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A de Rham decomposition type theorem for contact sub-Riemannian manifolds

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Abstract

In this paper we prove a result which can be regarded as a sub-Riemannian version of de Rham decomposition theorem. More precisely, suppose that (M, H, g) is a contact and oriented sub-Riemannian manifold such that the Reeb vector field ξ is an infinitesimal isometry. Under such assumptions there exists a unique metric and torsion-free connection on H. Suppose that there exists a point $q \in M$ such that the holonomy group $\Psi(q)$ acts reducibly on $H(q)$ yielding a decomposition $H(q) = H_1(q) \oplus \cdots \oplus H_m(q)$ into $\Psi(q)$-irreducible factors. Using parallel transport we obtain the decomposition $H = H_1 \oplus \cdots \oplus H_m$ of H into sub-distributions H_i. Unlike the Riemannian case, the distributions H_i are not integrable, however they induce integrable distributions Δ_i on M/ξ, which is locally a smooth manifold. As a result, every point in M has a neighborhood U such that $T(U/\xi) = \Delta_1 \oplus \cdots \oplus \Delta_m$, and the latter decomposition of $T(U/\xi)$ induces the decomposition of U/ξ into the product of Riemannian manifolds. One can restate this as follows: every contact sub-Riemannian manifold whose holonomy group acts reducibly has, at least locally, the structure of a fiber bundle over a product of Riemannian manifolds. We also give a version of the theorem for indefinite metrics.

Keywords Contact distributions · Connections · Sub-Riemannian geometry · De Rham decomposition theorem

1 Introduction and statement of results

Let M be a smooth (by smooth we mean of class C^∞) connected manifold. Suppose that H is a smooth bracket generating distribution on M of constant rank and g is a smooth

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Riemannian metric on H. The pair (H, g) is called a *sub-Riemannian metric* or a *sub-Riemannian structure* on M. The triple (M, H, g) is referred to as a *sub-Riemannian manifold*. Sub-Riemannian manifolds appear in many mathematical as well as physical problems and have been studied by many authors—see for instance [1–5,8,12,14] and the reference sections therein. Various problems in sub-Riemannian geometry like for instance the behavior of sub-Riemannian geodesics and their minimizing properties, conjugate and cut loci, sub-Riemannian spheres, isometries and conformal mappings, nilpotent approximations, differential properties of the sub-Riemannian distance etc. have been investigated in detail. In this paper we deal with *holonomy* determined by a class of connections introduced in [13] for contact sub-Riemannian manifolds, and prove a theorem that can be considered as a version of de Rham decomposition theorem for Riemannian manifolds. Different approaches to sub-Riemannian holonomy and some other problems involving it are treated, e.g., in [7,9].

By a *contact sub-Riemannian manifolds* we mean a sub-Riemannian manifold (M, H, g), where $\dim M = 2n + 1$, and H is a contact distribution on M. Given a contact connected sub-Riemannian manifold (M, H, g) it is natural to consider the bundle of orthonormal horizontal frames $O_{H,g}(M)$ associated with it:

$$O_{H,g}(M) = \{(q; v_1, \dots, v_{2n}) : \ v_1, \dots, v_{2n} \text{ is an orthonormal basis of } H(q),\ q \in M\}.$$

This is a principle bundle with structure group $O(2n)$. Moreover we will assume that H and TM are oriented, so the structure group can be reduced to $SO(2n)$. Let ξ be the Reeb vector field which is well defined in such a situation. We will assume that ξ is an infinitesimal isometry. Now it can be proved [13] that there exists a unique connection Γ on $O_{H,g}(M)$ which is torsion-free (the definition of the torsion in our case is presented below). In the usual way Γ defines the covariant differentiation

$$\nabla : \mathrm{Sec}(TM) \times \mathrm{Sec}(H) \longrightarrow \mathrm{Sec}(H),$$

where we use the following notation: if $E \longrightarrow M$ is a vector bundle then by $\mathrm{Sec}(E)$ we denote the $C^\infty(M)$-module of sections of E. Having a connection on the bundle $O_{H,g}(M)$ we can consider its holonomy group $\Psi(q)$ at a point $q \in M$. Since M is connected the groups $\Psi(q_1)$ and $\Psi(q_2)$ are isomorphic for any two points $q_1, q_2 \in M$. The holonomy group $\Psi(q)$ naturally acts on $H(q)$ (for H is an associated vector bundle to $O_{H,g}(M)$ with typical fiber $\mathbb{R}^{2n}$). Suppose that the action of $\Psi(q)$ on $H(q)$ is reducible. Then $H(q)$ decomposes into $\Psi(q)$-irreducible factors

$$H(q) = H_1(q) \oplus \cdots \oplus H_m(q) \tag{1.1}$$

which are mutually orthogonal with respect to g. By use of parallel translations we extend $H_i(q)$ to distributions H_i on M resulting in a global decomposition

$$H = H_1 \oplus \cdots \oplus H_m. \tag{1.2}$$

Next let us consider the set M/ξ of orbits of ξ. It is locally a smooth manifold of dimension $2n$.

If we fix an arbitrary point $q_0 \in M$ and a neighborhood U of q_0 such that U/ξ is a connected smooth manifold, then we can canonically identify U/ξ with a regular $2n$-dimensional submanifold B of M with $q_0 \in B$ (the details can be found below). Then the sub-Riemannian metric $(H_{|U}, g_{|U})$ induces a natural Riemannian metric g_B on B, and the connection ∇ induces a connection ∇^B on B which turns out to be the Levi-Civita connection with respect to g_B. Moreover, if we denote by $p : U \longrightarrow B$ the projection in the direction of ξ, then

$$d_q p_{|H(q)} : H(q) \longrightarrow T_{\pi(q)} B$$

is a linear isometry. Using this projection, the decomposition (1.2) induces a decomposition

$$TB = \Delta_1 \oplus \cdots \oplus \Delta_m \tag{1.3}$$

of TB into the Whitney sum of mutually orthogonal distributions. It is proved that Δ_i are integrable and parallel with respect to ∇^B, so in turn (1.3) induces a decomposition of B. The main theorem may be stated as follows.

Theorem 1.1 *Suppose that (M, H, g) is a contact oriented sub-Riemannian manifold such that the Reeb vector field ξ is an infinitesimal isometry. Denote by Γ the unique torsion-free connection on $O_{H,g}(M)$ and suppose that there exists a point $q \in M$ such that the holonomy group $\Psi(q)$ of Γ acts reducibly on $H(q)$ inducing the decomposition (1.1). Then every point in M has a neighborhood U such that the manifold U/ξ is isometric to the product $(B_1, g_1) \times \cdots \times (B_m, g_m)$ of Riemannian manifolds, where B_i is of dimension* rank H_i*, $i = 1, \dots, m$. More precisely, each B_i may be identified with a maximal integrable manifold for the distribution Δ_i, $i = 1, \dots, m$.*

In particular, suppose that (M_1, H_1, g_1), (M_2, H_2, g_2) are two sub-Riemannian manifolds satisfying the above assumptions. Let $f : (M_1, H_1, g_1) \longrightarrow (M_2, H_2, g_2)$ be an isometry and let ξ be the Reeb vector field on (M_1, H_1, g_1). Then for every sufficiently small open set $U \subset M_1$, which is convex with respect to ξ (that is to say every trajectory of ξ intersects U in a connected set), the Riemannian manifolds U/ξ and $f(U)/f_*\xi$ are isometric.

Using the results from [16] we can generalized the above theorem to contact sub-pseudo-Riemannian manifolds (e.g. sub-Lorentzian manifolds), i.e., when the metric g on H is not necessarily positive definite. We need only to assume that $\Psi(q)$ acts *nondegenerately and reducibly* on $H(q)$ which means that the decomposition (1.1) consists of subspaces $H_i(q)$ nondegenerate with respect to g.

Theorem 1.2 *Suppose that the assumptions of Theorem 1.1, where "sub-Riemannian manifold" is replaced with "sub-pseudo-Riemannian manifold" and "$\Psi(q)$ acts reducibly on $H(q)$" is replaced with "$\Psi(q)$ acts nondegenerately and reducibly on $H(q)$", are satisfied. Then every point in M has a neighborhood U such that the manifold U/ξ is isometric to the product $(B_1, g_1) \times \cdots \times (B_m, g_m)$ of pseudo-Riemannian manifolds, where B_i is of dimension* rank H_i *and, as above, may be identified with a maximal integrable manifold of the distribution Δ_i, $i = 1, \dots, m$.*

Of course the remark made after the statement of Theorem 1.1 remains true with obvious modifications.

Finally, we state the last theorem that we prove in the present paper. If (M, H, g) is a given contact and oriented sub-Riemannian manifold, $\dim M = 2n + 1$, denote by α the normalized contact 1-from (see Sect. 2 for details). Then we can define the operator $J : H \longrightarrow H$ by $d\alpha(X, Y) = g(X, J(Y))$. The operator J is a vector bundle morphism covering the identity. Furthermore, J is nondegenerate and antisymmetric with respect to g, therefore it has purely imaginary eigenvalues $\pm ib_j$, $j = 1, \dots, n$ (see [11] for further properties of J in the indefinite case). If the b_j's are pointwise mutually distinct, then each $b_j : M \longrightarrow \mathbb{R}$ is a smooth function. We say that the structure (H, g) is strongly nondegenerate at a point $q \in M$, if $b_1(q) < \cdots < b_n(q)$ under suitable numeration (cf. [1] where the numbers $b_1(q), \dots, b_n(q)$ are called fundamental frequencies).

Theorem 1.3 *Suppose that (M, H, g) is a contact oriented sub-Riemannian manifold. Suppose that (i) the Reeb vector field ξ is an infinitesimal isometry. Denote by Γ the unique torsion-free connection on $O_{H,g}(M)$. Suppose next that (ii) the operator J is parallel with respect to Γ. If J is strongly nondegenerate at a point q and U is a sufficiently small neighborhood of q, then U/ξ is isometric to a product of 2-dimensional Riemannian manifolds. Consequently, the conformal type of U/ξ depends neither on the choice of a metric g satisfying (i) and (ii) nor on a point q at which J is strongly nondegenerate.*

As the reader can see, all above theorems concern a decomposition of the quotient manifold U/ξ into a product of (pseudo-)Riemannian manifolds, provided that U is a sufficiently small neighborhood of a fixed point. However, it would be very interesting to know if the set U itself admits a decomposition into a product of sub-(pseudo-)Riemannian manifolds. In the sub-Riemannian case, for instance, the set U is a so-called geodesic metric space. Then we know [10] that U admits a decomposition into a product of metric spaces. Such a decomposition is unique (up to a permutation of factors) and it would be of high importance to explicate if the factors in the mentioned decomposition carry some natural sub-Riemannian structure.

Content of the paper

In Sect. 2 we recall basic notions from contact sub-Riemannian geometry. In particular, we present the theory of connections on H introduced in the paper [13]. In Sect. 3 we prove the theorems.

Throughout the paper we adopt the following convention. A vector $v \in TM$ which belongs to H will be called *horizontal*. On the other hand, if Γ is a distribution on $O_{H,g}(M)$, e.g., a connection, then a vector $V \in TO_{H,g}(M)$ belonging to Γ will be referred to as a Γ-*horizontal* vector.

2 Contact sub-Riemannian geometry

Suppose that (M, H, g) is a contact sub-Riemannian manifold, $\dim M = 2n+1$. We assume M to be connected. Let us suppose that M is oriented as a contact manifold which means that the vector bundles TM and H are oriented. This is equivalent to the existence of a globally defined contact form, i.e., a 1-form α on M with the property that $H = \ker\alpha$ (see [6,11]). In such a situation, i.e., when there exists a globally defined contact form, we will say that the *sub-Riemannian manifold* (M, H, g) *is oriented*. Such a contact form is not unique, so we normalize it as follows: we suppose that

$$\underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{n \text{ factors}}(X_1, \ldots, X_{2n}) = 1, \tag{2.1}$$

where $X_1, \ldots, X_{2n}$ is a fixed local positively oriented orthonormal frame for H. For n even we have two such forms α defined up to a sign, so we choose either of them. A form satisfying (2.1) will be referred to as the *normalized contact form*.

If α is the normalized contact form then we define the *Reeb vector field* ξ on (M, H, g) as the solution to the system of equations

$$d\alpha(\xi, \cdot) = 0, \;\; \alpha(\xi) = 1.$$

Such a field has the property that $[\xi, X] \in \mathrm{Sec}(H)$ whenever $X \in \mathrm{Sec}(H)$. In particular the (local) flow φ^t of ξ preserves the distribution H. Moreover, ξ defines a canonical decomposition $TM = H \oplus \mathrm{Span}\{\xi\}$. The projection defined by this decomposition will be denoted by

$$P : TM \longrightarrow H. \tag{2.2}$$

2.1 Geodesics

Suppose that $X_1, \ldots, X_{2n}$ be an orthonormal frame defined on an open set $U \subset M$. Let $\mathcal{H} : T^*M_{|U} \longrightarrow \mathbb{R}$ be defined by

$$\mathcal{H}(q, \lambda) = \frac{1}{2}\sum_{i=1}^{2n} \langle \lambda, X_i(q) \rangle^2. \tag{2.3}$$

Clearly, the value of (2.3) does not depend on the choice of an orthonormal basis, so $\mathcal{H}$ is in fact defined on the whole cotangent bundle: $\mathcal{H} : T^*M \longrightarrow \mathbb{R}$. We call $\mathcal{H}$ the *geodesic Hamiltonian*. By a *normal* or *Hamiltonian geodesic* we mean any curve being a projection onto M of the trajectory of the Hamiltonian vector field $\overrightarrow{\mathcal{H}}$. In other words, a curve $\sigma : [a, b] \longrightarrow M$ is a Hamiltonian geodesic if there exists $\lambda : [a, b] \longrightarrow T^*M$ such that

$$\lambda(t) \in T^*_{\sigma(t)}M \;\; \text{and} \;\; (\dot{\sigma}(t), \dot{\lambda}(t)) = \overrightarrow{\mathcal{H}}. \tag{2.4}$$

It can be proved that in the contact case, every geodesic, i.e., a curve which locally minimizes the sub-Riemannian distance, is a Hamiltonian geodesic. However we will not use this fact. Let $\sigma : [a, b] \longrightarrow M$ be a Hamiltonian geodesic and let $\lambda(t)$ be its lift to T^*M as in (2.4). Suppose that S is a submanifold in M such that $\sigma(a) \in S$. We say that σ satisfies the *(Pontryagin) transversality condition with respect to S* if $\lambda(a)_{|T_{\sigma(a)}S} = 0$.

For a point $q \in M$, denote by $\mathcal{D}_q$ the set of all covectors $\lambda \in T_q^*M$ such that the Hamiltonian geodesic with initial condition (q, λ) exists on the interval [0, 1]. Then we define the *exponential mapping* with pole at q as follows:

$$\exp_q : \mathcal{D}_q \longrightarrow M, \quad \exp_q(\lambda) = \sigma(1),$$

where $\sigma(t)$ is the Hamiltonian geodesic with initial condition (q, λ). One proves that $\exp_q$ is smooth.

2.2 Isometries and infinitesimal isometries

Given two contact sub-Riemannian manifolds (M_1, H_1, g_1), (M_2, H_2, g_2), a diffeomorphism $f : M_1 \longrightarrow M_2$ is called an *isometry* if $d_q f(H_1(q)) \subset H_2(f(q))$ and $d_q f : H_1(q) \longrightarrow H_2(f(q))$ is a linear isometry for every $q \in M$. In other words $g_2(d_q f(v), d_q f(w)) = g_1(v, w)$ for all $q \in M$ and $v, w \in H_1(q)$. If the manifolds (M_i, H_i, g_i), $i = 1, 2$, are oriented and $f : M_1 \longrightarrow M_2$ is an isometry, then $f^*\alpha_2 = \pm\alpha_1$, as well as $f_*\xi_1 = \pm\xi_2$, where α_i is the normalized contact form and ξ_i is the Reeb vector field on M_i, $i = 1, 2$. It can be also proved that isometries preserve Hamiltonian geodesics. More precisely, if f is an isometry and $\sigma : [a, b] \longrightarrow M$ is a Hamiltonian geodesic satisfying the transversality condition with respect to a submanifold S, then $f \circ \sigma$ is a Hamiltonian geodesic satisfying the transversality condition with respect to $f(S)$.

A vector field Z on a sub-Riemannian manifold (M, H, g) is called an *infinitesimal isometry* if its (local) flow consists of isometries. It can be shown that Z is an infinitesimal isometry if and only if (i) $[Z, Y] \in \mathrm{Sec}(H)$ and (ii) $Z(g(X, Y)) = g([Z, X], Y) + g(X, [Z, Y])$ for every $X, Y \in \mathrm{Sec}(H)$.

2.3 Connection on the bundle of horizontal frames

In this subsection we present the construction of the connection which agrees with a given sub-Riemannian structure. Details are described in [13]. Note that [13, Proposition 7.1] is not true (one needs to impose stronger assumptions).

Let (M, H, g) be an oriented contact sub-Riemannian manifold. Consider the *bundle of horizontal frames* determined by it:

$$L_H(M) = \{(q; v_1, \ldots, v_{2n}) : \ q \in M, \ H(q) = \mathrm{Span}\{v_1, \ldots, v_{2n}\}\};$$

by $\pi : L_H(M) \longrightarrow M$ we denote its projection, i.e., $\pi(q; v_1, \ldots, v_{2n}) = q$. This is a principle bundle with the structure group $GL(2n)$. Indeed, we have a natural right action: $(q; v_1, \ldots, v_{2n}).a = (q; a^i_1 v_i, \ldots, a^i_{2n} v_i)$, $a \in GL(2n)$ (here and below we use the Einstein summation convention). Moreover, if $X_1, \ldots, X_{2n}$ is a basis of sections of H defined on an open set $U \subset M$ then the local trivialization $\psi : \pi^{-1}(U) \longrightarrow U \times GL(2n)$ of $L_H(M)$ acts as follows. If $l = (q; v_1, \ldots, v_{2n})$ then

$$\psi(l) = (q, a(l)),$$

where $a(l) \in GL(2n)$ is such that $v_i = a^j_i(l) X_j(q)$. The metric g reduces $L_H(M)$ to the bundle

$$O_{H,g}(M) = \{(q; v_1, \ldots, v_{2n}) \in L_H(M) : \ g(v_i, v_j) = \delta_{ij}, i, j = 1, \ldots, 2n\}$$

of orthonormal horizontal frames. This is a principle $O(2n)$-bundle. Every $l = (q; v_1, \ldots, v_{2n}) \in O_{H,g}(M)$ defines the linear isomorphism $l : \mathbb{R}^{2n} \longrightarrow H(\pi(l)) = H(q)$ which is given by

$$l(r) = r^i v_i. \tag{2.5}$$

As usual, by a connection on $O_{H,g}(M)$ we mean a distribution $\Gamma \subset TO_{H,g}(M)$ such that $TO_{H,g}(M) = \Gamma \oplus V$ and which is $O(2n)$-invariant, i.e., $d_l R_a(\Gamma_l) = \Gamma_{l.a}$ for every $a \in O(2n)$ and $l \in O_{H,g}(M)$. Here V stands for the vertical distribution on $O_{H,g}(M)$: $V_l = \ker d_l\pi$, and $R_a : O_{H,g}(M) \longrightarrow O_{H,g}(M)$ is the right action of $O(2n)$. Note that if Γ is a connection on $O_{H,g}(M)$ then we have a natural splitting

$$\Gamma = \Gamma^H \oplus \Gamma^\xi, \tag{2.6}$$

where $\Gamma^H = (d\pi)^{-1}(H) \cap \Gamma$ and $\Gamma^\xi = (d\pi)^{-1}(\mathrm{Span}\{\xi\}) \cap \Gamma$; as above ξ stands for the Reeb vector field.

Given a connection Γ on $O_{H,g}(M)$ we want to define its torsion. First of all we need to specify the counter part of the canonical 1-form from the theory of linear frame bundles. We do it as follows. For every $l \in O_{H,g}(M)$ we define

$$\theta(l) = l^{-1} \circ P \circ d_l\pi : T_l O_{H,g}(M) \longrightarrow \mathbb{R}^{2n}, \tag{2.7}$$

where P is defined in (2.2). The object θ is a 1-form on $O_{H,g}(M)$ with values in $\mathbb{R}^{2n}$ and will be called the *canonical* 1*-form on* $O_{H,g}(M)$. Now by the *torsion form* of Γ we mean the 2-form Θ which is given by

$$\Theta = d\theta \circ (\mathrm{pr}, \mathrm{pr}), \tag{2.8}$$

where $\mathrm{pr} : TO_{H,g}(M) = \Gamma \oplus V \longrightarrow \Gamma$ stands for the projection. Due to the splitting (2.6), the torsion can be decomposed into the horizontal torsion and vertical torsion (see [13]). It can be proved [13] that there always exist connections on $O_{H,g}(M)$ with vanishing horizontal torsion. The class of connections with vanishing horizontal torsion is determined by a canonical choice of Γ^H. To be more precise, various connections with vanishing horizontal torsion have the same component Γ^H, while the component Γ^ξ may be different.

Suppose further that the Reeb field is an infinitesimal isometry. Under such assumptions one can prove [13] that there exist a unique connection on $O_{H,g}(M)$ which is torsion-free. In other words, under the mentioned assumptions, there exists a unique torsion-free and metric connection associated with the structure (H, g). Such a connection induces the covariant derivation $\nabla : \mathrm{Sec}(TM) \times \mathrm{Sec}(H) \longrightarrow \mathrm{Sec}(H)$. Being metric means that

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y),$$

moreover, the vanishing of the horizontal torsion means that

$$\nabla_X Y - \nabla_Y X = P([X, Y]), \tag{2.9}$$

whereas the vanishing of the vertical torsion is expressed by

$$\nabla_\xi X = [\xi, X], \tag{2.10}$$

whenever $Z \in \mathrm{Sec}(TM)$, $X, Y \in \mathrm{Sec}(H)$—cf. [13].

At the end of this section let us note that if Γ is the mentioned torsion-free connection on $O_{H,g}(M)$, then the component Γ^ξ in the splitting (2.6) is given by $\Gamma^\xi = \mathrm{Span}\{\xi^*\}$, where the vector field ξ^* (being the Γ-horizontal lift of ξ) is defined as follows. Take $q \in M$ and let $\varphi^t : U \longrightarrow M$ be the (local) flow of ξ, where U is a neighborhood of q. Then we can lift φ^t to the mapping $\Phi^t : \pi^{-1}(U) \longrightarrow O_{H,g}(M)$, $\Phi^t(q; v_1, \ldots, v_{2n}) = (\varphi^t(q); d_q\varphi^t(v_1), \ldots, d_q\varphi^t(v_{2n}))$, and for $l \in \pi^{-1}(q)$ we set

$$\xi^*(l) = \frac{d}{dt}\Big|_{t=0} \Phi^t(l).$$

2.4 Holonomy

Given a connection Γ on $O_{H,g}(M)$, we can define parallel displacement along curves on M and the holonomy group in the standard manner (see [15]). Consider a piecewise smooth curve $\gamma : [a, b] \longrightarrow M$. The curve γ induces the parallel displacement of fibers

$$\tau_\gamma : \pi^{-1}(\gamma(a)) \longrightarrow \pi^{-1}(\gamma(b))$$

which is defined as follows. Take $l \in \pi^{-1}(\gamma(a))$ and let $\gamma^* : [a, b] \longrightarrow O_{H,g}(M)$ be the Γ-horizontal lift of γ initiating at l, i.e., $\pi \circ \gamma^* = \gamma$, $\frac{d}{dt}\gamma^*(t) \in \Gamma_{\gamma^*(t)}$ whenever the derivative exists, and $\gamma^*(a) = l$. Then $\tau_\gamma(l) = \gamma^*(b)$. Moreover, if we are given two piecewise smooth curves $\gamma_i : [a_i, b_i] \longrightarrow M$, $i = 1, 2$, such that $\gamma_1(b_1) = \gamma_2(a_2)$, we have $\tau_{\gamma_2\cdot\gamma_1} = \tau_{\gamma_2} \circ \tau_{\gamma_1}$, where $\gamma_2 \cdot \gamma_1$ is the concatenation of γ_1 and γ_2. In particular, if $C(q)$ denotes the set of all piecewise smooth loops at a point $q \in M$, then

$$\Psi(q) = \{\tau_\gamma : \gamma \in C(q)\}$$

is a Lie group which is called the *holonomy group* at q and is denoted by $\Psi(q)$. Such a group can be realized as a subgroup $\Psi(l)$, $l \in \pi^{-1}(q)$, of the structure group $O(2n)$: if $\gamma \in C(q)$ then l and $\tau_\gamma(l)$ belong to the same fiber of $\pi : O_{H,g}(M) \longrightarrow M$, hence γ determines a unique element $a_\gamma \in O(2n)$ such that $\tau_\gamma(l) = l.a_\gamma$. In this way $\Psi(l) = \{a_\gamma : \gamma \in C(q)\}$ is a subgroup of $O(2n)$. It is proved that if M is connected then holonomy groups at any two points are isomorphic.

The connection Γ induces also the parallel displacement in every vector bundle associated with $O_{H,g}(M)$, so in particular in H. More precisely, if $\gamma : [a, b] \longrightarrow M$ is a curve then we can define the parallel displacement or translation along γ (we use the same notation as above)

$$\tau_\gamma : H(\gamma(a)) \longrightarrow H(\gamma(b))$$

as $\tau(v) = \gamma^*(b)(r)$, where $\gamma^* : [a, b] \longrightarrow O_{H,g}(M)$ is a Γ-horizontal lift of γ, $\pi(\gamma^*(a)) = \gamma(a)$, and $r \in \mathbb{R}^{2n}$ is such that $\gamma^*(a)(r) = v$. Notice that for every γ the map τ_γ is a linear isometry. In particular, the holonomy group $\Psi(q)$ acts on $H(q)$.

Suppose that $\Psi(q)$ acts reducibly on $H(q)$, and let

$$H(q) = H_1(q) \oplus \cdots \oplus H_m(q) \tag{2.11}$$

be the decomposition of $H(q)$ into $\Psi(q)$-irreducible and $\Psi(q)$-invariant mutually orthogonal subspaces. It is a standard observation that the decomposition (2.11) can be extended by the parallel displacement to the decomposition

$$H = H_1 \oplus \cdots \oplus H_m \tag{2.12}$$

of the distribution H. Indeed, if γ is a curve starting at q then $\tau_\gamma(H_i(q))$ does not depend on γ but only on its endpoints. To end this subsection, we note that, by the definition of the covariant derivation induced by Γ,

$$\nabla_Z(\mathrm{Sec}(H_i)) \subset \mathrm{Sec}(H_i) \tag{2.13}$$

for every $Z \in \mathrm{Sec}(TM)$ and $i = 1, \ldots, m$.

3 Proof of Theorems 1.1 and 1.2

In this section we assume that (M, H, g) is a fixed contact oriented and connected sub-Riemannian manifold, $\dim M = 2n + 1$. Suppose that the Reeb vector field ξ is an infinitesimal isometry and denote its (local) flow by φ^t. Moreover, let Γ be the unique torsion-free connection on $O_{H,g}(M)$. Suppose that the holonomy group acts reducibly on H and the corresponding decomposition is

$$H = H_1 \oplus \cdots \oplus H_m.$$

As above H_i are constant rank distributions which are pairwise orthogonal with respect to g, rank $H_i > 0$, $i = 1, \dots m$. By ∇ we will denote the covariant derivation induced by Γ.

3.1 Distributions $\widetilde{H}_i$

Distributions H_i need not be integrable, however their extensions are. For every $i = 1, \dots, m$ let us define

$$\widetilde{H}_i = H_i \oplus \mathrm{Span}\{\xi\}.$$

Proposition 3.1 *The distribution $\widetilde{H}_i$ is integrable, $i = 1, \dots, m$.*

Proof Indeed by (2.13) it follows that

$$\nabla_X Y \in \mathrm{Sec}(H_i) \;\; \text{and} \;\; \nabla_\xi X = [\xi, X] \in \mathrm{Sec}(H_i)$$

for every $X, Y \in \mathrm{Sec}(H_i)$. Consequently,

$$P([X, Y]) = \nabla_X Y - \nabla_Y X \in \mathrm{Sec}(H_i)$$

which in turn implies $[X, Y] \in \mathrm{Sec}(\widetilde{H}_i)$. □

In particular we see that the distributions H_i, as well as $\widetilde{H}_i$, are invariant by the flow of ξ.

3.2 The submanifold *B*: construction of the bundle over *B*

Fix a point $q_0 \in M$. Let $q_0 \in U$ where $U \subset M$ is an open set that will be specified below. We construct a regular submanifold B of M, $q_0 \in B$, which can be canonically identified with U/ξ.

We start by choosing a coordinate system around q_0 which will be convenient for our purposes. Denote by $\delta : (-\varepsilon, \varepsilon) \longrightarrow M$ the trajectory of the Reeb field ξ such that $\delta(0) = q_0$. Select a local basis $X_1, \dots, X_{2n}$ of section of H defined near q_0 and let g_i^t stand for the (local) flow of X_i, $i = 1, \dots, 2n$. We can assume that each g_i^t is defined on a neighborhood of q_0 and for $|t| < \varepsilon$. By shrinking U we can suppose that the mapping

$$(\tilde{x}^1, \dots, \tilde{x}^{2n}, z) \longrightarrow g_1^{\tilde{x}^1} \circ \cdots \circ g_{2n}^{\tilde{x}^{2n}} \circ \delta(z)$$

defines coordinates $(\tilde{x}^1, \dots, \tilde{x}^{2n}, z)$ on U such that $\tilde{x}^i(q_0) = z(q_0) = 0$,

$$H_{|\delta} = \mathrm{Span}\Big\{\frac{\partial}{\partial \tilde{x}^1}, \dots, \frac{\partial}{\partial \tilde{x}^{2n}}\Big\},$$

and $\xi_{|\delta} = \frac{\partial}{\partial z}$. Let $(\tilde{x}^1, \dots, \tilde{x}^{2n}, z, \tilde{p}_1, \dots, \tilde{p}_{2n}, r)$ be the Darboux coordinates on $T^*M_{|U}$ and let us set

$$A = \{(0, \dots, 0, z, \tilde{p}_1, \dots, \tilde{p}_{2n}, 0) : \ |z|, |\tilde{p}_1|, \dots, |\tilde{p}_{2n}| < \varepsilon\} \subset T^*M_{|U}.$$

The set A can be regarded as the set of initial conditions for sub-Riemannian geodesics satisfying the Pontryagin transversality conditions with respect to δ (cf. [1]). Now, the assignment

$$(x^1, \dots, x^{2n}, z) \longrightarrow \exp_{(0,\dots,0,z)}(x^1, \dots, x^{2n}, 0) \tag{3.1}$$

defines the desired coordinates $(x^1, \dots, x^{2n}, z)$ around q_0. We can suppose that they are defined on U (shrinking U again if necessary). Let us notice that

$$H_{|\delta} = \mathrm{Span}\Big\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{2n}}\Big\},$$

$\xi_{|\delta} = \frac{\partial}{\partial z}$, and straight lines $t \longrightarrow (tv, z)$ in these coordinates, where $v \in \mathbb{R}^{2n}$ and $\sum_{i=1}^n v_i^2 = 1$, are sub-Riemannian geodesics parameterized by arc length, which satisfy the transversality conditions with respect to δ. We define the following family of hypersurfaces

$$S_w = \{q \in U : \ z(q) = w\}$$

transverse to δ. We will identify U with an open subset of $\mathbb{R}^{2n+1}$ with coordinates (x, z), so we will also write $S_w = \{(x, z) : \ z = w\}$.

Proposition 3.2 $\varphi^t(S_w) = S_{w+t}$.

Proof S_w is the union of geodesics which in our coordinates have the form $\sigma(s) = (sv, w)$, $|v| = 1$. As we said above these are are exactly the geodesics that start from δ and satisfy the Pontryagin transversality conditions with respect to δ. Fix such a geodesic $\sigma(s) = (sv, w)$. Since φ^t is an isometry preserving δ, the curve $s \longrightarrow \varphi^t(\sigma(s))$ is again a geodesic that starts from δ and satisfies the transversality condition with respect to δ. Thus it must be of the form $s \longrightarrow (s\tilde{v}, \tilde{w})$. Because $\varphi^t(\sigma(0)) = (0, w + t)$ in (x, z)-coordinates, $\tilde{w} = w + t$ which ends the proof. □

Now, let us set $B = S_0$. Remark that B (or more precisely, its germ at q_0) is defined canonically and does not depend on coordinates. We define the projection

$$p : U \longrightarrow B$$

as the projection onto B in the direction of ξ:

$$p(q) = \varphi^{t(q)}(q),$$

where $t(q)$ is such that $\varphi^{t(q)}(q) \in B$. By construction, ξ is transverse to B, and $t(q)$ depends smoothly on q by the implicit function theorem. Obviously $t(q) = 0$ for $q \in B$ and, moreover,

$$p \circ \varphi^t = p. \tag{3.2}$$

We see that $p : U \longrightarrow B$ is a fiber bundle with fibers being trajectories of ξ.

3.3 Induced metric and connection on *B*

Our next aim is to endow B with a suitably induced Riemannian metric and a connection. Suppose that $X \in \mathrm{Sec}(TB)$. First we construct the canonical 'lift' of X to the field $\widetilde{X} \in \mathrm{Sec}(H)$ on U by formula

$$\widetilde{X}(\varphi^t(q)) = d_q\varphi^t(X(q) - \alpha_q(X)\xi(q)) \tag{3.3}$$

for every $q \in B$ and every t for which the above expression is defined. Recall that α stands for the normalized conatct form.

Proposition 3.3 *Suppose that $X \in \mathrm{Sec}(TB)$ and let $\widetilde{X} \in \mathrm{Sec}(H)$ be the horizontal lift defined above. For every $q \in U$*

$$d_q p(\widetilde{X}(q)) = X(p(q)).$$

Proof Let $q = \varphi^t(\bar{q})$, where $\bar{q} \in B$. Then using (3.3) and (3.2) we have

$$d_q p(\widetilde{X}(q)) = d_{\varphi^t(\bar{q})} p \circ d_{\bar{q}}\varphi^t(X(\bar{q}) - \alpha_{\bar{q}}(X)\xi(\bar{q})) = d_{\bar{q}} p(X(\bar{q}) - \alpha_{\bar{q}}(X)\xi(\bar{q}))$$

and it suffices to notice that $d_{\bar{q}} p(X(\bar{q})) = X(\bar{q})$ and $d_{\bar{q}} p(\xi) = 0$. The first equality follows from $p_{|B} = \mathrm{id}$ and the other from the definition of p. □

Now we define the announced Riemannian metric on B. For $q \in B$ and $X, Y \in \mathrm{Sec}(TB)$ we set

$$g_B(X(q), Y(q)) = g\big(X(q) - \alpha_q(X)\xi(q), Y(q) - \alpha_q(Y)\xi(q)\big). \tag{3.4}$$

The last equation can be rewritten as

$$g_B(X(q), Y(q)) = g_B(d_q p(\widetilde{X}(q)), d_q p(\widetilde{Y}(q))) = g(\widetilde{X}(q), \widetilde{X}(q)). \tag{3.5}$$

Note that if $X_1, \dots, X_{2n}$ is a basis of $T_q B$, then $\widetilde{X}_1 = X_1 - \alpha(X_1)\xi(q), \dots, \widetilde{X}_{2n} = X_{2n} - \alpha(X_{2n})\xi(q)$ is a basis of H_q and $d_q p(\widetilde{X}_i) = X_i$ for every i. Remembering (3.5) we obtain the following statement.

Corollary 3.1 *For every $q \in B$*

$$d_q p_{|H_q} : H_q \longrightarrow T_q B$$

is a linear isometry.

Notice that B carries a natural orientation determined by the orientation of H.

Remark 3.1 Let us remark that if we apply the same procedure to define the Riemannian metric on S_w, $w \neq 0$, then the resulting Riemannian manifold will be isometric to (B, g_B).

We proceed to define a connection on B. Denote by $O(B)$ the bundle of orthonormal frames of B. Let $\pi_B : O(B) \longrightarrow B$ be the corresponding projection and $V_B = \ker d\pi_B$ be the vertical distribution. By Corollary 3.1 we have the natural mapping $\hat{p} : O_{H,g}(U) \longrightarrow O(B)$, $\hat{p}(q; v_1, \ldots, v_{2n}) = (p(q); d_q p(v_1), \ldots, d_q p(v_{2n}))$. Of course the diagram

$$\begin{array}{ccc} O_{H,g}(U) & \xrightarrow{\hat{p}} & O(B) \\ \pi \downarrow & & \downarrow \pi_B \\ U & \xrightarrow{p} & B \end{array} \tag{3.6}$$

is commutative. Recall that we have the decomposition $\Gamma = \Gamma^H \oplus \Gamma^\xi$, where $\Gamma^\xi = \mathrm{Span}\{\xi^*\}$. Note that $\hat{p} \circ \Phi^t = \hat{p}$ (where Φ^t is the local flow of ξ^*). Indeed,

$$\begin{aligned} &\hat{p} \circ \Phi^t(q; v_1, \ldots, v_{2n}) = (p(\varphi^t(q)); d_q(p \circ \varphi^t)(v_1), \ldots, d_q(p \circ \varphi^t)(v_{2n})) \\ &\quad = (p(q); d_q p(v_1), \ldots, d_q p(v_{2n})). \end{aligned}$$

Proposition 3.4 *The mapping $\hat{p}$ defined above is a surjective submersion.*

Proof Evidently $\hat{p}$ is onto B. Fix $l = (q; v_1, \ldots, v_{2n}) \in O_{H,g}(U)$, $q \in B$, and take $w \in T_{\hat{p}(l)} O(B)$. Then $w = \bar{\sigma}^{\boldsymbol{\cdot}}(0)$, where $\bar{\sigma} : [-\varepsilon, \varepsilon] \longrightarrow O(B)$ is a suitable smooth curve. Clearly, $\bar{\sigma}(t) = (\sigma(t); w_1(t), \ldots, w_{2n}(t))$, $\bar{\sigma}(0) = \hat{p}(l)$, so in particular $w_i(0) = d_q p(v_i)$. For the curve $\sigma : [-\varepsilon, \varepsilon] \longrightarrow B$, $\sigma(0) = q$, let us construct its lift to a horizontal curve $\widetilde{\sigma} : [-\varepsilon, \varepsilon] \longrightarrow U$, $\widetilde{\sigma}(0) = \sigma(0)$, as follows. Supposing that $\varepsilon > 0$ is sufficiently small and σ is contained in a coordinate chart V, extend the field $\dot{\sigma}(t)$ to a vector field $Z \in \mathrm{Sec}(TB_{|V})$. Using (3.3), we obtain the field $\widetilde{Z} \in \mathrm{Sec}(H_{|p^{-1}(V)})$ and as $\widetilde{\sigma} : [-\varepsilon, \varepsilon] \longrightarrow U$ we simply take the trajectory of $\widetilde{Z}$ starting from $\sigma(0)$. By construction, $p \circ \widetilde{\sigma} = \sigma$. Now define

$$v_i(t) = (d_{\widetilde{\sigma}(t)} p_{|H(\widetilde{\sigma}(t))})^{-1} w_i(t), \quad i = 1, \ldots, 2n,$$

and set

$$c(t) = (\widetilde{\sigma}(t); v_1(t), \ldots, v_{2n}(t)).$$

It is easy to check that $d_l\hat{p}(\dot{c}(0)) = w$.

□

Proposition 3.5 (a) $d\hat{p}(\Gamma^{\xi}) = 0$;

(b) $d\hat{p}(V) = V_B$;

(c) *The distribution* $\Gamma^B = d\hat{p}(\Gamma^H)$ *is a connection on* $O(B)$.

Proof Part (a) and the inclusion $d\hat{p}(V) \subset V_B$ follow from the equation before Proposition 3.4, diagram (3.6) and Proposition 3.4. Now take $w \in V_B$. Then $w = d\hat{p}(v)$, $v \in TO_{H,g}(M)$, and since $dp(d\pi(v)) = d\pi_B(w) = 0$, it must be $v = \lambda\xi^* + v'$, $\lambda \in \mathbb{R}$, $v' \in V$. Consequently $w = d\hat{p}(v') \in d\hat{p}(V)$.

We will prove (c). First notice that

$$\hat{p} \circ R_a = R_a \circ \hat{p}. \tag{3.7}$$

Next, since $d_l R_a(\Gamma^H_l) = \Gamma^H_{l.a}$,

$$d_{\hat{p}(l)} R_a(\Gamma^B_{\hat{p}(l)}) = d_{\hat{p}(l)} R_a \circ d_l\hat{p}(\Gamma^H_l) = d_{l.a}\hat{p} \circ d_l R_a(\Gamma^H_l) = d_{l.a}\hat{p} \circ (\Gamma^H_{l.a}) = \Gamma^B_{\hat{p}(l).a}.$$

Moreover, $d\pi_B(\Gamma^B) = d\pi_B \circ d\hat{p}(\Gamma^H) = dp \circ d\pi(\Gamma^H) = dp(H) = TB$, so

$$TO(B) = \Gamma^B \oplus V_B. \tag{3.8}$$

□

Next we will compute the torsion of Γ^B. To this end denote by $\mathrm{pr}_B : TO(B) \longrightarrow \Gamma^B$ the projection corresponding to the decomposition (3.8).

Corollary 3.2 $\mathrm{pr}_B \circ d\hat{p} = d\hat{p} \circ \mathrm{pr}$.

Denote by θ_B the canonical 1-form on $O(B)$.

Lemma 3.1 *Let* θ *be the canonical* 1*-form on* $O_{H,g}(M)$ *defined in Sect.* 2.3. *Then*

$$\hat{p}^*\theta_B = \theta. \tag{3.9}$$

Proof Take $l \in O_{H,g}(M)$. We have

$$(\hat{p}^*\theta_B)(l) = \theta_B(\hat{p}(l)) \circ d_l\hat{p} = \hat{p}(l)^{-1} \circ d_{\hat{p}(l)}\pi_B \circ d_l\hat{p} = \hat{p}(l)^{-1} \circ d_{\pi(l)}p \circ d_l\pi$$

and recalling (2.7) it is enough to prove that

$$\hat{p}(l)^{-1} \circ d_{\pi(l)}p = l^{-1} \circ P. \tag{3.10}$$

Let $l = (q; v_1, \ldots, v_{2n})$ and $v \in TM$. Then $\hat{p}(l)^{-1} \circ d_q p(v) = r$ if and only if $d_q p(v) = d_q p(r^i v_i)$, which in turn is equivalent to $v = r^i v_i + \lambda\xi(q)$ for a certain $\lambda \in \mathbb{R}$. Now $l^{-1} \circ P(v) = r$ and (3.10) is proved. □

The torsion of Γ^B is equal to $\Theta_B = d\theta_B \circ (\mathrm{pr}_B, \mathrm{pr}_B)$. Take two vectors $v, w \in TO(B)$. Then $v = d\hat{p}(\hat{v})$, $w = d\hat{p}(\hat{w})$ for $\hat{v}, \hat{w} \in \Gamma^H \subset TO_{H,g}(M)$, and

$$\begin{aligned}\Theta_B(v, w) &= d\theta_B(\mathrm{pr}_B \circ d\hat{p}(\hat{v}), \mathrm{pr}_B \circ d\hat{p}(\hat{w})) = d\theta_B(d\hat{p} \circ \mathrm{pr}(\hat{v}), d\hat{p} \circ \mathrm{pr}(\hat{w}))\\ &= \hat{p}^*(d\theta_B)(\mathrm{pr}(\hat{v}), \mathrm{pr}(\hat{w})) = \Theta(\hat{v}, \hat{w}) = 0,\end{aligned}$$

where Θ is the torsion form of Γ (see (2.8)). We proved the following proposition.

Proposition 3.6 Γ^B *is the Levi-Civita connection with respect to the metric* g_B.

Denote by $\nabla^B : \mathrm{Sec}(TB) \times \mathrm{Sec}(TB) \longrightarrow \mathrm{Sec}(TB)$ the covariant derivation induced by Γ^B. We have an explicit formula for ∇^B.

Proposition 3.7 *For every* $X, Y \in \mathrm{Sec}(TB)$ *and* $q \in B$

$$(\nabla^B_X Y)(q) = d_q p(\nabla_{\widetilde{X}} \widetilde{Y})(q), \tag{3.11}$$

where $\widetilde{X}$, $\widetilde{Y}$ *are defined according to formula* (3.3).

We postpone the proof of this proposition until the appendix.

3.4 Distributions Δ_i and decomposition of *TB*

The decomposition of H induces the decomposition of TB into the distributions on B which are defined as

$$\Delta_i = dp(H_i) = dp(\widetilde{H}_i), \tag{3.12}$$

$i = 1, \ldots, m$. By Proposition 3.3 such a definition is correct.

For a curve $\gamma : [a, b] \longrightarrow B$ denote by $\tau^B_\gamma : H(\gamma(a)) \longrightarrow H(\gamma(b))$ the parallel translation along γ determined by the connection Γ^B. We prove the following lemma.

Lemma 3.2 *Suppose that* $\gamma : [a, b] \longrightarrow B$ *and* $\widetilde{\gamma} : [a, b] \longrightarrow M$ *are piecewise smooth curves such that* $\widetilde{\gamma}$ *is horizontal and* $p \circ \widetilde{\gamma} = \gamma$. *Then the diagram*

$$\begin{array}{ccc} H(\widetilde{\gamma}(a)) & \xrightarrow{\tau_{\widetilde{\gamma}}} & H(\widetilde{\gamma}(b)) \\ {\scriptstyle d_{\widetilde{\gamma}(a)}p}\downarrow & & \downarrow{\scriptstyle d_{\widetilde{\gamma}(b)}p} \\ T_{\gamma(a)}B & \xrightarrow{\tau^B_\gamma} & T_{\gamma(b)}B \end{array}$$

is commutative.

Proof Take $v \in T_{\gamma(a)}B$ and $\widetilde{v} \in H(\widetilde{\gamma}(a))$ such that $dp(\widetilde{v}) = v$. Suppose that $\widetilde{\gamma}^* : [a, b] \longrightarrow O_{H,g}(M)$ is a Γ-horizontal lift of $\widetilde{\gamma}$. Choose $r \in \mathbb{R}^{2n}$ such that $\widetilde{\gamma}^*(a)(r) = \widetilde{v}$. Then, by definition of the parallel transport, $\tau_\gamma(\widetilde{v}) = \widetilde{\gamma}^*(b)(r)$. Now the curve

$$t \longrightarrow \hat{p}(\widetilde{\gamma}^*(t))(r) = d_{\gamma(t)}p\big(\widetilde{\gamma}^*(t)(r)\big)$$

(again by definition) is parallel in TB, projects onto γ and initiates at v, therefore

$$\tau_\gamma^B(v) = d_{\gamma(b)}p\big(\widetilde{\gamma}^*(b)(r)\big).$$

This proves the commutativity of the above diagram. □

Proposition 3.8 *The distributions Δ_i are parallel with respect to the connection Γ^B, i.e., for every point $q \in B$, each Δ_i can be obtained from $\Delta_i(q)$ by parallel transport. Moreover, Δ_i are irreducible with respect to the holonomy group of Γ^B.*

Proof Fix an index i and take a piecewise smooth curve $\gamma : [a, b] \longrightarrow B$. Pick numbers $a = a_0 < a_1 < \cdots < a_m = b$ such that each $\gamma_j = \gamma_{|[a_{j-1},a_j]}$ admits a lift to a horizontal curve $\widetilde{\gamma}_j : [a_{j-1}, a_j] \longrightarrow U$, $\widetilde{\gamma}_j(a_{j-1}) = \gamma(a_{j-1})$, as it is described in the proof of Proposition 3.4. Obviously $\tau_\gamma^B = \tau_{\gamma_m}^B \circ \cdots \circ \tau_{\gamma_1}^B$ and by Lemma 3.2 each $\tau_{\gamma_j}^B$ preserves the distribution Δ_i. It follows that $\tau_\gamma^B(\Delta_i(\gamma(a)) = \Delta_i(\gamma(b))$ as desired.

Fix now a point $q_0 \in B$ and suppose that we have a decomposition

$$\Delta_i(q_0) = \Delta_i^{(1)}(q_0) \oplus \Delta_i^{(2)}(q_0) \tag{3.13}$$

into nontrivial components. Let $H_i^{(j)}(q_0) = (d_{q_0}p)^{-1}\big(\Delta_i^{(j)}(q_0)\big) \cap H_i(q_0)$, $j = 1, 2$. Clearly, $H_i(q_0) = H_i^{(1)}(q_0) \oplus H_i^{(2)}(q_0)$ and $H_i^{(j)}(q_0)$ are not $\Psi(q_0)$-invariant. Therefore there exists a nonzero $\hat{v} \in H_i^{(1)}(q_0)$ and a horizontal curve $\widetilde{\gamma} : [0, 1] \longrightarrow M$ such that $\widetilde{\gamma}(0) = \widetilde{\gamma}(1) = q_0$ and $\tau_{\widetilde{\gamma}}(\hat{v}) \in H_i^{(2)}(q_0)$. Let $\gamma = p \circ \widetilde{\gamma}$. Now $dp(\hat{v}) \in \Delta_i^{(1)}(q_0)$ and by Lemma 3.2 $\tau_\gamma^B(dp(\hat{v})) \in \Delta_i^{(2)}(q_0)$ which proves that $\Delta_i^{(j)}(q_0)$ are not invariant with respect to the holonomy group of Γ^B. □

In particular it follows that $\nabla_X^B(\mathrm{Sec}(\Delta_i)) \subset \mathrm{Sec}(\Delta_i)$ for every $X \in \mathrm{Sec}(TB)$ and, consequently, Δ_i are integrable. Indeed, if $X, Y \in \mathrm{Sec}(\Delta_i)$ then $\nabla_X^B Y - \nabla_Y^B X = [X, Y] \in \mathrm{Sec}(\Delta_i)$. Now, to finish the prove of Theorem 1.1 we just use de Rham decomposition theorem [15]. Let us note that the integrability of the distributions Δ_i can be also proved in the following way. For a point $q \in B$ denote by M_i the maximal integral manifold of $\widetilde{H}_i$ passing through q. Using, e.g., Corollary 3.1 we deduce that $p_{|M_i} : M_i \longrightarrow B$ is of constant rank and hence $p(M_i)$ is an integral manifold of Δ_i passing through q.

In the sub-pseudo-Riemannian case the proof goes along the same lines. The only difference is that the structure group of the bundle $O_{H,g}(M)$ is now $O(k, 2n - k)$ where k is the index of a metric g, and by an orthonormal frame we mean every frame $X_1, \ldots, X_{2n}$ such that $g(X_i, X_j) = 0$, $i \neq j$, $g(X_i, X_i) = -1$, $1 \leq i \leq k$, $g(X_j, X_j) = 1$, $k + 1 \leq j \leq 2n$. Moreover, a few words more about Hamiltonian geodesics in the indefinite case should be added, and we do it in the appendix. At the end we use the version of de Rham Theorem proved in [16].

4 Proof of Theorem 1.3

Replacing M with an open subset, if needed, we can suppose that our structure is strongly nondegenerate on M. Then the eigenvalues $\pm ib_j$ of J are smooth functions on M. For any point $q \in M$ there exists a neighborhood U of q and an orthonormal frame $X_1, \ldots, X_{2n} \in \mathrm{Sec}(H_{|U})$ such that $J(X_{2j-1}) = -b_j X_{2j}$ and $J(X_{2j}) = b_j X_{2j-1}$ on U, $j = 1, \ldots, n$. Let us define

$$H_{j|U} = \mathrm{Span}\{X_{2j-1}, X_{2j}\}. \tag{4.1}$$

Of course $H_{j|U}$ glue together to globally defined distributions on M and we obtain the decomposition of H

$$H = H_1 \oplus \cdots \oplus H_n \tag{4.2}$$

into the Whitney sum of pairwise orthogonal rank 2 sub-distributions.

Let $l = (q; v_1, \ldots, v_{2n}) \in O_{H,g}(M)$. Denote by $e_1, \ldots, e_{2n}$ the standard basis of $\mathbb{R}^{2n}$ and by $f^1, \ldots, f^{2n}$ the dual basis of $(\mathbb{R}^{2n})^*$. Let us recall that since H, H^* and $\mathrm{Hom}(H, H)$ are vector bundles associated with $O_{H,g}(M)$ with typical fiber equal to $\mathbb{R}^{2n}$, $(\mathbb{R}^{2n})^*$, $(\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$, respectively, then l acts as linear isomorphisms (cf. [15]) $l : \mathbb{R}^{2n} \longrightarrow H(q)$, $l : (\mathbb{R}^{2n})^* \longrightarrow H(q)^*$, $l : (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n} \longrightarrow \mathrm{Hom}(H(q), H(q))$ which are respectively defined by $l(e_j) = v_j$, $l(f^j) = v^{*j}$, $l(f^j \otimes e_k) = v^{*j} \otimes v_k$; here $v^{*j} \in H_q^*$ is the covector dual to $v_j \in H_q$.

Now fix a point $q \in M$. Choose an orthonormal basis $v_1, \ldots, v_{2n}$ of $H(q)$ such that

$$J_q(v_{2j-1}) = -b_j(q)v_{2j}, \quad J_q(v_{2j}) = b_j(q)v_{2j-1}, \quad j = 1, \ldots, n.$$

Let us define a $2n \times 2n$-matrix (A^i_j) by

$$A^i_j f^j \otimes e_i = \sum_{j=1}^{n} \big[- b_j(q) f^{2j-1} \otimes e_{2j} + b_j(q) f^{2j} \otimes e_{2j-1} \big].$$

If $l = (q; v_1, \ldots, v_{2n})$ then clearly $l(A^i_j f^j \otimes e_i) = J_q$. Further take an arbitrary smooth horizontal curve $\sigma : [0, 1] \longrightarrow M$ such that $\sigma(0) = q$. Denote by $\sigma^* : [0, 1] \longrightarrow O_{H,g}(M)$ the Γ-horizontal lift of σ which satisfies $\sigma^*(0) = l$. Then $\sigma^*(0)(A^i_j f^j \otimes e_i) = J_q$. By assumption the operator J is parallel, therefore

$$\sigma^*(t)(A^i_j f^j \otimes e_i) = J_{\sigma(t)} \tag{4.3}$$

for $t \in [0, 1]$.[1] Equation (4.3) means that if $\sigma^*(t) = (\sigma(t); v_1(t), \ldots, v_{2n}(t))$ then for every t

$$J_{\sigma(t)}(v_{2j-1}(t)) = -b_j(q)v_{2j}(t), \quad J_q(v_{2j}(t)) = b_j(q)v_{2j-1}(t), \quad j = 1, \ldots, n.$$

Since σ is an arbitrary horizontal curve, and any two points of M can be joined by a horizontal curve, this ends the proof of the following proposition.

Proposition 4.1 *Under the assumptions of Theorem 1.3, $b_j = const$, $j = 1, \ldots, n$, in a neighborhood of every point at which the structure is strongly nondegenerate. Moreover, the distributions $H_1, \ldots, H_n$ from (4.2) are parallel on such a neighborhood.*

To end the proof we proceed exactly as above which results in a decomposition $(B_1, g_1)\times\cdots\times(B_n, g_n)$ of B into the product of 2-dimensional Riemannian manifolds. It remains to recall the classical result saying that any two Riemannian manifolds of dimension 2 are locally conformally equivalent.

Availability of data and material Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflicts of interest The author declares that he has no conflict of interest.

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Appendix A: Hamiltonian geodesics in sub-pseudo-Riemannian case

Since sub-pseudo-Riemannian geometry is very little known as compared to the sub-Riemannian one, we give here some facts concerning Hamiltonian geodesics and prove that they are preserved by isometries. Suppose that (M, H, g) is a contact sub-pseudo-Riemannian manifold and suppose that g has index k. By a local orthonormal frame for (H, g) we mean a frame $X_1, \ldots, X_{2n}$ defined on an open set $U \subset M$ such that $g(X_i, X_j) = \varepsilon_i\delta_{ij}$, where

$$\varepsilon_i = \begin{cases} -1 & : i = 1, \ldots, k \\ +1 & : i = k+1, \ldots, 2n \end{cases}.$$

[1] Note [15] that parallel curves in $\mathrm{Hom}(H, H)$ covering σ are exactly of the form $t \longrightarrow \sigma^*(t)(A)$ with $A \in (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$.

We define the *geodesic Hamiltonian* $\mathcal{H} : T^*M \longrightarrow \mathbb{R}$. First we do so locally. Suppose that $U \subset M$ is an open subset such that there exists an orthonormal frame $X_1, \ldots, X_{2n}$ for $H_{|U}$. Then we set

$$\mathcal{H}(q, \lambda) = \frac{1}{2} \sum_{i=1}^{2n} \varepsilon_i \langle \lambda, X_i(q) \rangle^2$$

on $T^*M_{|U}$. Next we notice that such a definition is independent of the choice of an orthonormal frame. Indeed, if $X'_1, \ldots, X'_{2n}$ is any other orthonormal frame for $H_{|U}$, then $X'_i = a_i^j X_j$, where $(a_j^i) \in O(k, 2n-k)$. By the definition of $O(k, 2n-k)$ we obtain that

$$\begin{aligned}\sum_{i=1}^{2n} \varepsilon_i \langle \lambda, X'_i(q) \rangle^2 &= \sum_{i,j=1}^{2n} \varepsilon_i (a_i^j)^2 \langle \lambda, X_j \rangle^2 \\ &= \sum_{j=1}^{2n} \Big(\sum_{i=1}^{2n} \varepsilon_i (a_i^j)^2 \Big) \langle \lambda, X_j \rangle^2 = \sum_{j=1}^{2n} \varepsilon_j \langle \lambda, X_j \rangle^2\end{aligned}$$

as claimed. It follows that $\mathcal{H}$ is well defined on the whole T^*M. Recall that we have the canonical symplectic structure on T^*M which we will denote by Υ (Υ is the exterior differential of the Liouville 1-form on T^*M). Thus the geodesic Hamiltonian determines the Hamiltonian vector field $\overrightarrow{\mathcal{H}}$ on T^*M. Now by a *Hamiltonian geodesic* on the given sub-pseudo-Riemannian manifold we mean every curve on M which is a projection of a trajectory of the field $\overrightarrow{\mathcal{H}}$. Once we have the notion of Hamiltonian geodesics, we can define the exponential mapping with pole at a given point $q_0 \in M$ exactly as it is done in the sub-Riemannian case.

Suppose $f : M_1 \longrightarrow M_2$ is a diffeomorphism. Then we have the induced diffeomorphism $\hat{f} : T^*M_1 \longrightarrow T^*M_2$ which acts by $\hat{f}(q, \lambda) = (f(q), ((d_q f)^{-1})^* \lambda)$. It is well-known that $\hat{f}$ is a symplectomorphism with respect to the canonical symplectic structures on T^*M_i.

A diffeomorphism $f : (M_1, H_1, g_1) \longrightarrow (M_2, H_2, g_2)$ of two sub-pseudo-Riemannian manifolds is called an isometry if $d_q f(H_1(q)) \subset H_2(f(q))$ and $d_q f : H_1(q) \longrightarrow H_2(f(q))$ is a linear isometry for every $q \in M$. In particular the two metrics g_1, g_2 have the same index.

Lemma A.1 *Suppose that (M_i, H_i, g_i) is a sub-pseudo-Riemannian manifold and denote by $\mathcal{H}_i$ the geodesic Hamiltonian on (M_i, H_i, g_i), $i = 1, 2$. If $f : M_1 \longrightarrow M_2$ is an isometry then*

$$\mathcal{H}_2 \circ \hat{f} = \mathcal{H}_1 \tag{A.1}$$

and, moreover,

$$d\hat{f}(\overrightarrow{\mathcal{H}_1}) = \overrightarrow{\mathcal{H}_2}. \tag{A.2}$$

Proof Fix $(q,\lambda) \in T^*M_1$ and let $X_1,\dots,X_{2n}$ be an orthonormal frame for H_2 defined in a neighborhood of $f(q)$. Then

$$\mathcal{H}_2 \circ \hat{f}(q,\lambda) = \frac{1}{2}\sum_{j=1}^{2n}\langle((d_q f)^{-1})^*\lambda, X_j(f(q))\rangle$$
$$= \frac{1}{2}\sum_{j=1}^{2n}\langle\lambda, (d_q f)^{-1}X_j(f(q))\rangle = \mathcal{H}_1(q,\lambda),$$

since $(d_q f)^{-1}X_1,\dots,(d_q f)^{-1}X_{2n}$ is an orthonormal frame for H_1 around q.

In order to prove (A.2) we note that in addition to (A.1) we also have $\hat{f}^*\Upsilon_2 = \Upsilon_1$. □

Consequently, we have finished the proof of the following proposition.

Proposition A.1 *Isometries preserve Hamiltonian geodesics.*

Appendix B: Proof of Proposition 3.7

In this section we prove that the operator

$$(D_X Y)(q) = d_q p(\nabla_{\widetilde{X}}\widetilde{Y})(q),$$

where $X, Y \in \mathrm{Sec}(TB)$ and $q \in B$, is a Levi-Civita connection for the metric g_B. As above $\widetilde{X}, \widetilde{Y}$ are defined according to formula (3.3).

For a function $f \in C^\infty(B)$ let us define $\widetilde{f} \in C^\infty(U)$ by $\widetilde{f}(q) = f(p(q))$ and observe that

$$\widetilde{fX} = \widetilde{f}\widetilde{X} \tag{B.1}$$

whenever $X \in \mathrm{Sec}(TB)$. Further, we have

Lemma B.1 *Under the above notation, for* $X \in \mathrm{Sec}(TB)$*,* $f \in C^\infty(B)$*,* $q \in B$

$$\widetilde{X}(\widetilde{f}) = \widetilde{X(f)};$$

in particular, $\widetilde{X}(\widetilde{f})(q) = X(f)(q)$*. Moreover,*

$$\xi(\widetilde{f}) = 0.$$

Proof The second formula is obvious since $\widetilde{f}$ is constant along the trajectories of ξ. To prove the first part, fix an arbitrary point belonging to U. Such a point is of the form $\varphi^s(q)$ where $q \in B$ and $s \in \mathbb{R}$. We have

$$\begin{aligned}\widetilde{X}(\widetilde{f})(\varphi^s(q)) &= d_q\varphi^s\big(X(q) - \alpha_q(X)\xi(q)\big)(f \circ p)\\ &= d_{\varphi^s(q)} p \circ d_q\varphi^s\big(X(q) - \alpha_q(X)\xi(q)\big)(f) = d_q p\big(X(q) - \alpha_q(X)\xi(q)\big)(f)\\ &= X(f)(q) = \widetilde{X(f)}(\varphi^s(q)).\end{aligned}$$

□

We make sure that D defined above is indeed a connection. To this end take $X, Y \in \mathrm{Sec}(TB)$, $f \in C^\infty(B)$ and $q \in B$. Then

$$(D_{fX}Y)(q) = d_q p((\nabla_{\widetilde{fX}}\tilde{Y})(q)) = \tilde{f}(q) d_q p((\nabla_{\tilde{X}}\tilde{Y})(q)) = (fD_XY)(q),$$

and

$$\begin{aligned}\big(D_X(fY)\big)(q) &= d_q p((\nabla_{\tilde{X}}\widetilde{fY})(q)) = d_q p((\nabla_{\tilde{X}}\tilde{f}\tilde{Y})(q))\\ &= d_q p\big(\tilde{X}(\tilde{f})(q)\tilde{Y}(q) + \tilde{f}(q)\nabla_{\tilde{X}}\tilde{Y}(q)\big) = (X(f)Y + fD_XY)(q).\end{aligned}$$

Fix $X, Y, Z \in \mathrm{Sec}(TB)$ and $q \in B$. At first we will compute the torsion of D.

$$\begin{aligned}(D_XY - D_YX)(q) &= d_q p(\nabla_{\tilde{X}}\widetilde{Y} - \nabla_{\tilde{Y}}\widetilde{X})(q) = d_q p\big(P([\widetilde{X}, \widetilde{Y}])(q)\big)\\ &= d_q p\big([\widetilde{X}, \widetilde{Y}](q)\big),\end{aligned}$$

where the last equality follows from the fact that $dp(\xi) = 0$. Now, for any $f \in C^\infty(B)$

$$\begin{aligned}&d_q p\big([\widetilde{X}, \widetilde{Y}](q)\big)(f) = [\widetilde{X}, \widetilde{Y}](f \circ p)(q) = [\widetilde{X}, \widetilde{Y}](\widetilde{f})(q) = \big(\widetilde{X}(\widetilde{Y}(\widetilde{f})) - \widetilde{Y}(\widetilde{X}(\widetilde{f}))\big)(q)\\ &= \big(\widetilde{X}(\widetilde{Y(f)}) - \widetilde{Y}(\widetilde{X(f)})\big)(q) = \big(\widetilde{X(Y(f))} - \widetilde{Y(X(f))}\big)(q) = \big(X(Y(f)) - Y(X(f))\big)(q)\\ &= [X, Y](f)(q).\end{aligned}$$

It follows that

$$D_XY - D_YX = [X, Y]$$

and D is torsion-free. Next we prove that D is a metric connection. Recall that

$$\widetilde{Z}(g(\widetilde{X}, \widetilde{Y})) = g(\nabla_{\widetilde{Z}}\widetilde{X}, \widetilde{Y}) + g(\widetilde{X}, \nabla_{\widetilde{Z}}\widetilde{Y}). \tag{B.2}$$

In order to evaluate the left-hand side of (B.2) let us notice that for every s (for which it makes sense)

$$\begin{aligned}g(\widetilde{X}, \widetilde{Y})(\varphi^s(q)) &= g\Big(d_q\varphi^s\big(X(q) - \alpha_q(X)\xi(q)\big), d_q\varphi^s\big(Y(q) - \alpha_q(Y)\xi(q)\big)\Big)\\ &= g\big(X(q) - \alpha_q(X)\xi(q), Y(q) - \alpha_q(Y)\xi(q)\big) = g_B(X(q), Y(q))\\ &= g_B(X, Y)(p \circ \varphi^s(q)) = \widetilde{g_B(X, Y)}(\varphi^s(q)).\end{aligned}$$

Therefore, according to Lemma B.1, we have

$$\widetilde{Z}(g(\widetilde{X}, \widetilde{Y}))(q) = Z(g_B(X, Y))(q).$$

Now the first summand on the right-hand side of (B.2) evaluated at q is

$$g(\nabla_{\widetilde{Z}} \widetilde{X}, \widetilde{Y})(q) = g_B\Big(d_q p\big(\nabla_{\widetilde{Z}} \widetilde{X}\big), d_q p\big(\widetilde{Y}\big)\Big) = g_B(D_Z X, Y)(q)$$

(we use (3.5) here) and similarly for the second summand. Hence

$$Z(g_B(X, Y)) = g_B(D_Z X, Y) + g_B(X, D_Z Y)$$

which ends the proof of Proposition 3.7.

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