Spectral geometry on manifolds with fibred boundary metrics II: heat kernel asymptotics

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Abstract

In this paper we continue with the analysis of spectral problems in the setting of complete manifolds with fibred boundary metrics, also referred to as ϕ-metrics, as initiated in our previous work (Grieser et al. in Spectral geometry on manifolds with fibred boundary metrics I: Low energy resolvent, 2020). We consider the Hodge Laplacian for a ϕ-metric and construct the corresponding heat kernel as a polyhomogeneous conormal distribution on an appropriate manifold with corners. Our discussion is a generalization of an earlier work by Albin and Sher, and provides a fundamental first step towards analysis of Ray–Singer torsion, eta-invariants and index theorems in the setting.

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1 Introduction and statement of the main results

Consider a compact smooth Riemannian manifold $\overline{M}$ with boundary ∂M, which is the total space of a fibration $\phi : \partial M \to B$ over a closed manifold B with the fibre given by a closed manifold F. Consider a collar neighborhood $[0, \varepsilon]_x \times \partial M$ of the boundary, with a boundary defining function x. In the open interior M of such a manifold there are various possible complete Riemannian metrics, specified by their structure as $x \to 0$. We shall recall here the main three classes of these complete Riemannian metrics.

1.1 Fibred boundary and scattering metrics

In this work we are interested in the fibred boundary metrics, also called ϕ-metrics. Ignoring cross-terms for the purpose of a clear exposition, these metrics are asymptotically given near the boundary ∂M by

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_F,$$

where g_B is a Riemannian metric on the base B, and g_F is a symmetric bilinear form on ∂M, restricting to Riemannian metrics on fibres F. In case of trivial fibres, such a metric is called scattering. A trivial example of a scattering metric is the Euclidean space, with the metric written in polar coordinates as $dr^2 + r^2 d\theta^2$. After a change of variables $x = r^{-1}$ we obtain

$$g_{\mathrm{Sc}} = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2}.$$

Such metrics arise naturally in various geometric examples. Complete Ricci flat metrics are often ϕ-metrics. Scattering metrics include metrics of locally Euclidean (ALE) manifolds. Products of these spaces with any compact manifold provide natural examples of ϕ-metrics. Furthermore, common classes of gravitational instantons, such as the Taub-NUT metrics and reduced 2-monopole moduli space metric, are ϕ-metrics under appropriate coordinate change, cf. [9, p. 2].

While there are various approaches to Euclidean scattering theory, a microlocal approach has been taken by Melrose [19], where elliptic theory of scattering metrics has been developed. Elliptic theory of ϕ-metrics has been studied by Mazzeo and Melrose in [15]. This work later was generalized to the case of towers of fibrations with so called a-metrics by Grieser and Hunsicker [6]. Elliptic theory of [15] has also been extended by Grieser and Hunsicker [7] to include not fully elliptic operators. Hodge theory of ϕ-metrics has been developed by Hausel, Hunsicker and Mazzeo [9]. Index theory (bypassing usual heat operator approach and using adiabatic limit methods instead) in this setting has been addressed by Leichtnam et al. [11].

1.2 Fibred boundary cusp and *b*-metrics

Fibred (boundary) cusp metrics, also referred to as d-metrics are conformally equivalent to ϕ-metrics by a conformal factor x^2. Ignoring as before the cross-terms for the purpose of a clear exposition, these metrics are asymptotically given near the boundary ∂M by

$$g_d = \frac{dx^2}{x^2} + x^2 g_F + \phi^* g_B.$$

In case of trivial fibres, such a metric is called a b-metric. Under the coordinate change $x = e^{-t}$, a b-metric becomes a cylindrical metric $dt^2 + g_B$. Same change of coordinates turns a genuine fibred cusp metric into $dt^2 + \phi^* g_B + e^{-2t} g_F$, which is a Q-rank one cusp when ϕ is a fibration of tori over a torus. Other examples include products of compact manifolds with locally symmetric spaces with finite volume hyperbolic cusps.

Elliptic theory of b-metrics was pioneered by Melrose [17]. Since ϕ-metrics and fibred cusp metrics differ by a conformal change, elliptic theory of ϕ-metrics is suited for fibred cusp metrics as well. Vaillant [22] has utilized elliptic theory of ϕ-metrics as well as a microlocal heat kernel construction in order to establish an index theorem for fibred cusp metrics. We emphasize that his heat kernel construction refers to the Hodge Dirac and Hodge Laplacian of a fibred cusp metric, not a ϕ-metric.

1.3 Complete edge and conformally compact metrics

The third class of complete Riemannian metrics on manifolds with fibred boundary, that has been of focal relevance in recent geometric analysis developments are complete edge metrics that by definition are given asymptotically near the boundary ∂M by

$$g_e = \frac{dx^2 + \phi^* g_B}{x^2} + g_F.$$

In case of trivial fibres, such metrics are also called conformally compact with the classical example being the hyperbolic space $\mathbb{H}^n$. The edge metrics also generalize the b-metrics that arise as special case of edge metrics with trivial base. The significance of edge metrics also lies in their conformal equivalence to the incomplete singular wedge metrics. These metrics appeared prominently in the resolution of the Calabi-Yau conjecture on Fano manifolds, cf. Donaldson [3], Tian [21] as well as Jeffres et al. [10].

Elliptic theory of edge metrics has been developed by Mazzeo [13]. Prior to that, the zero-calculus containing geometric operators associated to conformally compact metrics, as well as Hodge theory have been studied by Mazzeo [12]. Meromorphic extension of the resolvent of conformally compact spaces is due to Mazzeo and Melrose [14]. Heat kernel and the (renormalized) Gauss Bonnet index theorem on general edge metrics is due to Albin [1]. Let us also mention the work by Mazzeo and Vertman [16] on analytic torsion and by Vertman [23] on incomplete wedge spaces, both of which are based on a microlocal heat kernel construction on wedge manifolds.

1.4 Main result and structure of the paper

While the previous overview of the various classes of metrics on fibred boundary spaces puts our work into context, we are dealing here only with fibred boundary ϕ-metrics and construct the heat kernel for finite times microlocally as a polyhomogeneous function on an appropriate heat space blowup. The construction is similar in spirit to [1, 16, 22]. Our main result is as follows (cf. Corollary 7.2 for the precise statement)

Theorem 1.1 *The heat kernel of the Hodge Laplacian for a ϕ-metric is a polyhomogeneous function on an appropriate heat space blowup for finite time. Same holds for Laplace-type operators with normal operator having the same structure as the scalar Laplacian of a ϕ-metric, times an identity matrix. This includes for instance square of the spin Dirac operator.*

The heat kernel construction in the special case of scattering metrics has been outlined in the appendix to Sher [20], where the author specifies an initial heat parametrix inside the heat calculus of Albin [1, Theorem 4.3]. The heat calculus of Albin [1, Theorem 4.3] refers to both scattering and ϕ-metrics, but concentrates only on the case of complete edge metrics in detail. Moreover, [1] does not deal with the actual heat kernel construction in the ϕ-case, as its focus lies on the complete edge case. Our paper closes this gap in the literature.

The main difference between [1, 20] and our presentation here is threefold. First, our initial heat parametrix construction generalizes the appendix in [20] to the more general case of fibered boundaries. Second, we do not use the heat calculus of [1, Theorem 4.3], but rather a simpler version. Third is rather a minor but subtle point, that we work out the Volterra series argument. Note that this argument is spelled out in detail in Melrose's book [17, Proposition 7.17].

The structure of the paper is as follows. In Sect. 2 we introduce the basic geometric preliminaries of ϕ-metrics, including the ϕ-vector fields and differential ϕ-operators. Moreover, we recall the basic definition of polyhomogeneous functions on manifolds with corners. In Sect. 3 we review the ϕ-double space, which arises in the microlocal description of the resolvent for the Hodge Laplacian of a ϕ-metric. That space is built upon in the heat calculus construction as outlined in the subsequent section Sect. 4. The heat kernel construction then proceeds in three steps. We first construct the heat space and define the heat calculus in Sect. 5. We then construct and initial heat parametrix, solving the heat equation to first order in Sect. 6. Our final chapter Sect. 7 is concerned with the triple space argument, which finishes the heat kernel construction and proves the main theorem.

1.5 Outlook and upcoming work

In the upcoming work by the first author, the presented heat kernel asymptotics together with the low energy resolvent, as constructed in our previous work [8] jointly with Daniel Grieser, is applied to define the renormalized heat trace and study its asymptotics. This leads to the definition of a renormalized analytic torsion for ϕ-manifolds.

We also intend to apply our analysis to establish an index theorem for ϕ-metrics by direct heat kernel methods, following the ideas of [1] instead of adiabatic arguments of [11]. Note that in view of the heat kernel construction, presented here, Albin [1, Theorem 6.2] now applies to ϕ-manifolds and provides an important first step towards an index theorem in this setting by heat kernel methods. However, an index theorem will not be a mere Corollary of our work here: without the image of the Dirac operator being closed, and without Getzler rescaling techniques, two of the three terms in the formula of [1, Theorem 6.2] remain mysterious.

2 Preliminaries on fibred boundary manifolds and ϕ-metrics

2.1 ϕ-Metrics

Let $\overline{M} = M \cup \partial M$ be a compact smooth manifold with boundary ∂M. Assume that the boundary ∂M is the total space of the fibration $\phi : \partial M \longrightarrow B$ with typical fibre F, where both F and base B are smooth compact manifolds without boundary.

Assumption 2.1 Throughout the paper we assume that the boundary fibration $(\partial M, g_{\partial M})$ and its base (B, g_B) are equipped with Riemannian metrics such that $\phi : (\partial M, g_{\partial M}) \to (B, g_B)$ is a Riemannian submersion: we split the tangent bundle $T \partial M$ into the vertical subbundle $T^V \partial M$ and its orthogonal complement with respect to $g_{\partial M}$—the horizontal subbundle $T^H \partial M$. Then ϕ is a Riemannian submersion if the restriction $d\phi : T^H \partial M \to TB$ is an isometry. In this case we may write $g_{\partial M} = g_F + \phi^* g_B$, where g_F equals $g_{\partial M}$ on $T^V \partial M$ and vanishes on $T^H \partial M$.

Let $x \in C^\infty(\overline{M})$ be a boundary defining function i.e $x \geq 0$, $\partial M = \{x = 0\}$ and $dx \neq 0$ at ∂M. By the collar neighborhood theorem there always exists a collar $[0, \varepsilon)_x \times \partial M \subset \overline{M}$ of the boundary. Replacing $g_F + \phi^* g_B$ by $\frac{\phi^* g_B}{x^2} + g_F$, for any constant positive value of x, still defines a Riemannian submersion. We can now define ϕ-metrics.

Definition 2.2 We call a Riemannian metric g_ϕ on the open interior M a ϕ-metric, if in the collar neighborhood $\mathcal{U} = (0, \varepsilon)_x \times \partial M$ of the boundary, g_ϕ takes the following form

$$g_\phi \restriction \mathcal{U} = \frac{dx^2}{x^4} + \frac{\phi^* g_B}{x^2} + g_F + h =: g_0 + h, \tag{2.1}$$

where h is a higher order term, i.e. satisfies $|h|_{g_0} = O(x)$ as $x \to 0$.

The Assumption 2.1 is used in order to deduce the structure (2.9) for the Hodge Laplacian, which in turn is essential in the second step of the heat kernel construction in (6.3).

Example 2.3 The Euclidean plane $\mathbb{R}^m$ is a particular example of a ϕ-manifold (a scattering manifold) with the fibre F being a single point and the base $B = \mathbb{S}^{m-1}$. Indeed,

choosing polar coordinates (r, θ) and writing $x = 1/r$, the Euclidean metric on $\mathbb{R}^m$ can be written away from the origin as

$$g_{\mathbb{R}^m} = \frac{dx^2}{x^4} + \frac{g_{\mathbb{S}^{m-1}}}{x^2}.$$

2.2 ϕ-Vector fields and differential ϕ-operators

Definition 2.4 The ϕ-vector fields $\mathcal{V}_\phi \equiv \mathcal{V}_\phi(M)$ are by definition smooth vector fields over $\overline{M}$, tangent to the fibres of ϕ and such that for $x \in C^\infty(\overline{M})$ we have $Vx \in x^2 C^\infty(\overline{M})$. Any ϕ-vector field is then locally generated near ∂M by

$$x^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_j},$$

where $\{x, y_i, z_j\}$ are local coordinates on $[0, \varepsilon) \times \partial M$ with $y = \{y_i\}_i$ being local coordinates on the base B, lifted to ∂M and extended to $[0, \varepsilon) \times \partial M$, and $z = \{z_j\}_j$ restricting to local coordinates on the fibres F. We introduce the so called ϕ-tangent space by requiring $\mathcal{V}_\phi(M)$ to be its smooth sections

$$C^\infty(\overline{M}, {}^\phi TM) = \mathcal{V}_\phi(M) = C^\infty(\overline{M})\text{-span}\left\langle x^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_j} \right\rangle.$$

The dual ${}^\phi T^*M$, the so-called ϕ-cotangent space, satisfies

$$C^\infty(\overline{M}, {}^\phi T^*M) = C^\infty(\overline{M})\text{-span}\left\langle \frac{dx}{x^2}, \frac{dy_i}{x}, dz_j \right\rangle.$$

The space of ϕ-vector fields $\mathcal{V}_\phi(M)$ has a Lie-Algebra structure and is a $C^\infty(\overline{M})$-module. Therefore one may introduce $\mathrm{Diff}_\phi^k(M)$ as an graded algebra. Explicitly, $P \in \mathrm{Diff}_\phi^k(M)$ if it is a k-th order differential operator in the open interior M of the following structure near the boundary

$$P = \sum_{|\alpha|+|\beta|+q \leq k} P_{\alpha,\beta,q}(x, y, z)(x^2 D_x)^q (x D_y)^\beta D_z^\alpha, \tag{2.2}$$

with coefficients $P_{\alpha,\beta,q} \in C^\infty(\overline{M})$ smooth up to the boundary. Its ϕ-symbol $\sigma_{\phi,k}(P)$ is then locally given for any cotangent vector $(\xi, \eta, \zeta) \in {}^\phi T^*M$ over the base point $(x, y, z) \in \overline{M}$ by the homogeneous polynomial

$$\sigma_{\phi,k}(P)(x, y, z; \xi, \eta, \zeta) = \sum_{|\alpha|+|\beta|+q=k} P_{\alpha,\beta,q}(x, y, z)\xi^q \eta^\beta \zeta^\alpha.$$

We say that P is ϕ-elliptic if $\sigma_{\phi,k}(P)$ is invertible off the zero-section of ${}^\phi T^*M$. Writing $P^k({}^\phi T^*M)$ for the space of homogeneus polynomial of degree k on the fibres

of $^\phi T^* M$, the ϕ-symbol map defines a short exact sequence

$$0 \longrightarrow \text{Diff}_\phi^{k-1}(M) \hookrightarrow \text{Diff}_\phi^k(M) \xrightarrow{\sigma_{\phi,k}} P^k(^\phi T^* M) \longrightarrow 0. \tag{2.3}$$

Same constructions extend to case of differential operators acting on sections of a flat vector bundle (L, ∇) over $\overline{M}$, compactly supported in the interior M. In that case, coefficients $P_{\alpha,\beta,q}$ are smooth sections of the endomorphism bundle $\text{End}(L)$ and each derivative $X \in \mathcal{V}_\phi$ in (2.2) is replaced by ∇_X. We write $\text{Diff}_\phi^*(M, L)$ for differential ϕ-operators acting on sections, and the notion of ellipticity carries over verbatim.

2.3 Hodge–Laplacian of a ϕ metric

The Hodge Laplacian Δ_ϕ is an element of $\text{Diff}_\phi^2(M, \Lambda^{*\phi} T^* M)$ and this section is devoted to writing out its explicit structure, following Hausel, Hunsicker and Mazzeo [9, §5.3.2].

Assume for the moment that the fibration ϕ is trivial, so that we can identify $\partial M \cong B \times F$. The exact model ϕ-metric is given in this case in a collar neighborhood $\mathcal{U} = (0, \varepsilon) \times \partial M$ of the boundary by

$$g_0 \restriction \mathcal{U} = \frac{dx^2}{x^4} + \frac{g_B}{x^2} + g_F,$$

where in this model case g_F is assumed to be constant along B. We write $b = \dim B$. In this case the Laplace Beltrami operator of (M, g_0) is of the following explicit form in a collar neighborhood $\mathcal{U}$ of the boundary

$$\Delta_\phi \restriction \mathcal{U} = -x^4 \partial_x^2 + x^2 \Delta_B + \Delta_F - (2 - b)x^3 \partial_x. \tag{2.4}$$

where Δ_B is the Laplace Beltrami operator of (B, g_B) and Δ_F is the family of Laplace Beltrami operator on the fibres (F, g_F). We want to explain in what way the structure of the Hodge Laplacian acting on differential forms for a general ϕ-metric admits a similar structure as above.

Under the Assumption 2.1 we may split $T \partial M$ and its dual $T^* \partial M$ orthogonally with respect to $\phi^* g_B + g_F$ into vertical and horizontal parts. Writing $\mathcal{V}$ for the canonical vertical bundle, and $\phi^* T B$ for the horizontal bundle as in Definition 2.2 (we also write $\phi^* T^* B$ for its dual), we obtain

$$T \partial M = \phi^* T B \oplus \mathcal{V}, \quad T^* \partial M = \phi^* T^* B \oplus \mathcal{V}^*. \tag{2.5}$$

This splitting induces an orthogonal splitting of the ϕ-cotangent bundle $^\phi T^* M$ in the collar neighborhood $\mathcal{U}$ of the boundary

$$^\phi T^* M \restriction \mathcal{U} = \text{span} \left\{ \frac{dx}{x^2} \right\} \oplus x^{-1} \phi^* T^* B \oplus \mathcal{V}^*. \tag{2.6}$$

Let us assume that the higher order term $h \equiv 0$ for the moment. With respect to the corresponding decomposition of $\Lambda^{*\phi} T^* M$ over $\mathcal{U}$ we compute for the exterior derivative, cf. [9, §5.3.2]

$$D_\phi = x^2 D_x + x\mathbb{A} + D_F + x D_B - x^2 \mathcal{R}. \tag{2.7}$$

We shall now explain the individual terms in (2.7). The term $x^2 D_x$ acts for any section ω of the bundle $\Lambda^\ell \left(x^{-1} \phi^* T^* B \right) \oplus \Lambda \mathcal{V}^*$ as follows

$$\left(x^2 D_x \right) \omega = \frac{dx}{x^2} \wedge \left(x^2 \partial_x \right) \omega, \quad x^2 D_x \left(\frac{dx}{x^2} \wedge \omega \right) = - \left(x^2 \partial_x \right) \omega.$$

The term $\mathbb{A}$ is given by $A + A^*$, where A is a 0-th order differential operator, acting for any section ω of $\Lambda^\ell \left(x^{-1} \phi^* T^* B \right) \oplus \Lambda \mathcal{V}^*$ by

$$A\omega = -\ell \cdot \frac{dx}{x^2} \wedge \omega, \quad A \left(\frac{dx}{x^2} \wedge \omega \right) = (b - \ell) \cdot \omega.$$

The term $D_F = d_F + d_F^*$, acts as the Gauss Bonnet operator on the $\Lambda \mathcal{V}^*$ component, cf. the first displayed equation in [9, p. 527]. The term D_B is given by

$$D_B = (d_B - \mathbb{I}) + (d_B - \mathbb{I})^* . \tag{2.8}$$

Here, $\mathbb{I}$ is the second fundamental form, d_B is the lift of the exterior derivative on B to ∂M plus the action of the derivative in the B-direction on the $\mathcal{V}^*$-components of the form, cf. the second displayed equation in [9, p. 527]. Finally, $\mathcal{R} = R + R^*$, where R is the curvature of the Riemannian submersion ϕ. We can now take the square of D_ϕ to compute the Hodge Laplacian over $\mathcal{U}$

$$\Delta_\phi = D_\phi^2 = -(x^2 D_x)^2 + D_F^2 + x^2 D_B^2 + Q, \ Q \in x \cdot \mathrm{Diff}_\phi^2(\mathcal{U}, \Lambda^{*\phi} T^* \mathcal{U}). \tag{2.9}$$

Note that D_B^2 equals Δ_B (the Hodge Laplacian on B) up to additional higher order terms in $x \cdot \mathrm{Diff}_\phi^2(\mathcal{U}, \Lambda^{*\phi} T^* \mathcal{U})$. For the general higher order term h with $|h|_{g_0} = O(x)$ as $x \to 0$, the arguments carry over up to higher order and the statement (2.9) is still true. This replaces (2.4) in the general case.

Remark 2.5 The arguments of this paper apply to a more general class of Laplace-type operators in $\mathrm{Diff}_\phi^2(M, L)$, provided the structure of (2.4) (times the identity matrix) holds in the collar neighborhood $\mathcal{U}$ up to higher order terms. This includes squares of geometric Dirac operators, such as the spin Dirac operator.

2.4 Polyhomogeneous functions on manifolds with corners

The contribution of this paper is a heat kernel construction for the Hodge Laplacian Δ_ϕ such that the heat kernel lifts to a polyhomogeneous distribution on an appropriate

manifold with corners. In this subsection we provide brief definitions of manifolds with corners, polyhomogeneous functions on and maps between these spaces. We refer the reader to [4, 17] for more details.

Definition 2.6 An n-dimensional compact manifold X with corners is by definition locally modelled near each $p \in X$ diffeomorphically by $(\mathbb{R}^+)^k \times \mathbb{R}^{N-k}$ for some $k \in \mathbb{N}_0$, where we write $\mathbb{R}^+ = [0, \infty)$. The index k is called the codimension of p. A boundary face of X is the closure of a connected component of the set of points of codimension 1. We assume that all boundary faces are embedded, i.e. each boundary hypersurface $H \subset X$ is given by $\{\rho_H = 0\}$ for some boundary defining function $\rho_H \in C^\infty(X)$, where $d_{\rho_H} \neq 0$ and $\rho_H \geq 0$. A corner is the closure of a connected component of the set of points of codimension at least two.

We now define polyhomogeneous functions on manifolds with corners. Polyhomogeneous sections valued in vector bundles over manifolds with corners are defined analogously.

Definition 2.7 Let X be a manifold with corners and $\{(H_i, \rho_i)\}_{i=1}^N$ an enumeration of its (embedded) boundaries with the corresponding defining functions. For any multi-index $\beta = (b_1, \ldots, b_N) \in \mathbb{C}^N$ we write $\rho^\beta = \rho_1^{b_1} \ldots \rho_N^{b_N}$. Denote by $\mathcal{V}_b(X)$ the space of smooth vector fields on X which lie tangent to all boundary faces. An index set $E_i = \{(\gamma, p)\} \subset \mathbb{C} \times \mathbb{N}_0$ satisfies the following hypotheses:

(1) the real parts $\mathrm{Re}(\gamma)$ accumulate only at $+\infty$,
(2) for each γ there exists $P_\gamma \in \mathbb{N}_0$, such that $p \leq P_\gamma$ for all $(\gamma, p) \in E_i$.
(3) if $(\gamma, p) \in E_i$, then $(\gamma + j, p') \in E_i$ for all $j \in \mathbb{N}_0$ and $0 \leq p' \leq p$.

An index family $E = (E_1, \ldots, E_N)$ is an N-tuple of index sets. Finally, we say that a smooth function ω on the interior of X is polyhomogeneous with index family E, we write $\omega \in \mathcal{A}_{\mathrm{phg}}^E(X)$, if near each H_i,

$$\omega \sim \sum_{(\gamma, p) \in E_i} a_{\gamma, p} \rho_i^\gamma (\log \rho_i)^p, \quad \text{as } \rho_i \to 0,$$

with coefficients $a_{\gamma, p}$ polyhomogeneous on H_i with index E_j at any intersection $H_i \cap H_j$ of hypersurfaces. We require the asymptotic expansion to be preserved under repeated application of $\mathcal{V}_b(X)$. Since $\dim H_i < \dim X$, this is an inductive definition in the dimension of X.

There are following classes of morphisms between manifolds with corners.

Definition 2.8 Let X and X' be two manifolds with corners and $f : X \longrightarrow X'$ a smooth map, i.e. in each local chart f can be extended to a smooth map between open (in $\mathbb{R}^n$) domains containing the charts.

(1) f is called a b-map if for any collection $\{\rho_j\}_{j \in J}, \{\rho_i'\}_{i \in I}$ of boundary defining functions on X and X', respectively, there exist non-negative numbers $\{\alpha_{ij}\}_{ij}$ such that for any index $i \in I$

$$f^* \rho_i' = a_i \prod_{j \in J} \rho_j^{\alpha_{ij}}, \quad 0 < a_i \in C^\infty(X). \tag{2.10}$$

We define b-tangent bundles as follows: in any local chart $(\mathbb{R}^+)^k_{(x)} \times \mathbb{R}^{n-k}_{(y)}$ with local coordinates $(x) = \{x_1, \ldots, x_k\}$ and $(y) = \{y_1, \ldots, y_{n-k}\}$ the b-tangent bundle is defined by the spanning sections being tangent to the boundaries of the chart, i.e.

$$x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_{n-k}}.$$

If f is a b-map, restriction of the total differential df to vector fields that are tangent to the boundaries of the chart above, is valued in vector fields that are tangent to the boundaries of the chart again. Hence, the total differential of a b-map induces

$$^b df : {}^b TX \to {}^b TX'.$$

(2) A b-map f is called b-submersion if its total differential df induces a surjective map $^b df$ between b-tangent bundles.
(3) A b-submersion is called b-fibration if for each $j \in J$ there is at most one $i \in I$ such that $\alpha_{ij} \neq 0$. The numbers α_{ij} are defined by (2.10).

The significance of these maps is that polyhomogeneous functions pull back to polyhomogeneous functions under b-maps. Moreover, push forward under b-fibrations of densities with coefficients given by polyhomogeneous functions is again a density with polyhomogeneous coefficients, see [18] for the explicit statement of the pullback and the pushforward theorems and the explicit form of the index families.

3 Construction of the ϕ-double space M^2_ϕ

The material of this section is drawn from [15], where a calculus of pseudo-differential operators on M, containing parametrices of elliptic $P \in \mathrm{Diff}^m_\phi(M, L)$, is developed. The Schwartz kernels of these parametrices are built of polyhomogeneous conormal distributions on a certain manifold with corners, with the usual singularity along the diagonal. In this section we only provide the definition of that manifold with corners rather than explaining other elements of the ϕ-calculus.

Since $\overline{M}$ is a manifold with boundary, $\overline{M}^2$ has two boundary hypersurfaces, intersecting at $B = \partial M \times \partial M \subset \overline{M}^2$. Consider near $B \subset \overline{M}^2$ local coordinates $(x, y, z), (x', y', z')$, which are just two copies of the local coordinates introduced in Definition 2.4. In these coordinates, $B = \{x = x' = 0\}$.

3.1 b-Double space $M^2_b := [\overline{M}^2, B]$

The blowup $[\overline{M}^2, B]$ is defined as the disjoint union of $M^2 \setminus B$ with the interior spherical normal bundle of $B \subset \overline{M}^2$, under an identification explained in [17], cf. also [4]. The blowup $[\overline{M}^2, B]$ is equipped with the minimal differential structure such that smooth functions in the interior of $\overline{M}^2$ and polar coordinates around B are smooth. The interior

Fig. 1 The b-double space M_b^2
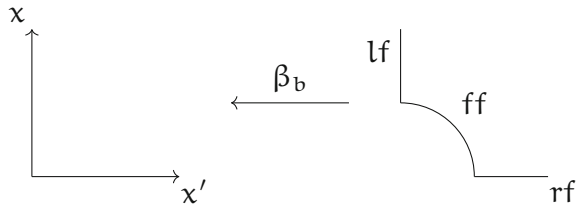
spherical normal bundle of B defines a new boundary hypersurface of $[\overline{M}^2, B]$, the front face ff, in addition to the previous boundary faces $\{x = 0\}$ (the right face) and $\{x' = 0\}$ (the left face). The front face is itself a quarter circle fibration over B. $M_b^2 = [\overline{M}^2, B]$ is equipped with the obvious blowdown map $\beta_b : M_b^2 \to \overline{M}^2$ and is illustrated in Fig. 1.

We now introduce projective coordinates in a neighborhood of the front face ff, that are very convenient in the computations, but are not globally defined over all of ff. Near the corner intersection of ff and rf, away from lf, we have projective coordinates

$$s = \frac{x}{x'}, \ y, \ z, \ x', \ y', \ z', \tag{3.1}$$

where s is a defining function of rf and x' a defining function of ff. Near the corner intersection of ff and lf, away from rf, we get projective coordinates by interchanging the roles of x and x'. The pullback by the blowdown map is locally simply a change of coordinates.

3.2 ϕ-Double space M_ϕ^2

We now blow up the fibre diagonal $\phi \subset M_b^2$

$$\phi := \{(h, h') \in B = \partial M \times \partial M; \phi(h) = \phi(h')\},$$

lifted up the b-double space, where $\phi : \partial M \to B$ is the fibration of the boundary. Its lift $\beta_b^*(\phi) \subset M_b^2$ is given in the local coordinates (3.1) by $\{y = y', s = 1, x' = 0\}$. We define the ϕ-double space by a similar procedure as above, by blowing up $\beta_b^*(\phi)$
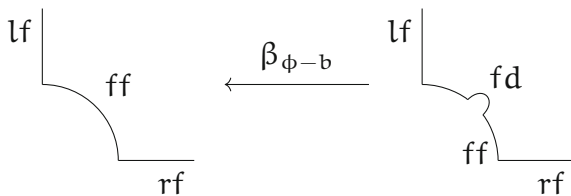
$$M_\phi^2 := [M_b^2; \beta_b^*(\phi)], \quad \beta_{\phi-b} : M_\phi^2 \longrightarrow M_b^2.$$

The new boundary hypersurface is denoted by fd and the blowup is illustrated in Fig. 2. We also define the full blowdown map $\beta_\phi := \beta_b \circ \beta_{\phi-b} : M_\phi^2 \to \overline{M}^2$.

In local coordinates we have the following projective coordinates near fd (here the roles of x and x' can be interchanged, leading to equivalent projective coordinate systems)

$$S := \frac{s - 1}{x'}, \ U := \frac{y - y'}{x'}, \ x', \ y', \ z, \ z', \tag{3.2}$$

Fig. 2 The ϕ-double space M_ϕ^2

where x' is the defining function of fd, and the rest of the original front face ff lies in the limit $|(S, U)| \to \infty$. As before, the pullback by the blowdown map is locally simply a change of coordinates.

Remark 3.1 The centrality of the ϕ-double space M_ϕ^2 stems from the fact that Schwartz kernels of parametrices to elliptic differential ϕ-operators, studied in the ϕ-calculus by Mazzeo and Melrose [15], lift to polyhomogeneous conormal distributions on M_ϕ^2 with conormal singularity along the lifted diagonal.

4 Outline of the heat kernel construction for a ϕ-metric

Let Δ_ϕ be the unique self-adjoint extension of the Hodge Laplacian on the ϕ-manifold M with fibred boundary ∂M and a ϕ-metric g_ϕ. As noted in Remark 2.5, we can replace Δ_ϕ by any Laplace-type operator in $\mathrm{Diff}_\phi^2(M, L)$, provided the structure of (2.4) (times the identity matrix) holds in the collar neighbourhood $\mathcal{U}$ up to higher order terms. This includes squares of geometric Dirac operators, such as the spin Dirac operator.

The heat operator of Δ_ϕ is denoted by $e^{-t\Delta_\phi}$ and solves for any given section ω_0 of $\Lambda^{*\phi}T^*M$ in the domain of Δ_ϕ by definition the homogeneous heat problem

$$
\begin{aligned}
(\partial_t + \Delta_\phi)\omega(t, p) &= 0, \quad (t, p) \in [0, \infty) \times M, \\
\omega(0, p) &= \omega_0(p), \quad p \in M,
\end{aligned}
\tag{4.1}
$$

with $\omega = e^{-t\Delta_\phi}\omega_0$, a section of $\Lambda^{*\phi}T^*M$ for any fixed $t \in [0, \infty)$. The heat operator is an integral operator

$$
e^{-t\Delta_\phi}\omega_0(p) = \int_M H\left(t, p, \widetilde{p}\right) \omega_0(\widetilde{p})\mathrm{dvol}_{g_\phi}(\widetilde{p}),
\tag{4.2}
$$

with the heat kernel $H(t, \cdot, \cdot)$ being a section of $\Lambda^{*\phi}T^*M \boxtimes \left(\Lambda^{*\phi}T^*M\right)^*$ for any $t \in [0, \infty)$. Here, $\boxtimes$ denotes the external tensor product of vector bundles.

Consider local coordinates $(t, (x, y, z), (\widetilde{x}, \widetilde{y}, \widetilde{z}))$ near the highest codimension corner in the heat space M_h^2, where (x, y, z) and $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ are two copies of the local coordinates on M near the boundary ∂M, as before. Then the heat kernel H has a non-uniform behaviour at the diagonal $D = \{(0, p, p) \mid p \in \overline{M}\}$ and at the submanifold

$$
A = \{(t, (0, y, z), (0, \widetilde{y}, \widetilde{z})) \in M_h^2 : y = \widetilde{y}\}.
$$
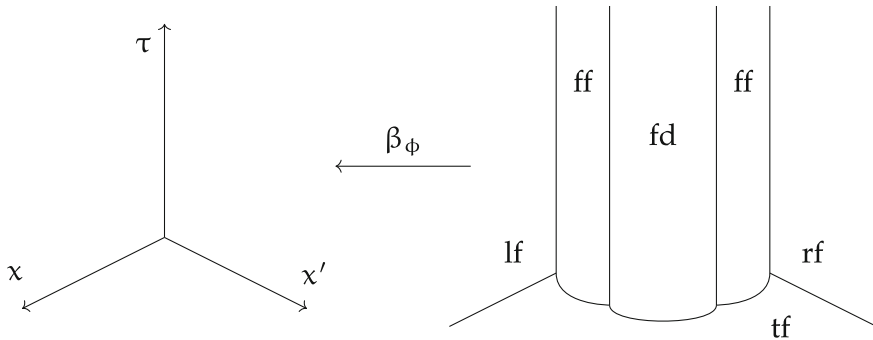
Fig. 3 Intermediate heat blowup space $[0, \infty) \times M_\phi^2$

The asymptotic behaviour of H near the submanifolds D and A of M_h^2 is conveniently studied using the blowup procedure of Sect. 3. We proceed in the remainder of this paper with the following 3 steps:

Step 1 We construct the "heat blowup space" $H M_\phi$ by an additional blowup in $[0, \infty) \times M_\phi^2$, where M_ϕ^2 is the ϕ-double space introduced in Sect. 3. More specifically, we lift the diagonal $D \in M_h^2$ to $[0, \infty) \times M_\phi^2$ and blow it up, treating $\sqrt{t}$ as a smooth variable[1] (i.e. we extend $t \in (0, 1)$ to $[0, \infty]$ and consider the smooth structure of functions smooth in $\sqrt{t}$). We then may define the heat calculus of smoothing operators with Schwartz kernels that lift to polyhomogeneous functions on the heat blowup space $H M_\phi$.

Step 2 We obtain an initial parametrix for H inside the heat calculus, solving the heat equation up to first order. This requires us to lift the heat equation to $H M_\phi$ and to solve the resulting equations (normal problems) at the various boundary faces of the heat blowup space.

Step 3 The exact heat kernel is then obtained by a Volterra series argument, which requires the triple space construction and the composition formula of the final section Sect. 7.

5 Step 1: Construction of the heat blowup space

Consider the ϕ-double space M_ϕ^2 with the blowdown map $\beta_\phi : M_\phi^2 \to \overline{M}^2$. We obtain an intermediate heat blowup space by taking its product with the time axis $[0, \infty)$. We treat the square root of the time variable $\tau := \sqrt{t}$ as a smooth variable. The resulting intermediate heat blowup space is illustrated in Fig. 3.

Let us explain the abbreviations for the boundary hypersurfaces in the intermediate heat blowup space: ff stands for front face, fd—the fibre diagonal, lf and rf—the left and right faces, respectively, and finally tf stands for the temporal face. In view of

---

[1] If X is any manifold, possibly with corners, and t is a boundary defining function for $X \times \{0\}$ in $X \times [0, \infty)$ then $\sqrt{t}$ is a boundary defining function for the pre-image of $X \times \{0\}$ in $[X \times [0, \infty), X \times \{0\}, \langle dt \rangle]$, the parabolic blow-up of $X \times \{0\}$ in the direction of dt. More correctly, the pull-back of the boundary defining function of the latter space squares to a boundary defining function of the former.

(3.1) and (3.2), the projective coordinates on $[0, \infty) \times M_\phi^2$ near the various boundary hypersurfaces are as follows.

5.1 Projective coordinates near the intersection of rf with ff, away from fd

In view of the projective coordinates (3.1) on the ϕ double space, we have the following projective coordinates, which are valid uniformly up to tf, away from an open neighborhood of fd.

$$s = \frac{x}{x'}, \; y, \; z, \; x', \; y', \; z', \; \tau = \sqrt{t}. \tag{5.1}$$

In these coordinates, s is a defining function of rf, τ is a defining function of tf, x' is a defining function of ff. Interchanging the roles of x and x' yields projective coordinates near the intersection of lf with ff, where $s' = x'/x$ is a defining function of lf.

5.2 Projective coordinates near the fd up to tf

In view of the projective coordinates (3.2) on the ϕ double space, we have the following projective coordinates, which are valid in open neighborhood of fd uniformly up to tf, away from lf and rf. Here the roles of x and x' can be interchanged, leading to an equivalent system of coordinates.

$$S = \frac{s - 1}{x'}, \; U = \frac{y - y'}{x'}, \; z, \; x', \; y', \; z', \; \tau = \sqrt{t}. \tag{5.2}$$

In these coordinates, τ is a defining function of tf, x' is a defining function of fd, and ff lies in the limit $|(S, U)| \to \infty$.

5.3 Heat blowup space as a blowup of temporal diagonal

The final heat blowup space HM_ϕ is obtained by blowing up the lift of the diagonal $D = \{(0, p, p) \mid p \in \overline{M}\}$ to $[0, \infty) \times M_\phi^2$. In the local coordinates in an open neighborhood $\mathcal{U}_{\mathrm{fd}}$ of fd, its lift is given by

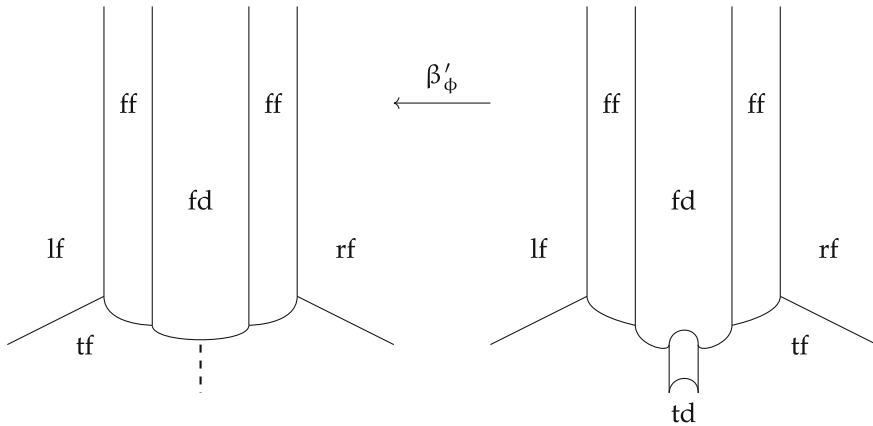$$\beta_\phi^*(D) \upharpoonright \mathcal{U}_{\mathrm{fd}} = \{S = 0, U = 0, z = z', \tau = 0\}. \tag{5.3}$$

The heat blowup space HM_ϕ is then defined as a parabolic blowup

$$HM_\phi := \left[[0, \infty) \times M_\phi^2, \beta_\phi^*(D) \right]. \tag{5.4}$$

This blowup space is illustrated in Fig. 4.

The full blow down map is defined by

$$\beta := \beta_\phi \circ \beta_\phi' : HM_\phi \to [0, \infty) \times \overline{M}^2.$$

Fig. 4 The heat blowup space HM_ϕ

Let us now describe the resulting heat blowup space in projective coordinates. The previous coordinate systems (5.1) and (5.2) remain valid away from an open neighborhood of the new boundary face td. Near td we have up to the intersection with fd the following projective coordinates (with respect to the notation of (5.2))

$$S = \frac{S}{\tau} = \frac{x - x'}{(x')^2 \sqrt{t}}, \; \mathcal{U} = \frac{U}{\tau} = \frac{y - y'}{x'\sqrt{t}}, \; \mathcal{Z} = \frac{z - z'}{\sqrt{t}}, \; x', \; y', \; z', \; \tau = \sqrt{t}. \quad (5.5)$$

In these coordinates, x' is a defining function of fd, τ is a defining function of td, and tf lies in the limit $|(S, \mathcal{U}, \mathcal{Z})| \to \infty$. In these coordinates, the roles of x and x' may be interchanged freely. The pullback by the blowdown map β is locally simply a change between standard and projective coordinates (5.1), (5.2) and (5.2).

We conclude the heat blow up space construction by singling out a class of polyhomogeneous functions on it, that define the "heat calculus" in our setting.

Definition 5.1 (*Heat calculus*) We write $m = \dim M$. Consider a vector bundle L over $\overline{M}$ and the projection $\pi : [0, \infty) \times \overline{M}^2 \to \overline{M}^2$ onto the second component. Define the pullback bundle over HM_ϕ by

$$E := (\pi \circ \beta)^* \Big(L \boxtimes L^* \Big). \quad (5.6)$$

For any $a, \ell \in \mathbb{R}$, the space $\mathcal{H}_\phi^{a,\ell}(M, L)$ is defined as the space of linear operators A acting on smooth sections of L that are compactly supported in M

$$A : C_0^\infty(M, L) \to C_0^\infty(M, L),$$

with Schwartz kernels K_A lifting to polyhomogeneus sections $\beta^* K_A$ of E such that for any defining functions ρ_{fd} and ρ_{td} of fd and td, respectively, we have

$$\beta^* K_A = \rho_{fd}^{-3+a} \cdot \rho_{td}^{-m+\ell} G_A.$$

Here G_A is polyhomogeneous, smooth at fd and td, and vanishing to infinite order at lf, rf, ff and tf. We further define

$$\mathcal{H}_\phi^\infty(M, L) := \bigcap_{a,\ell,\in\mathbb{R}} \mathcal{H}_\phi^{a,\ell}(M, L).$$

Our main example is $L = \Lambda^{*\phi} T^* M$. We shall abbreviate $\Lambda_\phi^* := \Lambda^{*\phi} T^* M$ and denote the corresponding heat calculus by $\mathcal{H}_\phi^{a,\ell}(M, \Lambda_\phi^*)$.

Example 5.2 Consider the Example 2.3 and the scalar Euclidean heat kernel H on $\mathbb{R}^m$. We want to explain that H lies inside the heat calculus defined above, more precisely $H \in \mathcal{H}_\phi^{3,0}(M, \Lambda_\phi^0)$. Write for any pair $p, p' \in \mathbb{R}^m$

$$p =: \|p\|\eta =: \eta/x, \quad p' =: \|p'\|\eta' =: \eta'/x',$$

where $\eta, \eta' \in \mathbb{S}^{m-1}$. For simplicity, let us assume $\eta = \eta'$. We obtain from the standard formula for the scalar Euclidean heat kernel, using the projective coordinates (5.5), where τ and x' are defining functions of td and fd, respectively

$$H(t, p, p') = \frac{1}{(4\pi)^{\frac{m}{2}}} \tau^{-m} \exp\left(-\frac{|\mathcal{S}|}{4|(1+x'\mathcal{S})|} \|\eta\|^2\right).$$

This corresponds to the claim $H \in \mathcal{H}_\phi^{3,0}(M, \Lambda_\phi^0)$. Indeed, H is smooth at $\{x' = 0\}$, i.e. at fd. H behaves as τ^{-m} at td. It vanishes to infinite order at ff, i.e. as $|\mathcal{S}| \to \infty$. The general case $\eta \neq \eta'$ is similar, albeit with more cumbersome computations.

Remark 5.3 We point out that our Schwartz kernels are not multiplied with half-densities here, which is common in many other references. This simplifies our presentation here, but leads to some shifts in the asymptotics later on, when we study compositions of the Schwartz kernels in Sect. 7.

6 Step 2: Construction of an initial heat kernel parametrix

We construct an initial heat parametrix for Δ_ϕ by solving the heat equation, lifted to HM_ϕ, to leading order at fd and td. The solutions of the heat equation at fd and td can be extended off these boundary faces with any power of the respective defining functions. The correct powers are determined by studying the lift of the delta distribution at fd and td as well.

6.1 Solving the heat equation near fd

Let us consider the relevant geometric quantities written in projective coordinates near fd. Recall the projective coordinates (5.2), that are valid near fd, away from td. We compute in these coordinates

$$\beta^* \partial_x = (x')^{-2} \partial_S, \quad \beta^* \partial_y = (x')^{-1} \partial_U, \quad \beta^* \partial_z = \partial_z. \tag{6.1}$$

Let us point out that fd is the total space of fibration over B with fibres $\mathbb{R} \times \mathbb{R}^b \times F^2$. Here, $y' \in B$ denotes the base point of the fibration, $(S, U, z, z') \in \mathbb{R} \times \mathbb{R}^b \times F^2$ coordinates on the fibres. Recall the formula (2.9) for the Hodge Laplacian Δ_ϕ near the boundary. Note that the term D_B^2 in (2.9) equals Δ_B (the Hodge Laplacian on B) plus terms in $x \cdot \text{Diff}_\phi^2(\mathcal{U})$. Thus, in view of (6.1) we compute

$$\beta^* \Delta_\phi \restriction \text{fd} = \Delta_{S,U,y'} + \Delta_{F,y'}, \tag{6.2}$$

where $\Delta_{S,U,y}$ is the Euclidean Hodge Laplacian on $\mathbb{R}^{b+1} \cong \mathbb{R} \times T_{y'}B$ with Euclidean coordinates (S, U), where the scalar product on $T_{y'}B$ is defined with respect to $g_B(y')$. The second summand, $\Delta_{F,y'}$ is given by $D_{F_{y'}}^2$ in the notation of (2.9), which is a Laplace type operator acting on sections of $\Lambda \mathcal{V}^*$. We conclude

$$\beta^* \left(t \left(\partial_t + \Delta_\phi \right) \right) \restriction \text{fd} = \frac{1}{2} \tau \partial_\tau + \tau^2 \left(\Delta_{S,U,y'} + \Delta_{F,y'} \right) =: \mathcal{L}_{\text{fd}}.$$

Note that the parameter y' simply indicates the base point of the fibration fd, and for each fixed base point, the equation $\mathcal{L}_{\text{fd}} u = 0$ is an partial differential equation on the fibres of fd. A solution to $\mathcal{L}_{\text{fd}} u = 0$ is given by

$$N_{\text{fd}}(H)(\tau, S, U, z, z'; y') := H_{\Delta_{S,U,y'}}(\tau, S, U, 0) H_{\Delta_{F,y'}}(\tau, z, z'), \tag{6.3}$$

where $H_{\Delta_{S,U,y'}}$ and $H_{\Delta_{F,y'}}$ are the heat kernels of $\Delta_{S,U,y'}$ and $\Delta_{F,y'}$, respectively. We extend it off the front face fd as

$$\beta^* H_0' := N_{\text{fd}}(H)(\tau, S, U, z, z'; y') \, \psi(x), \tag{6.4}$$

with a cutoff function $\psi \in C_0^\infty[0, \infty)$ with compact support in $[0, 2\varepsilon)$ for some $\varepsilon \in (0, 1)$ such that $\psi \equiv 1$ on $[0, \varepsilon]$. Then H_0' solves the heat equation to first order at fd, i.e.

$$\beta^* \left(t \left(\partial_t + \Delta_\phi \right) H_0' \right) \restriction \text{fd} = 0.$$

6.2 Solving the heat equation near *td* ∪ *fd*

Our heat parametrix H_0' does not solve the heat equation to any order at td. Here we explain the standard procedure how H_0' is corrected to provide a heat parametrix,

solving the heat equation to higher order at td as well. Recall, near td we have up to the intersection with fd the coordinates (5.5)

$$\mathcal{S} = \frac{S}{\tau} = \frac{x - x'}{(x')^2 \sqrt{t}}, \quad \mathcal{U} = \frac{U}{\tau} = \frac{y - y'}{x' \sqrt{t}}, \quad \mathcal{Z} = \frac{z - z'}{\sqrt{t}}, \quad x', \ y', \ z', \ \tau = \sqrt{t}. \quad (6.5)$$

In these coordinates, x' is a defining function of fd, τ is a defining function of td, and tf lies in the limit $|(\mathcal{S}, \mathcal{U}, \mathcal{Z})| \to \infty$. In these coordinates, the roles of x and x' may be interchanged freely. We fix coordinates near $td \cap fd$. In these coordinates the individual partial derivatives are written as follows

$$\beta^* \partial_x = \frac{1}{\tau x'^2} \partial_{\mathcal{S}}, \quad \beta^* \partial_y = \frac{1}{\tau x'} \partial_{\mathcal{U}}, \quad \beta^* \partial_z = \frac{1}{\tau} \partial_{\mathcal{Z}}. \quad (6.6)$$

Let us point out that td is the total space of fibration over $\overline{M}$ with fibres $\mathbb{R} \times \mathbb{R}^b \times \mathbb{R}^f$. Here, $p' = (x', y', z') \in \overline{M}$ denotes the base point of the fibration, $(\mathcal{S}, \mathcal{U}, \mathcal{Z}) \in \mathbb{R} \times \mathbb{R}^b \times \mathbb{R}^f$ coordinates on the fibres. In view of (6.6) and (2.9) we compute

$$\beta^* t \Delta_\phi \restriction td = \Delta_{\mathcal{S}, \mathcal{U}, \mathcal{Z}, x', y', z'},$$

where on the right hand side we have for each fixed (x', y', z') the Hodge Laplacian on $\mathbb{R} \times \mathbb{R}^b \times \mathbb{R}^f \cong \mathbb{R} \times T_{y'} B \times \mathcal{V}_{(y', z')}$, with Euclidean coordinates $(\mathcal{S}, \mathcal{U}, \mathcal{Z})$, defined with respect to the metric $g_B(y')$ at the $T_{y'} B \cong \mathbb{R}^b$ component, and the metric $g_F(y', z')$ at the $\mathcal{V}_{(y', z')} \cong \mathbb{R}^f$ component. Away from fd, this is simply the Hodge Laplacian on $T_{p'} M \cong \mathbb{R}^m$ with respect to the metric g_ϕ on $T_{p'} M$. Denote the corresponding heat kernel, evaluated at $\tau = 1$, by $N_{td}(H)$. We extend it off td, using the cutoff function ψ as above, and define

$$\beta^* H_0'' := \tau^{-m} N_{td}(H)(\tau, \mathcal{S}, \mathcal{U}, \mathcal{Z}, x', y', z') \psi(\tau). \quad (6.7)$$

By construction, H_0'' solves the heat equation to first order at td

$$\beta^* \left(t \left(\partial_t + \Delta_\phi \right) H_0'' \right) \restriction td = O(\tau^{-m+1}), \quad \text{as } \tau \to 0.$$

Since by construction $\beta^* H_0'$ and $\beta^* H_0''$ agree on $td \cap fd$, there exists a well-defined operator $H_0 \in \mathcal{H}_\phi^{3,0}(M, \Lambda_\phi^*)$, solving the heat equation to first order at fd and td.

6.3 Constructing an initial heat parametrix

In the final step below, one improves the parametrix H_0 to solve away the error at td to infinite order. This is done by the usual argument, which is outlined in various cases, cf. Melrose [17], and Grieser [5] as a basic reference, as well as Albin [1], Mazzeo and Vertman [16] for the same argument in different geometric settings. This defines a new heat parametrix, still denoted by H_0 in $\mathcal{H}_\phi^{3,0}(M, \Lambda_\phi^*)$ such that

$$(\partial_t + \Delta_\phi) H_0 \in \mathcal{H}_\phi^{4,\infty}(M, \Lambda_\phi^*).$$

Same arguments apply when Δ_ϕ is replaced the square of a geometric Dirac operator, such as the spin Dirac operator, or more generally by any Laplace-type operator in $\mathrm{Diff}_\phi^2(M, L)$, provided the structure of (2.4) (times the identity matrix) holds in the collar neighborhood $\mathcal{U}$ up to higher order terms. This yields the same structure as in (6.2), so that the rest of the argument applies. This proves the following result.

Theorem 6.1 *There exists an initial heat parametrix $H_0 \in \mathcal{H}_\phi^{3,0}(M, \Lambda_\phi^*)$ for the Hodge Laplacian Δ_ϕ, solving the heat equation to first order at fd, and to infinite order at td, i.e. $(\partial_t + \Delta_\phi)H_0 \in \mathcal{H}_\phi^{4,\infty}(M, \Lambda_\phi^*)$. The restriction of H_0 to fd is given by $N_{\mathrm{fd}}(H)$. The leading term in the asymptotic expansion of H_0 at td is given by $N_{\mathrm{td}}(H)$.*

If Δ_ϕ is replaced by a Laplace-type operator in $\mathrm{Diff}_\phi^2(M, L)$ such that (2.4) (times the identity matrix) holds in the collar neighborhood $\mathcal{U}$ up to higher order terms, the same statement holds with $\Lambda_\phi^* \equiv \Lambda^{*\phi} T^* M$ replaced by L.

7 Step 3: Triple space construction and composition of operators

In this section we use the initial heat parametrix H_0 in Theorem 6.1 to construct the exact heat kernel as a polyhomogeneous section of E in (5.6) on the heat space HM_ϕ. The construction is based on the following composition result, which is the main technical result of this section.

Theorem 7.1 (Composition theorem) *Assume that, $A \in \mathcal{H}_\phi^{a,\ell}(M, \Lambda_\phi^*)$ and $B \in \mathcal{H}_\phi^{a',\infty}(M, \Lambda_\phi^*)$. We denote the corresponding Schwartz kernels of A and B by K_A and K_B, respectively. Then the composition $A \circ B$ with the Schwartz kernel given by*

$$K_{A \circ B}(t, p, p') := \int_M \int_0^t K_A(t - t', p, p'') \cdot K_B(t', p'', p')dt'\, \mathrm{dvol}_{g_\phi}(p''), \quad (7.1)$$

is well defined and $A \circ B \in \mathcal{H}_\phi^{a+a',\infty}(M, \Lambda_\phi^*)$. Same statement holds for Λ_ϕ^* replaced by any vector bundle L over $\overline{M}$.

The dot in (7.1) is defined fibrewise as follows

$$\begin{aligned}
\cdot : \left(\Lambda_{\phi,p}^* \boxtimes (\Lambda_{\phi,p''}^*)^*\right) \times \left(\Lambda_{\phi,p''}^* \boxtimes (\Lambda_{\phi,p'}^*)^*\right) &\to \Lambda_{\phi,p}^* \boxtimes (\Lambda_{\phi,p'}^*)^*, \\
\left((s, s''), (r'', r')\right) &\mapsto (s''(r'')s, s''(r'')r').
\end{aligned} \quad (7.2)$$

We will prove this theorem below, and assuming it for the moment we conclude our main result (Theorem 1.1) as a corollary.

Corollary 7.2 *The heat kernel of Δ_ϕ lifts to a polyhomogeneous section of E in (5.6) on the heat space HM_ϕ, vanishing to infinite order at ff, tf, rf and lf, smooth at fd, and of order $(-m)$ at td. More precisely, $e^{-t\Delta_\phi} \in \mathcal{H}_\phi^{3,0}(M, \Lambda_\phi^*)$. Same holds for Laplace-type operators as in Remark 2.5, with Λ_ϕ^* replaced by L.*

Proof Consider the initial parametrix H_0 of Theorem 6.1. When viewed as operators acting by an additional convolution in time, we find

$$(\partial_t + \Delta_\phi)H_0 = (\mathrm{Id} + P), \quad P \in \mathcal{H}_\phi^{4,\infty}(M, \Lambda_\phi^*).$$

Formally, the heat kernel is obtained by inverting the error term $(\mathrm{Id} + P)$

$$e^{-t\Delta_\phi} = H_0 \circ (\mathrm{Id} + P)^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell H_0 \circ P^\ell. \tag{7.3}$$

However, convergence of the Neumann series in $\mathcal{H}_\phi^*(M, \Lambda_\phi^*)$ is intricate, since the lift of the Schwartz kernel for error term P to HM_ϕ does not vanish to infinite order at fd. We remedy this by correcting the fd asymptotics of H_0, asymptotically summing the lifts $\beta^*((-1)^\ell H_0 \circ P^\ell)$ over $\ell \in \mathbb{N}_0$. This defines a new heat parametrix $H_0' \in \mathcal{H}_\phi^{3,\infty}(M, \Lambda_\phi^*)$, such that

$$H_0' \sim \sum_{\ell=0}^{\infty} (-1)^\ell N_{\mathrm{fd}}(H_0 \circ P^\ell) \rho_{\mathrm{fd}}^{-3+4\ell}. \tag{7.4}$$

As a consequence, we obtain a much better error term whose Schwartz kernel, lifted to HM_ϕ, by construction vanishes to infinite order at all boundary faces of HM_ϕ

$$(\partial_t + \Delta_\phi)H_0' = (\mathrm{Id} + P'), \quad P' \in \mathcal{H}_\phi^{\infty,\infty}(M, \Lambda_\phi^*). \tag{7.5}$$

Now we can invert $(\mathrm{Id} + P')$ and obtain the heat kernel as a Neumann series

$$e^{-t\Delta_\phi} \equiv H = H_0' \circ (\mathrm{Id} + P')^{-1} = H_0' + \sum_{\ell=1}^{\infty} (-1)^\ell H_0' \circ P'^\ell. \tag{7.6}$$

The series can be shown to converge in $\mathcal{H}_\phi^{\infty,\infty}(M, \Lambda_\phi^*)$ by a Volterra series argument cf. [2, Theorem 2.19], and [5, Proposition 2.10].

We shall be precise: let $K_{P'}$ be the Schwartz kernel of P', and $K_{P'^\ell}$ the Schwartz kernel of P'^ℓ. For any fixed $(p, p', t) \in \overline{M}^2 \times \mathbb{R}^+$ we set $(p_0, t_0) = (p, t)$ and $(p_\ell, t_\ell) = (p', 0)$, and then write $K_{P'^\ell}(p, p', t)$ as follows

$$K_{P'^\ell}(p, p', t) = \int_{\triangle} \int_{M^{\ell-1}} \prod_{k=0}^{\ell-1} K_{P'}(p_k, p_{k+1}, t_k - t_{k+1}) \prod_{k=0}^{\ell-2} dt_{k+1} \mathrm{dvol}_{g_\phi}(p_{k+1}), \tag{7.7}$$

where the integration region $\triangle \in [0, t]^{\ell-1}$ is a simplex given by

$$\triangle = \left\{ (t_1, \ldots, t_{\ell-1}) \in [0, t]^{\ell-1} \mid t \geq t_1 \geq \cdots \geq t_{\ell-1} \right\}.$$

The lift $\beta^* K_{P'}$ to HM_ϕ vanishes to infinite order at all boundary faces. Thus, for any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ depending only on N such that (we denote by $\| \cdot \|_\infty$ the supremum over $\partial M \times \partial M$ of the pointwise (g_ϕ-induced) norms on the fibres of $\Lambda_\phi^* \boxtimes (\Lambda_\phi^*)^*$)

$$\|K_{P'}(x, x', t)\|_\infty \leq C_N (t\, x\, x')^N (x')^{2+b}.$$

Recall now the notation of Definition 2.2 and consider a Riemannian metric $\overline{g}$ on M, such that $\overline{g} \upharpoonright \mathcal{U} = dx^2 + g_F + \phi^* g_B$. Note that M has finite volume $\mathrm{vol}(M, \overline{g})$ with respect to $\overline{g}$. Moreover, we obtain for the volume forms of g_ϕ and $\overline{g}$ (up to a bounded nowhere vanishing factor)

$$\mathrm{dvol}_{g_\phi} = (x')^{-2-b} \mathrm{dvol}_{\overline{g}}.$$

Consequently, we obtain from (7.7) for the pointwise (g_ϕ-induced) norms on the fibres of $\Lambda_\phi^* \boxtimes (\Lambda_\phi^*)^*$, exactly as in [5, Proposition 2.10]

$$\|K_{P'^\ell}(x, y, z, x', y', z', t)\| \leq C_N (t\, x\, x')^N \mathrm{vol}(\triangle) \mathrm{vol}(M, \overline{g}))^{\ell-1}$$
$$= (t\, x\, x')^N \frac{(t\, C_N \mathrm{vol}(M, \overline{g}))^{\ell-1}}{(\ell-1)!}. \tag{7.8}$$

Similarly, we obtain an estimate for the Schwartz kernel of $H_0' \circ P'^\ell$

$$\|K_{H_0' \circ P'^\ell}(x, y, z, x', y', z', t)\| \leq C (t\, x\, x')^K \frac{(t\, C_N \mathrm{vol}(M, \overline{g}))^{\ell-1}}{(\ell-1)!}, \tag{7.9}$$

for some constant $C > 0$, depending only on H_0'. Consequently

$$\left\|\left(K_H - K_{H_0'}\right)(x, y, z, x', y', z', t)\right\| \leq C (t\, x\, x')^N e^{t\, C_N \mathrm{vol}(M, \overline{g})}. \tag{7.10}$$

Since $K \in \mathbb{N}$ was arbitrary, this proves that the infinite sum in (7.6) converges in $\mathcal{H}_\phi^*(M, \Lambda_\phi^*)$. This concludes the proof. $\qquad\qquad\square$

7.1 Proof of the composition theorem

In this subsection we prove Theorem 7.1, where $\beta^* K_B$ is assumed to be vanishing to infinite order at td, but $\beta^* K_A$ is not necessarily. We write the composition integral (7.1) using pullback and push forward as follows. We write $\mathbb{R}_+ \equiv \mathbb{R}^+ := [0, \infty)$ and define the maps

$$\pi_C : \overline{M}^3 \times \mathbb{R}_{t'}^+ \times \mathbb{R}_{t''}^+ \longrightarrow \overline{M}^2 \times \mathbb{R}_{t'+t''}^+, \quad (p, p', p'', t', t'') \rightarrow (p, p'', t' + t''),$$
$$\pi_L : \overline{M}^3 \times \mathbb{R}_{t'}^+ \times \mathbb{R}_{t''}^+ \longrightarrow \overline{M}^2 \times \mathbb{R}_{t''}^+, \quad (p, p', p'', t', t'') \rightarrow (p, p', t'),$$
$$\pi_R : \overline{M}^3 \times \mathbb{R}_{t''}^+ \times \mathbb{R}_{t''}^+ \longrightarrow \overline{M}^2 \times \mathbb{R}_{t'}^+, \quad (p, p', p'', t', t'') \rightarrow (p', p'', t').$$

Then we can write (7.1) by pulling back K_A and K_B to $\overline{M}^3 \times \mathbb{R}_+^2$ via π_L, π_R and pushing forward (integrate) along $t = t' + t''$ and in p' with respect to dvol_{g_ϕ}

$$K_C = (\pi_C)_* \left((\pi_L^* K_A) \cdot (\pi_R^* K_B) \right), \tag{7.11}$$

where the dot is defined fibrewise as in (7.2). We prove the composition theorem by constructing the "heat triple space" HM_ϕ^3 by a resolution process from $\overline{M}^3 \times \mathbb{R}_+^2$, with blow down map

$$\beta_3 : HM_\phi^3 \longrightarrow \overline{M}^3 \times \mathbb{R}_+^2.$$

We show that the maps π_C, π_L, π_R lift to b-fibrations Π_C, Π_L, Π_R in the sense of Melrose [17], i.e. in the commutative diagram diagram (7.12)

$$\begin{array}{ccc}
HM_\phi^3 & \longrightarrow & M_\phi^2 \times \mathbb{R}^+ \\
\beta_{\mathrm{Tr}} \downarrow & & \downarrow \beta_\phi \\
\overline{M}^3 \times \mathbb{R}_+^2 & \longrightarrow & \overline{M}^2 \times \mathbb{R}_+
\end{array} \tag{7.12}$$

the maps π_C, π_L, π_R in bottom arrow lift to b-fibrations Π_C, Π_L, Π_R in the upper arrow. Here we use the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$, introduced in Fig. 3, together with the corresponding blowdown map β_ϕ, since the additional blowup of the temporal diagonal will not be necessary due to Proposition 7.1. Defining $\kappa_{A,B,C} := \beta_\phi^*(K_{A,B,C})$, we obtain using the commutativity of diagram (7.12) a new version of the relation (7.11)

$$\kappa_C \equiv \beta_\phi^*(K_C) = (\Pi_C)_*(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B). \tag{7.13}$$

The idea is now to see that $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B$ is indeed polyhomogeneous and the pushforward under Π_C preserves the polyhomogeneity. In the rest of this section we follow this strategy more concretely, first construct the triple space HM_ϕ^3, compute the lift of boundary defining functions under projections to compute explicitly the asymptotics of $\Pi_L^* \kappa_A, \Pi_R^* \kappa_B$ and also the pushfoward, $(\Pi_C)_*(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B)$.

7.2 Construction of the triple space

In order to apply Melrose's pushforward theorem [18], to conclude polyhomogeneity κ_C, the maps Π_C, Π_L, Π_R need to be b-fibrations. This dictates the construction of the triple space HM_ϕ^3 as a blowup of $\overline{M}^3 \times \mathbb{R}_+^2$. We describe the blowups using local coordinates $p = (x, y, z)$ and their copies $p' = (x', y', z')$ and $p'' = (x'', y'', z'')$ on M. The time coordinates on each $\mathbb{R}_+^2$ are written as t and t'. The first submanifold to blow up is then

$$F := \{t' = t'' = 0, x = x' = x'' = 0\} \subset \overline{M}^3 \times \mathbb{R}_+^2.$$

Fig. 5 Illustration of M_b^3 in spatial direction
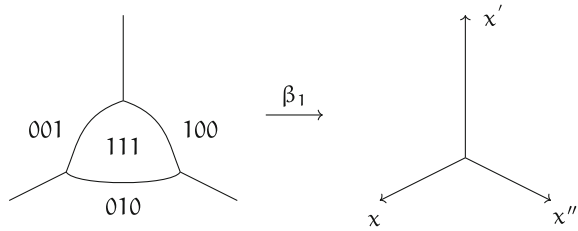
We refer the reader to § 3 for the basic elements of the blowup procedure. As before we blow up parabolically in the time direction, i.e. we treat $\sqrt{t}$ and $\sqrt{t'}$ as smooth. The resulting blowup space $M_b^3 = [\overline{M}^3 \times \mathbb{R}_+^2; F]$ is illustrated in Fig. 5 and comes with the blow down map

$$\beta_1 : M_b^3 \longrightarrow \overline{M}^3 \times \mathbb{R}_+^2.$$

We denote the resulting new boundary face, which is the inward spherical normal bundle of $F \subset \overline{M}^3 \times \mathbb{R}_+^2$ by (111), where the first 1 indicates that the boundary face corresponds to $x = 0$, the second 1 corresponds to $x' = 0$, and the third 1 to $x'' = 0$. This principle is also used in the namesgiving for other boundary faces, e.g. (100) is the lift of $\{x = 0\}$.

We then blow up M_b^3 at

$$F_O := \beta_1^* \{t' = t'' = 0\}.$$

The next submanifolds to blow up are submanifolds of the codimension 2 in spatial direction corresponding to each projection π_C, π_L, π_R. Accordingly we denote these submanifolds as F_C, F_L, F_R, which are defined as

$$F_C := \beta_1^* \{t' = t'' = 0, x = x'' = 0\},$$
$$F_L := \beta_1^* \{t'' = 0, x' = x'' = 0\},$$
$$F_R := \beta_1^* \{t' = 0, x = x' = 0\}.$$

We point out that the order of blowing of submanifolds F_C, F_L, F_R after blowing up of F and F_O is immaterial as they become disjoint. As before we blow up parabolically in the time direction. The resulting blowup space

$$M_{b,t}^3 = [[M_b^3; F_O]; F_C, F_L, F_R],$$

is illustrated in Fig. 6 and comes with the blow down map

$$\beta_2 : M_{b,t}^3 \longrightarrow M_b^3.$$

The triple elliptic space of the ϕ-calculus, see Grieser and Hunsicker [6], includes the fibre-diagonal blow up in each direction. Here, we need to perform the same
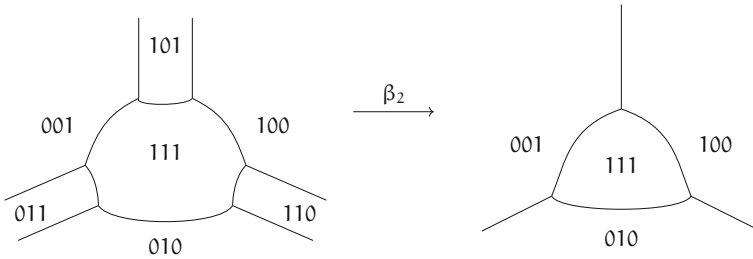
Fig. 6 Illustration of $M_{b,t}^3$ in spatial direction

blowups combined with blowing up the time direction. More precisely, using local coordinates, we blow up the following submanifolds

$$
\begin{aligned}
F_{C,Sc} &:= (F \cup F_C) \cap (\beta_2 \circ \beta_1)^* \{ x = x'', y = y'' \}, \\
F_{L,Sc} &:= (F \cup F_L) \cap (\beta_2 \circ \beta_1)^* \{ x' = x'', y' = y'' \}, \\
F_{R,Sc} &:= (F \cup F_R) \cap (\beta_2 \circ \beta_1)^* \{ x = x', y = y' \}.
\end{aligned}
$$

as well as their intersection

$$
O := F_{C,Sc} \cap F_{L,Sc} \cap F_{R,Sc}.
$$

This defines the triple space in the heat calculus, illustrated in Fig. 7 as

$$
HM_\phi^3 := \left[\overline{M}_{b,t}^3, O; F_{C,Sc}, F_{L,Sc}, F_{R,Sc} \right].
$$
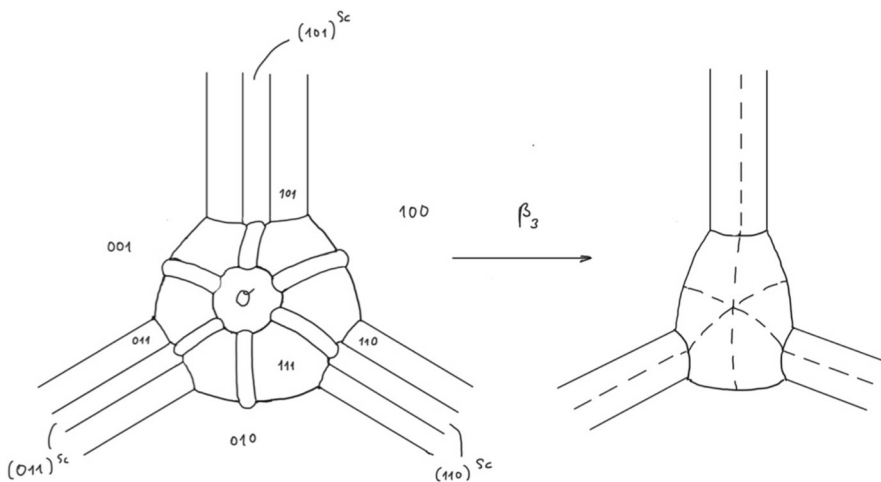
Fig. 7 Illustration of HM_ϕ^3 in spatial direction

The triple space comes with the intermediate blow down map $\beta_3 : HM_\phi^3 \to M_{b,t}^3$ and we define the total blowdown map as

$$\beta_{\mathrm{Tr}} := \beta_3 \circ \beta_2 \circ \beta_1 : HM_\phi^3 \longrightarrow \overline{M}^3 \times \mathbb{R}_+^2.$$

From the construction, we compute immediately

$$\begin{aligned}
\beta_{Tr}^*(x) &= \rho_{111}\rho_O\rho_{101}\rho_{101}^{sc}\rho_{110}\rho_{110}^{sc}\rho_{100}, \\
\beta_{Tr}^*(x') &= \rho_{111}\rho_O\rho_{011}\rho_{011}^{sc}\rho_{110}\rho_{110}^{sc}\rho_{010}, \\
\beta_{Tr}^*(x'') &= \rho_{111}\rho_O\rho_{101}\rho_{101}^{sc}\rho_{011}\rho_{011}^{sc}\rho_{001}.
\end{aligned} \tag{7.14}$$

7.3 Lifts of boundary defining functions under various projections

We shall now study the lifts of boundary defining functions on the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$ to the triple heat space under the maps $\Pi_C, \Pi_L, \Pi_R : HM_\phi^3 \to M_\phi^2 \times \mathbb{R}^+$. We use the following notation: we denote a boundary defining function of any boundary face (ijk), in HM_ϕ^3 by ρ_{ijk}; $i, j, k \in \{0, 1\}$. Boundary defining functions of $(110)^{\mathrm{Sc}}$, $(101)^{\mathrm{Sc}}$ and $(011)^{\mathrm{Sc}}$ are denoted by ρ_{110}^{Sc}, ρ_{101}^{Sc} and ρ_{011}^{Sc}, respectively. The boundary face $\mathcal{O}$, arising from the blowup of O, comes with a defining function ρ_O. The boundary face, arising from the blowup of F_O, comes with a defining function τ_O. Let τ and τ' be defining functions for the two boundary faces in HM_ϕ^3, corresponding to $\{t' = 0\}$ and $\{t'' = 0\}$, respectively. The notation is according to the labels in Figs. 7 and 8.

We rename the boundary faces lf, rf and ff in the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$ as (ij), $i, j \in \{0, 1\}$, according to the labels in Fig. 8 and write for the corresponding boundary defining functions ρ_{ij}. The boundary face fd is renamed $(11)^{\mathrm{Sc}}$ and its

Fig. 8 Illustration of projections in spatial direction
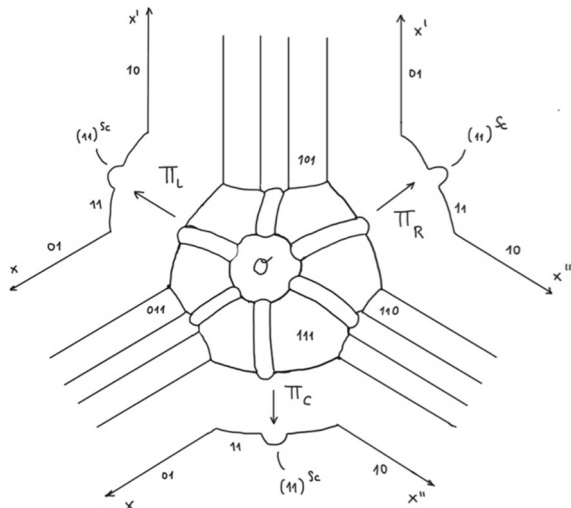
defining function is written as ρ_{11}^{Sc}. Defining function of tf in $M_\phi^2 \times \mathbb{R}^+$ is denoted by τ. The notation is according to the labels in Fig. 8, and corresponds closely to the notation of boundary faces on the triple space.

We compute the pullback of $\rho_{11}, \rho_{01}, \rho_{10}, \rho_{11}^{\mathrm{sc}}$ under Π_C, Π_L, Π_R. Here, Fig. 8 provides a helpful orientation.

$$
\begin{aligned}
\Pi_C^*(\rho_{11}) &= \rho_{111} \cdot \rho_{101} & \Pi_L^*(\rho_{11}) &= \rho_{111} \cdot \rho_{110} & \Pi_R^*(\rho_{11}) &= \rho_{111} \cdot \rho_{011} \\
\Pi_C^*(\rho_{01}) &= \rho_{011} \cdot \rho_{011}^{\mathrm{Sc}} \cdot \rho_{001} & \Pi_L^*(\rho_{01}) &= \rho_{011} \cdot \rho_{011}^{\mathrm{Sc}} \cdot \rho_{010} & \Pi_R^*(\rho_{01}) &= \rho_{101} \cdot \rho_{101}^{\mathrm{Sc}} \cdot \rho_{001} \\
\Pi_C^*(\rho_{10}) &= \rho_{110} \cdot \rho_{110}^{\mathrm{Sc}} \cdot \rho_{100} & \Pi_L^*(\rho_{10}) &= \rho_{101} \cdot \rho_{101}^{\mathrm{Sc}} \cdot \rho_{100} & \Pi_R^*(\rho_{10}) &= \rho_{110} \cdot \rho_{110}^{\mathrm{Sc}} \cdot \rho_{010} \\
\Pi_C^*(\rho_{11}^{\mathrm{Sc}}) &= \rho_{101}^{\mathrm{Sc}} \cdot \rho_O & \Pi_L^*(\rho_{11}^{\mathrm{Sc}}) &= \rho_{110}^{\mathrm{Sc}} \cdot \rho_O & \Pi_R^*(\rho_{11}^{\mathrm{Sc}}) &= \rho_{011}^{\mathrm{Sc}} \cdot \rho_O
\end{aligned}
\tag{7.15}
$$

Now the lifts of the time variable τ are somewhat more intricate to argue. Let us first compute the lifts of time direction boundary defining functions under the blow down map β_{Tr}. We find

$$
\begin{aligned}
\beta_{\mathrm{Tr}}^*(t') &= \tau' \cdot \tau_O \cdot \rho_{110}^2 \cdot (\rho_{110}^{\mathrm{Sc}})^2 \cdot \rho_{101}^2 \cdot (\rho_{101}^{\mathrm{Sc}})^2 \cdot \rho_{111}^2 \cdot \rho_O^2, \\
\beta_{\mathrm{Tr}}^*(t'') &= \tau'' \cdot \tau_O \cdot \rho_{101}^2 \cdot (\rho_{101}^{\mathrm{Sc}})^2 \cdot \rho_{011}^2 \cdot (\rho_{011}^{\mathrm{Sc}})^2 \cdot \rho_{111}^2 \cdot \rho_O^2, \\
\beta_{\mathrm{Tr}}^*(t' + t'') &= \tau_O \cdot \rho_{111}^2 \cdot \rho_{101}^2 \cdot \rho_{F_0}^2 \cdot (\rho_{101}^{\mathrm{Sc}})^2
\end{aligned}
\tag{7.16}
$$

From the commutative diagram (7.12), we have

$$
\pi_C \circ \beta_{\mathrm{Tr}} = \beta_\phi \circ \Pi_C. \tag{7.17}
$$

Now the lifts $\beta_\phi^*(t'), \beta_\phi^*(t''), \beta_\phi^*(t' + t'')$ to $\Pi_L(HM_\phi^3), \Pi_R(HM_\phi^3), \Pi_C(HM_\phi^3)$, respectively, are equal to $\tau \cdot (\rho_{11}^{Sc})^2 \cdot \rho_{11}^2$. Therefore we compute, in view of (7.16) and (7.17)

$$
\begin{aligned}
\Pi_C^*(\tau (\rho_{11}^{\mathrm{Sc}})^2 \rho_{11}^2) &= \Pi_C^*(\tau) \rho_{111}^2 (\rho_{101}^{\mathrm{Sc}})^2 \cdot \rho_O^2, \\
&= \tau_O \cdot \rho_{111}^2 \cdot (\rho_{101}^{\mathrm{Sc}})^2 \cdot \rho_O^2.
\end{aligned}
$$

From here we conclude $\Pi_C^*(\tau) = \tau_O$. Similarly one can compute the other lifts and we arrive at the following identities

$$
\begin{aligned}
\Pi_L^*(\tau) &= \tau' \cdot \tau_O \cdot (\rho_{101}^{\mathrm{Sc}})^2 \cdot \rho_{101}^2, \\
\Pi_R^*(\tau) &= \tau' \cdot \tau_O \cdot (\rho_{101}^{\mathrm{Sc}})^2 \cdot \rho_{101}^2, \\
\Pi_C^*(\tau) &= \tau_O.
\end{aligned}
\tag{7.18}
$$

7.4 Projections Π_C, Π_L, Π_R are b-fibrations

Condition for the application of Melrose's pushforward theorem [18] is that the maps Π_C, Π_L, Π_R are b-fibrations, recall Definition 2.8. By discussion above, (7.15) and (7.18), Π_C, Π_L, Π_R are indeed b-fibrations.

7.5 Lifts of kernels and densities to the triple space

Consider $A \in \mathcal{H}_\phi^{\ell,q}(M, \Lambda_\phi^*)$, $B \in \mathcal{H}_\phi^{\ell',\infty}(M, \Lambda_\phi^*)$. We write K_A and K_B for the Schwartz kernels of A and B, respectively. We consider the composition $C = A \circ B$ with Schwartz kernel K_C. We have by construction

$$\begin{aligned}
\pi_L^* K_A &= K_A(t', x, y, z, x', y', z'), \\
\pi_R^* K_B &= K_B(t'', x', y', z', x'', y'', z'').
\end{aligned} \tag{7.19}$$

We also write $K_C \equiv K_C(t, x, y, z, x'', y'', z'')$ and set

$$\begin{aligned}
\nu_3 &:= dt' dt'' \mathrm{dvol}_{g_\phi}(x, y, z) \, \mathrm{dvol}_{g_\phi}(x', y', z') \, \mathrm{dvol}_{g_\phi}(x'', y'', z''), \\
\nu_2 &:= dt \, \mathrm{dvol}_{g_\phi}(x, y, z) \, \mathrm{dvol}_{g_\phi}(x'', y'', z'').
\end{aligned} \tag{7.20}$$

Then we obtain by construction (making the relation (7.11) precise)

$$K_C \cdot \nu_2 = (\pi_C)_* \left(\pi_L^* K_A \cdot \pi_R^* K_A \cdot \beta_{Tr}^* \nu_3 \right).$$

Writing this relation in terms of the lifts $\kappa_{A,B,C} = \beta_\phi^* K_{A,B,C}$, we obtain

$$\kappa_C \cdot \beta_\phi^* \nu_2 = (\Pi_C)_* \left(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_A \cdot \beta_{Tr}^* \nu_3 \right). \tag{7.21}$$

This formula makes clear how to proceed

Step (1): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_A$.
Step (2): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{Tr}^* \nu_3$.
Step (3): Apply Pushfoward Theorem to study pushforward by $(\Pi_C)_*$
Step (4): Compute the asymptotics of $\beta_\phi^* \nu_2$
Step (5): Compare the asymptotics of both sides in (7.21) to study κ_C.

7.5.1 Step (1): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_A$

Since $B \in \mathcal{H}_\phi^{\ell',\infty}(M, \Lambda_\phi^*)$, we find in view of the (7.15) and (7.18) after an explicit counting of exponents

$$\kappa_B \sim \rho_{\mathrm{fd}}^{\ell'-3} \left(\rho_{\mathrm{ff}} \rho_{\mathrm{rf}} \rho_{\mathrm{lf}} \rho_{\mathrm{tf}} \right)^\infty \equiv \left(\rho_{11}^{\mathrm{Sc}} \right)^{\ell'-3} \left(\tau \rho_{11} \rho_{10} \rho_{01} \right)^\infty, \quad \Pi_R^* \kappa_B \sim \left(\rho_{011}^{\mathrm{Sc}} \rho_O \right)^{\ell'-3},$$

where we adopted a convention that for kernels on the triple space HM_ϕ^3, we only write out asymptotics at those boundary faces, where the kernel is not vanishing to infinite order, suppressing the other boundary faces from the formula.

We would like to write down a similar expansion for $A \in \mathcal{H}_\phi^{\ell,p}(M, \Lambda_\phi^*)$. However, since in general $p \neq \infty$, the asymptotics of κ_A is not uniform as $\tau \to 0$. Nevertheless, by (7.18) the asymptotics of the lifts of τ under Π_L and Π_R coincide. Since $\Pi_R^* \kappa_B$ contributes $\Pi_R(\tau^\infty)$, the composition $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B$ is still polyhomogeneous and

vanishing to infinite order at the faces $\mathcal{O}$, $(101)^{\mathrm{Sc}}$, (101) and the lift of $\{t'' = 0\}$. Thus, we may assume $p = \infty$ without loss of generality and obtain

$$\kappa_A \sim \rho_{\mathrm{fd}}^{\ell-3} \left(\rho_{\mathrm{ff}} \rho_{\mathrm{rf}} \rho_{\mathrm{lf}} \rho_{\mathrm{tf}} \right)^{\infty} \equiv (\rho_{11}^{\mathrm{Sc}})^{\ell-3} \left(\tau \rho_{11} \rho_{10} \rho_{01} \right)^{\infty}, \quad \Pi_L^* \kappa_A \sim \left(\rho_{110}^{\mathrm{Sc}} \rho_O \right)^{\ell-3}.$$

Consequently, we arrive under the notation which suppresses defining functions of infinite order

$$\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \sim \rho_O^{\ell+\ell'-6}. \tag{7.22}$$

7.5.2 Step (2): Compute the asymptotics of $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{\mathrm{Tr}}^* \nu_3$

The density ν_3 is given in local coordinates by

$$
\begin{aligned}
\nu_3 &= dt' dt'' \left(x^{-b-2} dx\, dy\, dz \right) \left(x'^{-b-2} dx'\, dy'\, dz' \right) \left(x''^{-b-2} dx''\, dy''\, dz'' \right) \\
&= t' t'' (xx'x'')^{-1} \frac{dt'}{t'} \frac{dt''}{t''} \left(\frac{dx}{x} \frac{dy}{x^b} dz \right) \left(\frac{dx'}{x'} \frac{dy'}{x'^b} dz' \right) \left(\frac{dx''}{x''} \frac{dy''}{x''^b} dz'' \right) \\
&=: t' t'' (xx'x'')^{-1} \nu_3',
\end{aligned}
$$

up to a smooth bounded function. The lift $\beta_{\mathrm{Tr}}^* \nu_3'$ equals $h \cdot \nu_b^{(3)}$, where $\nu_b^{(3)}$ is a b-density on HM_ϕ^3, that is a smooth density on HM_ϕ^3, divided by a product of all its boundary defining functions. The factor h is a polyhomogeneous function on HM_ϕ^3, smooth at the boundary face $\mathcal{O}$; its asymptotics at other boundary faces is irrelevant, since $\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B$ vanishes to infinite order there. Thus we compute in view of (7.14), (7.16) and (7.22)

$$
\begin{aligned}
&\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{\mathrm{Tr}}^* \nu_3 \\
&= \Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{\mathrm{Tr}}^* \left(t' t'' (xx'x'')^{-1} \right) \beta_{\mathrm{Tr}}^* \nu_3' \\
&\sim \rho_O^{\ell+\ell'-5} \beta_{\mathrm{Tr}}^* \nu_3' \sim \rho_O^{\ell+\ell'-5} \nu_b^{(3)},
\end{aligned}
\tag{7.23}
$$

where as before we suppressed the other boundary defining functions from the notation, where the kernels vanish to infinite order.

7.5.3 Step (3): Apply Pushfoward Theorem to study pushforward by $(\Pi_C)_*$

The next step is applying the pushforward theorem of Melrose [18]. We use the notation of Definition 2.8. Then the pushforward theorem of Melrose says the following.

Theorem 7.3 *Let M, M' be two compact manifolds with corners and ν_b, ν_b' are b-densities on M, M', respectively. Let u be a polyhomogeneous function on M with index sets E_j at the faces H_j of M. Suppose that each $(z, p) \in E_j$ has $Re(z) > 0$ if the index j satisfies $e(i, j) = 0$ for all j. Then the pushforward $f_*(u\nu_b)$ is well-defined and equals $h\nu_b'$ where h is polyhomogeneous on M' and has an index family $f_b(\mathcal{E})$ given by an explicit formula in terms of the index family $\mathcal{E}$ for M.*

We refer the reader to [17, 18] for the explicit definition of the index family $f_b(\mathcal{E})$, and just say that in our specific case we have for any $\alpha > 0$ (all other suppressed boundary functions enter with infinite order)

$$(\Pi_C)_* \left(\rho_O^\alpha v_b^{(3)} \right) = (\rho_{11}^{\text{Sc}})^\alpha v_b^{(2)},$$

where $v_b^{(2)}$ is a b-density on the intermediate heat space $M_\phi^2 \times \mathbb{R}^+$, that is a smooth density on $M_\phi^2 \times \mathbb{R}^+$, divided by a product of all its boundary defining functions. Hence we arrive in view of (7.23) at

$$\begin{aligned}
(\Pi_C)_* \left(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{\text{Tr}}^* v_3 \right) &= (\rho_{11}^{\text{Sc}})^{\ell+\ell'-5} v_b^{(2)} \\
&\equiv (\rho_{11}^{\text{Sc}})^{\ell+\ell'-5} (\tau \rho_{11} \rho_{10} \rho_{01})^\infty v_b^{(2)}.
\end{aligned} \tag{7.24}$$

7.5.4 Step (4): Compute the asymptotics of $\beta_\phi^* v_2$

The density v_2 is given in local coordinates by

$$\begin{aligned}
v_2 &= dt \left(x^{-b-2} dx\, dy\, dz \right) \left(x''^{-b-2} dx''\, dy''\, dz'' \right) \\
&= t' t'' (xx'')^{-1} \frac{dt}{t} \left(\frac{dx}{x} \frac{dy}{x^b} dz \right) \left(\frac{dx''}{x''} \frac{dy''}{x''^b} dz'' \right) \\
&=: t(xx'')^{-1} v_2',
\end{aligned}$$

up to a smooth bounded function. The lift $\beta_\phi^* v_2'$ equals $h' \cdot v_b^{(2)}$, where h' is a polyhomogeneous function on $M_\phi^2 \times \mathbb{R}^+$, smooth at the boundary face fd; its asymptotics at other boundary faces is irrelevant, since it will be multiplied with a kernel that vanishes to infinite order there. Hence we find

$$\begin{aligned}
\beta_\phi^* v_2 &= \beta_\phi^* \left(t(xx'')^{-1} \right) v_b^{(2)} \\
&= \tau \left(\rho_{11}^{\text{Sc}} \rho_{11} \right)^{-2} (\rho_{10} \rho_{01})^{-1} h' v_b^{(2)}
\end{aligned} \tag{7.25}$$

7.5.5 Step (5): Compare the asymptotics of both sides in (7.21) to study κ_C

Combining (7.24) and (7.25), we find

$$\begin{aligned}
(\Pi_C)_* \left(\Pi_L^* \kappa_A \cdot \Pi_R^* \kappa_B \cdot \beta_{\text{Tr}}^* v_3 \right) &= (\rho_{11}^{\text{Sc}})^{\ell+\ell'-5} (\tau \rho_{11} \rho_{10} \rho_{01})^\infty v_b^{(2)} \\
&= (\rho_{11}^{\text{Sc}})^{\ell+\ell'-3} (\tau \rho_{11} \rho_{10} \rho_{01})^\infty \beta_\phi^* v_2 \\
&= \kappa_C \cdot \beta_\phi^* v_2.
\end{aligned} \tag{7.26}$$

This proves the composition Theorem 7.1.

Conflict of interest The authors declare that they have no conflicts of interest

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