e_2, e_4] = \gamma_1 e_1 + \eta_2 e_2 + \gamma_3 e_3,$$
$$[e_3, e_4] = \gamma_2 e_1 + \gamma_3 e_2 + \eta_3 e_3,$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{r}^3 \rtimes \mathbb{R}$ with e_3 timelike. After a straightforward calculation we get the following polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$:

$$\tilde{\mathfrak{P}}_{11} = -\eta_1(\eta_1 + \eta_2 + \eta_3 - 2X_4) - \mu,\ \tilde{\mathfrak{P}}_{22} = -\eta_2(\eta_1 + \eta_2 + \eta_3 - 2X_4) - \mu,$$
$$\tilde{\mathfrak{P}}_{33} = \eta_3(\eta_1 + \eta_2 + \eta_3 - 2X_4) + \mu,\quad \tilde{\mathfrak{P}}_{44} = -\eta_1^2 - \eta_2^2 - \eta_3^2 - \mu\,.$$

Hence, $\eta_2\tilde{\mathfrak{P}}_{11} - \eta_1\tilde{\mathfrak{P}}_{22} = (\eta_1 - \eta_2)\mu$ and $\eta_3\tilde{\mathfrak{P}}_{11} + \eta_1\tilde{\mathfrak{P}}_{33} = (\eta_1 - \eta_3)\mu$. These relations, together with the expression of $\tilde{\mathfrak{P}}_{44}$, imply that $\eta_1 = \eta_2 = \eta_3 = \kappa$ and a standard calculation shows that the corresponding left-invariant metric has constant sectional curvature $-\kappa^2$.

2.1.4.2. *The self-adjoint part of the derivation D_{sad} has complex eigenvalues.*

If the self-dual part of the derivation, D_{sad}, has complex eigenvalues then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}^3$, with e_3 timelike, so that

$$D_{sad} = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \delta & \nu \\ 0 & -\nu & \delta \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 \\ -\gamma_1 & 0 & \gamma_3 \\ \gamma_2 & \gamma_3 & 0 \end{pmatrix},$$

where $\nu \neq 0$. The corresponding left-invariant metrics are described by

$$[e_1, e_4] = \eta e_1 - \gamma_1 e_2 + \gamma_2 e_3,\qquad [e_2, e_4] = \gamma_1 e_1 + \delta e_2 + (\gamma_3 - \nu)e_3,$$
$$[e_3, e_4] = \gamma_2 e_1 + (\gamma_3 + \nu)e_2 + \delta e_3,$$

and a standard calculation shows that the polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
&\tilde{\mathfrak{P}}_{11} = -\eta^2 - 2(\delta - X_4)\eta - \mu, && \tilde{\mathfrak{P}}_{12} = \gamma_1(\delta - \eta) - \gamma_2\nu,\\
&\tilde{\mathfrak{P}}_{13} = \gamma_2(\delta - \eta) + \gamma_1\nu, && \tilde{\mathfrak{P}}_{14} = -X_1\eta - X_2\gamma_1 - X_3\gamma_2,\\
&\tilde{\mathfrak{P}}_{22} = -2\delta^2 - (\eta - 2X_4)\delta - 2\gamma_3\nu - \mu, && \tilde{\mathfrak{P}}_{23} = -(2\delta + \eta - 2X_4)\nu,\\
&\tilde{\mathfrak{P}}_{24} = X_1\gamma_1 - X_2\delta - X_3(\nu + \gamma_3), && \tilde{\mathfrak{P}}_{33} = 2\delta^2 + (\eta - 2X_4)\delta - 2\gamma_3\nu + \mu,\\
&\tilde{\mathfrak{P}}_{34} = X_1\gamma_2 - X_2(\nu - \gamma_3) + X_3\delta, && \tilde{\mathfrak{P}}_{44} = -2\delta^2 - \eta^2 + 2\nu^2 - \mu\,.
\end{aligned}$$

We work with the homothetic metric determined by $\hat{e}_i = \frac{1}{\nu}e_i$. Since $\gamma_2\tilde{\mathfrak{P}}_{12} - \gamma_1\tilde{\mathfrak{P}}_{13} = -\gamma_1^2 - \gamma_2^2$ and $\tilde{\mathfrak{P}}_{22} + \tilde{\mathfrak{P}}_{33} = -4\gamma_3$, it follows that $\gamma_1 = \gamma_2 = \gamma_3 = 0$. Now, $\tilde{\mathfrak{P}}_{23} = -2\delta - \eta + 2X_4$, $\tilde{\mathfrak{P}}_{24} - \delta\,\tilde{\mathfrak{P}}_{34} = -X_3(\delta^2 + 1)$, and $\delta\,\tilde{\mathfrak{P}}_{24} + \tilde{\mathfrak{P}}_{34} = -X_2(\delta^2 + 1)$ lead to $X_2 = X_3 = 0$ and $X_4 = \delta + \frac{1}{2}\eta$. Thus, the system of polynomial equations $\{\tilde{\mathfrak{P}}_{ij} = 0\}$ reduces to

$$\tilde{\mathfrak{P}}_{11} = \tilde{\mathfrak{P}}_{22} = -\tilde{\mathfrak{P}}_{33} = -\mu = 0,\quad \tilde{\mathfrak{P}}_{14} = -X_1\eta = 0,$$
$$\tilde{\mathfrak{P}}_{44} = -2\delta^2 - \eta^2 - \mu + 2 = 0,$$

which shows that $X_1\eta = 0$ and the left-invariant metric given by

$$[e_1, e_4] = \eta e_1, \quad [e_2, e_4] = \varepsilon\sqrt{1 - \tfrac{1}{2}\eta^2}\, e_2 - e_3, \quad [e_3, e_4] = e_2 + \varepsilon\sqrt{1 - \tfrac{1}{2}\eta^2}\, e_3,$$

with $-\sqrt{2} \le \eta \le \sqrt{2}$ and $\varepsilon^2 = 1$ is a left-invariant steady Ricci soliton which corresponds to Assertion (i) in Theorem 1.2. Moreover, the left-invariant soliton vector field is given by $X = X_1 e_1 + \varepsilon\, e_4$ if $\eta = 0$, and $X = \frac{1}{2}\left(\eta + \varepsilon\sqrt{4 - 2\eta^2}\right) e_4$ if $\eta \neq 0$.

Note that $(e_1, e_2, e_3, e_4) \mapsto (e_1, e_2, -e_3, -e_4)$ defines an isometry interchanging (η, ε) and $(-\eta, -\varepsilon)$, and hence we may assume $0 \le \eta \le \sqrt{2}$. Moreover, for $\eta = 0$, the same isometry interchanges $\varepsilon = 1$ and $\varepsilon = -1$. A straightforward calculation shows that the above metrics are never symmetric and they are Einstein if and only if $\eta = -\frac{2\varepsilon}{\sqrt{3}}$, in which case corresponds to Assertion (i) in Theorem 1.1.

2.1.4.3. *The self-adjoint part of the derivation D_{sad} has a double root.*

In this case, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathfrak{r}^3$, with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$, so that

$$D_{sad} = \begin{pmatrix} \eta_1 & 0 & 0 \\ \varepsilon & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 \\ 0 & -\gamma_1 & \gamma_3 \\ -\gamma_3 & -\gamma_2 & 0 \end{pmatrix},$$

where $\varepsilon^2 = 1$. Thus, corresponding left-invariant metrics are described by

$$\begin{aligned} &[u_1, u_4] = (\eta_1 + \gamma_1)u_1 + \varepsilon u_2 - \gamma_3 u_3,\ [u_2, u_4] = (\eta_1 - \gamma_1)u_2 - \gamma_2 u_3, \\ &[u_3, u_4] = \gamma_2 u_1 + \gamma_3 u_2 + \eta_2 u_3, \end{aligned}$$

where $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$. We will consider the following polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$:

$$\begin{aligned} &\tilde{\mathfrak{P}}_{11} = -\varepsilon(2\eta_1 + \eta_2 + 2\gamma_1 - 2X_4), \quad \tilde{\mathfrak{P}}_{12} = -\eta_1(2\eta_1 + \eta_2) + 2X_4\eta_1 - \mu, \\ &\tilde{\mathfrak{P}}_{13} = -\gamma_3(\eta_1 - \eta_2) - \varepsilon\gamma_2, \qquad\qquad \tilde{\mathfrak{P}}_{23} = -\gamma_2(\eta_1 - \eta_2), \\ &\tilde{\mathfrak{P}}_{33} = -\eta_2(2\eta_1 + \eta_2) + 2X_4\eta_2 - \mu,\ \tilde{\mathfrak{P}}_{44} = -2\eta_1^2 - \eta_2^2 - \mu\,. \end{aligned}$$

One easily checks that

$$\begin{aligned} &\gamma_2\tilde{\mathfrak{P}}_{13} - \gamma_3\tilde{\mathfrak{P}}_{23} = -\varepsilon\gamma_2^2, \qquad \eta_2\tilde{\mathfrak{P}}_{12} - \eta_1\tilde{\mathfrak{P}}_{33} = (\eta_1 - \eta_2)\mu, \\ &\varepsilon\eta_1\tilde{\mathfrak{P}}_{11} - \tilde{\mathfrak{P}}_{33} + \tilde{\mathfrak{P}}_{44} = -\eta_1(4\eta_1 - \eta_2 + 2\gamma_1) + 2X_4(\eta_1 - \eta_2), \end{aligned}$$

and since $\tilde{\mathfrak{P}}_{44} = -2\eta_1^2 - \eta_2^2 - \mu$ it follows that $\gamma_2 = 0$, $\eta_2 = \eta_1$ and $\eta_1(3\eta_1 + 2\gamma_1) = 0$.

If $3\eta_1 + 2\gamma_1 = 0$ then the resulting left-invariant metric is Einstein and it corresponds to Assertion (ii) in Theorem 1.1. Finally, if $\eta_1 = \gamma_2 = \eta_2 = 0$ and $\gamma_1 \neq 0$, then the left-invariant metric corresponds to

$$[u_1, u_4] = \gamma_1 u_1 + \varepsilon u_2 - \gamma_3 u_3, \quad [u_2, u_4] = -\gamma_1 u_2, \quad [u_3, u_4] = \gamma_3 u_2, \tag{2}$$

and u_2 is a recurrent null vector. Furthermore, a straightforward calculation shows that the curvature tensor is transversally flat (i.e., $R(Y, Z) = 0$ for all $Y, Z \in u_2^\perp$) and the Ricci operator is isotropic ($\rho_{11} = -2\varepsilon\gamma_1$ is the only non-zero component of the Ricci tensor). Hence the underlying structure is that of a pp-wave which is neither symmetric nor locally conformally flat.

2.1.4.4. *The self-adjoint part of the derivation D_{sad} has a triple root.*
Let $\{u_1, u_2, u_3\}$ be a pseudo-orthonormal basis of $\mathfrak{r}^3$, with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$, so that

$$D_{sad} = \begin{pmatrix} \eta & 0 & 1 \\ 0 & \eta & 0 \\ 0 & 1 & \eta \end{pmatrix}, \quad D_{asad} = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 \\ 0 & -\gamma_1 & \gamma_3 \\ -\gamma_3 & -\gamma_2 & 0 \end{pmatrix}.$$

Therefore the corresponding left-invariant metrics are given by

$$\begin{aligned} &[u_1, u_4] = (\eta + \gamma_1)u_1 - \gamma_3 u_3, \qquad [u_2, u_4] = (\eta - \gamma_1)u_2 - (\gamma_2 - 1)u_3, \\ &[u_3, u_4] = (\gamma_2 + 1)u_1 + \gamma_3 u_2 + \eta u_3, \end{aligned}$$

where $\{u_1, u_2, u_3, u_4\}$ is a pseudo-orthonormal basis of of $\mathfrak{r}^3 \rtimes \mathbb{R}$, with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$. A straightforward calculation shows that the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\begin{aligned} &\tilde{\mathfrak{P}}_{12} = -3\eta^2 + 2X_4\eta + \gamma_3 - \mu, && \tilde{\mathfrak{P}}_{14} = -X_2(\eta - \gamma_1) - X_3\gamma_3, \\ &\tilde{\mathfrak{P}}_{22} = 2\gamma_2, && \tilde{\mathfrak{P}}_{23} = -3\eta + \gamma_1 + 2X_4, \\ &\tilde{\mathfrak{P}}_{24} = -X_1(\eta + \gamma_1) - X_3(\gamma_2 + 1), && \tilde{\mathfrak{P}}_{33} = -3\eta^2 + 2X_4\eta - 2\gamma_3 - \mu, \\ &\tilde{\mathfrak{P}}_{34} = -X_3\eta + X_2(\gamma_2 - 1) + X_1\gamma_3, && \tilde{\mathfrak{P}}_{44} = -3\eta^2 - \mu\,. \end{aligned}$$

Since $\tilde{\mathfrak{P}}_{22} = 2\gamma_2$, $\tilde{\mathfrak{P}}_{12} - \tilde{\mathfrak{P}}_{33} = 3\gamma_3$ and $\tilde{\mathfrak{P}}_{12} - \eta\,\tilde{\mathfrak{P}}_{23} - \tilde{\mathfrak{P}}_{44} = \eta(3\eta - \gamma_1) + \gamma_3$, it follows that $\gamma_2 = \gamma_3 = 0$ and $\eta(3\eta - \gamma_1) = 0$.

Now, if $3\eta - \gamma_1 = 0$ then the corresponding left-invariant metric is Einstein, and it corresponds to Assertion (iii) in Theorem 1.1 if $\gamma_1 = 3\eta \neq 0$. (The case where $\eta = \gamma_1 = 0$ corresponds to a Ricci-flat plane wave).

If $\eta = 0$ and $\gamma_1 \neq 0$, then a straightforward calculation shows that left-invariant metrics, which are given by

$$[u_1, u_4] = \gamma_1 u_1, \quad [u_2, u_4] = -\gamma_1 u_2 + u_3, \quad [u_3, u_4] = u_1,$$

are neither Einstein nor symmetric. Moreover, the system of polynomial equations $\{\tilde{\mathfrak{P}}_{ij} = 0\}$ reduces to

$$\begin{aligned} &\tilde{\mathfrak{P}}_{12} = \tilde{\mathfrak{P}}_{33} = \tilde{\mathfrak{P}}_{44} = -\mu = 0, \quad \tilde{\mathfrak{P}}_{14} = X_2\gamma_1 = 0, \quad \tilde{\mathfrak{P}}_{23} = \gamma_1 + 2X_4 = 0, \\ &\tilde{\mathfrak{P}}_{24} = -X_1\gamma_1 - X_3 = 0, \qquad \tilde{\mathfrak{P}}_{34} = -X_2 = 0, \end{aligned}$$

and it defines a left-invariant steady Ricci soliton with left-invariant soliton vector field $X = X_1u_1 - X_1\gamma_1u_3 - \frac{1}{2}\gamma_1u_4$.

Finally, note that $(u_1, u_2, u_3, u_4) \mapsto (-u_1, -u_2, u_3, -u_4)$ defines an isometry interchanging γ_1 and $-\gamma_1$ and hence, without loss of generality, we can restrict the parameter γ_1 to $\gamma_1 > 0$. Setting $\alpha = \gamma_1$, this case corresponds to Assertion (ii) in Theorem 1.2.

2.2 The structure operator *L* has complex eigenvalues

If the structure operator L is of type Ib then there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with e_3 timelike, where $\mathfrak{g}_3 = \operatorname{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \operatorname{span}\{e_4\}$, so that

$$[e_1, e_2] = -\beta e_2 - \alpha e_3, \quad [e_1, e_3] = -\alpha e_2 + \beta e_3, \quad [e_2, e_3] = \lambda e_1, \quad \underset{(i=1,2,3)}{[e_i, e_4]} = \sum_{j=1}^{3} \alpha_i^j e_j,$$

with $\beta \neq 0$, for certain $\alpha_i^j \in \mathbb{R}$. Next we consider separately the cases when the real eigenvalue $\lambda = 0$ and $\lambda \neq 0$.

2.2.1 Case of zero real eigenvalue: metrics on $E(1, 1) \rtimes \mathbb{R}$

If $\lambda = 0$ then the corresponding metrics are given by

$$\begin{aligned} &[e_1, e_2] = -\beta e_2 - \alpha e_3, \qquad [e_1, e_3] = -\alpha e_2 + \beta e_3, \qquad [e_1, e_4] = \gamma_1 e_2 + \gamma_2 e_3, \\ &[e_2, e_4] = 2\gamma_3 \beta e_2 + (\gamma_3 - \gamma_4)\alpha e_3, \qquad [e_3, e_4] = (\gamma_3 - \gamma_4)\alpha e_2 + 2\gamma_4 \beta e_3, \end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{e}(1, 1) \rtimes \mathfrak{r}$ with e_3 timelike. A straightforward calculation shows that the polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
&\mathfrak{P}_{11} = -4\beta^2 - \gamma_1^2 + \gamma_2^2 - 2\mu, \qquad \mathfrak{P}_{23} = -4((\gamma_3 - \gamma_4)^2 + 1)\alpha\beta - \gamma_1\gamma_2, \\
&\mathfrak{P}_{12} = (\gamma_2(\gamma_3 - \gamma_4) - 2X_3)\alpha - 2(\gamma_1(2\gamma_3 + \gamma_4) + X_2)\beta + 2X_4\gamma_1, \\
&\mathfrak{P}_{13} = -(\gamma_1(\gamma_3 - \gamma_4) - 2X_2)\alpha + 2(\gamma_2(\gamma_3 + 2\gamma_4) - X_3)\beta - 2X_4\gamma_2, \\
&\mathfrak{P}_{14} = -4(\gamma_3 - \gamma_4)\beta^2, \qquad \mathfrak{P}_{44} = -8(\gamma_3^2 + \gamma_4^2)\beta^2 - \gamma_1^2 + \gamma_2^2 - 2\mu, \\
&\mathfrak{P}_{22} = -8\gamma_3(\gamma_3 + \gamma_4)\beta^2 + 4(2X_4\gamma_3 + X_1)\beta + \gamma_1^2 - 2\mu, \\
&\mathfrak{P}_{24} = -(\gamma_2 + 2X_3(\gamma_3 - \gamma_4))\alpha + (\gamma_1 - 4X_2\gamma_3)\beta - 2X_1\gamma_1, \\
&\mathfrak{P}_{33} = 8(\gamma_3 + \gamma_4)\gamma_4\beta^2 - 4(2X_4\gamma_4 - X_1)\beta + \gamma_2^2 + 2\mu, \\
&\mathfrak{P}_{34} = (\gamma_1 + 2X_2(\gamma_3 - \gamma_4))\alpha + (\gamma_2 + 4X_3\gamma_4)\beta + 2X_1\gamma_2 .
\end{aligned}$$

Since $\beta \neq 0$, we may assume $\beta = 1$ just working with the homothetic metric determined by $\hat{e}_i = \frac{1}{\beta} e_i$. Using the expressions above for $\mathfrak{P}_{14}$, $\mathfrak{P}_{11}$, $\mathfrak{P}_{23}$ and $\mathfrak{P}_{44}$, together with $\mathfrak{P}_{22} + \mathfrak{P}_{33} = \gamma_1^2 + \gamma_2^2 - 8(\gamma_3^2 - \gamma_4^2 - X_4(\gamma_3 - \gamma_4) - X_1)$, we get

$$\gamma_4 = \gamma_3, \quad \mu = -\tfrac{1}{2}(\gamma_1^2 - \gamma_2^2 + 4), \quad \alpha = -\tfrac{1}{4}\gamma_1\gamma_2, \quad \gamma_3 = \tfrac{\varepsilon_1}{2}, \quad X_1 = -\tfrac{1}{8}(\gamma_1^2 + \gamma_2^2),$$

where $\varepsilon_1^2 = 1$. Now, one easily checks that

$$\begin{aligned}
\varepsilon_1 \mathfrak{P}_{12} - \mathfrak{P}_{24} - \tfrac{1}{4}\gamma_1\gamma_2\mathfrak{P}_{34} + \tfrac{1}{2}\gamma_1\mathfrak{P}_{33} &= \tfrac{1}{16}\gamma_1(\gamma_2^2 - 8)(\gamma_2^2 + 2\gamma_1^2 + 8),\\
\varepsilon_1 \mathfrak{P}_{13} - \tfrac{1}{4}\gamma_1\gamma_2\mathfrak{P}_{24} + \mathfrak{P}_{34} - \tfrac{1}{2}\gamma_2\mathfrak{P}_{33} &= -\tfrac{1}{16}\gamma_2(\gamma_1^2 + 8)(2\gamma_2^2 + \gamma_1^2 - 8),
\end{aligned}$$

from where it follows that $\gamma_1 = 0$ and $\gamma_2 \in \{-2, 0, 2\}$. A standard calculation shows that the corresponding left-invariant metric, which is given by

$$[e_1, e_2] = -e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = \gamma_2 e_3, \quad [e_2, e_4] = \varepsilon_1 e_2, \quad [e_3, e_4] = \varepsilon_1 e_3,$$

is Einstein if and only if $\gamma_2 = 0$ (and locally isometric to a product of two surfaces with the same constant curvature). Hence we take $\gamma_2 = 2\varepsilon_2$, with $\varepsilon_2^2 = 1$, and the system of polynomial equations $\{\mathfrak{P}_{ij} = 0\}$ reduces to

$$\begin{aligned}
&\mathfrak{P}_{12} = -2X_2 = 0, \quad \mathfrak{P}_{13} = -2(X_3 + 2\varepsilon_2 X_4) + 6\varepsilon_1\varepsilon_2 = 0, \mathfrak{P}_{22} = 4\varepsilon_1 X_4 - 6 = 0,\\
&\mathfrak{P}_{24} = -2\varepsilon_1 X_2 = 0, \mathfrak{P}_{33} = -4\varepsilon_1 X_4 + 6 = 0, \qquad \mathfrak{P}_{34} = 2\varepsilon_1 X_3 = 0,
\end{aligned}$$

which shows that $X_2 = X_3 = 0$, $X_4 = \frac{3\varepsilon_1}{2}$, and the left-invariant metric given by

$$[e_1, e_2] = -e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = 2\varepsilon_2 e_3, \quad [e_2, e_4] = \varepsilon_1 e_2, \quad [e_3, e_4] = \varepsilon_1 e_3,$$

is an steady Ricci soliton with left-invariant soliton vector field $X = -\frac{1}{2}e_1 + \frac{3\varepsilon_1}{2}e_4$.

Note that $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_2, -e_3, -e_4)$ is an isometry interchanging $\varepsilon_1 = 1$ and $\varepsilon_1 = -1$, and $(e_1, e_2, e_3, e_4) \mapsto (e_1, -e_2, -e_3, e_4)$ defines an isometry which interchanges $\varepsilon_2 = 1$ and $\varepsilon_2 = -1$. Hence we can set $\varepsilon_1 = \varepsilon_2 = 1$ obtaining Assertion (iii) in Theorem 1.2.

2.2.2 Case of non-zero real eigenvalue: metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$

If $\lambda \neq 0$ then the corresponding left-invariant metrics are given by

$$\begin{aligned}
&[e_1, e_2] = -\beta e_2 - \alpha e_3, \qquad [e_1, e_3] = -\alpha e_2 + \beta e_3, \qquad [e_2, e_3] = \lambda e_1,\\
&[e_1, e_4] = (\alpha^2 + \beta^2)(\gamma_1 e_2 + \gamma_2 e_3), \quad [e_2, e_4] = -(\gamma_1\alpha - \gamma_2\beta)\lambda e_1 + \gamma_3\beta e_2 + \gamma_3\alpha e_3,\\
&[e_3, e_4] = (\gamma_2\alpha + \gamma_1\beta)\lambda e_1 + \gamma_3\alpha e_2 - \gamma_3\beta e_3,
\end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathfrak{r}$ with e_3 timelike.

A straightforward calculation shows that the polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned}
\mathfrak{P}_{11} &= -((\alpha^2 + \beta^2)^2 - (\alpha^2 - \beta^2)\lambda^2)(\gamma_1^2 - \gamma_2^2) - 4\alpha\beta\lambda^2\gamma_1\gamma_2 - 4\beta^2 - \lambda^2 - 2\mu,\\
\mathfrak{P}_{12} &= (2X_4(\alpha^2 + \beta^2 - \alpha\lambda) - (\alpha^2 + \beta^2 + 2\alpha\lambda)\beta\gamma_3)\gamma_1\\
&\quad + (((\alpha^2 + \beta^2)\alpha - (\alpha^2 - \beta^2)\lambda)\gamma_3 + 2X_4\beta\lambda)\gamma_2 - 2(X_3(\alpha - \lambda) + X_2\beta),\\
\mathfrak{P}_{13} &= -(((\alpha^2 + \beta^2)\alpha - (\alpha^2 - \beta^2)\lambda)\gamma_3 - 2X_4\beta\lambda)\gamma_1\\
&\quad - (2X_4(\alpha^2 + \beta^2 - \alpha\lambda) + (\alpha^2 + \beta^2 + 2\alpha\lambda)\beta\gamma_3)\gamma_2 + 2(X_2(\alpha - \lambda) - X_3\beta),\\
\mathfrak{P}_{14} &= 2(X_2\alpha - X_3\beta)\lambda\gamma_1 - 2(X_3\alpha + X_2\beta)\lambda\gamma_2 - 4\beta^2\gamma_3,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{22} &= ((\alpha^2+\beta^2)^2-\alpha^2\lambda^2)\gamma_1^2-\beta^2\lambda^2\gamma_2^2+2\alpha\beta\lambda^2\gamma_1\gamma_2+4X_4\beta\gamma_3\\
&\quad +4X_1\beta-(2\alpha-\lambda)\lambda-2\mu,\\
\mathfrak{P}_{23} &= \alpha\beta(\lambda^2(\gamma_1^2-\gamma_2^2)-4\gamma_3^2)-((\alpha^2+\beta^2)^2-(\alpha^2-\beta^2)\lambda^2)\gamma_1\gamma_2-2(2\alpha-\lambda)\beta,\\
\mathfrak{P}_{24} &= ((\alpha^2+\beta^2)(\beta-2X_1)-\beta\lambda^2)\gamma_1-((\alpha^2+\beta^2)(\alpha-2\lambda)+\alpha\lambda^2)\gamma_2-2(X_3\alpha+X_2\beta)\gamma_3,\\
\mathfrak{P}_{33} &= -\beta^2\lambda^2\gamma_1^2+((\alpha^2+\beta^2)^2-\alpha^2\lambda^2)\gamma_2^2-2\alpha\beta\lambda^2\gamma_1\gamma_2+4X_4\beta\gamma_3\\
&\quad +4X_1\beta+(2\alpha-\lambda)\lambda+2\mu,\\
\mathfrak{P}_{34} &= ((\alpha^2+\beta^2)(\alpha-2\lambda)+\alpha\lambda^2)\gamma_1+((\alpha^2+\beta^2)(2X_1+\beta)-\beta\lambda^2)\gamma_2\\
&\quad +(2X_2\alpha-2X_3\beta)\gamma_3,\\
\mathfrak{P}_{44} &= -(\alpha^2+\beta^2-(\alpha+\beta)\lambda)(\alpha^2-\alpha\lambda+(\beta+\lambda)\beta)(\gamma_1^2-\gamma_2^2)-4\beta^2\gamma_3^2\\
&\quad -4(\alpha^2+\beta^2-\alpha\lambda)\beta\lambda\gamma_1\gamma_2-2\mu\,.
\end{aligned}$$

In this case we make use of Gröbner basis again, but due to the difficulty in getting such a basis using the above polynomials $\mathfrak{P}_{ij}$, we reduce the number of variables as follows. After a straightforward calculation, the expressions of $\mathfrak{P}_{11}$, $\mathfrak{P}_{22}$, $\beta\,\mathfrak{P}_{12}-(\alpha-\lambda)\mathfrak{P}_{13}$ and $(\alpha-\lambda)\mathfrak{P}_{12}+\beta\,\mathfrak{P}_{13}$ let us to clear μ, X_1, X_2 and X_3, respectively, obtaining:

$$\begin{aligned}
\mu &= -\tfrac{1}{2}((\alpha^2+\beta^2)^2-(\alpha^2-\beta^2)\lambda^2)(\gamma_1^2-\gamma_2^2)-2\alpha\beta\lambda^2\gamma_1\gamma_2-2\beta^2-\tfrac{1}{2}\lambda^2,\\
X_1 &= -\tfrac{1}{4\beta}((\alpha^2+\beta^2)^2-\alpha^2\lambda^2)\gamma_1^2+\tfrac{1}{4}\beta\lambda^2\gamma_2^2-\tfrac{1}{2}\alpha\lambda^2\gamma_1\gamma_2-X_4\gamma_3-\tfrac{1}{4\beta}((\lambda-2\alpha)\lambda-2\mu),\\
X_2 &= \left(\tfrac{1}{2}\left(\alpha^2-\beta^2-\tfrac{4\alpha\beta^2\lambda}{(\alpha-\lambda)^2+\beta^2}\right)\gamma_3+X_4\beta\right)\gamma_1+\left(\tfrac{\alpha\beta(\alpha^2+\beta^2-\lambda^2)}{(\alpha-\lambda)^2+\beta^2}\gamma_3+X_4\alpha\right)\gamma_2,\\
X_3 &= -\left(\tfrac{\alpha\beta(\alpha^2+\beta^2-\lambda^2)}{(\alpha-\lambda)^2+\beta^2}\gamma_3-X_4\alpha\right)\gamma_1+\left(\tfrac{1}{2}\left(\alpha^2-\beta^2-\tfrac{4\alpha\beta^2\lambda}{(\alpha-\lambda)^2+\beta^2}\right)\gamma_3-X_4\beta\right)\gamma_2\,.
\end{aligned}$$

Hence we can eliminate the above variables from the polynomials $\mathfrak{P}_{ij}$ and, as a consequence, X_4 is also eliminated. Let us denote by $\mathfrak{Q}_{ij}$ the expressions obtained from the polynomials $\mathfrak{P}_{ij}$ after substituting μ, X_1, X_2 and X_3. These expressions $\mathfrak{Q}_{ij}$ are not directly polynomials since they contain variables in denominators. We avoid this problem considering $\mathfrak{Q}'_{ij}$ given by

$$\begin{aligned}
&\mathfrak{Q}'_{14}=\Big((\alpha-\lambda)^2+\beta^2\Big)\mathfrak{Q}_{14},\quad \mathfrak{Q}'_{23}=\mathfrak{Q}_{23},\quad \mathfrak{Q}'_{24}=2\Big((\alpha-\lambda)^2+\beta^2\Big)\beta\,\mathfrak{Q}_{24},\\
&\mathfrak{Q}'_{33}=\mathfrak{Q}_{33},\quad \mathfrak{Q}'_{34}=2\Big((\alpha-\lambda)^2+\beta^2\Big)\beta\,\mathfrak{Q}_{34},\quad \mathfrak{Q}'_{44}=\mathfrak{Q}_{44},
\end{aligned}$$

the remaining ones being zero. Thus, $\mathfrak{Q}'_{ij}$ are polynomials in $\mathbb{R}[\gamma_1,\gamma_2,\gamma_3,\lambda,\alpha,\beta]$. Now, let $\mathcal{I}\subset\mathbb{R}[\gamma_1,\gamma_2,\gamma_3,\lambda,\alpha,\beta]$ be the ideal generated by the polynomials $\mathfrak{Q}'_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the lexicographical order and one gets that the polynomial $\mathbf{g}=(\alpha^2+\beta^2)^2\beta^2$ belongs to $\mathcal{G}$. Since $\beta\neq 0$, one has that there are no left-invariant Ricci solitons in this case.

2.3 The structure operator *L* has a double root of its minimal polynomial

If the structure operator L is of type II then there exists a pseudo-orthonormal basis $\{u_1,u_2,u_3,u_4\}$ of $\mathfrak{g}=\mathfrak{g}_3\rtimes\mathfrak{r}$, with $\langle u_1,u_2\rangle=\langle u_3,u_3\rangle=\langle u_4,u_4\rangle=1$, where $\mathfrak{g}_3=\operatorname{span}\{u_1,u_2,u_3\}$ and $\mathfrak{r}=\operatorname{span}\{u_4\}$, so that

$$[u_1,u_2]=\lambda_2 u_3,\quad [u_1,u_3]=-\lambda_1 u_1-\varepsilon u_2,\quad [u_2,u_3]=\lambda_1 u_2,\quad \underset{(i=1,2,3)}{[u_i,u_4]}=\sum_{j=1}^{3}\alpha_i^j u_j,$$

with $\varepsilon^2=1$, for certain $\alpha_i^j\in\mathbb{R}$. Next, depending on the eigenvalues λ_i, we are led to the following different possibilities.

2.3.1 Case of zero eigenvalues: metrics on $H^3\rtimes\mathbb{R}$

If $\lambda_1=\lambda_2=0$ then the corresponding metrics are determined by

$$\begin{aligned}&[u_1,u_3]=-\varepsilon u_2,\ [u_1,u_4]=\gamma_1 u_1+\gamma_2 u_2+\gamma_3 u_3,\\&[u_2,u_4]=\gamma_4 u_2,\ \ [u_3,u_4]=\gamma_5 u_1+\gamma_6 u_2-(\gamma_1-\gamma_4)u_3,\end{aligned}$$

and the following polynomials $\mathfrak{P}_{ij}$ are obtained:

$$\begin{aligned}&\mathfrak{P}_{12}=-2\gamma_4^2-2\gamma_1\gamma_4+\gamma_5\gamma_6+2X_4\gamma_1+2X_4\gamma_4-2\mu,\qquad \mathfrak{P}_{22}=\gamma_5^2,\\&\mathfrak{P}_{14}=-2X_1\gamma_2-2X_2\gamma_4-2X_3\gamma_6-\varepsilon\gamma_5,\qquad \mathfrak{P}_{34}=2(X_3\gamma_1-X_1\gamma_3-X_3\gamma_4),\\&\mathfrak{P}_{33}=-2(2\gamma_4^2-2\gamma_1\gamma_4+\gamma_5\gamma_6+2X_4\gamma_1-2X_4\gamma_4+\mu),\\&\mathfrak{P}_{44}=-3\gamma_1^2-3\gamma_4^2+2\gamma_1\gamma_4-2\gamma_3\gamma_5-2\gamma_5\gamma_6-2\mu,\qquad \mathfrak{P}_{24}=-2(X_1\gamma_1+X_3\gamma_5)\,.\end{aligned}$$

Note that γ_5 must vanish and hence $2(\gamma_1-\gamma_4)\mathfrak{P}_{12}+(\gamma_1+\gamma_4)\mathfrak{P}_{33}=2(\gamma_4-3\gamma_1)\mu$. Thus, either $\mu=0$ or $\gamma_4=3\gamma_1$. If $\mu=0$ then $\mathfrak{P}_{44}=-2\gamma_1^2-2\gamma_4^2-(\gamma_1-\gamma_4)^2$ and if $\gamma_4=3\gamma_1$ then one easily checks that

$$\begin{aligned}\mathfrak{P}_{24}&=-2X_1\gamma_1,\\2\gamma_1^2\mathfrak{P}_{14}-(2\gamma_1\gamma_2-\gamma_3\gamma_6)\mathfrak{P}_{24}-\gamma_1\gamma_6\mathfrak{P}_{34}&=-12X_2\gamma_1^3,\\\gamma_1\mathfrak{P}_{34}-\gamma_3\mathfrak{P}_{24}&=-4X_3\gamma_1^2,\\\mathfrak{P}_{12}-\mathfrak{P}_{44}&=8X_4\gamma_1\,.\end{aligned}$$

Hence, in any case, $\gamma_1=\gamma_4=0$, and the left-invariant metric is given by

$$[u_1,u_3]=-\varepsilon u_2,\quad [u_1,u_4]=\gamma_2 u_2+\gamma_3 u_3,\quad [u_3,u_4]=\gamma_6 u_2\,. \tag{3}$$

A straightforward calculation shows that u_2 is parallel and the curvature tensor satisfies $R(Y,Z)=0$ and $\nabla_Y R=0$ for all $Y,Z\in u_2^\perp$. Thus, the underlying structure is a plane wave.

2.3.2 Case $\lambda_1=0$, $\lambda_2\neq 0$: metrics on $\tilde{E}(2)\rtimes\mathbb{R}$ or $E(1,1)\rtimes\mathbb{R}$

In this case one has

$$\begin{aligned}&[u_1,u_2]=\lambda_2 u_3, &&[u_1,u_3]=-\varepsilon u_2, &&[u_1,u_4]=\gamma_1 u_2+\gamma_2 u_3,\\&[u_2,u_4]=\gamma_3 u_2+\gamma_4\lambda_2 u_3, &&[u_3,u_4]=-\varepsilon\gamma_4 u_2+\gamma_3 u_3,\end{aligned}$$

and the following polynomials $\mathfrak{P}_{ij}$ are obtained after a straightforward calculation:

$$\mathfrak{P}_{12}=\lambda_2^2-\gamma_2\gamma_4\lambda_2-2(\gamma_3^2-X_4\gamma_3+\mu),\ \mathfrak{P}_{24}=-\gamma_4\lambda_2^2,$$
$$\mathfrak{P}_{44}=2(\gamma_4\varepsilon-\gamma_2)\gamma_4\lambda_2-3\gamma_3^2-2\mu, \qquad \mathfrak{P}_{33}=2\gamma_2\gamma_4\lambda_2-\lambda_2^2-2(2\gamma_3^2-2X_4\gamma_3+\mu)\,.$$

It now follows that $2\,\mathfrak{P}_{12}-\frac{2(\gamma_2+\varepsilon\gamma_4)}{\lambda_2}\mathfrak{P}_{24}-\mathfrak{P}_{33}-\mathfrak{P}_{44}=3(\lambda_2^2+\gamma_3^2)$ and, since $\lambda_2\neq 0$, there are no left-invariant Ricci solitons in this case.

2.3.3 Case $\lambda_1\neq 0, \lambda_2=0$: metrics on $E(1,1)\rtimes\mathbb{R}$

If $\lambda_1\neq 0$ and $\lambda_2=0$ then

$$[u_1,u_3]=-\lambda_1u_1-\varepsilon u_2, \qquad [u_1,u_4]=\gamma_1u_1+\gamma_2u_2,\ [u_2,u_3]=\lambda_1u_2,$$
$$[u_2,u_4]=-(2\varepsilon\gamma_2\lambda_1-\gamma_1)u_2,\ [u_3,u_4]=\gamma_3u_1+\gamma_4u_2,$$

and straightforward calculations show that the non-zero polynomials $\mathfrak{P}_{ij}$ are given by

$$\mathfrak{P}_{11}=-4\varepsilon\lambda_1+\gamma_4^2-4\gamma_1\gamma_2+4X_4\gamma_2-4\varepsilon X_3, \qquad \mathfrak{P}_{22}=\gamma_3^2,$$
$$\mathfrak{P}_{12}=-4\gamma_2^2\lambda_1^2+4\varepsilon(2\gamma_1-X_4)\gamma_2\lambda_1-4\gamma_1^2+\gamma_3\gamma_4+4X_4\gamma_1-2\mu,$$
$$\mathfrak{P}_{13}=(2\varepsilon\gamma_2\gamma_4-2X_2)\lambda_1-3\gamma_1\gamma_4-\gamma_2\gamma_3+2X_4\gamma_4+2\varepsilon X_1,$$
$$\mathfrak{P}_{14}=(4\varepsilon X_2\gamma_2-\gamma_4)\lambda_1-2X_2\gamma_1-2X_1\gamma_2-\varepsilon\gamma_3-2X_3\gamma_4,$$
$$\mathfrak{P}_{23}=2(2\varepsilon\gamma_2\gamma_3+X_1)\lambda_1-3\gamma_1\gamma_3+2X_4\gamma_3, \qquad \mathfrak{P}_{24}=\gamma_3\lambda_1-2X_1\gamma_1-2X_3\gamma_3,$$
$$\mathfrak{P}_{44}=-4\gamma_2^2\lambda_1^2+8\varepsilon\gamma_1\gamma_2\lambda_1-4\gamma_1^2-2\gamma_3\gamma_4-2\mu\,, \quad \mathfrak{P}_{33}=-2(\gamma_3\gamma_4+\mu)\,.$$

Since γ_3 must be zero, it follows that $\mathfrak{P}_{23}=2X_1\lambda_1$ and $\mathfrak{P}_{33}=-2\mu$, and hence $X_1=\mu=0$. Now, $\mathfrak{P}_{44}=-4(\varepsilon\gamma_2\lambda_1-\gamma_1)^2$ implies $\gamma_1=\varepsilon\gamma_2\lambda_1$ and thus $\mathfrak{P}_{13}=-(\varepsilon\gamma_2\gamma_4+2X_2)\lambda_1+2X_4\gamma_4$, from where we get $X_2=-\frac{\varepsilon}{2}\gamma_2\gamma_4+X_4\frac{\gamma_4}{\lambda_1}$. At this point, the left-invariant metric is given by

$$[u_1,u_3]=-\lambda_1u_1-\varepsilon u_2,\ [u_1,u_4]=\varepsilon\gamma_2\lambda_1u_1+\gamma_2u_2,\ [u_2,u_3]=\lambda_1u_2,$$
$$[u_2,u_4]=-\varepsilon\gamma_2\lambda_1u_2, \qquad [u_3,u_4]=\gamma_4u_2,$$

and the system of polynomial equations $\{\mathfrak{P}_{ij}=0\}$ reduces to

$$\mathfrak{P}_{11}=-4\varepsilon(\gamma_2^2+1)\lambda_1+\gamma_4^2+4X_4\gamma_2-4\varepsilon X_3=0,$$
$$\mathfrak{P}_{14}=-\gamma_4\{(\gamma_2^2+1)\lambda_1+2(X_3-\varepsilon X_4\gamma_2)\}=0\,.$$

Set $v_1=u_1$, $v_2=\frac{1}{2}u_2$, $v_3=\varepsilon\gamma_2u_3+u_4$ and $v_4=u_3$. A straightforward calculation shows that $[v_i,v_j]=0$ for all $i,j\in\{1,2,3\}$ and $[v_4,v_i]\in\operatorname{span}\{v_1,v_2,v_3\}$. Hence any left-invariant metric above is isometric to some left-invariant metric on $\mathbb{R}^3\rtimes\mathbb{R}$ as discussed in Sect. 2.1.4.

2.3.4 Case of non-zero eigenvalues: metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$

In this case one has the metric expressed in terms of the Lie brackets

$$
\begin{aligned}
&[u_1, u_2] = \lambda_2 u_3, \quad [u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \; [u_2, u_3] = \lambda_1 u_2, \\
&[u_1, u_4] = \lambda_1\gamma_1 u_1 + \varepsilon\gamma_1 u_2 + \gamma_2\lambda_2 u_3, \qquad [u_2, u_4] = -\gamma_1\lambda_1 u_2 + \gamma_3\lambda_2 u_3, \\
&[u_3, u_4] = -\gamma_3\lambda_1 u_1 - (\gamma_2\lambda_1 + \varepsilon\gamma_3) u_2,
\end{aligned}
$$

and a straightforward calculation shows that the polynomials $\mathfrak{P}_{ij}$ are given by

$$
\begin{aligned}
\mathfrak{P}_{11} &= \gamma_2^2(\lambda_1^2 - \lambda_2^2) - 2\varepsilon(2\gamma_1^2 - \gamma_2\gamma_3 + 2)\lambda_1 + 2\varepsilon\lambda_2 + \gamma_3^2 + 4\varepsilon X_4\gamma_1 - 4\varepsilon X_3, \\
\mathfrak{P}_{12} &= \gamma_2\gamma_3\lambda_1^2 - (\gamma_2\gamma_3 - 1)\lambda_2^2 - 2\lambda_1\lambda_2 + \varepsilon\gamma_3^2\lambda_1 - 2\mu, \\
\mathfrak{P}_{13} &= \gamma_1\gamma_2(\lambda_1^2 - \lambda_1\lambda_2) + 2(\varepsilon\gamma_1\gamma_3 - X_4\gamma_2 - X_2)\lambda_1 \\
&\quad + (\varepsilon\gamma_1\gamma_3 + 2X_4\gamma_2 + 2X_2)\lambda_2 - 2\varepsilon\gamma_3 X_4 + 2\varepsilon X_1, \\
\mathfrak{P}_{14} &= \gamma_2(\lambda_1 - \lambda_2)^2 + 2(X_2\gamma_1 + X_3\gamma_2 + \varepsilon\gamma_3)\lambda_1 - 2\varepsilon\gamma_3\lambda_2 - 2\varepsilon X_1\gamma_1 + 2\varepsilon X_3\gamma_3, \\
\mathfrak{P}_{22} &= \gamma_3^2(\lambda_1^2 - \lambda_2^2), \qquad \mathfrak{P}_{34} = -2(X_1\gamma_2 + X_2\gamma_3)\lambda_2, \\
\mathfrak{P}_{23} &= -\gamma_1\gamma_3(\lambda_1^2 - \lambda_1\lambda_2) - 2(X_4\gamma_3 - X_1)(\lambda_1 - \lambda_2), \\
\mathfrak{P}_{24} &= -\gamma_3(\lambda_1^2 + \lambda_2^2) + 2\gamma_3\lambda_1\lambda_2 - 2(X_1\gamma_1 - X_3\gamma_3)\lambda_1, \\
\mathfrak{P}_{33} &= -2\gamma_2\gamma_3\lambda_1^2 + (2\gamma_2\gamma_3 - 1)\lambda_2^2 - 2\varepsilon\gamma_3^2\lambda_1 - 2\mu, \\
\mathfrak{P}_{44} &= -2\gamma_2\gamma_3(\lambda_1 - \lambda_2)^2 - 2\varepsilon\gamma_3^2(\lambda_1 - \lambda_2) - 2\mu\,.
\end{aligned}
$$

Let $\mathcal{I} \subset \mathbb{R}[\gamma_1, \gamma_2, \gamma_3, \varepsilon, \lambda_1, \lambda_2, \mu, X_1, X_2, X_3, X_4]$ be the ideal generated by the polynomials $\mathfrak{P}_{ij}$. We compute a Gröbner basis $\mathcal{G}$ of $\mathcal{I}$ with respect to the graded reverse lexicographical order and obtain that the polynomial $\mathbf{g} = \lambda_2^3$ belongs to $\mathcal{G}$. Since $\lambda_2 \neq 0$, there are no left-invariant Ricci solitons in this case.

2.4 The structure operator *L* has a triple root of its minimal polynomial

If the structure operator L is of type III then there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, so that

$$
[u_1, u_2] = u_1 + \lambda u_3, \quad [u_1, u_3] = -\lambda u_1, \quad [u_2, u_3] = \lambda u_2 + u_3, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,
$$

for certain $\alpha_i^j \in \mathbb{R}$. Next we consider separately the cases $\lambda = 0$ and $\lambda \neq 0$.

2.4.1 Case of zero eigenvalue: metrics on $E(1, 1) \rtimes \mathbb{R}$

If $\lambda = 0$, then

$$
\begin{aligned}
&[u_1, u_2] = u_1, \qquad [u_1, u_4] = \gamma_1 u_1, \; [u_2, u_3] = u_3, \\
&[u_2, u_4] = \gamma_2 u_1 + \gamma_3 u_3, \; [u_3, u_4] = \gamma_4 u_3,
\end{aligned}
$$

and a straightforward calculation shows that the non-zero polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{aligned} \mathfrak{P}_{12} &= -\gamma_1^2 - \gamma_1\gamma_4 + 2X_4\gamma_1 + 2X_2 - 2\mu, & \mathfrak{P}_{23} &= -2\gamma_3\gamma_4 + 2X_4\gamma_3 + 2X_3, \\ \mathfrak{P}_{22} &= -\gamma_3^2 - 2\gamma_2\gamma_4 + 4X_4\gamma_2 - 4X_1 - 4, & \mathfrak{P}_{34} &= -2(X_2\gamma_3 + X_3\gamma_4), \\ \mathfrak{P}_{24} &= -(2X_1 + 1)\gamma_1 - 2X_2\gamma_2 + 2\gamma_4, & \mathfrak{P}_{44} &= -\gamma_1^2 - 2\gamma_4^2 - 2\mu, \\ \mathfrak{P}_{33} &= -2(\gamma_4^2 + \gamma_1\gamma_4 - 2X_4\gamma_4 + 2X_2 + \mu)\,. \end{aligned}$$

From the expressions of $\mathfrak{P}_{22}$, $\mathfrak{P}_{23}$ and $\mathfrak{P}_{44}$ we get

$$X_1 = -\tfrac{1}{4}\gamma_3^2 - \tfrac{1}{2}(\gamma_4 - 2X_4)\gamma_2 - 1, \quad X_3 = (\gamma_4 - X_4)\gamma_3, \quad \mu = -\tfrac{1}{2}\gamma_1^2 - \gamma_4^2,$$

and thus $\mathfrak{P}_{12} = 2\gamma_4^2 - (\gamma_4 - 2X_4)\gamma_1 + 2X_2$ which implies $X_2 = -\gamma_4^2 + \frac{1}{2}(\gamma_4 - 2X_4)\gamma_1$. Now, $\mathfrak{P}_{24} = \frac{1}{2}(\gamma_3^2 + 2)\gamma_1 + 2(\gamma_2\gamma_4 + 1)\gamma_4$ and hence $\gamma_1 = -\frac{4(\gamma_2\gamma_4+1)\gamma_4}{\gamma_3^2+2}$. At this point, the system of polynomial equations $\{\mathfrak{P}_{ij} = 0\}$ reduces to

$$\begin{aligned} \mathfrak{P}_{33} &= \tfrac{4}{(\gamma_3^2+2)^2}\left\{(\gamma_3^2 + 2\gamma_2\gamma_4 + 4)^2\gamma_4 + X_4(\gamma_3^2 + 2)(\gamma_3^2 - 4\gamma_2\gamma_4 - 2)\right\}\gamma_4 = 0, \\ \mathfrak{P}_{34} &= \tfrac{2}{\gamma_3^2+2}\left\{2(\gamma_2\gamma_4 + 1)\gamma_4 + (\gamma_3^2 - 4\gamma_2\gamma_4 - 2)X_4\right\}\gamma_3\gamma_4 = 0\,. \end{aligned}$$

One easily checks that

$$\tfrac{\gamma_3}{2}\mathfrak{P}_{33} - \mathfrak{P}_{34} = \tfrac{1}{2(\gamma_3^2+2)^2}\left\{3\gamma_3^4 + 12\gamma_3^2 + (\gamma_3^2 + 4\gamma_2\gamma_4 + 6)^2 + 12\right\}\gamma_3\gamma_4^2$$

and therefore $\gamma_3\gamma_4$=0.

If $\gamma_4 = 0$ (which implies $\gamma_1 = 0$), the left-invariant metric is given by

$$[u_1, u_2] = u_1, \quad [u_2, u_3] = u_3, \quad [u_2, u_4] = \gamma_2 u_1 + \gamma_3 u_3, \tag{4}$$

and a standard calculation shows that u_1 is a recurrent null vector. Moreover, the only non-zero component of the Ricci tensor $\rho_{22} = -2 - \frac{1}{2}\gamma_3^2$ shows that the Ricci operator is isotropic, $R(Y, Z) = 0$, and $\nabla_Y R = 0$ for all $Y, Z \in u_1^\perp$. Hence the underlying structure corresponds to a plane wave.

If $\gamma_4 \neq 0$ then $\gamma_3 = 0$, and $\mathfrak{P}_{33} = 4\left\{(\gamma_2\gamma_4 + 2)^2\gamma_4 - X_4(2\gamma_2\gamma_4 + 1)\right\}\gamma_4$. Note that if $2\gamma_2\gamma_4 + 1 = 0$ then $\mathfrak{P}_{33} \neq 0$. Hence the left-invariant metric is given by

$$\begin{aligned} &[u_1, u_2] = u_1, \quad [u_1, u_4] = -2(\gamma_2\gamma_4 + 1)\gamma_4 u_1, \ [u_2, u_3] = u_3, \\ &[u_2, u_4] = \gamma_2 u_1, \ [u_3, u_4] = \gamma_4 u_3, \end{aligned}$$

and it is an expanding left-invariant Ricci soliton with $\mu = -(2(\gamma_2\gamma_4 + 1)^2 + 1)\gamma_4^2$ and left-invariant soliton vector field $X = X_1u_1 + X_2u_2 + X_4u_4$, where

$$\begin{aligned} X_1 &= \tfrac{1}{2(2\gamma_2\gamma_4+1)}(\gamma_2\gamma_4 + 2)(2(\gamma_2\gamma_4 + 1)\gamma_2\gamma_4 - 1), \\ X_2 &= \tfrac{1}{2\gamma_2\gamma_4+1}(\gamma_2\gamma_4 + 2)(2(\gamma_2\gamma_4 + 2)\gamma_2\gamma_4 + 3)\gamma_4^2, \\ X_4 &= \tfrac{1}{2\gamma_2\gamma_4+1}(\gamma_2\gamma_4 + 2)^2\,\gamma_4\,. \end{aligned}$$

A straightforward calculation shows that the above metric is symmetric if and only if $(\gamma_2\gamma_4 + 1)(\gamma_2\gamma_4 + 2) = 0$. Moreover, it is Einstein if and only if $\gamma_2\gamma_4 + 2 = 0$, in which case the sectional curvature is constant. Otherwise, if $\gamma_2\gamma_4 + 1 = 0$, then the metric is locally a product $\mathbb{L}^2 \times N(c)$, where $\mathbb{L}^2$ is the Minkowskian plane and $N(c)$ a surface of constant curvature c. Finally, note that $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_3, -u_4)$ defines an isometry interchanging (γ_4, γ_2) and $(-\gamma_4, -\gamma_2)$ and hence, without loss of generality, we can restrict the parameter γ_4 to $\gamma_4 > 0$. Setting $\alpha = \gamma_4$ and $\beta = \gamma_2$, this case corresponds to Assertion (iv) in Theorem 1.2 and Remark 1.5.

2.4.2 Case of non-zero eigenvalue: metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$

If $\lambda \neq 0$, then

$$\begin{aligned}
&[u_1, u_2] = u_1 + \lambda u_3, \quad [u_1, u_3] = -\lambda u_1, \quad [u_2, u_3] = \lambda u_2 + u_3,\\
&[u_1, u_4] = \gamma_1\lambda u_1 + \gamma_2\lambda^2 u_3, \quad [u_3, u_4] = -\gamma_3\lambda u_1 - \gamma_2\lambda^2 u_2 - \gamma_2\lambda u_3,\\
&[u_2, u_4] = \gamma_3 u_1 - (\gamma_1 - \gamma_2)\lambda u_2 - (\gamma_1 - \gamma_2 - \gamma_3\lambda)u_3,
\end{aligned}$$

and the following polynomials $\mathfrak{P}_{ij}$ are obtained:

$$\begin{aligned}
&\mathfrak{P}_{12} = -(\gamma_2^2 - \gamma_1\gamma_2 + 1)\lambda^2 + 2X_4\gamma_2\lambda + 2X_2 - 2\mu, \quad \mathfrak{P}_{13} = 3\gamma_2^2\lambda^3,\\
&\mathfrak{P}_{33} = (2\gamma_2^2 - 2\gamma_1\gamma_2 - 1)\lambda^2 - 4X_4\gamma_2\lambda - 4X_2 - 2\mu, \quad \mathfrak{P}_{44} = -3\gamma_2^2\lambda^2 - 2\mu\,.
\end{aligned}$$

One easily checks that $2\,\mathfrak{P}_{12} - \frac{3}{\lambda}\,\mathfrak{P}_{13} + \mathfrak{P}_{33} - 3\,\mathfrak{P}_{44} = -3\lambda^2$. Since $\lambda \neq 0$, there are no left-invariant Ricci solitons in this case.

3 Extensions of Riemannian Lie groups

In this section we analyze left-invariant Lorentzian metrics which are extensions of three-dimensional unimodular Riemannian Lie groups. In particular, we show that any left-invariant Ricci soliton in this setting is trivial.

Lemma 3.1 *A four-dimensional Lie group $G = G_3 \rtimes \mathbb{R}$ equipped with a left-invariant Lorentzian metric whose restriction to G_3 is Riemannian, is a left-invariant Ricci soliton if and only if it is a space of non-negative constant sectional curvature.*

Let $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ and let L be the structure operator of $\mathfrak{g}_3$. L is self-adjoint and diagonalizable, so there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$, with e_4 timelike, where $\mathfrak{g}_3 = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{r} = \text{span}\{e_4\}$, so that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1, \quad \underset{(i=1,2,3)}{[e_i, e_4]} = \sum_{j=1}^{3} \alpha_i^j e_j,$$

for certain $\alpha_i^j \in \mathbb{R}$. Next, depending on the eigenvalues λ_i and imposing the Jacobi identity, we are led to the following different possibilities.

3.1 Structure operator with non-zero eigenvalues: metrics on $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ and $SU(2)\times\mathbb{R}$

If $\lambda_1\lambda_2\lambda_3\neq 0$, left-invariant Lorentzian metrics are described by

$$\begin{array}{lll}[e_1,e_2]=\lambda_3 e_3, & [e_1,e_3]=-\lambda_2 e_2, & [e_1,e_4]=\gamma_1\lambda_2 e_2+\gamma_2\lambda_3 e_3,\\ [e_2,e_3]=\lambda_1 e_1, & [e_2,e_4]=-\gamma_1\lambda_1 e_1+\gamma_3\lambda_3 e_3, & [e_3,e_4]=-\gamma_2\lambda_1 e_1-\gamma_3\lambda_2 e_2,\end{array}$$

and proceeding as in Sect. 2.1.1 a straightforward calculation shows that there are no left-invariant Ricci solitons in this case.

3.2 Structure operator with a zero eigenvalue: metrics on $\tilde{E}(2)\rtimes\mathbb{R}$ and $E(1,1)\rtimes\mathbb{R}$

Without loss of generality, we assume $\lambda_3=0$ and $\lambda_1\lambda_2\neq 0$. Then Lorentzian left-invariant metrics on $\tilde{E}(2)\rtimes\mathbb{R}$ or $E(1,1)\rtimes\mathbb{R}$ are given by

$$\begin{array}{lll}[e_1,e_3]=-\lambda_2 e_2, & [e_1,e_4]=\gamma_1 e_1+\gamma_2\lambda_2 e_2, & [e_2,e_3]=\lambda_1 e_1,\\ [e_2,e_4]=-\gamma_2\lambda_1 e_1+\gamma_1 e_2, & [e_3,e_4]=\gamma_3 e_1+\gamma_4 e_2\,. & \end{array}$$

Proceeding as in Sect. 2.1.2.2 one has that the existence of left-invariant Ricci solitons leads to $\lambda_2=\lambda_1$, $\gamma_1=\gamma_3=\gamma_4=0$ and hence to flat metrics on $\tilde{E}(2)\rtimes\mathbb{R}$.

3.3 Structure operator of rank one: metrics on $H^3\rtimes\mathbb{R}$

Set $\lambda_1=\lambda_2=0$ and $\lambda_3\neq 0$ to express left-invariant Lorentzian metrics as

$$\begin{array}{ll}[e_1,e_2]=\lambda_3 e_3, & [e_1,e_4]=\gamma_1 e_1+\gamma_2 e_2+\gamma_3 e_3,\\ [e_2,e_4]=\gamma_4 e_1+\gamma_5 e_2+\gamma_6 e_3, & [e_3,e_4]=(\gamma_1+\gamma_5)e_3\,.\end{array}$$

A straightforward calculation as in Sect. 2.1.3.1 shows that there are no left-invariant Ricci solitons in this case.

3.4 Case of zero eigenvalues: metrics on $\mathbb{R}^3\rtimes\mathbb{R}$

Proceeding as in Sect. 2.1.4.1 one has that left-invariant metrics are described by

$$\begin{array}{ll}[e_1,e_4]=\eta_1 e_1-\gamma_1 e_2-\gamma_2 e_3, & [e_2,e_4]=\gamma_1 e_1+\eta_2 e_2-\gamma_3 e_3,\\ [e_3,e_4]=\gamma_2 e_1+\gamma_3 e_2+\eta_3 e_3\,. & \end{array}$$

Analogous calculations to those in Sect. 2.1.4.1 show that $\mathbb{R}^3\rtimes\mathbb{R}$ is a left-invariant Ricci soliton if and only if $\eta_1=\eta_2=\eta_3=\kappa$, in which case the sectional curvature is constant κ^2.

4 Extensions of degenerate Lie groups

In this section we study left-invariant Lorentzian metrics which are extensions of three-dimensional unimodular Lie groups with degenerate metric. We show that the underlying structure of any non-Einstein soliton is either a plane wave (Sect. 4.1 and Sect. 4.2.1.1) or a symmetric product $\mathbb{L}^2 \times N(c)$ (Sect. 4.2.2.3.). While products $\mathbb{L}^2 \times N(c)$ discussed in Sect. 4.2.2.3 are left-invariant Ricci solitons, the case of plane waves is more complicated and we analyze it in Sect. 5.

Let $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ be a four-dimensional Lie algebra with a Lorentzian inner product $\langle , \rangle$ which restricts to a degenerate inner product on the subalgebra $\mathfrak{g}_3$. Let $\mathfrak{g}_3' = [\mathfrak{g}_3, \mathfrak{g}_3]$ be the derived subalgebra of $\mathfrak{g}_3$. We consider separately the different cases for $\dim \mathfrak{g}_3' \in \{0, 1, 2, 3\}$.

4.1 dim $\mathfrak{g}_3' = 0$: left-invariant metrics on $\mathbb{R}^3 \rtimes \mathbb{R}$

The Lie algebra $\mathfrak{g}_3$ is Abelian, since $\dim \mathfrak{g}_3' = 0$. In this case there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \operatorname{span}\{u_4\}$, with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, so that

$$\begin{aligned}
&[u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3,\ [u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2 + \gamma_6 u_3,\\
&[u_3, u_4] = \gamma_7 u_1 + \gamma_8 u_2 + \gamma_9 u_3,
\end{aligned}$$

where $\gamma_i \in \mathbb{R}$. A straightforward calculation leads to the polynomials

$$\begin{aligned}
&\mathfrak{P}_{11} = -\gamma_7^2 + 4X_4\gamma_1 - 2\mu, &&\mathfrak{P}_{13} = 2X_4\gamma_7,\\
&\mathfrak{P}_{34} = \gamma_7^2 + \gamma_8^2 + 2X_4\gamma_9 - 2\mu, &&\mathfrak{P}_{23} = 2X_4\gamma_8\,.
\end{aligned}$$

It follows from the expressions of $\mathfrak{P}_{13}$ and $\mathfrak{P}_{23}$, together with $\mathfrak{P}_{11} - \mathfrak{P}_{34} = -2\gamma_7^2 - \gamma_8^2 + 2X_4(2\gamma_1 - \gamma_9)$, that $\gamma_7 = \gamma_8 = 0$. Hence the left-invariant metric is given by

$$[u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3, \quad [u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2 + \gamma_6 u_3, \quad [u_3, u_4] = \gamma_9 u_3, \tag{5}$$

and a standard calculation shows that u_3 is a recurrent null vector such that $R(Y, Z) = 0$ and $\nabla_Y R = 0$ for all $Y, Z \in u_3^{\perp}$. Moreover, the only non-zero component of the Ricci tensor is $\rho_{44} = -\gamma_1^2 - \frac{1}{2}(\gamma_2 + \gamma_4)^2 - \gamma_5^2 + (\gamma_1 + \gamma_5)\gamma_9$ which shows that the Ricci operator is isotropic, and thus the underlying structure is a plane wave.

4.2 dim $\mathfrak{g}_3' = 1$: left-invariant metrics on $H^3 \rtimes \mathbb{R}$

Since the restriction of the metric to $\mathfrak{g}_3$ has signature $(+, +, 0)$ then $\mathfrak{g}_3' = \operatorname{span}\{v\}$ can be a null or a spacelike subspace. We analyse those two possibilities separately.

4.2.1 $\mathfrak{g}_3' = \operatorname{span}\{v\}$ is a null subspace

In this case, setting $u_3 = v$ there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \operatorname{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \operatorname{span}\{u_4\}$, so that

$$[u_1, u_2] = \lambda_1 u_3, \quad [u_1, u_3] = \lambda_2 u_3, \quad [u_2, u_3] = \lambda_3 u_3, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ and where at least one of λ_1, λ_2 and λ_3 is non-zero. Next, depending on the λ_i's, we are led to the following different possibilities.

4.2.1.1. *Case* $\lambda_2 = \lambda_3 = 0$.

If $\lambda_2 = \lambda_3 = 0$, then necessarily $\lambda_1 \neq 0$ and one gets

$$\begin{array}{ll} [u_1, u_2] = \lambda_1 u_3, & [u_1, u_4] = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3, \\ [u_2, u_4] = \gamma_4 u_1 + \gamma_5 u_2 + \gamma_6 u_3, & [u_3, u_4] = (\gamma_1 + \gamma_5) u_3 . \end{array} \tag{6}$$

A standard calculation shows that u_3 is a recurrent vector field and the curvature tensor satisfies $R(Y, Z) = 0$ and $\nabla_Y R = 0$ for all $Y, Z \in u_3^\perp$. The only non-zero component of the Ricci tensor is $\rho_{44} = \frac{1}{2}\{\lambda_1^2 + 4\gamma_1\gamma_5 - (\gamma_2 + \gamma_4)^2\}$ which shows that the Ricci operator is isotropic and hence the underlying structure is a plane wave.

4.2.1.2. *Case* $\lambda_2 = 0,\ \lambda_3 \neq 0$.

In this case one has

$$\begin{array}{lll} [u_1, u_2] = \lambda_1 u_3, & [u_1, u_4] = \gamma_1\lambda_3 u_1 + (\gamma_1 - \gamma_2)\lambda_1 u_3, & [u_2, u_3] = \lambda_3 u_3, \\ [u_2, u_4] = \gamma_3 u_1 + \gamma_4 u_3, & [u_3, u_4] = \gamma_2\lambda_3 u_3, & \end{array}$$

and the non-zero polynomials $\mathfrak{P}_{ij}$ are given by

$$\begin{array}{l} \mathfrak{P}_{11} = 4X_4\gamma_1\lambda_3 - 2\mu, \qquad \mathfrak{P}_{12} = 2X_4\gamma_3, \qquad \mathfrak{P}_{22} = -\lambda_3^2 - 2\mu, \\ \mathfrak{P}_{14} = \lambda_1\lambda_3 + 2(X_4(\gamma_1 - \gamma_2) + X_2)\lambda_1 - 2X_1\gamma_1\lambda_3 - 2X_2\gamma_3, \\ \mathfrak{P}_{24} = -\gamma_1\lambda_3^2 - 2X_1\lambda_1 + 2X_3\lambda_3 + 2X_4\gamma_4, \qquad \mathfrak{P}_{34} = (2X_4\gamma_2 - 2X_2)\lambda_3 - \lambda_3^2 - 2\mu, \\ \mathfrak{P}_{44} = \lambda_1^2 - 2\gamma_1(\gamma_1 - \gamma_2)\lambda_3^2 - 4X_1(\gamma_1 - \gamma_2)\lambda_1 - 4X_3\gamma_2\lambda_3 - 4X_2\gamma_4 - \gamma_3^2 . \end{array}$$

Since $\lambda_3 \neq 0$, $\mathfrak{P}_{11} - \mathfrak{P}_{22} = (\lambda_3 + 4X_4\gamma_1)\lambda_3$ implies that $X_4 \neq 0$. Hence this expression, together with the expressions of $\mathfrak{P}_{12}$ and $\mathfrak{P}_{22}$, lead to

$$\gamma_3 = 0, \quad \gamma_1 = -\tfrac{1}{4X_4}\lambda_3, \quad \mu = -\tfrac{1}{2}\lambda_3^2,$$

and a direct calculation shows that

$$2\lambda_1\mathfrak{P}_{14} - 2\gamma_2\lambda_3\mathfrak{P}_{24} + \tfrac{2}{\lambda_3}(\lambda_1^2 - \gamma_3\lambda_1 + \gamma_4\lambda_3)\mathfrak{P}_{34} - \lambda_3\mathfrak{P}_{44} = \frac{1}{8X_4^2}\lambda_3^5 .$$

Since $\lambda_3 \neq 0$, there are no left-invariant Ricci solitons in this case.

4.2.1.3. *Case* $\lambda_2 \neq 0$.
If $\lambda_2 \neq 0$, then the Lie algebra structure is given by

$$\begin{aligned}
&[u_1, u_2] = \lambda_1 u_3, \quad [u_1, u_3] = \lambda_2 u_3, \quad [u_2, u_3] = \lambda_3 u_3, \quad [u_3, u_4] = \gamma_4 \lambda_2 u_3,\\
&[u_1, u_4] = -\gamma_1 \lambda_2 \lambda_3 u_1 + \gamma_1 \lambda_2^2 u_2 + \gamma_2 \lambda_2 u_3,\\
&[u_2, u_4] = -\gamma_3 \lambda_3 u_1 + \gamma_3 \lambda_2 u_2 + (\gamma_1 \lambda_1 \lambda_3 - (\gamma_3 - \gamma_4)\lambda_1 + \gamma_2 \lambda_3) u_3,
\end{aligned}$$

and the non-zero polynomials $\mathfrak{P}_{ij}$ are as follows:

$$\begin{aligned}
\mathfrak{P}_{11} &= -\lambda_2^2 - 4X_4\gamma_1\lambda_2\lambda_3 - 2\mu, \qquad \mathfrak{P}_{12} = 2X_4\gamma_1\lambda_2^2 - \lambda_2\lambda_3 - 2X_4\gamma_3\lambda_3,\\
\mathfrak{P}_{14} &= \gamma_1\lambda_2^2\lambda_3 - \gamma_3\lambda_2^2 + \lambda_1\lambda_3 + 2X_2(\lambda_1 + \gamma_3\lambda_3) + 2(X_4\gamma_2 + X_3 + X_1\gamma_1\lambda_3)\lambda_2,\\
\mathfrak{P}_{22} &= -\lambda_3^2 + 4X_4\gamma_3\lambda_2 - 2\mu,\\
\mathfrak{P}_{24} &= \gamma_1\lambda_2\lambda_3^2 - 2X_1\gamma_1\lambda_2^2 - \lambda_1\lambda_2 + 2X_4\gamma_1\lambda_1\lambda_3 - \gamma_3\lambda_2\lambda_3\\
&\quad - 2(X_4(\gamma_3 - \gamma_4) + X_1)\lambda_1 - 2X_2\gamma_3\lambda_2 + 2(X_4\gamma_2 + X_3)\lambda_3,\\
\mathfrak{P}_{34} &= -\lambda_2^2 - \lambda_3^2 + 2(X_4\gamma_4 - X_1)\lambda_2 - 2X_2\lambda_3 - 2\mu,\\
\mathfrak{P}_{44} &= -\gamma_1^2\lambda_2^4 - 2\gamma_1^2\lambda_2^2\lambda_3^2 + 2\gamma_1(\gamma_3 - \gamma_4)\lambda_2^2\lambda_3 + \lambda_1^2 - 2\gamma_3(\gamma_3 - \gamma_4)\lambda_2^2 - \gamma_3^2\lambda_3^2\\
&\quad - 4X_2\gamma_1\lambda_1\lambda_3 + 4X_2(\gamma_3 - \gamma_4)\lambda_1 - 4(X_1\gamma_2 + X_3\gamma_4)\lambda_2 - 4X_2\gamma_2\lambda_3 .
\end{aligned}$$

Since $\lambda_2 \neq 0$, then

$$\mathfrak{P}_{11} - \mathfrak{P}_{22} = -\lambda_2^2 + \lambda_3^2 - 4X_4(\gamma_1\lambda_3 + \gamma_3)\lambda_2, \quad \mathfrak{P}_{12} = -\lambda_2\lambda_3 + 2X_4(\gamma_1\lambda_2^2 - \gamma_3\lambda_3),$$

imply that $X_4 \neq 0$. Now, from the expressions of $\mathfrak{P}_{12}$, $\mathfrak{P}_{11}$ and $\mathfrak{P}_{22}$ we obtain

$$\gamma_1 = \frac{(\lambda_2 + 2X_4\gamma_3)\lambda_3}{2X_4\lambda_2^2}, \quad \mu = -\frac{\lambda_2^3 + 2(\lambda_2 + 2X_4\gamma_3)\lambda_3^2}{2\lambda_2}, \quad \gamma_3 = -\frac{\lambda_2}{4X_4},$$

and a direct calculation shows that

$$\begin{aligned}
&2(\gamma_4\lambda_2^2 - \lambda_1\lambda_3)\mathfrak{P}_{14} + 2(\lambda_1 + \gamma_4\lambda_3)\lambda_2\mathfrak{P}_{24}\\
&\quad -2(\lambda_1^2 + (\gamma_1\lambda_1 + \gamma_2)(\lambda_2^2 + \lambda_3^2) + \gamma_4\lambda_1\lambda_3)\mathfrak{P}_{34} + (\lambda_2^2 + \lambda_3^2)\mathfrak{P}_{44} = -\frac{(\lambda_2^2 + \lambda_3^2)^3}{8X_4^2} .
\end{aligned}$$

Since $\lambda_2 \neq 0$, there are no left-invariant Ricci solitons in this case.

4.2.2 $\mathfrak{g}_3' = \text{span}\{v\}$ is a spacelike subspace

Setting $u_1 = \frac{v}{\|v\|}$, there exists a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \text{span}\{u_1, u_2, u_3\}$ and

$\mathfrak{r} = \mathrm{span}\{u_4\}$, so that

$$[u_1, u_2] = \lambda_1 u_1, \quad [u_1, u_3] = \lambda_2 u_1, \quad [u_2, u_3] = \lambda_3 u_1, \quad \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\alpha_i^j \in \mathbb{R}$ and where at least one of λ_1, λ_2 and λ_3 is non-zero. Depending on the λ_i's we are led to the following different possibilities.

4.2.2.1. *Case* $\lambda_1 = \lambda_2 = 0$.
Since $\lambda_3 \neq 0$ one has the Lie algebra structure

$$\begin{aligned}&[u_1, u_4] = \gamma_1 u_1, \quad [u_2, u_3] = \lambda_3 u_1, \quad [u_2, u_4] = \gamma_2 u_1 + \gamma_3 u_2 + \gamma_4 u_3,\\ &[u_3, u_4] = \gamma_5 u_1 + \gamma_6 u_2 + (\gamma_1 - \gamma_3) u_3\,.\end{aligned}$$

A direct calculation shows $\mathfrak{P}_{33} = -\lambda_3^2$; hence there are no left-invariant Ricci solitons in this case.

4.2.2.2 *Case* $\lambda_1 = 0, \ \lambda_2 \neq 0$.
In this case one has

$$\begin{aligned}&[u_1, u_3] = \lambda_2 u_1, && [u_1, u_4] = \gamma_1 \lambda_2 u_1, \qquad [u_2, u_3] = \lambda_3 u_1,\\ &[u_2, u_4] = (\gamma_1 - \gamma_2)\lambda_3 u_1 + \gamma_2 \lambda_2 u_2, && [u_3, u_4] = \gamma_3 u_1 + \gamma_4 u_2\,.\end{aligned}$$

It now follows from $\mathfrak{P}_{33} = -2\lambda_2^2 - \lambda_3^2$ that there are no left-invariant Ricci solitons in this case.

4.2.2.3. *Case* $\lambda_1 \neq 0$.
If $\lambda_1 \neq 0$ then the Lie algebra structure becomes

$$\begin{aligned}&[u_1, u_2] = \lambda_1 u_1, \quad [u_1, u_3] = \lambda_2 u_1, \quad [u_1, u_4] = \gamma_1 \lambda_1 u_1, \quad [u_2, u_3] = \lambda_3 u_1,\\ &[u_2, u_4] = \lambda_1 \gamma_2 u_1 - \gamma_3 \lambda_1 \lambda_2 u_2 + \gamma_3 \lambda_1^2 u_3,\\ &[u_3, u_4] = -(\gamma_3 \lambda_2 \lambda_3 - \gamma_2 \lambda_2 + (\gamma_1 - \gamma_4)\lambda_3) u_1 - \gamma_4 \lambda_2 u_2 + \gamma_4 \lambda_1 u_3,\end{aligned}$$

and one has the polynomials $\mathfrak{P}_{ij}$ as follows:

$$\begin{aligned}\mathfrak{P}_{11} &= -\gamma_3^2 \lambda_2^2 \lambda_3^2 + 2\gamma_3 \lambda_1 \lambda_2^2 + 2\gamma_2 \gamma_3 \lambda_2^2 \lambda_3 - 2(\gamma_1 - \gamma_4)\gamma_3 \lambda_2 \lambda_3^2 - 2\lambda_1^2 - \gamma_2^2 \lambda_2^2\\ &\quad - (\gamma_1 - \gamma_4)^2 \lambda_3^2 - 2(2\gamma_1 + \gamma_4)\lambda_1 \lambda_2 + 2\gamma_2 \lambda_1 \lambda_3 + 2(\gamma_1 - \gamma_4)\gamma_2 \lambda_2 \lambda_3\\ &\quad + 4(X_4 \gamma_1 + X_2)\lambda_1 + 4X_3 \lambda_2 - 2\mu,\\ \mathfrak{P}_{12} &= -\gamma_3 \gamma_4 \lambda_2^2 \lambda_3 + \gamma_2 \gamma_4 \lambda_2^2 - 2\gamma_2 \lambda_1 \lambda_2 - (2\gamma_1 + \gamma_4)\lambda_1 \lambda_3 - (\gamma_1 - \gamma_4)\gamma_4 \lambda_2 \lambda_3\\ &\quad + 2(X_4 \gamma_2 - X_1)\lambda_1 + 2X_3 \lambda_3,\\ \mathfrak{P}_{13} &= 2\gamma_3 \lambda_2^2 \lambda_3 - 2\gamma_2 \lambda_2^2 + 2\lambda_1 \lambda_3 + 2(\gamma_1 - \gamma_4 - X_4 \gamma_3)\lambda_2 \lambda_3 + (2X_4 \gamma_2 - 2X_1)\lambda_2\\ &\quad - 2(X_4(\gamma_1 - \gamma_4) + X_2)\lambda_3,\\ \mathfrak{P}_{14} &= \gamma_3^2 \lambda_1 \lambda_2^2 \lambda_3 + \gamma_3 \lambda_1^2 \lambda_3 - \gamma_2 \gamma_3 \lambda_1 \lambda_2^2 - (\gamma_1 + \gamma_4)\gamma_3 \lambda_1 \lambda_2 \lambda_3 + 2\gamma_2 \lambda_1^2 + 2\gamma_1 \gamma_2 \lambda_1 \lambda_2\end{aligned}$$

$$
\begin{aligned}
&- 2\gamma_1(\gamma_1 - \gamma_4)\lambda_1\lambda_3 + 2X_3\gamma_3\lambda_2\lambda_3 - 2(X_1\gamma_1 + X_2\gamma_2)\lambda_1 - 2X_3\gamma_2\lambda_2\\
&+ 2X_3(\gamma_1 - \gamma_4)\lambda_3,\\
\mathfrak{P}_{22} &= 2\gamma_3\lambda_1\lambda_2^2 - 2\lambda_1^2 - \gamma_4^2\lambda_2^2 - 4X_4\gamma_3\lambda_1\lambda_2 - 2\gamma_2\lambda_1\lambda_3 - 2\mu,\\
\mathfrak{P}_{23} &= \gamma_3\lambda_2\lambda_3^2 + \gamma_4\lambda_2^2 + (\gamma_1 - \gamma_4)\lambda_3^2 - 2\lambda_1\lambda_2 - \gamma_2\lambda_2\lambda_3 - 2X_4\gamma_4\lambda_2,\\
\mathfrak{P}_{24} &= -2\gamma_3\lambda_1^2\lambda_2 + 2\gamma_3\gamma_4\lambda_1\lambda_2^2 - \gamma_2\gamma_3\lambda_1\lambda_2\lambda_3 - 2(\gamma_1 - X_4\gamma_3)\lambda_1^2\\
&+ (\gamma_2^2 - \gamma_1\gamma_4 + 2X_2\gamma_3)\lambda_1\lambda_2 - (\gamma_1 - \gamma_4)\gamma_2\lambda_1\lambda_3 + 2X_3\gamma_4\lambda_2,\\
\mathfrak{P}_{33} &= -2\lambda_2^2 - \lambda_3^2,\\
\mathfrak{P}_{34} &= \gamma_3^2\lambda_2^2\lambda_3^2 - 2\gamma_2\gamma_3\lambda_2^2\lambda_3 + 2(\gamma_1 - \gamma_4)\gamma_3\lambda_2\lambda_3^2 + (\gamma_2^2 + \gamma_4^2)\lambda_2^2 + (\gamma_1 - \gamma_4)^2\lambda_3^2\\
&- (2\gamma_1 + \gamma_4)\lambda_1\lambda_2 - \gamma_2\lambda_1\lambda_3 - 2(\gamma_1 - \gamma_4)\gamma_2\lambda_2\lambda_3 + 2X_4\gamma_4\lambda_1 - 2\mu,\\
\mathfrak{P}_{44} &= -2\gamma_3^2\lambda_1^2\lambda_2^2 + 2\gamma_3\lambda_1^3 - (2\gamma_1^2 + \gamma_2^2 - 2\gamma_1\gamma_4 + 4X_2\gamma_3)\lambda_1^2 - 4X_3\gamma_4\lambda_1\,.
\end{aligned}
$$

The polynomial $\mathfrak{P}_{33}$ gives $\lambda_2 = \lambda_3 = 0$, and thus $\mathfrak{P}_{22} - \mathfrak{P}_{34} = -2(\lambda_1 + X_4\gamma_4)\lambda_1$ implies $X_4 \neq 0$ and $\gamma_4 \neq 0$. Hence

$$
\begin{aligned}
\gamma_2\mathfrak{P}_{11} - 2\gamma_1\mathfrak{P}_{12} + 2\,\mathfrak{P}_{14} - \gamma_2\mathfrak{P}_{22} &= 4\gamma_2\lambda_1^2,\\
\gamma_1\mathfrak{P}_{22} - \mathfrak{P}_{24} - \gamma_1\mathfrak{P}_{34} &= -2X_4(\gamma_3\lambda_1 + \gamma_1\gamma_4)\lambda_1,
\end{aligned}
$$

lead to $\gamma_2 = 0$, and $\gamma_1 = -\frac{\gamma_3\lambda_1}{\gamma_4}$. Finally, a standard calculation shows that the left-invariant metric, given by

$$
[u_1, u_2] = \lambda_1 u_1, \quad [u_1, u_4] = -\frac{\gamma_3\lambda_1^2}{\gamma_4}u_1, \quad [u_2, u_4] = \gamma_3\lambda_1^2 u_3, \quad [u_3, u_4] = \gamma_4\lambda_1 u_3,
$$

is symmetric and locally isometric to a product $\mathbb{L}^2 \times N(c)$ where N is a surface of constant curvature c. Furthermore, it is a expanding Ricci soliton with $\mu = -\lambda_1^2$ and left-invariant soliton vector field $X = -\frac{\gamma_3\lambda_1^2}{\gamma_4^2}u_2 + \frac{\gamma_3^2\lambda_1^3}{2\gamma_4^3}u_3 - \frac{\lambda_1}{\gamma_4}u_4$.

4.3 dim $\mathfrak{g}_3' = 2$: left-invariant metrics on $\tilde{E}(2) \rtimes \mathbb{R}$ and $E(1,1) \rtimes \mathbb{R}$

Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ be the derived subalgebra of $\mathfrak{g}$. Without loss of generality we may assume $\mathfrak{g}' = \mathfrak{g}_3$. Indeed, if $\dim \mathfrak{g}' < 3$ then there exist two linearly independent vectors $x_1, x_2 \in \mathfrak{g}$ acting as derivations on $\mathfrak{g}$. Since $\mathfrak{g}$ is Lorentzian, we can choose a non-null vector $y \in \operatorname{span}\{x_1, x_2\}$ so that $\mathfrak{g} = \mathfrak{h} \rtimes \operatorname{span}\{y\}$, where the restriction of the metric to the three-dimensional subalgebra $\mathfrak{h}$ is non-degenerate. Thus, $\mathfrak{g}$ corresponds to one of the cases already studied in Sects. 2 and 3.

Let $\mathfrak{g}_3' = \operatorname{span}\{w_1, w_2\}$, where $w_i = v_i + \xi_i u_3$, with v_i spacelike and u_3 null and orthogonal to v_1 and v_2.

If $\{v_1, v_2\}$ are linearly independent, i.e., $\mathfrak{g}_3'$ is a spacelike subspace, we choose an orthonormal basis $\{u_1, u_2\}$ for $\operatorname{span}\{v_1, v_2\}$ which can be completed to a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$ with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle =$

1, where $\mathfrak{g}_3 = \text{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \text{span}\{u_4\}$, so that

$$[u_1, u_2] = \gamma_1 u_1 + \gamma_2 u_2,\ [u_1, u_3] = \gamma_3 u_1 + \gamma_4 u_2,$$
$$[u_2, u_3] = \gamma_5 u_1 + \gamma_6 u_2,\ \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\gamma_i, \alpha_i^j \in \mathbb{R}$.

If $\{v_1, v_2\}$ are linearly dependent, i.e., the restriction of the metric to $\mathfrak{g}_3'$ is degenerate, then $\{u_1 = \frac{v_1}{\|v_1\|}, u_3\}$ is a basis of $\mathfrak{g}_3'$ which can be completed to a pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$, where $\mathfrak{g}_3 = \text{span}\{u_1, u_2, u_3\}$ and $\mathfrak{r} = \text{span}\{u_4\}$, so that

$$[u_1, u_2] = \gamma_1 u_1 + \gamma_2 u_3,\ [u_1, u_3] = \gamma_3 u_1 + \gamma_4 u_3,$$
$$[u_2, u_3] = \gamma_5 u_1 + \gamma_6 u_3,\ \underset{(i=1,2,3)}{[u_i, u_4]} = \sum_{j=1}^{3} \alpha_i^j u_j,$$

for certain $\gamma_i, \alpha_i^j \in \mathbb{R}$.

In any of the two cases above, a straightforward calculation shows that the Jacobi identity is not satisfied since $\dim \mathfrak{g}_3' = 2$ and $\dim \mathfrak{g}' = 3$. Hence there are no left-invariant Ricci solitons in this case.

4.4 $\dim \mathfrak{g}_3' = 3$: left-invariant metrics on $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$

In this case, $\mathfrak{g}_3' = \mathfrak{g}_3$ and we consider the pseudo-orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \text{span}\{u_4\}$ with $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle$ and $\text{ad}_{u_3} : \mathfrak{g}_3 \to \mathfrak{g}_3$. Since $\mathfrak{g}_3' = \mathfrak{g}_3$, ad_{u_3} must be of rank 2 and, apart from 0, it must have either two real eigenvalues or two conjugate complex eigenvalues. Moreover, writing $u_3 = [x_1, x_2]$, $x_1, x_2 \in \mathfrak{g}_3$, we have $\text{ad}_{u_3} = \text{ad}_{x_1} \circ \text{ad}_{x_2} - \text{ad}_{x_2} \circ \text{ad}_{x_1}$, which implies $\text{tr}(\text{ad}_{u_3}) = 0$. Thus, two possibilities may occur, none of them supporting left-invariant Ricci solitons.

4.4.1 ad_{u_3} has real eigenvalues $\{0, \lambda, -\lambda\}$, with $\lambda \neq 0$

Let v_1 and v_2 be unit eigenvectors, i.e., $[v_1, u_3] = \lambda v_1$ and $[v_2, u_3] = -\lambda v_2$. The Jacobi identity implies $[v_1, v_2] \in \text{span}\{u_3\}$. Thus, rescaling u_3 if necessary, we get a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_3, v_4\rangle = 1$, $\langle v_1, v_2\rangle = \kappa \neq \pm 1$, where $\mathfrak{g}_3 = \text{span}\{v_1, v_2, v_3\}$ and $\mathfrak{r} = \text{span}\{v_4\}$, so that

$$\begin{aligned} &[v_1, v_2] = v_3, && [v_1, v_3] = \lambda v_1, && [v_1, v_4] = \gamma_1 v_1 + \gamma_2 v_3, \\ &[v_2, v_3] = -\lambda v_2, && [v_2, v_4] = -\gamma_1 v_2 + \gamma_3 v_3, && [v_3, v_4] = \gamma_3 \lambda v_1 + \gamma_2 \lambda v_2 . \end{aligned}$$

We compute $\mathfrak{P}_{33} = \frac{4\lambda^2}{\kappa^2 - 1}$ and, since $\lambda \neq 0$, there are no left-invariant Ricci solitons in this case.

4.4.2 ad_{u_3} has complex eigenvalues $\{0, i\beta, -i\beta\}$, with $\beta \neq 0$

Let v_1 and v_2 be unit vectors so that $[v_1, u_3] = \beta v_2$ and $[v_2, u_3] = -\beta v_1$. The Jacobi identity implies $[v_1, v_2] \in \mathrm{span}\{u_3\}$. Thus, rescaling u_3 if necessary, we get a basis $\{v_1, v_2, v_3, v_4\}$ of $\mathfrak{g} = \mathfrak{g}_3 \rtimes \mathfrak{r}$, with $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_3, v_4\rangle = 1$, $\langle v_1, v_2\rangle = \kappa \neq \pm 1$, where $\mathfrak{g}_3 = \mathrm{span}\{v_1, v_2, v_3\}$ and $\mathfrak{r} = \mathrm{span}\{v_4\}$, so that

$$\begin{aligned}
&[v_1, v_2] = v_3, && [v_1, v_3] = \beta v_2, && [v_1, v_4] = \gamma_1 v_2 + \gamma_2 v_3,\\
&[v_2, v_3] = -\beta v_1, && [v_2, v_4] = -\gamma_1 v_1 + \gamma_3 v_3, && [v_3, v_4] = \gamma_2 \beta v_1 + \gamma_3 \beta v_2 .
\end{aligned}$$

We will make use of the following polynomials $\tilde{\mathfrak{P}}_{ij} = (\kappa^2 - 1)\mathfrak{P}_{ij}$:

$$\begin{aligned}
\tilde{\mathfrak{P}}_{11} &= -(\kappa^2-1)\beta^2(\gamma_2+\kappa\gamma_3)^2 - 4(2\kappa\beta - X_4(\kappa^2-1))\kappa\gamma_1\\
&\quad + 2(2X_3\kappa^3 - \kappa^2 - 2X_3\kappa + 1)\beta - 2(\kappa^2-1)\mu,\\
\tilde{\mathfrak{P}}_{12} &= -(\kappa^2-1)\kappa\beta^2(\gamma_2^2+\gamma_3^2) - (\kappa^4-1)\beta^2\gamma_2\gamma_3 - 8\kappa\beta\gamma_1 - 2(\kappa^2-1)\kappa\mu,\\
\tilde{\mathfrak{P}}_{22} &= -(\kappa^2-1)\beta^2(\kappa\gamma_2+\gamma_3)^2 - 4(2\kappa\beta + X_4(\kappa^2-1))\kappa\gamma_1\\
&\quad - 2(2X_3\kappa^3 + \kappa^2 - 2\kappa X_3 - 1)\beta - 2(\kappa^2-1)\mu,\\
\tilde{\mathfrak{P}}_{33} &= 4\beta^2\kappa^2,\\
\tilde{\mathfrak{P}}_{44} &= 4\kappa^2\gamma_1^2 - 2(\kappa^2-1)\beta(\gamma_2^2+\gamma_3^2) - 4(\kappa^2-1)(X_1\gamma_2+X_2\gamma_3) - 1 .
\end{aligned}$$

Since $\beta \neq 0$, $\tilde{\mathfrak{P}}_{33}$ shows that $\kappa = 0$, and by a direct calculation one has

$$\gamma_3\tilde{\mathfrak{P}}_{11} - \gamma_2\tilde{\mathfrak{P}}_{12} - \gamma_3\tilde{\mathfrak{P}}_{22} = -\beta^2\gamma_3^3, \qquad \gamma_2\tilde{\mathfrak{P}}_{11} + \gamma_3\tilde{\mathfrak{P}}_{12} - \gamma_2\tilde{\mathfrak{P}}_{22} = \beta^2\gamma_2^3 .$$

Hence $\gamma_2 = \gamma_3 = 0$ and thus $\tilde{\mathfrak{P}}_{44} = -1$, which shows that there are no left-invariant Ricci solitons in this case.

5 Left-invariant Ricci solitons on pp-wave Lie groups

Based on the analysis of previous sections, left-invariant Ricci solitons on pp-wave Lie groups split naturally into two distinct possibilities as they are plane waves or not. The case of pp-wave Lie groups which are not plane waves is as follows:

Theorem 5.1 *A four-dimensional Lorentzian pp-wave Lie group which is not a plane wave is a non-trivial left-invariant Ricci soliton if and only it is homothetic to* $G = \mathbb{R}^3 \rtimes \mathbb{R}$ *with left-invariant metric given by the Lie algebra*

$$[u_1, u_4] = \gamma_1 u_1 + \varepsilon u_2, \quad [u_2, u_4] = -\gamma_1 u_2,$$

where $\gamma_1 \neq 0$, $\varepsilon = \pm 1$, *and* $\{u_1, \ldots, u_4\}$ *is a pseudo-orthonormal basis with* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$.

Proof Lorentzian Lie groups as above are extensions of unimodular Lorentzian Lie groups and have been discussed in Sect. 2.1.4.3. Since the sectional curvature is independent of the structure constant γ_3, we set $\gamma_3 = 0$ in Equation (2) and work at the homothetic level ([21, 22]). Now, the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ reduce to

$$\tilde{\mathfrak{P}}_{11} = 2\varepsilon(X_4 - \gamma_1),\ \tilde{\mathfrak{P}}_{12} = \tilde{\mathfrak{P}}_{33} = \tilde{\mathfrak{P}}_{44} = -\mu,\ \tilde{\mathfrak{P}}_{14} = X_2\gamma_1 - \varepsilon X_1,\ \tilde{\mathfrak{P}}_{24} = -X_1\gamma_1,$$

so we get a left-invariant steady Ricci soliton with left-invariant soliton vector field $X = X_3u_3 + \gamma_1u_4$, for any $\gamma_1 \neq 0$.

□

Remark 5.2 Globke and Leistner proved in [17] that four-dimensional Ricci-flat homogeneous pp-waves are plane waves. Examples in Theorem 5.1 show that the result above cannot be extended to steady Ricci soliton pp-waves. Moreover, pp-wave Lie groups in Theorem 5.1 are conformal C-spaces, but not conformally Einstein (see [3, 18] for more information).

Theorem 5.3 *A four-dimensional Lorentzian plane wave Lie group is a non-trivial left-invariant Ricci soliton if and only if it is homothetic to one of the following:*

(i) $G = H^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_3] = u_2, \quad [u_1, u_4] = \gamma_3u_3,$$

where $\gamma_3 \neq 0$ *and* $\{u_1, \dots, u_4\}$ *is pseudo-orthonormal with* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$.

(ii) $G = E(1, 1) \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_2] = u_1, \quad [u_2, u_3] = u_3, \quad [u_2, u_4] = \gamma_3u_3,$$

and $\{u_1, \dots, u_4\}$ *is pseudo-orthonormal with* $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = \langle u_4, u_4\rangle = 1$.

(iii) $G = \mathbb{R}^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$[u_1, u_4] = \gamma_1u_1 + \gamma_2u_2, \quad [u_2, u_4] = \gamma_4u_1 + \gamma_5u_2, \quad [u_3, u_4] = u_3,$$

where $\{u_1, \dots, u_4\}$ *is pseudo-orthonormal with* $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$.

(iv) $G = H^3 \rtimes \mathbb{R}$ *with Lie algebra given by*

$$\begin{aligned} &[u_1, u_2] = \lambda_1u_3, && [u_1, u_4] = \gamma_1u_1 + \gamma_2u_2,\\ &[u_2, u_4] = \gamma_4u_1 + \gamma_5u_2, && [u_3, u_4] = (\gamma_1 + \gamma_5)u_3, \end{aligned}$$

where $\gamma_1 + \gamma_5 \neq 0$, *and* $\{u_1, \dots, u_4\}$ *is pseudo-orthonormal with* $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_3, u_4\rangle = 1$.

Proof Lie groups in Assertions (i) and (ii) are Lorentzian extensions of unimodular Lorentzian Lie groups. Assertion (i) was considered in Sect. 2.3.1 and a straightforward calculation shows that the curvature tensor does not involve the structure constants ε

and γ_2, so we can take $\varepsilon = -1$ and $\gamma_2 = 0$ in Equation (3) (see [21, 22]). Now, the only component of the Ricci tensor is $\rho_{11} = -\frac{1}{2}(\gamma_3^2 - \gamma_6^2)$ and the non-zero polynomials $\mathfrak{P}_{ij}$ reduce to

$$\begin{array}{l}\mathfrak{P}_{11} = -\frac{1}{2}\{\gamma_3^2 - \gamma_6^2 - 4X_3\},\ \mathfrak{P}_{12} = \mathfrak{P}_{33} = \mathfrak{P}_{44} = -\mu,\\ \mathfrak{P}_{13} = X_4(\gamma_3 + \gamma_6) - X_1, \quad \mathfrak{P}_{14} = -X_3\gamma_6, \quad \mathfrak{P}_{34} = -X_1\gamma_3\,.\end{array}$$

Hence, we get a left-invariant steady Ricci soliton if and only if it is Ricci-flat ($\gamma_3^2 = \gamma_6^2$) or, otherwise, $\gamma_6 = 0$ and $\gamma_3 \neq 0$. In this latter case, the left-invariant soliton vector field is given by $X = X_2u_2 + \frac{1}{4}\gamma_3^2 u_3$. Assertion (ii) was treated in Sect. 2.4.1 and since the curvature tensor does not depend on the structure constant γ_2, one can eliminate it in Equation (4) remaining in the same homothetic class due to the work of Kulkarni [22] (see also [21]). The non-zero polynomials $\mathfrak{P}_{ij}$ now reduce to

$$\begin{array}{lll}\mathfrak{P}_{12} = X_2 - \mu, & \mathfrak{P}_{22} = -\frac{1}{2}\gamma_3^2 - 2(X_1 + 1), & \mathfrak{P}_{23} = X_4\gamma_3 + X_3,\\ \mathfrak{P}_{33} = -2X_2 - \mu, & \mathfrak{P}_{34} = -X_2\gamma_3, & \mathfrak{P}_{44} = -\mu,\end{array}$$

so we get a left-invariant steady Ricci soliton with left-invariant soliton vector field $X = -(\frac{1}{4}\gamma_3^2 + 1)u_1 - X_4\gamma_3u_3 + X_4u_4$.

Plane wave Lie groups in Assertion (iii) are Lorentzian extensions of unimodular degenerate Lie groups and correspond to Sect. 4.1. First of all, observe that proceeding as in the previous cases, one can eliminate the structure constants γ_3 and γ_6 in Equation (5) and remain in the same homothetic class. A straightforward calculation shows that the Ricci tensor vanishes if and only if $\rho_{44} = -\gamma_1^2 - \frac{1}{2}(\gamma_2 + \gamma_4)^2 - \gamma_5^2 + (\gamma_1 + \gamma_5)\gamma_9 = 0$, and the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\begin{array}{llll}\tilde{\mathfrak{P}}_{11} = 2X_4\gamma_1 - \mu, & \tilde{\mathfrak{P}}_{14} = -X_1\gamma_1 - X_2\gamma_4, & \tilde{\mathfrak{P}}_{12} = X_4(\gamma_2 + \gamma_4), & \tilde{\mathfrak{P}}_{34} = X_4\gamma_9 - \mu,\\ \tilde{\mathfrak{P}}_{22} = 2X_4\gamma_5 - \mu, & \tilde{\mathfrak{P}}_{24} = -X_1\gamma_2 - X_2\gamma_5, & \tilde{\mathfrak{P}}_{44} = \rho_{44} - 2X_3\gamma_9\,. & \end{array}$$

We consider separately the cases $\gamma_9 \neq 0$ and $\gamma_9 = 0$. Assuming $\gamma_9 \neq 0$, we can take $\gamma_9 = 1$ without loss of generality. If $\gamma_2 = -\gamma_4$ and $\gamma_1 = \gamma_5 = \frac{1}{2}$, then $X = \frac{1}{2}\rho_{44}u_3 + \mu u_4$ is a locally conformally flat expanding, steady or shrinking left-invariant Ricci soliton. Otherwise, if $\gamma_2 \neq -\gamma_4$, or $\gamma_1 \neq \frac{1}{2}$, or $\gamma_5 \neq \frac{1}{2}$, then $X = \frac{1}{2}\rho_{44}u_3$ is a steady left-invariant Ricci soliton. Finally, if $\gamma_9 = 0$, then G is a left-invariant Ricci soliton if and only if it is Ricci-flat.

Plane wave Lie groups in Assertion (iv) are Lorentzian extensions of unimodular degenerate Lie groups and correspond to Sect. 4.2.1.1. We proceed as in the previous case and eliminate the structure constants γ_3 and γ_6, so that the Ricci tensor vanishes if and only if $\rho_{44} = \frac{1}{2}\{\lambda_1^2 + 4\gamma_1\gamma_5 - (\gamma_2 + \gamma_4)^2\} = 0$, and the non-zero polynomials $\tilde{\mathfrak{P}}_{ij} = \frac{1}{2}\mathfrak{P}_{ij}$ are given by

$$\begin{array}{lll}\tilde{\mathfrak{P}}_{11} = 2X_4\gamma_1 - \mu, & \tilde{\mathfrak{P}}_{14} = -X_1\gamma_1 + X_2(\lambda_1 - \gamma_4), & \tilde{\mathfrak{P}}_{12} = X_4(\gamma_2 + \gamma_4),\\ \tilde{\mathfrak{P}}_{22} = 2X_4\gamma_5 - \mu, & \tilde{\mathfrak{P}}_{24} = -X_1(\lambda_1 + \gamma_2) - X_2\gamma_5, & \tilde{\mathfrak{P}}_{34} = X_4(\gamma_1 + \gamma_5) - \mu,\\ \tilde{\mathfrak{P}}_{44} = \rho_{44} - 2X_3(\gamma_1 + \gamma_5)\,. & & \end{array}$$

If $\gamma_1+\gamma_5=0$, then the existence of left-invariant Ricci solitons reduces to the Ricci-flat case. Assuming $\gamma_1+\gamma_5\neq 0$ one has two distinct possibilities. If $\gamma_2+\gamma_4=0$ and $\gamma_1-\gamma_5=0$, then $X=\frac{1}{2(\gamma_1+\gamma_5)}\rho_{44}u_3+\frac{1}{\gamma_1+\gamma_5}\mu u_4$ is a locally conformally flat expanding, steady or shrinking left-invariant Ricci soliton. Otherwise, if $\gamma_2+\gamma_4\neq 0$ or $\gamma_1-\gamma_5\neq 0$, then $X=\frac{1}{2(\gamma_1+\gamma_5)}\rho_{44}u_3$ is a steady left-invariant Ricci soliton. □

Remark 5.4 Plane wave Lie groups in Theorem 5.3 have vanishing Cotton tensor, and thus they are conformally Einstein [3]. Plane wave Lie groups corresponding to Assertion (iii) and Assertion (iv) which admit expanding, steady and shrinking left-invariant Ricci solitons are locally conformally flat.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

Data availability Not applicable in the manuscript as no datasets were generated or analysed during the current study.

References

1. Arroyo, R., Lafuente, R.: Homogeneous Ricci solitons in low dimensions. Int. Math. Res. Not. IMRN **13**, 4901–4932 (2015)
2. Blau, M., O'Loughlin, M.: Homogeneous plane waves. Nucl. Phys. B **654**(1–2), 135–176 (2003)
3. Brozos-Vázquez, M., García-Río, E., Valle-Regueiro, X.: Isotropic quasi-Einstein manifolds. Class. Quantum Gravity **36**(24), 245005 (2019)
4. Brozos-Vázquez, M., Calvaruso, G., García-Río, E., Gavino-Fernández, S.: Three-dimensional Lorentzian homogeneous Ricci solitons. Israel J. Math. **188**, 385–403 (2012)
5. Brozos-Vázquez, M., García-Río, E., Gavino-Fernández, S.: Locally conformally flat Lorentzian gradient Ricci solitons. J. Geom. Anal. **23**, 1196–1212 (2013)
6. Brozos-Vázquez, M., Caeiro-Oliveira, S., García-Río, E.: Homogeneous Ricci solitons and critical metrics (to appear)
7. Cahen, M., Leroy, J., Parker, M., Tricerri, F., Vanhecke, L.: Lorentz manifolds modelled on a Lorentz symmetric space. J. Geom. Phys. **7**(4), 571–581 (1990)
8. Calvaruso, G., Castrillón-López, M.: Cyclic Lorentzian Lie groups. Geom. Dedicata **181**, 119–136 (2016)
9. Calvaruso, G., Zaeim, A.: Four-dimensional Lorentzian Lie groups. Differ. Geom. Appl. **31**, 496–509 (2013)
10. Calvaruso, G., Storm, R., van der Veken, J.: Parallel and totally geodesic hypersurfaces of non-reductive homogeneous four-manifolds. Math. Nachr. **293**, 1707–1729 (2020)

11. Chow, B., Chu, S.-C., Glickenstein, D., Guenther, Ch., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci Flow: Techniques and Applications. Part I. Geometric Aspects. Mathematical Surveys and Monographs, vol. 135. American Mathematical Societ, Providence (2007)
12. Cox, D., Little, D., O'Shea, D.: Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra. Undergraduate Texts in Mathematics, Springer, Cham (2015)
13. Decker, W., Greuel, G.-M., Pfister, G., Schönemann, H.: SINGULAR 4-1-0—a computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2016)
14. di Cerbo, L.: Generic properties of homogeneous Ricci solitons. Adv. Geom. **14**, 225–237 (2014)
15. Fels, M.E., Renner, A.G.: Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four. Can. J. Math. **58**(2), 282–311 (2006)
16. García-Río, E., Gilkey, P., Nikcevic, S.: Homogeneity of Lorentzian three-manifolds with recurrent curvature. Math. Nachr. **287**, 32–47 (2014)
17. Globke, W., Leistner, Th.: Locally homogeneous pp-waves. J. Geom. Phys. **108**, 83–101 (2016)
18. Gover, A.R., Nagy, P.A.: Four-dimensional conformal C-spaces. Q. J. Math. **58**, 443–462 (2007)
19. Jensen, G.R.: Homogeneous Einstein spaces of dimension four. J. Diff. Geom. **3**, 309–349 (1969)
20. Kozameh, C.N., Newman, E.T., Tod, K.P.: Conformal Einstein spaces. Gen. Relativ. Gravit. **17**, 343–352 (1985)
21. Kowalski, O.: On regular curvature structures. Math. Z. **125**, 129–138 (1972)
22. Kulkarni, R.S.: Curvature and metric. Ann. Math. (2) **91**, 311–331 (1970)
23. Kühnel, W., Rademacher, H.B.: Conformal transformations of pseudo-Riemannian manifolds, recent developments in pseudo-Riemannian geometry. In: ESI Lectures in Mathematics and Physics. European Mathematical Society, Zürich, pp. 261–298 (2008)
24. Lauret, J.: Ricci soliton homogeneous nilmanifolds. Math. Ann. **319**, 715–733 (2001)
25. Leistner, Th.: Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds. Differ. Geom. Appl. **24**, 458–478 (2006)
26. Milnor, J.: Curvature of left invariant metrics on Lie groups. Adv. Math. **21**, 293–329 (1976)
27. O'Neill, B.: Semi-Riemannian geometry with applications to relativity. Academic Press, New York (1983)
28. Otero-Casal, T.: Métricas de Einstein en grupos de Lie lorentzianos, Publicaciones del Departamento de Geometría y Topología, Universidad de Santiago de Compostela
29. Rahmani, S.: Metriques de Lorentz sur les groupes de Lie unimodulaires, de dimension trois. J. Geom. Phys. **9**, 295–302 (1992)
30. Yan, Z., Deng, S.: Double extensions on Riemannian Ricci nilsolitons. J. Geom. Anal. **31**, 9996–10023 (2021)