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Uniform approximation by multivariate quasi-projection operators

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Abstract
Approximation properties of quasi-projection operators $Q_j(f, \varphi, \widetilde{\varphi})$ are studied. These operators are associated with a function φ satisfying the Strang–Fix conditions and a tempered distribution $\widetilde{\varphi}$ such that compatibility conditions with φ hold. Error estimates in the uniform norm are obtained for a wide class of quasi-projection operators defined on the space of uniformly continuous functions and on the anisotropic Besov-type spaces. Under additional assumptions on φ and $\widetilde{\varphi}$, two-sided estimates in terms of realizations of the K-functional are also established.

Keywords Quasi-projection operator · Anisotropic Besov-type space · Error estimate · Best approximation · Moduli of smoothness · Realization of K-functional

Mathematics Subject Classification 41A25 · 41A17 · 41A15 · 42B10 · 94A20 · 97N50

1 Introduction

Quasi-projection operators are a generalisation of the so-called scaling expansions

$$Q_j(f, \varphi, \widetilde{\varphi})(x) = 2^j \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}(2^j \cdot -k)\rangle \varphi(2^j x - k), \quad f, \varphi, \widetilde{\varphi} \in L_2(\mathbb{R}),$$

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playing an important role in the wavelet theory (see, e.g., [11, 13, 25, 26, 34]). Such expansions are also well defined for other classes of functions φ and $\widetilde{\varphi}$ whenever the inner product $\langle f, \widetilde{\varphi}(2^j \cdot -k)\rangle$ has meaning and the series converges in some sense. In [13], Jia considered a larger class of quasi-projection operators $Q_j(f, \varphi, \widetilde{\varphi})$ with compactly supported functions φ and $\widetilde{\varphi}$ and obtained error estimates in L_p and other function spaces for these operators. The classical Kantorovich-Kotelnikov operators (see, e.g., [9, 10, 19, 21, 30, 38]) are operators of the same form $Q_j(f, \varphi, \widetilde{\varphi})$ with the characteristic function of [0, 1] as $\widetilde{\varphi}$. Another classical special case of quasi-projection operators is the sampling expansion

$$Q_j(f, \varphi, \delta)(x) = \sum_{k\in\mathbb{Z}} f(2^{-j}k)\varphi(2^j x - k) = 2^j \sum_{k\in\mathbb{Z}} \langle f, \delta(2^j \cdot -k)\rangle \varphi(2^j x - k),$$

where δ is the Dirac delta-function. In the case $\varphi(x) = \operatorname{sinc} x := \sin \pi x/\pi x$, it is the classical Kotelnikov–Shannon expansion. Since δ is a tempered distribution, under the usual notation $\langle f, \delta\rangle := \delta(f)$, the operator $Q_j(f, \varphi, \delta)$ is defined only for functions f from the Schwartz class, but to extend this class, one can set $\langle f, \delta\rangle := \langle \widehat{f}, \widehat{\delta}\rangle$. The sampling expansion is of great applied importance, it is especially actively used by engineers working in signal processing. Approximation properties of the sampling operators $Q_j(f, \varphi, \delta)$ associated with different functions φ were studied by a lot of authors (see, e.g., [2–7, 14, 28, 33, 37]).

Given a matrix M, we define the multivariate quasi-projection operator $Q_j(f, \varphi, \widetilde{\varphi})$ associated with a function φ and a distribution/function $\widetilde{\varphi}$ as follows

$$Q_j(f, \varphi, \widetilde{\varphi})(x) = |\det M|^j \sum_{n\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}(M^j \cdot -n)\rangle \varphi(M^j x - n),$$

where the "inner product" $\langle f, \widetilde{\varphi}(M^j \cdot -n)\rangle$ has meaning in some sense. If the Fourier transform of f has enough decay, then the operators $Q_j(f, \varphi, \widetilde{\varphi})$ with $\langle f, \widetilde{\varphi}\rangle := \langle \widehat{f}, \widehat{\widetilde{\varphi}}\rangle$ are well defined for a wide class of tempered distributions $\widetilde{\varphi}$ and appropriate functions φ. For fast decaying functions φ, approximation properties of such quasi-projection operators were studied in [26]. Namely, the error estimates in the L_p-norm, $2 \le p \le \infty$, were given in terms of the Fourier transform of f, and the approximation order of $Q_j(f, \varphi, \widetilde{\varphi})$ was found for the isotropic matrices M. Similar results for a class of bandlimited functions φ and $p < \infty$ were obtained in [16], and then the class of functions φ was essentially extended in [8]. These results were improved in several directions in [20, 22] for fast decaying and bandlimited functions φ respectively. Namely, the error estimates in L_p-norm, $1 \le p < \infty$, were given in terms of the moduli of smoothness and the errors of best approximation. Moreover, the class of approximated functions f was extended and the requirement on smoothness of $\widehat{\varphi}$ and $\widehat{\widetilde{\varphi}}$ was weakened in [22] due to using the Fourier multipliers method.

The technique developed in [20, 22] does not work appropriately for $p = \infty$ because a function from L_∞ cannot be uniformly approximated by functions from L_2. In [22], the case $p = \infty$ is considered, but the estimates are obtained only for functions f satisfying the additional assumption $f(x) \to 0$ as $|x| \to \infty$. The goal of the present paper is to fix this drawback. To this end, we use a new technique based on convolution representations. In particular, we determine an "inner product" $\langle f, \widetilde{\varphi}(M^j \cdot +n)\rangle$ as a limit of some convolutions, so that the operator $Q_j(f, \varphi, \widetilde{\varphi})$ is well defined on the space of uniformly continuous functions and on the anisotropic Besov-type spaces whenever the series $\sum_{k\in\mathbb{Z}^d} |\varphi(x-k)|$ converges uniformly on any compact set. Using this approach, we obtain error estimates in the uniform norm for $Q_j(f, \varphi, \widetilde{\varphi})$ under the assumptions of the Strang–Fix conditions for φ, compatibility conditions for φ and $\widetilde{\varphi}$, and belonging of some special functions associated with φ and $\widetilde{\varphi}$ to Wiener's algebra. These results are similar to those for $p < \infty$ in [22], where some Fourier multiplier conditions are assumed instead of belonging to Wiener's algebra. Under additional assumptions on φ and $\widetilde{\varphi}$, two-sided estimates in terms of realizations of the K-functional are presented. A new Whittaker–Nyquist–Kotelnikov–Shannon-type theorem is also proved.

2 Notation

We use the standard multi-index notation. Let $x = (x_1, \dots, x_d)^T$ and $y = (y_1, \dots, y_d)^T$ be column vectors in $\mathbb{R}^d$, then $(x, y) := x_1y_1 + \dots + x_dy_d$, $|x| := \sqrt{(x, x)}$; $\mathbf{0} = (0, \dots, 0)^T \in \mathbb{R}^d$; $\mathbb{Z}_+^d := \{x \in \mathbb{Z}^d : x_k \geq 0, k = 1, \dots, d\}$, $\mathbb{T}^d = [-1/2, 1/2]^d$. If $r > 0$, then B_r denotes the ball of radius r with the center in $\mathbf{0}$. If $\alpha \in \mathbb{Z}_+^d$, $a, b \in \mathbb{R}^d$, we set

$$[\alpha] = \sum_{j=1}^d \alpha_j, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x^\alpha} = \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d},$$

$$a^b = \prod_{j=1}^d a_j^{b_j}, \quad \alpha! = \prod_{j=1}^d \alpha_j!.$$

If M is a $d \times d$ matrix, then $\|M\|$ denotes its operator norm in $\mathbb{R}^d$; M^* denotes the conjugate matrix to M, $m = |\det M|$; the identity matrix is denoted by I. A $d \times d$ matrix M whose eigenvalues are bigger than 1 in modulus is called a dilation matrix. We denote the set of all dilation matrices by $\mathfrak{M}$. It is well known that $\lim_{j\to\infty} \|M^{-j}\| = 0$ for any dilation matrix M. A matrix M is called isotropic if it is similar to a diagonal matrix such that numbers $\lambda_1, \dots, \lambda_d$ are placed on the main diagonal and $|\lambda_1| = \dots = |\lambda_d|$.

Let L_p denote the space $L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, with the norm $\|\cdot\|_p = \|\cdot\|_{L_p(\mathbb{R}^d)}$. As usual, C denotes the space of all uniformly continuous bounded functions on $\mathbb{R}^d$ equipped with the norm $\|f\| = \max_{x\in\mathbb{R}^d} |f(x)|$. We use W_p^n, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on $\mathbb{R}^d$, i.e. the set of functions whose derivatives up to order n are in L_p, with the usual Sobolev norm. Let $\mathcal{S}$ denote the Schwartz class of functions defined on $\mathbb{R}^d$. The dual space of $\mathcal{S}$ is $\mathcal{S}'$, i.e. $\mathcal{S}'$ is the space of tempered distributions.

If f, g are functions defined on $\mathbb{R}^d$ and $f\overline{g} \in L_1$, then $\langle f, g\rangle$ denotes the usual inner product; the convolution of appropriate functions f and g is denoted by $f * g$; the Fourier transform of $f \in \mathcal{S}'$ is denoted by $\mathcal{F}f = \widehat{f}$. We also set $f^-(x) = \overline{f(-x)}$.

We say that $f \in \mathcal{S}'$ belongs to Wiener's algebra W_0 if there exists a function $g \in L_1$ such that

$$f(x) = \int_{\mathbb{R}^d} g(\xi) e^{2\pi i(x,\xi)} d\xi. \tag{1}$$

The corresponding norm is given by $\|f\|_{W_0} = \|g\|_1$.

For a fixed matrix $M \in \mathfrak{M}$ and a function φ defined on $\mathbb{R}^d$, we set

$$\varphi_{jk}(x) := m^{j/2}\varphi(M^j x + k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{R}^d.$$

If $\widetilde{\varphi} \in \mathcal{S}'$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, then we define $\widetilde{\varphi}_{jk}$ by $\langle f, \widetilde{\varphi}_{jk}\rangle := \langle f_{-j,-M^{-j}k}, \widetilde{\varphi}\rangle$, $f \in \mathcal{S}$.

Let $\mathcal{L}_\infty$ denote the set of functions $\varphi \in L_\infty$ such that $\sum_{k\in\mathbb{Z}^d} |\varphi(\cdot + k)| \in L_\infty(\mathbb{T}^d)$, and let $\mathcal{L}C$ denote the set of continuous functions φ such that the series $\sum_{k\in\mathbb{Z}^d} |\varphi(\cdot + k)|$ converges uniformly on any compact set. Both $\mathcal{L}_\infty$ and $\mathcal{L}C$ are Banach spaces with the norm

$$\|\varphi\|_{\mathcal{L}C} = \|\varphi\|_{\mathcal{L}_\infty} := \left\| \sum_{k\in\mathbb{Z}^d} |\varphi(\cdot + k)| \right\|_{L_\infty(\mathbb{T}^d)}.$$

For any $d \times d$ matrix A, we introduce the space $\mathcal{B}_A := \{g \in L_\infty \,:\, \operatorname{supp} \widehat{g} \subset A^*\mathbb{T}^d\}$ and the corresponding error of best approximation

$$E_A(f) := \inf\{\|f - g\| \,:\, g \in \mathcal{B}_A\}.$$

Let α be a positive function defined on the set of all $d \times d$ matrices A. We introduce the anisotropic Besov-type space $\mathbb{B}_A^{\alpha(\cdot)}$ associated with a matrix A and α as the set of all functions f such that

$$\|f\|_{\mathbb{B}_A^{\alpha(\cdot)}} := \|f\| + \sum_{\nu=1}^{\infty} \alpha(A^\nu) E_{A^\nu}(f) < \infty.$$

Note that in the case $A = 2I_d$ and $\alpha(t) \equiv t^{\alpha_0}$, $\alpha_0 > 0$, the space $\mathbb{B}_A^{\alpha(\cdot)}$ coincides with the classical Besov space $B_{\infty,1}^{\alpha_0}(\mathbb{R}^d)$.

For any matrix $M \in \mathfrak{M}$, we denote by $\mathcal{A}_M$ the set of all positive functions $\alpha : \mathbb{R}^{d\times d} \to \mathbb{R}_+$ that satisfy the condition $\alpha(M^{\mu+1}) \le c(M)\alpha(M^\mu)$ for all $\mu \in \mathbb{Z}_+$.

For any $d \times d$ matrix A, we also introduce the anisotropic fractional modulus of smoothness of order s, $s > 0$,

$$\Omega_s(f, A) := \sup_{|A^{-1}t|<1, t\in\mathbb{R}^d} \|\Delta_t^s f\|,$$

where

$$\Delta_t^s f(x) := \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{s}{\nu} f(x + t\nu).$$

Recall that the standard fractional modulus of smoothness of order s, $s > 0$, is defined by

$$\omega_s(f, h) := \sup_{|t|<h} \|\Delta_t^s f\|, \quad h > 0. \tag{2}$$

We refer to [23] for the collection of basic properties of moduli of smoothness in $L_p(\mathbb{R}^d)$.

Let η denote a real-valued function in $C^\infty(\mathbb{R}^d)$ such that $\eta(\xi) = 1$ for $\xi \in \mathbb{T}^d$ and $\eta(\xi) = 0$ for $\xi \notin 2\mathbb{T}^d$. For $\delta > 0$ and a $d \times d$ matrix A, we set

$$\eta_\delta = \eta(\delta^{-1}\cdot) \quad \text{and} \quad \mathcal{N}_\delta = \mathcal{F}^{-1}\eta_\delta,$$
$$\eta_A = \eta(A^{*-1}\cdot) \quad \text{and} \quad \mathcal{N}_A = \mathcal{F}^{-1}\eta_A.$$

3 Preliminary information and main definitions

In what follows, we discuss the quasi-projection operators

$$Q_j(f, \varphi, \widetilde{\varphi}) := \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk},$$

where the "inner product" $\langle f, \widetilde{\varphi}_{jk}\rangle$ is defined in a special way and the series converges in some sense. The expansions $\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$ are elements of the shift-invariant spaces generated by φ. It is well known that a function f can be approximated by elements of such spaces only if φ satisfies the Strang–Fix conditions.

Definition 1 A function φ is said to satisfy *the Strang–Fix conditions* of order s if $D^\beta \widehat{\varphi}(k) = 0$ for every $\beta \in \mathbb{Z}_+^d$, $[\beta] < s$, and for all $k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$.

Certain compatibility conditions for a distribution $\widetilde{\varphi}$ and a function φ are also required to provide good approximation properties of the operator $Q_j(f, \varphi, \widetilde{\varphi})$. For our purposes, we use the following conditions.

Definition 2 A tempered distribution $\widetilde{\varphi}$ and a function φ are said to be *weakly compatible of order s* if $D^\beta(1 - \overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}})(\mathbf{0}) = 0$ for every $\beta \in \mathbb{Z}_+^d$, $[\beta] < s$.

Definition 3 A tempered distribution $\widetilde{\varphi}$ and a function φ are said to be *strictly compatible* if there exists $\delta > 0$ such that $\overline{\widehat{\varphi}}(\xi)\widehat{\widetilde{\varphi}}(\xi) = 1$ a.e. on $\delta\mathbb{T}^d$.

Denote by $\mathcal{S}'_N$, $N \geq 0$, the set of tempered distributions $\widetilde{\varphi}$ whose Fourier transforms $\widehat{\widetilde{\varphi}}$ are measurable functions on $\mathbb{R}^d$ such that $|\widehat{\widetilde{\varphi}}(\xi)| \leq C_{\widetilde{\varphi}}(1 + |\xi|)^N$ for almost all

$\xi \in \mathbb{R}^d$. Note that for $\widetilde{\varphi} \in \mathcal{S}'_N$ and appropriate classes of functions φ, such quasi-projection operators $Q_j(f, \varphi, \widetilde{\varphi})$ with $\langle f, \widetilde{\varphi}_{jk}\rangle := \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle$ were studied in [16, 21, 26, 34]. In particular, approximation by these operators in the uniform norm was considered in [26]. The following statement can be derived from Theorems 4 and 5 in [26].

Theorem A *Let $s \in \mathbb{N}$, $N \ge 0$, $\delta \in (0, 1/2)$, and $M \in \mathfrak{M}$. Suppose*

1) $\varphi, \widehat{\varphi} \in \mathcal{L}_\infty$;
2) $\widehat{\varphi}(\cdot + l) \in C^s(B_\delta)$ *for all* $l \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ *and* $\sum_{l\ne\mathbf{0}} \sum_{\|\beta\|_1=s} \sup_{|\xi|<\delta} |D^\beta\widehat{\varphi}(\xi + l)| < \infty$;
3) *the Strang–Fix conditions of order s are satisfied for φ;*
4) $\widetilde{\varphi} \in \mathcal{S}'_N$ *and* $\widehat{\widetilde{\varphi}}\widehat{\varphi} \in C^s(B_\delta)$;
5) φ *and* $\widetilde{\varphi}$ *are weakly compatible of order s.*

If $f \in L_\infty$ is such that $\widehat{f} \in L_1$ and $\widehat{f}(\xi) = \mathcal{O}(|\xi|^{-N-d-\varepsilon})$, $\varepsilon > 0$, as $|\xi| \to \infty$, then

$$\Big\| f - \sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle\varphi_{jk}\Big\|_\infty \le C_1\|M^{*-j}\|^N \int\limits_{|M^{*-j}\xi|\ge\delta} |\xi|^N|\widehat{f}(\xi)|\,d\xi + C_2\|M^{*-j}\|^s \int\limits_{|M^{*-j}\xi|\le\delta} |\xi|^s|\widehat{f}(\xi)|\,d\xi,$$

where the constants C_1 and C_2 do not depend on f and j. If, moreover, M is an isotropic matrix, then

$$\Big\| f - \sum_{k\in\mathbb{Z}^d} \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}}\rangle\varphi_{jk}\Big\|_\infty \le C_3 \begin{cases} |\lambda|^{-j(N+\varepsilon)} & \text{if } s > N+\varepsilon \\ (j+1)|\lambda|^{-js} & \text{if } s = N+\varepsilon \\ |\lambda|^{-js} & \text{if } s < N+\varepsilon \end{cases},$$

where λ is an eigenvalue of M and C_3 does not depend on j.

Unfortunately, there is a restriction on the decay of $\widehat{f}$ in Theorem A. Obviously, such a restriction is redundant for some special cases. Namely, the inner product $\langle f, \widetilde{\varphi}_{jk}\rangle$ has meaning for any $f \in L_\infty$ whenever $\widetilde{\varphi}$ is an integrable function. Moreover, Theorem A provides approximation order for $Q_j(f, \varphi, \widetilde{\varphi})$ only for isotropic matrices M, and even for this case, more accurate error estimates in terms of smoothness of f were not obtained in [26]. The mentioned drawbacks were avoided in [19,Theorem 17′], where the uniform approximation by quasi-projection operators $Q_j(f, \varphi, \widetilde{\varphi})$ associated with a summable function $\widetilde{\varphi}$ and a bandlimited function φ was investigated. To formulate this result, we need to introduce the space $\mathcal{B}$ consisting of functions φ such that $\varphi = \mathcal{F}^{-1}\theta$, where the function θ is supported in a rectangle $R \subset \mathbb{R}^d$ and $\theta\big|_R \in C^d(R)$.

Theorem B *Let $s \in \mathbb{N}$, $\delta, \delta' > 0$, $M \in \mathfrak{M}$, and $f \in L_\infty$. Suppose*

1) $\varphi \in \mathcal{B} \cap \mathcal{L}_\infty$, $\operatorname{supp}\widehat{\varphi} \subset B_{1-\delta'}$, *and* $\widehat{\varphi} \in C^{s+d+1}(B_\delta)$;
2) $\widetilde{\varphi} \in L_1$ *and* $\widehat{\widetilde{\varphi}} \in C^{s+d+1}(B_\delta)$;

3) *φ and $\widetilde{\varphi}$ are weakly compatible of order s.*

Then

$$\Big\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \Big\|_\infty \le C\,\omega_s\left(f, \|M^{-j}\|\right),$$

where C does not depend on f and j.

In what follows, we consider quasi-projection operators $Q_j(f, \varphi, \widetilde{\varphi})$ associated with a tempered distribution $\widetilde{\varphi}$ belonging to the class $\mathcal{S}'_{\alpha;M}$, $M \in \mathfrak{M}$, $\alpha \in \mathcal{A}_M$. This class is defined as follows. We say that $\widetilde{\varphi} \in \mathcal{S}'$ belongs to $\mathcal{S}'_{\alpha;M}$ if $\widehat{\widetilde{\varphi}}$ is a measurable locally bounded function and the function $\mathcal{N}_{M^\nu} * \widetilde{\varphi}^-$ is summable and

$$\|\mathcal{N}_{M^\nu} * \widetilde{\varphi}^-\|_1 \le c\,\alpha(M^\nu) \quad \text{for all} \quad \nu \in \mathbb{Z}_+, \tag{3}$$

where the constant c is independent of ν.

Let us show that inequality (3) is satisfied for the most important special cases of $\widetilde{\varphi}$. Since $\|\mathcal{N}_{M^\nu}\|_1 = \|\mathcal{N}_1\|_1 = \|\widehat{\eta}\|_1$, one can easily see that (3) with $\alpha \equiv 1$ holds true if $\widetilde{\varphi} \in L_1$ or $\widetilde{\varphi}$ is the Dirac delta-function δ. Now let $\widetilde{\varphi}$ be a distribution associated with the differential operator D^β, $\beta \in \mathbb{Z}^d_+$, i.e., $\widetilde{\varphi}(x) = (-1)^{[\beta]} D^\beta \delta(x)$ (see [16]). In this case, we have

$$\mathcal{N}_{M^\nu} * \widetilde{\varphi}^- = \mathcal{F}^{-1}(\widehat{\mathcal{N}_{M^\nu}}\,\widehat{D^\beta\delta}) = \mathcal{F}^{-1}(\widehat{D^\beta \mathcal{N}_{M^\nu}}) = D^\beta \mathcal{N}_{M^\nu}.$$

If $M = \operatorname{diag}(m_1, \dots, m_d)$ and $\alpha(M) = m_1^{\beta_1} \dots m_d^{\beta_d}$, then using Bernstein's inequality (see, e.g., [35,p. 252]), we get

$$\|\mathcal{N}_{M^\nu} * \widetilde{\varphi}^-\|_1 = \|D^\beta \mathcal{N}_{M^\nu}\|_1 \le m_1^{\beta_1} \dots m_d^{\beta_d} \|\mathcal{N}_{M^\nu}\|_1 = \alpha(M)\|\widehat{\eta}\|_1.$$

Similarly, if M is an isotropic matrix, then $\widetilde{\varphi}$ belongs to the class $\mathcal{S}'_{\alpha;M}$ with $\alpha(M) = m^{[\beta]/d}$.

To extend the operator $Q_j(f, \varphi, \widetilde{\varphi})$ associated with $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$ onto the space $\mathbb{B}_M^{\alpha(\cdot)}$ and onto C, we need to define the "inner product" $\langle f, \widetilde{\varphi}_{jk}\rangle$ properly. Note that a similar extension in the case $p < \infty$ was implemented in [20, 22], but the definition of $\langle f, \widetilde{\varphi}_{jk}\rangle$ given there is not suitable for us now.

Definition 4 Let $M \in \mathfrak{M}$, $\alpha \in \mathcal{A}_M$, $\delta \in (0, 1]$, $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$, and $T_\mu \in \mathcal{B}_{\delta M^\mu}$, $\mu \in \mathbb{Z}_+$, be such that

$$\|f - T_\mu\| \le c(d) E_{\delta M^\mu}(f). \tag{4}$$

For every $f \in \mathbb{B}_M^{\alpha(\cdot)}$ ($f \in C$ in the case $\alpha \equiv \text{const}$), we set

$$\langle f, \widetilde{\varphi}_{0k}\rangle := \lim_{\mu\to\infty} T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)(k), \quad k \in \mathbb{Z}^d, \tag{5}$$

and

$$\langle f, \widetilde{\varphi}_{jk}\rangle := m^{-j/2}\langle f(M^{-j}\cdot), \widetilde{\varphi}_{0k}\rangle, \quad j \in \mathbb{Z}_+.$$

To approve this definition, we note that, according to Lemma 11 (see below), the limit in (5) exists and does not depend on the choice of δ and functions T_μ.

Remark 5 Since $\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-$ belongs to L_2, the convolution in (5) is also well-defined for all $T_\mu \in L_2$ and

$$T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)(k) = \langle \widehat{T_\mu}, \widehat{\widetilde{\varphi}_{0,-k}}\rangle. \tag{6}$$

Further, if a function f satisfies the conditions of Theorem A, then the functional $\langle f, \widetilde{\varphi}_{0k}\rangle$ defined in the sense of Definition 4 coincides with the functional $\langle \widehat{f}, \widehat{\widetilde{\varphi}_{0,-k}}\rangle$ from Theorem A. Indeed, since $\widehat{f} \in L_1$, we have $\lim_{|x|\to\infty} f(x) = 0$, and, due to Lemma 15 in [22], one can choose functions $T_\mu \in \mathcal{B}_{\delta M^\mu} \cap L_2$ satisfying (4). Then repeating the arguments in [20,Remark 12], we obtain

$$\langle \widehat{f}, \widehat{\widetilde{\varphi}_{0k}}\rangle = \lim_{\mu\to\infty} \langle \widehat{T_\mu}, \widehat{\widetilde{\varphi}_{0,-k}}\rangle,$$

which together with (6) yields the equality $\langle f, \widetilde{\varphi}_{0k}\rangle = \langle \widehat{f}, \widehat{\widetilde{\varphi}_{0,-k}}\rangle$.

Finally we note that the main results of this paper are given in terms of Wiener's algebra W_0. Various conditions of belonging to W_0 are overviewed in detail in the survey [27] (see also [15, 17, 18], for some new efficient sufficient conditions).

4 Main results

Let $M \in \mathfrak{M}$, $\alpha \in \mathcal{A}_M$, $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$, and $\varphi \in \mathcal{LC}$. Suppose that $f \in \mathbb{B}_M^{\alpha(\cdot)}$ or $f \in C$ in the case $\alpha \equiv \text{const}$. In what follows, we understand $\langle f, \widetilde{\varphi}_{jk}\rangle$ in the sense of Definition 4. By Lemmas 11 and 13 below, we have that $\{\langle f, \widetilde{\varphi}_{jk}\rangle\}_k \in \ell_\infty$. This together with Lemma 12 implies that the quasi-projection operators

$$Q_j(f, \varphi, \widetilde{\varphi}) = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$$

are well defined.

Theorem 6 *Let $M \in \mathfrak{M}$, $\alpha \in \mathcal{A}_M$, $s \in \mathbb{N}$, and $\delta \in (0, 1]$. Suppose*

1) *$\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$ and $\varphi \in \mathcal{LC}$;*
2) *the Strang–Fix condition of order s holds for φ;*
3) *φ and $\widetilde{\varphi}$ are weakly compatible of order s;*
4) *$\eta_\delta D^\beta \overline{\widehat{\widetilde{\varphi}}}\widehat{\varphi} \in W_0$ and $\eta_\delta D^\beta \widehat{\varphi}(\cdot + l) \in W_0$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] = s$, and $l \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$;*
5) *$\sum_{l\neq\mathbf{0}} \|\eta_\delta D^\beta \widehat{\varphi}(\cdot + l)\|_{W_0} < \infty$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] = s$.*

Then, for any $f \in \mathbb{B}_M^{\alpha(\cdot)}$, *we have*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c \left(\Omega_s(f, M^{-j}) + \sum_{\nu=j}^{\infty} \alpha(M^{\nu-j}) E_{M^\nu}(f) \right). \tag{7}$$

Moreover, if $\widetilde{\varphi} \in \mathcal{S}'_{\mathrm{const};M}$ *and* $f \in C$, *then*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c\, \Omega_s(f, M^{-j}). \tag{8}$$

In the above inequalities, the constant c does not depend on f *and* j.

Remark 7 Assumptions 4) and 5) in Theorem 6 may be given in terms of the Fourier multipliers as in [22,Theorem 20]. In this case, Theorem 6 holds true for any $p > 1$ after rewriting some basic notation in a more general form. Indeed, although the definition of the "inner products" $\langle f, \widetilde{\varphi}_{jk} \rangle$ for the operators Q_j in [22] is different from Definition 4, it may also be used for $p < \infty$ (see Remark 5).

Theorem 6 and the Beurling-type sufficient condition for belonging to Wiener's algebra given in [27,Theorem 6.1] imply the following more convenient to use statement.

Corollary 8 *Let* $s \in \mathbb{N}$, $\delta \in (0, 1]$, $M \in \mathfrak{M}$, $\alpha \in \mathcal{A}_M$, $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$, *and* $\varphi \in \mathcal{L}C$. *Suppose that conditions 2) and 3) of Theorem 6 are satisfied and, additionally, for some* $k \in \mathbb{N}$, $k > d/2$,

$$\widehat{\widetilde{\varphi}}\widehat{\varphi} \in W_2^{s+k}(2\delta\mathbb{T}^d), \quad \widehat{\varphi}(\cdot + l) \in W_2^{s+k}(2\delta\mathbb{T}^d) \ \ \textit{for all} \ \ l \in \mathbb{Z}^d \setminus \{\mathbf{0}\},$$

and

$$\sum_{l \ne \mathbf{0}} \|D^\beta \widehat{\varphi}(\cdot + l)\|_{L_2(2\delta\mathbb{T}^d)}^{1-\frac{d}{2k}} < \infty \ \ \textit{for all} \ \ \beta \in \mathbb{Z}_+^d, \ \ [\beta] = s.$$

Then inequalities (7) *and* (8) *hold true.*

Note that more general efficient sufficient conditions on smoothness of $\widehat{\varphi}$ and $\widehat{\widetilde{\varphi}}$ can be obtained by exploiting the results of the papers [15, 17, 18].

Theorem 9 *Let* $\delta \in (0, 1]$, $M \in \mathfrak{M}$, *and* $\alpha \in \mathcal{A}_M$. *Suppose that* $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$ *and* $\varphi \in \mathcal{L}C$ *satisfy the following conditions:*

1) $\operatorname{supp} \widehat{\varphi} \subset \mathbb{T}^d$;
2) φ *and* $\widetilde{\varphi}$ *are strictly compatible with respect to* δ.

Then, for any $f \in \mathbb{B}_M^{\alpha(\cdot)}$, *we have*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c \sum_{\nu=j}^{\infty} \alpha(M^{\nu-j}) E_{\delta M^\nu}(f). \tag{9}$$

Moreover, if $\widetilde{\varphi} \in \mathcal{S}'_{\mathrm{const};M}$ *and* $f \in C$, *then*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c\, E_{\delta M^j}(f). \tag{10}$$

In the above inequalities, the constant c *does not depend on* f *and* j.

As a corollary from this theorem, we obtain the following new Whittaker–Nyquist–Kotelnikov–Shannon-type formula.

Corollary 10 *Let* $M \in \mathfrak{M}$ *be such that* $\mathbb{T}^d \subset M^*\mathbb{T}^d$, $\delta \in (0, 1]$, *and let* φ, $\widetilde{\varphi}$ *be as in Theorem 9. If* $f \in \mathcal{B}_{\delta M^j}$, *then*

$$f(x) = \sum_{k\in\mathbb{Z}^d} (f(M^{-j}\cdot) * \mathcal{N}_1 * \widetilde{\varphi}^-)(k)\varphi(M^j x - k) \quad \text{for all} \quad x \in \mathbb{R}^d.$$

5 Auxiliary results

Lemma 11 *Let* $M \in \mathfrak{M}$, $n \in \mathbb{N}$, $\delta \in (0, 1]$, *and* $\alpha \in \mathcal{A}_M$. *Suppose that* $\widetilde{\varphi}$, f, *and* T_μ, $\mu \in \mathbb{Z}_+$, *are as in Definition 4 and*

$$q_\mu(k) := T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)(k), \quad k \in \mathbb{Z}^d.$$

Then the sequence $\{\{q_\mu(k)\}_k\}_{\mu=1}^\infty$ *converges in* ℓ_∞ *as* $\mu \to \infty$ *and its limit does not depend on the choice of* T_μ *and* δ; *a fortiori for every* $k \in \mathbb{Z}^d$ *there exists a limit* $\lim_{\mu\to\infty} q_\mu(k)$ *independent on the choice of* T_μ *and* δ. *Moreover, for all* $f \in \mathbb{B}_M^{\alpha(\cdot)}$, *we have*

$$\sum_{\mu=n}^{\infty} \|\{q_{\mu+1}(k) - q_\mu(k)\}_k\|_{\ell_\infty} \le c \sum_{\mu=n}^{\infty} \alpha(M^\mu) E_{\delta M^\mu}(f), \tag{11}$$

where the constant c *depends only on* d *and* M.

Proof Denote $\mu_0 = \min\{\mu \in \mathbb{N} : \mathbb{T}^d \subset \frac{1}{2} M^{*\nu}\mathbb{T}^d \quad \text{for all} \quad \nu \ge \mu - 1\}$. It is easy to see that $\eta(M^{*-\mu}\cdot)\eta(M^{*-\mu-\mu_0}\cdot) = \eta(M^{*-\mu}\cdot)$, and hence $\mathcal{N}_{M^\mu} = \mathcal{N}_{M^\mu} * \mathcal{N}_{M^{\mu+\mu_0}}$. Similarly, $\mathcal{N}_{M^{\mu+1}} = \mathcal{N}_{M^{\mu+1}} * \mathcal{N}_{M^{\mu+\mu_0}}$. Thus, taking into account that $T_\mu = T_\mu * \mathcal{N}_{M^\mu}$ and $T_{\mu+1} = T_{\mu+1} * \mathcal{N}_{M^{\mu+1}}$, we have

$$T_\nu * \mathcal{N}_{M^\nu} * \widetilde{\varphi}^- = T_\nu * \mathcal{N}_{M^\nu} * \mathcal{N}_{M^{\mu+\mu_0}} * \widetilde{\varphi}^- = T_\nu * \mathcal{N}_{M^{\mu+\mu_0}} * \widetilde{\varphi}^-, \quad \nu = \mu, \mu + 1.$$

It follows that

$$q_{\mu+1}(k) - q_\mu(k) = (T_{\mu+1} - T_\mu) * (\mathcal{N}_{M^{\mu+\mu_0}} * \widetilde{\varphi}^-)(k).$$

Then, using (3), we obtain

$$\begin{aligned}\|\{q_{\mu+1}(k)-q_\mu(k)\}_k\|_{\ell_\infty} &\le \|(T_{\mu+1}-T_\mu)*(\mathcal{N}_{M^{\mu+\mu_0}}*\widetilde{\varphi}^-)\|\\ &\le \|\mathcal{N}_{M^{\mu+\mu_0}}*\widetilde{\varphi}^-\|_1\|T_{\mu+1}-T_\mu\| \le c\alpha(M^{\mu+\mu_0})\|T_{\mu+1}-T_\mu\| \\ &\le c_1\big(\alpha(M^\mu)E_{\delta M^\mu}(f)+\alpha(M^{\mu+1})E_{\delta M^{\mu+1}}(f)\big),\end{aligned} \tag{12}$$

which after the corresponding summation implies (11).

Next, it is clear that there exists $\nu(\delta)\in\mathbb{N}$ such that $E_{\delta M^\mu}(f)\le E_{M^{\mu-\nu(\delta)}}(f)$ and $\alpha(M^\mu)\le C(\delta)\alpha(M^{\mu-\nu(\delta)})$ for big enough μ. Thus, if $f\in\mathbb{B}_M^{\alpha(\cdot)}$, then it follows from (11) that $\{\{q_\mu(k)\}_k\}_{\mu=1}^\infty$ is a Cauchy sequence in ℓ_∞. Fortiori, for every $k\in\mathbb{Z}^d$, the sequence $\{q_\mu(k)\}_{\mu=1}^\infty$ has a limit.

Now let $\alpha=$ const and $f\in C$. For every $\mu',\mu''\in\mathbb{N}$, there exists $\nu\in\mathbb{N}$ such that both $\widehat{T_{\mu'}}$ and $\widehat{T_{\mu''}}$ are supported in $M^{*\nu}\mathbb{T}^d$, and similarly to (12), we have

$$\begin{aligned}\|\{q_{\mu'}(k)-q_{\mu''}(k)\}_k\|_{\ell_\infty} &\le \|(T_{\mu'}-T_{\mu''})*(\mathcal{N}_{M^\nu}*\widetilde{\varphi}^-)\|\\ &\le c_2\big(E_{\delta M^{\mu'}}(f)+E_{\delta M^{\mu''}}(f)\big).\end{aligned}$$

Thus, again $\{\{q_\mu(k)\}_k\}_{\mu=1}^\infty$ is a Cauchy sequence in ℓ_∞, and every sequence $\{q_\mu(k)\}_{\mu=1}^\infty$ has a limit.

Let us check that the limit of $\{\{q_\mu(k)\}_k\}_{\mu=1}^\infty$ in ℓ_∞ does not depend on the choice of T_μ and δ. Let $\delta'\in(0,1]$ and $T'_\mu\in\mathcal{B}_{\delta' M^\mu}$ be such that $\|f-T'_\mu\|\le c'(d)E_{\delta' M^\mu}(f)$ and $q'_\mu(k)=T'_\mu*(\mathcal{N}_{M^\mu}*\widetilde{\varphi}^-)(k)$. Since both T_μ and T'_μ belong to $\mathcal{B}_{M^\mu}$, repeating the arguments of the proof of inequality (12) with T'_μ instead of $T_{\mu+1}$ and 0 instead of μ_0, we obtain

$$\|\{q'_\mu(k)-q_\mu(k)\}_k\|_{\ell_\infty}\le c_3\alpha(M^\mu)\|T'_\mu-T_\mu\|\le c_4\alpha(M^\mu)(E_{\delta M^\mu}(f)+E_{\delta' M^\mu}(f)).$$

It follows that $\|\{q'_\mu(k)-q_\mu(k)\}_k\|_{\ell_\infty}\to 0$ as $\mu\to\infty$, which yields the independence on the choice of T_μ and δ. ◇

The proof of the next lemma is obvious.

Lemma 12 *Let $\varphi\in\mathcal{L}_\infty$ and $\{a_k\}_{k\in\mathbb{Z}^d}\in\ell_\infty$. Then*

$$\left\|\sum_{k\in\mathbb{Z}^d}a_k\varphi_{0k}\right\|\le\|\varphi\|_{\mathcal{L}_\infty}\|\{a_k\}_k\|_{\ell_\infty}.$$

Lemma 13 *Let $f\in C$, $M\in\mathfrak{M}$, and $\widetilde{\varphi}\in\mathcal{S}'_{\mathrm{const};M}$. Then*

$$\|\{\langle f,\widetilde{\varphi}_{0k}\rangle\}_k\|_{\ell_\infty}\le c\,\|f\|,$$

where the constant c does not depend on f.

Proof By Lemma 11, for any $\varepsilon > 0$, there exists a function $T_\mu \in \mathcal{B}_{M^\mu}$ such that $\|f - T_\mu\| \le c_1 E_{M^\mu}(f)$ and

$$\|\{\langle f, \widetilde{\varphi}_{0k}\rangle - T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)(k)\}_k\|_{\ell_\infty} < \varepsilon. \tag{13}$$

Moreover, due to (3), we have

$$|T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)(k)| \le \|T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)\| \le c_2\|T_\mu\| \le c_3\|f\|.$$

Combining this with (13), we prove the lemma. ◇

In the next two lemmas, we recall some basic inequalities for the error of best approximation and the modulus of smoothness.

Lemma 14 (SEE *[29,5.2.1 (7)]*OR *[35,5.3.3]) Let $f \in C$ and $s \in \mathbb{N}$. Then*

$$E_I(f) \le c\,\omega_s(f, 1),$$

where the constant c does not depend on f.

Lemma 15 (SEE *[39]* OR *[23]) Let $s \in \mathbb{N}$ and $T \in \mathcal{B}_I$. Then*

$$\sum_{[\beta]=s} \|D^\beta T\| \le c\,\omega_s(T, 1),$$

where the constant c does not depend on T.

6 Estimates in terms of *K*-functionals

In Theorem 6, the uniform error estimates for the operators $Q_j(f, \varphi, \widetilde{\varphi})$ are given in terms of the classical moduli of smoothness. Similar estimates in the L_p-norm, $p < \infty$, were obtained in our recent paper [22], where we also established the corresponding lower estimates of approximation in terms of the same moduli of smoothness. It turns out that in the uniform metric the classical moduli of smoothness are not suitable to obtain sharp estimates of approximation (or two-sided inequalities) in the general case (see, e.g, [36,Ch. 9]). However, this situation can be improved by using K-functionals and their realizations instead of the classical moduli of smoothness (see, e.g., [36,Ch. 9], [1, 24]). For our purposes, it is convenient to use realizations of K-functionals.

In what follows, we need additional notation. We say that a function ρ belongs to the class $\mathcal{H}_s$, $s > 0$, if $\rho \in C^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\})$ is homogeneous of degree s, i.e., $\rho(\tau\xi) = \tau^s\rho(\xi)$, $\xi \in \mathbb{R}^d$. Any function $\rho \in \mathcal{H}_s$ generates a Weyl-type differentiation operator as follows:

$$\mathcal{D}(\rho)g := \mathcal{F}^{-1}(\rho\widehat{g}), \quad g \in \mathcal{S}.$$

For functions $T \in \mathcal{B}_{M^j}$, we define $\mathcal{D}(\rho)T$ by

$$\mathcal{D}(\rho)T := (\mathcal{D}(\rho)\mathcal{N}_{M^j}) * T.$$

Note that this operator is well defined because the function $\mathcal{F}^{-1}(\mathcal{D}(\rho)\mathcal{N}_{M^j})$ is summable on $\mathbb{R}^d$, see, e.g., [31].

Important examples of the Weyl-type operators are the linear differential operator

$$P_m(D)f = \sum_{[k]=m} a_k \frac{\partial^{k_1+\dots+k_d}}{\partial x_1^{k_1}\dots\partial x_d^{k_d}} f,$$

which corresponds to $\rho(\xi) = \sum_{[k]=m} a_k (i\xi_1)^{k_1}\dots(i\xi_d)^{k_d}$; the fractional Laplacian $(-\Delta)^{s/2}f$ (here $\rho(\xi) = |\xi|^s$, $\xi \in \mathbb{R}^d$); the classical Weyl derivative $f^{(s)}$ (here $\rho(\xi) = (i\xi)^s$, $\xi \in \mathbb{R}$).

The realization of the K-functional generated by a function $\rho \in \mathcal{H}_s$ is defined by

$$\mathcal{R}_\rho(f, M^{-j}) = \inf_{T\in\mathcal{B}_{M^j}} \{\|f - T\| + \|\mathcal{D}(\rho(M^{*-j}\cdot))T\|\}.$$

In many cases, the realization $\mathcal{R}_\rho(f, M^{-j})$ is equivalent to the corresponding modulus of smoothness or K-functional (see, e.g., [1, 23, 24]). For example, if $d = 1$ and $\rho(\xi) = (i\xi)^s$, then

$$\mathcal{R}_\rho(f, M^{-j}) \asymp \omega_s(f, M^{-j}),$$

where $\omega_s(f, M^{-j})$ is the fractional modulus of smoothness defined in (2). If $d = 1$ and $\rho(\xi) = |\xi|^s$, then

$$\mathcal{R}_\rho(f, M^{-j}) \asymp \sup_{|M^j h|\le 1} \left\| \sum_{\nu\ne 0} \frac{f(\cdot+\nu h) - f(\cdot)}{|\nu|^{s+1}} \right\|.$$

In the case $d \ge 1$ and $\rho(\xi) = |\xi|^2$, we have

$$\mathcal{R}_\rho(f, M^{-j}) \asymp \sup_{|M^j h|\le 1} \left\| \sum_{j=1}^{d} \big(f(\cdot + h\mathrm{e}_j) - 2f(\cdot) + f(\cdot - h\mathrm{e}_j)\big) \right\|,$$

where $\{\mathrm{e}_j\}_{j=1}^d$ is the standard basis in $\mathbb{R}^d$.

Theorem 16 *Let $s > 0$, $\rho \in \mathcal{H}_s$, $\delta \in (0, 1/2)$, $M \in \mathfrak{M}$, and $\alpha \in \mathcal{A}_M$. Suppose that $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$ and $\varphi \in \mathcal{LC}$ satisfy the following conditions:*

1) $\operatorname{supp}\widehat{\varphi} \subset \mathbb{T}^d$;
2) $\eta_\delta \frac{1-\widehat{\varphi}\widehat{\widetilde{\varphi}}}{\rho} \in W_0$.

Then, for any $f \in \mathbb{B}_M^{\alpha(\cdot)}$, *we have*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c \left(\mathcal{R}_\rho(f, M^{-j}) + \sum_{\nu=j}^{\infty} \alpha(M^{\nu-j}) E_{M^\nu}(f) \right).$$

Moreover, if $\widetilde{\varphi} \in \mathcal{S}'_{\mathrm{const};M}$ *and* $f \in C$, *then*

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c\, \mathcal{R}_\rho(f, M^{-j}).$$

In the above inequalities, the constant c does not depend on f *and* j.

In the next theorem, we obtain lower estimates of the approximation error by the quasi-projection operators $Q_j(f, \varphi, \widetilde{\varphi})$. Note that such type of estimates are also called strong converse inequalities, see, e.g., [12].

Theorem 17 *Let* $s > 0$, $\rho \in \mathcal{H}_s$, $M \in \mathfrak{M}$, *and* $\alpha \in \mathcal{A}_M$. *Suppose that* $\widetilde{\varphi} \in \mathcal{S}'_{\alpha;M}$ *and* $\varphi \in \mathcal{LC}$ *satisfy the following conditions:*

1) $\mathrm{supp}\, \widehat{\varphi} \subset \mathbb{T}^d$;
2) $\eta \frac{\rho}{1-\widehat{\varphi}\widehat{\widetilde{\varphi}}} \in W_0$.

Then, for any $f \in \mathbb{B}_M^{\alpha(\cdot)}$, *we have*

$$\mathcal{R}_\rho(f, M^{-j}) \le c\, \|f - Q_j(f, \varphi, \widetilde{\varphi})\| + c \sum_{\nu=j}^{\infty} \alpha(M^{\nu-j}) E_{M^\nu}(f). \tag{14}$$

Moreover, if $\widetilde{\varphi} \in \mathcal{S}'_{\mathrm{const};M}$ *and* $f \in C$, *then*

$$\mathcal{R}_\rho(f, M^{-j}) \le c\, \|f - Q_j(f, \varphi, \widetilde{\varphi})\|. \tag{15}$$

In the above inequalities, the constant c does not depend on f *and* j.

7 Remarks and examples

Applying Lemma 14, we can present estimate (7) in more traditional form for direct theorems of approximation theory, which are usually given only in terms of moduli of smoothness. The same concerns the formulation of Theorem 16.

Remark 18 Under the conditions of Theorem 6, we have that for each $f \in \mathbb{B}_M^{\alpha(\cdot)}$ the following inequality holds

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c \sum_{\nu=j}^{\infty} \alpha(M^{\nu-j}) \Omega_s(f, M^{-\nu}),$$

where the constant c does not depend on f and j.

Remark 19 The conditions on φ and $\widetilde{\varphi}$ in Theorems 16 and 17 can also be given in terms of smoothness of $\widehat{\varphi}$ and $\widehat{\widetilde{\varphi}}$, similarly to those given in Corollary 8. For this, one can use the sufficient conditions for belonging to Wiener's algebra given in [27] (see also [15, 18]).

Example 1 Let $s = 2$, $f \in C$, and let $Q_j(f, \varphi, \widetilde{\varphi})$ be a mixed sampling-Kantorovich quasi-projection operator associated with $\varphi(x) = 4^{-d}\prod_{l=1}^{d} \mathrm{sinc}^3(x_l/4)$ and

$$\widetilde{\varphi}(x) = \prod_{l=1}^{d'} \chi_{[-1/2,1/2]}(x_l) \prod_{l=d'+1}^{d} \delta(x_l).$$

Thus,

$$Q_j(f, \varphi, \widetilde{\varphi})(x) = \sum_{k\in\mathbb{Z}^d} \int_{k_1-1/2}^{k_1+1/2} dt_1 \dots \int_{k_{d'}-1/2}^{k_{d'}+1/2} dt_{d'}\, f(M^{-j}(t+k))\Big|_{t_{d'+1}=\dots=t_d=0} \varphi(Mx-k).$$

It is easy to see that all assumptions of Theorem 6 for the case $\widetilde{\varphi} \in \mathcal{S}'_{\mathrm{const};\,M}$ are satisfied, which implies

$$\|f - Q_j(f, \varphi, \widetilde{\varphi})\| \le c\,\Omega_2(f, M^{-j}).$$

Example 2 Let $d = 1$, $\varphi(x) = \mathrm{sinc}^4(x)$, and $\widetilde{\varphi}(x) = \chi_{\mathbb{T}}(x)$ (the characteristic function of $\mathbb{T}$). Then all conditions of Theorems 16 and 17 are satisfied. Therefore, for any $f \in C$, we have

$$\left\| f - \sum_{k\in\mathbb{Z}} M^j \left(\int_{M^{-j}\mathbb{T}} f(M^{-j}k - t)dt \right) \mathrm{sinc}^4(M^j \cdot -k) \right\| \asymp \omega_2(f, M^{-j}), \quad (16)$$

where $\asymp$ is a two-sided inequality with constants independent of f and j.

Now we consider approximation by quasi-projection operators generated by the Bochner-Riesz kernel of fractional order.

Example 3 Let $\varphi(x) = R_s^\gamma(x) := \mathcal{F}^{-1}\left((1-|3\xi|^s)_+^\gamma\right)(x)$, $s > 0$, $\gamma > \frac{d-1}{2}$, and $\rho(\xi) = |\xi|^s$.

1) If $\widetilde{\varphi}(x) = \delta(x)$, then for any $f \in C$, we have

$$\left\| f - m^j \sum_{k\in\mathbb{Z}^d} f(M^{-j}k) R_s^\gamma(M^j \cdot -k) \right\| \asymp \mathcal{R}_\rho(f, M^{-j}). \quad (17)$$

2) If $\widetilde{\varphi}(x) = \chi_{\mathbb{T}^d}(x)$, then for any $f \in C$ and $s \in (0, 2]$, we have

$$\left\| f - m^j \sum_{k\in\mathbb{Z}^d} \left(\int_{M^{-j}\mathbb{T}^d} f(M^{-j}k - t)dt \right) R_s^\gamma(M^j \cdot -k) \right\| \asymp \mathcal{R}_\rho(f, M^{-j}). \quad (18)$$

In the above two relations, the symbol $\asymp$ denotes a two-sided inequality with positive constants independent of f and j. Note that estimate (17) follows from Theorems 16 and 17 and the fact that

$$\frac{\eta(\xi)(1-(1-|3\xi|^s)_+^\gamma)}{|\xi|^s} \in W_0 \quad \text{and} \quad \frac{\eta(\xi)|\xi|^s}{1-(1-|3\xi|^s)_+^\gamma} \in W_0. \tag{19}$$

The proof of relations (19) can be found, e.g., in [32]. The proof of (18) is similar. In this case, instead of (19), we use the following relations:

$$\frac{\eta(\xi)(1-\operatorname{sinc}(\xi)(1-|3\xi|^s)_+^\gamma)}{|\xi|^s} \in W_0 \quad \text{and} \quad \frac{\eta(\xi)|\xi|^s}{1-\operatorname{sinc}(x)(1-|3\xi|^s)_+^\gamma} \in W_0,$$

which can be verified using the same arguments as in [32].

8 Proofs

Proof of Theorem 6 First we prove the inequality

$$\left\| T - \sum_{k\in\mathbb{Z}^d} \langle T, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} \right\| \le c_1 \sum_{[\beta]=s} \|D^\beta T\|, \quad T \in \mathcal{B}_{\delta I}. \tag{20}$$

Set $\widetilde{\Phi}(x) = T * (\mathcal{N}_\delta * \widetilde{\varphi}^-)(-x)$. Since $\widetilde{\Phi} \in L_\infty$ and $\varphi \in \mathcal{LC} \subset L_1$, the function

$$g_x(y) = \sum_{\nu\in\mathbb{Z}^d} \widetilde{\Phi}(-y+\nu)\varphi(x-y+\nu), \quad x \in \mathbb{R}^d,$$

is continuous on $\mathbb{R}^d$ and summable on $\mathbb{T}^d$. Let us check that the Fourier series of g_x is absolutely convergent, i.e.,

$$\sum_{k\in\mathbb{Z}^d} |\widehat{g_x}(k)| = \sum_{k\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} \widetilde{\Phi}(-y)\varphi(x-y)e^{-2\pi i(y,k)}dy \right| < \infty. \tag{21}$$

Let $k \in \mathbb{Z}^d$, $k \ne \mathbf{0}$, be fixed. Denoting $\Phi_k(y) = \varphi(y)e^{-2\pi i(y,k)}$, $e_k(x) = e^{2\pi i(k,x)}$, we get

$$\begin{aligned} \int_{\mathbb{R}^d} \widetilde{\Phi}(-y)\varphi(x-y)e^{-2\pi i(y,k)}dy &= \int_{\mathbb{R}^d} T * (\mathcal{N}_\delta * \widetilde{\varphi}^-)(y)\varphi(x-y)e^{-2\pi i(y,k)}dy \\ &= e_k(x)((T*\mathcal{N}_\delta) * (\mathcal{N}_\delta * \widetilde{\varphi}^-) * \Phi_k(x) = e_k(x)(T * (\mathcal{N}_\delta * \mathcal{N}_\delta * \widetilde{\varphi}^-) * \Phi_k)(x). \end{aligned} \tag{22}$$

By condition 2) and Taylor's formula, we have

$$\widehat{\varphi}(\xi+l)=\sum_{[\beta]=s}\frac{s}{\beta!}\xi^\beta\int_0^1(1-t)^{s-1}D^\beta\widehat{\varphi}(t\xi+l)dt.$$

Using this equality and setting $K_{t,\delta,k}(x):=\mathcal{F}^{-1}\left(\eta_\delta(t\xi)D^\beta\widehat{\varphi}(t\xi+k)\right)(x)$, we obtain

$$\begin{aligned}
&(\mathcal{N}_\delta*\mathcal{N}_\delta*\widetilde{\varphi}^-)*\Phi_k(x)\\
&=\int_{\mathbb{R}^d}\eta_\delta^2(\xi)\widehat{\varphi}(\xi+k)\overline{\widehat{\widetilde{\varphi}}}(\xi)e^{2\pi i(\xi,x)}d\xi\\
&=\sum_{[\beta]=s}\frac{s}{\beta!}\int_0^1(1-t)^{s-1}\int_{\mathbb{R}^d}\xi^\beta\eta_\delta^2(\xi)\overline{\widehat{\widetilde{\varphi}}}(\xi)\eta_{2\delta}(t\xi)D^\beta\widehat{\varphi}(t\xi+k)e^{2\pi i(\xi,x)}d\xi dt\\
&=\sum_{[\beta]=s}\frac{s}{\beta!(2\pi i)^\beta}\int_0^1(1-t)^{s-1}\int_{\mathbb{R}^d}\widehat{D^\beta\mathcal{N}_\delta}(\xi)\eta_\delta(\xi)\overline{\widehat{\widetilde{\varphi}}}(\xi)\widehat{K_{t,2\delta,k}}(\xi)e^{2\pi i(\xi,x)}d\xi dt\\
&=\sum_{[\beta]=s}\frac{s}{\beta!(2\pi i)^\beta}\int_0^1(1-t)^{s-1}(D^\beta\mathcal{N}_\delta*(N_\delta*\widetilde{\varphi}^-)*K_{t,2\delta,k})(x)dt. \qquad (23)
\end{aligned}$$

Since the function $\mathcal{N}_\delta*\widetilde{\varphi}^-$ is summable, it follows that

$$\begin{aligned}
&\|T*(\mathcal{N}_\delta*\mathcal{N}_\delta*\widetilde{\varphi}^-)*\Phi_k\|\\
&\le c_2\sum_{[\beta]=s}\int_0^1(1-t)^{s-1}\|T*D^\beta\mathcal{N}_\delta*(N_\delta*\widetilde{\varphi}^-)*K_{t,2\delta,k}\|dt\\
&\le c_3\sum_{[\beta]=s}\sup_{t\in(0,1)}\|T*D^\beta\mathcal{N}_\delta*K_{t,2\delta,k}\|.
\end{aligned} \qquad (24)$$

Due to condition 4), we get

$$\begin{aligned}
\|T*D^\beta\mathcal{N}_\delta*K_{t,2\delta,k}\|&=\|D^\beta T*\mathcal{N}_\delta*K_{t,2\delta,k}\|\le c_4\|D^\beta T*K_{t,2\delta,k}\|\\
&=c_4\|D^\beta T*\mathcal{N}_{t^{-1}\delta}*K_{t,\delta,k}\|\le c_4\|D^\beta T*K_{t,\delta,k}\| \qquad (25)\\
&\le c_4\|K_{t,\delta,k}\|_1\|D^\beta T\|\le c_5\|\eta_\delta D^\beta\widehat{\varphi}(\cdot+k)\|_{W_0}\|D^\beta T\|.
\end{aligned}$$

Thus, inequalities (24), (25), and condition 5) yield

$$\Big\|\sum_{k\ne\mathbf{0}}e_k(T*(\mathcal{N}_\delta*\mathcal{N}_\delta*\widetilde{\varphi}^-)*\Phi_k)\Big\|\le c_6\sum_{[\beta]=s}\|D^\beta T\|, \qquad (26)$$

which completes the proof of (21). Therefore, by the Poisson summation formula and (22), we have

$$\sum_{k\in\mathbb{Z}^d}(T*(\mathcal{N}_\delta*\widetilde{\varphi}^-))(k)\varphi_{0k}(x)=\sum_{k\in\mathbb{Z}^d}\widetilde{\Phi}(k)\varphi(x+k)$$
$$=\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\widetilde{\Phi}(-y)\varphi(x-y)e^{-2\pi i(y,k)}dy=\sum_{k\in\mathbb{Z}^d}e_k(T*(\mathcal{N}_\delta*\mathcal{N}_\delta*\widetilde{\varphi}^-)*\Phi_k), \tag{27}$$

and hence

$$T-\sum_{k\in\mathbb{Z}^d}(T*(\mathcal{N}_\delta*\widetilde{\varphi}^-))(k)\varphi_{0k}=T*(\mathcal{N}_\delta-(\mathcal{N}_\delta*\widetilde{\varphi}^-)*\varphi)$$
$$+\sum_{k\neq\mathbf{0}}e_k(T*(\mathcal{N}_\delta*\mathcal{N}_\delta*\widetilde{\varphi}^-)*\Phi_k)=:I_1+I_2. \tag{28}$$

The term I_2 is already estimated in (26). Consider I_1. By condition 4) and Taylor's formula, there holds

$$\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}=1+\sum_{[\beta]=s}\frac{s}{\beta!}\xi^\beta\int_0^1(1-t)^{s-1}D^\beta\widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}}(t\xi)dt.$$

Applying this equality, we obtain

$$\mathcal{N}_\delta(x)-((\mathcal{N}_\delta*\widetilde{\varphi}^-)*\varphi)(x)=\int_{\mathbb{R}^d}\eta_\delta(\xi)(1-\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}}(\xi))e^{2\pi i(\xi,x)}d\xi$$
$$=\sum_{[\beta]=s}\frac{s}{\beta!}\int_0^1(1-t)^{s-1}\int_{\mathbb{R}^d}\xi^\beta\eta_\delta(\xi)\eta_{2\delta}(t\xi)D^\beta\widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}}(t\xi)e^{2\pi i(\xi,x)}d\xi dt$$
$$=\sum_{[\beta]=s}\frac{s}{\beta!(2\pi i)^\beta}\int_0^1(1-t)^{s-1}\int_{\mathbb{R}^d}\widehat{D^\beta\mathcal{N}_\delta}(\xi)\widehat{K_{t,2\delta}}(\xi)e^{2\pi i(\xi,x)}d\xi dt \tag{29}$$
$$=\sum_{[\beta]=s}\frac{s}{\beta!(2\pi i)^\beta}\int_0^1(1-t)^{s-1}(D^\beta\mathcal{N}_\delta*K_{t,2\delta})(x)dt,$$

where $K_{t,\delta}(x)=\mathcal{F}^{-1}\left(\eta_\delta(t\xi)D^\beta\widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}}(t\xi)\right)(x)$. Next, using condition 4) and the same arguments as in (25), we have

$$\|I_1\|\le c_7\sum_{[\beta]=s}\int_0^1(1-t)^{s-1}\|T*D^\beta\mathcal{N}_\delta*K_{t,2\delta}\|dt$$
$$\le c_8\sum_{[\beta]=s}\sup_{t\in(0,1)}\|D^\beta T*K_{t,\delta}\|\le c_9\sum_{[\beta]=s}\|D^\beta T\|. \tag{30}$$

Combining this with (28) and (26), we get (20).

Now let T_μ, $\mu \in \mathbb{Z}_+$, be as in Definition 4 and $T = T_0$. It follows from (20) and Lemma 15 that

$$\left\| T - \sum_{k\in\mathbb{Z}^d} \langle T, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} \right\| \le c_{10}\omega_s(T, 1) \le c_{11}\left(\omega_s(f, 1) + E_{\delta I}(f)\right). \tag{31}$$

If $f \in \mathbb{B}_M^{\alpha(\cdot)}$, then Definition 4 and Lemmas 11 and 12 yield

$$\begin{aligned} \|Q_0(f - T, \varphi, \widetilde{\varphi})\| &\le c_{12}\|\{\langle f - T, \widetilde{\varphi}_{0k}\rangle\}_k\|_{\ell_\infty} \le c_{13}\sum_{\mu=0}^{\infty} \|\{\langle T_{\mu+1} - T_\mu, \widetilde{\varphi}_{0k}\rangle\}_k\|_{\ell_\infty} \\ &\le c_{14}\sum_{\mu=\nu}^{\infty} \alpha(M^\mu) E_{\delta M^\mu}(f). \end{aligned} \tag{32}$$

Applying (31) and (32) to the inequality

$$\|f - Q_0(f, \varphi, \widetilde{\varphi})\| \le \|T - Q_0(T, \varphi, \widetilde{\varphi})\| + \|Q_0(f - T, \varphi, \widetilde{\varphi})\| + \|f - T\| \tag{33}$$

and taking into account that $\|f - T\| \le c(d)E_{\delta I}(f)$, we obtain

$$\|f - Q_0(f, \varphi, \widetilde{\varphi})\| \le c_{15}\left(\omega_s(f, 1) + \sum_{\nu=0}^{\infty} \alpha(M^\nu) E_{\delta M^\nu}(f)\right). \tag{34}$$

Since there exists $\nu_0 = \nu(\delta) \in \mathbb{N}$ such that $E_{\delta M^\nu}(f) \le E_{M^{\nu-\nu_0}}(f)$ and $\alpha(M^\nu) \le c(\delta)\alpha(M^{\nu-\nu_0})$ for all $\nu > \nu_0$, using Lemma 14 and the inequality $\omega_s(f, \lambda) \le (1 + \lambda)^s\omega_s(f, 1)$ (see, e.g., [23]) to the first ν_0 terms of the sum, we get (7) for $j = 0$.

If $\widetilde{\varphi} \in \mathcal{S}'_{\mathrm{const};M}$ and $f \in C$, then, by Definition 4 and Lemma 13, we have

$$\|Q_0(f - T, \varphi, \widetilde{\varphi})\| \le c_{16}\|\{\langle f - T, \widetilde{\varphi}_{0k}\rangle\}_k\|_{\ell_\infty} \le c_{17}E_{\delta I}(f). \tag{35}$$

Combining this with (31) and (33) and applying Lemma 14 and the properties of moduli of smoothness, we get (8) for $j = 0$.

Thus, our theorem is proved for the case $j = 0$. To prove (7) and (8) for arbitrary j, it remains to note that

$$\begin{aligned} &\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\| = \left\| f(M^{-j}\cdot) - \sum_{k\in\mathbb{Z}^d} \langle f(M^{-j}\cdot), \widetilde{\varphi}_{0k}\rangle \varphi_{0k} \right\|, \\ &E_{M^\nu}(f(M^{-j}\cdot)) = E_{M^{\nu+j}}(f), \\ &\omega_s(f(M^{-j}\cdot), 1) = \Omega_s(f, M^{-j}), \end{aligned}$$

and that $f(M^{-j}\cdot) \in \mathbb{B}_M^{\alpha(\cdot)}$ whenever $f \in \mathbb{B}_M^{\alpha(\cdot)}$. ◇

Proof of Theorem 9 As above, it suffices to consider only the case $j = 0$. Repeating the arguments of the proof of Theorem 6, we obtain from (28) that

$$T - \sum_{k\in\mathbb{Z}^d} \langle T, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} = 0,$$

Thus, using (33), (32), and (35), we prove both the statements. ◇

Proof of Corollary 10 Since obviously $E_{\delta M^\nu}(f) = 0$ for any $\nu \ge j$, and both the functions f and $Q_j(f, \varphi, \widetilde{\varphi})$ are continuous, it follows from Theorem 9 that $f = Q_j(f, \varphi, \widetilde{\varphi})$ at each point. Thus, by Definition 4, for every $x \in \mathbb{R}^d$, we have

$$f(x) = m^{-j/2} \sum_{k\in\mathbb{Z}^d} \langle f(M^{-j}\cdot), \widetilde{\varphi}_{0k}\rangle \varphi_{jk} = m^{-j/2} \sum_{k\in\mathbb{Z}^d} \lim_{\mu\to\infty} T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-)(k)\varphi_{jk},$$

where $T_\mu \in \mathcal{B}_{\delta M^\mu}$ is such that

$$\|f(M^{-j}\cdot) - T_\mu\| \le c(d) E_{\delta M^\mu} f(M^{-j}\cdot).$$

It remains to note that for sufficiently large μ we have

$$T_\mu * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-) = f(M^{-j}\cdot) * (\mathcal{N}_{M^\mu} * \widetilde{\varphi}^-) = f(M^{-j}\cdot) * (\mathcal{N}_1 * \widetilde{\varphi}^-).$$

◇

Proof of Theorem 16 Analyzing the arguments of the proof of Theorem 6, one can see that it suffices to verify that

$$\|T * (\mathcal{N}_\delta * \mathcal{N}_\delta - (\mathcal{N}_\delta * \mathcal{N}_\delta * \widetilde{\varphi}) * \varphi)\| \le c_1 \|\mathcal{D}(\rho) T\| \tag{36}$$

for any $T \in \mathcal{B}_{\delta I}$ such that $\|f - T\| \le c(d) E_{\delta I}(f)$.

We have

$$\begin{aligned}
\mathcal{N}_\delta * \mathcal{N}_\delta(x) - (\mathcal{N}_\delta * \mathcal{N}_\delta * \widetilde{\varphi}) * \varphi(x) &= \int_{\mathbb{R}^d} \eta_\delta^2(\xi) \left(1 - \widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}\right) e^{2\pi i(\xi,x)} d\xi \\
&= \int_{\mathbb{R}^d} \eta_\delta(\xi) \frac{1 - \widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}}{\rho(\xi)} \rho(\xi)\eta_\delta(\xi) e^{2\pi i(\xi,x)} d\xi \\
&= (\mathcal{D}(\rho)\mathcal{N}_\delta) * K_\delta(x),
\end{aligned}$$

where $K_\delta(x) = \mathcal{F}^{-1}\left(\eta_\delta \frac{1-\widehat{\varphi}(\xi)\overline{\widehat{\widetilde{\varphi}}(\xi)}}{\rho(\xi)}\right)(x)$. Thus, using condition 2), we get

$$\begin{aligned}
\|T * (\mathcal{N}_\delta * \mathcal{N}_\delta - (\mathcal{N}_\delta * \mathcal{N}_\delta * \widetilde{\varphi}) * \varphi)\| &= \|T * (\mathcal{D}(\rho)\mathcal{N}_\delta) * K_\delta\| \\
&\le c_2 \|T * (\mathcal{D}(\rho)\mathcal{N}_\delta)\| = c_2 \|\mathcal{D}(\rho) T\|,
\end{aligned}$$

which completes the proof. ◇

Proof of Theorem 17 As in the proof of the previous theorems, it suffices to consider only the case $j = 0$. Let $T \in \mathcal{B}_I$ be such that $\|f - T\| \le c(d)E_I(f)$. Due to the same arguments as in the proof of Theorem 6, we have (28), which now takes the following form

$$\begin{aligned} T &- \sum_{k\in\mathbb{Z}^d} \langle T, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} \\ &= T - \sum_{k\in\mathbb{Z}^d} (T * (\mathcal{N}_1 * \widetilde{\varphi}^-))(k)\varphi_{0k} = T * (\mathcal{N}_1 - (\mathcal{N}_1 * \widetilde{\varphi}^-) * \varphi). \end{aligned}$$

Thus, denoting $K = \mathcal{F}^{-1}(\eta \frac{\rho}{1-\widehat{\widetilde{\varphi}}\widehat{\varphi}})$ and using condition 2), we obtain

$$\begin{aligned} \|\mathcal{D}(\rho)T\| &= \|T * \mathcal{D}(\rho)\mathcal{N}_1\| = \|T * \mathcal{F}^{-1}(\rho\eta)\| = \left\| T * \mathcal{F}^{-1}\left(\eta \frac{\rho}{1-\widehat{\widetilde{\varphi}}\widehat{\varphi}}(1-\widehat{\widetilde{\varphi}}\widehat{\varphi})\eta\right)\right\| \\ &= \|T * (\mathcal{N}_1 - (\mathcal{N}_1 * \widetilde{\varphi}^-) * \varphi) * K\| \le c_1 \|T * (\mathcal{N}_1 - (\mathcal{N}_1 * \widetilde{\varphi}^-) * \varphi)\| \\ &= c_1 \|T - Q_0(T, \varphi, \widetilde{\varphi})\|. \end{aligned}$$

Now, by the definition of the realization, we get

$$\begin{aligned} \mathcal{R}_\rho(f, I) &\le \|\mathcal{D}(\rho)T\|_p + E_I(f) \\ &\le c_1 \left(\|T - Q_0(T, \varphi, \widetilde{\varphi})\| + E_I(f)\right) \\ &\le c_1 \left(\|f - Q_0(f, \varphi, \widetilde{\varphi})\| + \|T - f\| + \|Q_0(T - f, \varphi, \widetilde{\varphi})\| + E_I(f)\right) \\ &\le c_2 \left(\|f - Q_0(f, \varphi, \widetilde{\varphi})\| + E_I(f) + \|Q_0(f - T, \varphi, \widetilde{\varphi})\|\right) \\ &\le c_3 \left(\|f - Q_0(f, \varphi, \widetilde{\varphi})\| + E_I(f)\right), \end{aligned}$$

where the last inequality follows from Lemmas 12 and 13. Thus, to prove (15), it remains to note that in view of the inclusion $\operatorname{supp} \mathcal{F}(Q_0(f, \varphi, \widetilde{\varphi})) \subset \operatorname{supp} \widehat{\varphi} \subset \mathbb{T}^d$, we have $E_I(f) \le \|f - Q_0(f, \varphi, \widetilde{\varphi})\|$.

Similarly, using (32), one can prove (14).

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