On an extension of Nunokawa's lemma

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Abstract

Jack's Lemma says that if $f(z)$ is regular in the disc $|z| \leq r$, $f(0) = 0$, and $|f(z)|$ assumes its maximum at z_0 on the circle $|z| = r$, then $z_0 f'(z)_0 / f(z_0) \geq 1$. This Lemma was generalized in several directions. In this paper we consider an improvement of some first author's results of this type.

Keywords Analytic functions · Bounded boundary rotation · Length problem · Starlike functions

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1 Introduction

Let $\mathcal{H}$ be the family of analytic functions in the region $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A} \subset \mathcal{H}$ denote the set of all functions $f(z)$ that are analytic in $\mathbb{D}$ with the series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \quad (1.1)$$

Also by $\mathcal{S}$ we means a subfamily of the set $\mathcal{A}$ which contains univalent functions. We next denote by $\mathcal{P}$ the class of analytic functions $p(z)$ which are normalized by

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$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad (1.2)$$

such that $\Re\{p(z)\} > 0$, $z \in \mathbb{D}$. Furthermore, by using the set $\mathcal{P}$, let $\mathcal{S}^*$ and $\mathcal{C}$ denote the families of starlike and convex functions in $\mathbb{D}$ which are defined as

$$\begin{aligned} \mathcal{S}^* &= \left\{ f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \in \mathcal{P}, z \in \mathbb{D} \right\}, \\ \mathcal{C} &= \left\{ f : f \in \mathcal{S} \text{ and } \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}, z \in \mathbb{D} \right\}. \end{aligned} \quad (1.3)$$

The above subfamilies of the set $\mathcal{S}$ are among the most studied families. Recall also, that if $f \in \mathcal{A}$ satisfies

$$\Re \left\{ \frac{zf'(z)}{e^{i\alpha}g(z)} \right\} > 0, \quad z \in \mathbb{D} \quad (1.4)$$

for some $g \in \mathcal{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then f is said to be close-to-convex in $\mathbb{D}$ and denoted by $f \in \mathcal{K}$. An univalent function $f \in \mathcal{A}$ belongs to $\mathcal{K}$ if and only if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays). We note that if $g(z) = z$, then the class $\mathcal{K}$ reduces to the set $\mathcal{R}$ of bounded turning functions.

Jack's Lemma [2], says that if $f(z)$ is regular in the disc $|z| \leq r$, $f(0) = 0$, and $|f(z)|$ assumes its maximum at z_0 on the circle $|z| = r$, then $z_0 f'(z_0)/f(z_0) \geq 1$. Now we will consider a lemma, which is a small extension of Jack's Lemma.

We will test behavior of $z_0 f'(z_0)/f(z_0)$ when $|f(z)|$ assumes its local maximum at z_0 on the following neighborhood

$$N_\varepsilon(z_0) = \{z : |z| \leq |z_0|, \quad |z - z_0| \leq \varepsilon\}.$$

Lemma 1.1 [1] Assume that $p(z) = a_k z^k + a_{k+1} z^{k+1} + \dots$, $a_k \neq 0$, $k \geq 1$ is analytic in $\mathbb{D}$. If there exists $z_0 \in \mathbb{D}$ such that

$$\max_{z \in N_\varepsilon(z_0)} |p(z)| = |p(z_0)| \quad (1.5)$$

for some $\varepsilon > 0$, then $z_0 p'(z_0)/p(z_0)$ is a real number and

$$\frac{z_0 p'(z_0)}{p(z_0)} \geq k \geq 0. \quad (1.6)$$

Proof Let us put

$$p(z) = z^k \varphi(z).$$

Then $\varphi(z)$ is analytic in $\mathbb{D}$ and $|\varphi(z)|$ takes its local maximum value at the point $z = z_0$ on the circular arc $z = |z_0|e^{i\theta}$, $\arg z_0 - \varepsilon < \theta < \arg z_0 + \varepsilon$. When z moves on the circle

$|z| = |z_0|$ with positive direction, then $\arg \varphi(z)$ is increasing at $z = z_0$. Therefore, we have

$$\left[\frac{d \arg \varphi(|z_0|e^{i\theta})}{d\theta} \right]_{\theta=\theta_0} = \Re \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} \geq 0$$

and so

$$\Re \frac{z_0 p'(z_0)}{p(z_0)} = \Re \left(k + \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} \right) \geq k \geq 0. \quad (1.7)$$

From the hypothesis, we have

$$\left[\frac{d|p(|z_0|e^{i\theta})|}{d\theta} \right]_{\theta=\theta_0} = 0,$$

so

$$\left[\frac{d \log |p(|z_0|e^{i\theta})|}{d\theta} \right]_{\theta=\theta_0} = 0$$

too. Therefore, we have

$$\begin{aligned} \left[\Im \frac{d \log p(z)}{d \log z} \right]_{z=z_0} &= \left[- \frac{d \log |p(|z_0|e^{i\theta})|}{d\theta} \right]_{\theta=\theta_0} \\ &= \left[- \left(\frac{d|p(z)|}{d\theta} \right) \frac{1}{|p(z)|} \right]_{\theta=\theta_0} \\ &= 0 = \Im \frac{z_0 p'(z_0)}{p(z_0)}. \end{aligned}$$

This shows

$$\frac{z_0 p'(z_0)}{p(z_0)} \geq k \geq 0. \quad (1.8)$$

□

Theorem 1.2 Assume that $p(z) = 1 + a_k z^k + a_{k+1} z^{k+1} + \dots$, $a_k \neq 0$, $k \geq 1$ is analytic in $\mathbb{D}$. Assume there exist two points $z_1, z_2 \in \mathbb{D}$ such that $|z_1| = |z_2|$ and

$$\arg p(z_2) = -\frac{\pi\beta}{2}, \quad \arg p(z_1) = \frac{\pi\alpha}{2} \quad (1.9)$$

for some $\alpha, \beta, \alpha, \beta \in (0, 2)$. If

$$-\frac{\pi\beta}{2} < \arg p(z) < \frac{\pi\alpha}{2}, \quad |z| < |z_1| = |z_2| \quad (1.10)$$

and

$$\max_{z \in N_\varepsilon(z_1)} |\arg\{p(z)\}| = |\arg\{p(z_1)\}| \quad (1.11)$$

$$\max_{z \in N_\varepsilon(z_2)} |\arg\{p(z)\}| = |\arg\{p(z_2)\}| \quad (1.12)$$

for some $\varepsilon > 0$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = \frac{i\alpha k}{2} \left(a + \frac{1}{a}\right), \quad \frac{\alpha k}{2} \left(a + \frac{1}{a}\right) \geq \alpha k, \quad (1.13)$$

where $p^{1/\alpha}(z_1) = ia$, $a > 0$, and

$$\frac{z_2 p'(z_2)}{p(z_2)} = -\frac{i\beta k}{2} \left(b + \frac{1}{b}\right), \quad -\frac{\beta k}{2} \left(b + \frac{1}{b}\right) \leq -\beta k, \quad (1.14)$$

where $p^{1/\beta}(z_2) = -ib$, $b > 0$.

Proof Let us check the first case $\arg p(z_1) = \pi\alpha/2$ and put

$$q(z) = [p(z)]^{1/\alpha}, \quad \varphi(z) = \frac{1 - q(z)}{1 + q(z)},$$

then it follows that

$$q(0) = 1, q'(0) = q''(0) = \dots = q^{(k-1)}(0) = 0 \quad \text{and} \quad q^{(k)}(0) \neq 0$$

and so

$$\varphi(0) = \varphi'(0) = \varphi''(0) = \dots = \varphi^{(k-1)}(0) = 0 \quad \text{and} \quad \varphi^{(k)}(0) \neq 0.$$

From the hypothesis, we have

$$\arg q(z) < \frac{\pi}{2}, \quad z \in D_\varepsilon(z_1) = N_\varepsilon(z_1) \setminus \{z_1\}$$

for some $\varepsilon > 0$ and

$$\arg q(z_1) = \frac{\pi}{2}.$$

Therefore, we have

$$|\varphi(z)| < 1, \quad z \in D_\varepsilon(z_1)$$

and

$$|\varphi(z_1)| = 1.$$

This shows that $|\varphi(z)|$ takes its local maximum value at the point $z = z_1$ in the domain $N_\varepsilon(z_1)$ and applying Lemma 1.1, we have

$$\begin{aligned} \frac{z_1 \varphi'(z_1)}{\varphi(z_1)} &= -\frac{z_1 q'(z_1)}{1 - q(z_1)} - \frac{z_1 q'(z_1)}{1 + q(z_1)} \\ &= -\frac{2z_1 q'(z_1)}{(1 - q(z_1))(1 + q(z_1))} \\ &= -\frac{2z_1 q'(z_1)}{(1 - ia)(1 + ia)} \end{aligned}$$

$$= -\frac{2z_1 q'(z_1)}{1+a^2} \geq k$$

where $k \geq 1$ and $q(z_1) = ia$ and $a > 0$. This shows that $z_1 q'(z_1)$ is a negative real number. This gives

$$\Im \frac{z_1 q'(z_1)}{q(z_1)} = \Im \frac{z_1 q'(z_1)}{ia} \geq \frac{k}{2} \left(a + \frac{1}{a} \right) \geq k.$$

On the other hand, we have

$$\frac{z_1 q'(z_1)}{q(z_1)} = \frac{1}{\alpha} \frac{z_1 p'(z_1)}{p(z_1)}$$

and putting $z = |z_1|e^{i\theta}$ in the section $\arg z_1 - \varepsilon < \theta < \arg z_1 + \varepsilon$, then $\arg\{q(z_1)\}$ takes its local maximum $\pi/2$ at the point $\theta = \arg\{z_1\}$ in this section and so, we have

$$\Re \frac{z_1 q'(z_1)}{q(z_1)} = \left[\Re \frac{d \log q(z)}{dz} \right]_{z=z_1} = \left[\frac{d \arg q(z)}{d\theta} \right]_{\theta=\arg z_1} = 0.$$

Therefore, we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = i \frac{\alpha k}{2} \left(a + \frac{1}{a} \right), \quad \frac{\alpha k}{2} \left(a + \frac{1}{a} \right) \geq \alpha k$$

where $p^{1/\alpha}(z_1) = ia$, $a > 0$. This completes the proof of (1.13).

For the next case $\arg p(z_2) = -\pi\beta/2$ let us put

$$\rho(z) = [p(z)]^{1/\beta}, \quad \psi(z) = \frac{1 - \rho(z)}{1 + \rho(z)},$$

then it follows that

$$\rho(0) = 1, \rho'(0) = \rho''(0) = \dots = \rho^{(k-1)}(0) = 0 \quad \text{and} \quad \rho^{(k)}(0) \neq 0$$

and so

$$\psi(0) = \psi'(0) = \psi''(0) = \dots = \psi^{(k-1)}(0) = 0 \quad \text{and} \quad \psi^{(k)}(0) \neq 0.$$

From the hypothesis, we have

$$\arg \rho(z) > -\frac{\pi}{2}, \quad z \in D_\varepsilon(z_2) = N_\varepsilon(z_2) \setminus \{z_2\}$$

for some $\varepsilon > 0$ and

$$\arg \rho(z_2) = -\frac{\pi}{2}.$$

Therefore, we have

$$|\psi(z)| < 1, \quad z \in D_\varepsilon(z_2)$$

and

$$|\psi(z_2)| = 1.$$

This shows that $|\psi(z)|$ takes its local maximum value at the point $z = z_2$ in the domain $N_\varepsilon(z_2)$ and applying Lemma 1.1, we have

$$\begin{aligned} \frac{z_2 \psi'(z_2)}{\psi(z_2)} &= -\frac{z_2 \rho'(z_2)}{1 - \rho(z_2)} - \frac{z_2 \rho'(z_2)}{1 + \rho(z_2)} \\ &= -\frac{2z_2 \rho'(z_2)}{(1 - \rho(z_2))(1 + \rho(z_2))} \\ &= -\frac{2z_2 \rho'(z_2)}{(1 - ib)(1 + ib)} \\ &= -\frac{2z_2 \rho'(z_2)}{1 + b^2} \\ &\geq k, \end{aligned}$$

where $k \geq 1$ and $\rho(z_2) = -ib$ with $b > 0$. This shows that $z_2 \rho'(z_2)$ is a negative real number. This gives

$$\Im \frac{z_2 \rho'(z_2)}{\rho(z_2)} = \Im \frac{z_2 \rho'(z_2)}{-ib} \leq \frac{k}{2} \left(b + \frac{1}{b} \right) \geq k.$$

On the other hand, we have

$$\frac{z_2 \rho'(z_2)}{\rho(z_2)} = \frac{1}{\beta} \frac{z_2 p'(z_2)}{p(z_2)}$$

and putting $z = |z_2|e^{i\theta}$ in the section $\arg z_2 - \varepsilon < \theta < \arg z_2 + \varepsilon$, then $\arg\{\rho(z_2)\}$ takes its local minimum $-\pi/2$ at the point $\theta = \arg\{z_2\}$ in this section and so, we have

$$\Re \frac{z_2 \rho'(z_2)}{\rho(z_2)} = \left[\Re \frac{d \log \rho(z)}{dz} \right]_{z=z_2} = \left[\frac{d \arg \rho(z)}{d\theta} \right]_{\theta=\arg z_2} = 0.$$

Therefore, we have

$$\frac{z_2 p'(z_2)}{p(z_2)} = -i \frac{\beta k}{2} \left(b + \frac{1}{b} \right)$$

is a pure imaginary number, where $p^{1/\beta}(z_2) = -ib$, $b > 0$ so

$$-\frac{\beta k}{2} \left(b + \frac{1}{b} \right) \leq -\beta k.$$

This completes the proof of (1.14). $\square$

Theorem 1.2 is a generalization of a result in [4,5].

Corollary 1.3 *Under the assumptions of Theorem 1.2 we have*

$$\int_{z_2}^{z_1} \left\{ \frac{zp'(z)}{p(z)} \right\}' dz \geq (\alpha + \beta)k.$$

Corollary 1.3 implies that the length of image curve under $zp'(z)/p(z)$ of the circular arc $|z| = |z_1| = |z_2|$ from z_2 to z_1 is great or equal to $(\alpha + \beta)k$.

Corollary 1.4 *Under the assumptions of Theorem 1.2 and if $\arg z_2 < \arg z_1$ and*

$$\left| \Re \left[\left\{ \frac{p'(z)}{p(z)} + \frac{zp''(z)}{p'(z)} - z \left(\frac{p'(z)}{p(z)} \right)^2 \right\} iz \right] \right| < \frac{1}{2\pi}, \quad z \in \mathbb{D}, \quad (1.15)$$

then we have

$$\Re\{p(z)\} > 0, \quad z \in \mathbb{D}.$$

Proof From Lemma 1.1 we have

$$0 < \alpha + \beta \leq (\alpha + \beta)k \geq \alpha + \beta \leq \left| \int_{z_2}^{z_1} \left\{ \frac{zp'(z)}{p(z)} \right\}' dz \right|.$$

On the other hand and applying (1.15), we have

$$\begin{aligned} 0 < \alpha + \beta &\leq \left| \int_{z_2}^{z_1} \left\{ \frac{zp'(z)}{p(z)} \right\}' dz \right| = \left| \Re \int_{\theta_2}^{\theta_1} \left\{ \frac{p'(z)}{p(z)} + \frac{zp''(z)}{p'(z)} - z \left(\frac{p'(z)}{p(z)} \right)^2 \right\} iz d\theta \right| \\ &\leq \int_{\theta_2}^{\theta_1} \left| \Re \left[\left\{ \frac{p'(z)}{p(z)} + \frac{zp''(z)}{p'(z)} - z \left(\frac{p'(z)}{p(z)} \right)^2 \right\} iz \right] \right| d\theta \\ &\leq \frac{\theta_1 - \theta_2}{2\pi} < 1, \end{aligned}$$

where $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$ and $\theta_2 < \theta_1$. This shows that

$$0 < \alpha + \beta < 1$$

which implies $0 < \alpha < 1$ and $0 < \beta < 1$ or

$$\Re\{p(z)\} > 0, \quad z \in \mathbb{D}.$$

□

In [8, p.54] we can find the following result for the convolution of power series

$$\sum_{n=0}^{\infty} a_n z^n * \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Lemma 1.5 *If $g(z) \in \mathcal{C}$, $f(z) \in \mathcal{S}^*$ and $F(z)$ is analytic in $\mathbb{D}$, then for all $z \in \mathbb{D}$*

$$\frac{g(z) * F(z)f(z)}{g(z) * f(z)} \in \overline{co} F(\mathbb{D}),$$

where $\overline{co}A$ denotes the closed convex hull of A .

Theorem 1.6 *If $f(z) = a_0 + a_1z + a_2z^2 + \dots$, $h(z) = b_0 + b_1z + b_2z^2 + \dots \in \mathcal{H}$, $zh''(z)/(2b_2) \in \mathcal{S}^*$ and*

$$q(z) = \frac{f'(z) - a_1}{h'(z) - b_1}, \quad p(z) = \frac{f''(z)}{h''(z)}, \quad z \in \mathbb{D},$$

are analytic functions in $\mathbb{D}$, then

$$q(\mathbb{D}) \subseteq \overline{co} p(\mathbb{D}). \quad (1.16)$$

Proof Denote

$$s(z) = \sum_{n=2}^{\infty} \frac{z^{n-1}}{n-1} = -\log(1-z), \quad z \in \mathbb{D},$$

then $s(z) \in \mathcal{C}$.

Applying Lemma 1.5, we have

$$\begin{aligned} \frac{f'(z) - a_1}{h'(z) - b_1} &= \frac{\sum_{n=2}^{\infty} na_n z^{n-1}}{\sum_{n=2}^{\infty} nb_n z^{n-1}} \\ &= \frac{\sum_{n=2}^{\infty} na_n z^{n-2}}{\sum_{n=2}^{\infty} nb_n z^{n-2}} \\ &= \frac{\sum_{n=2}^{\infty} \frac{z^{n-2}}{n-1} * \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}}{\sum_{n=2}^{\infty} \frac{z^{n-2}}{n-1} * \sum_{n=2}^{\infty} n(n-1)b_n z^{n-2}} \\ &= \frac{\frac{s(z)}{z} * f''(z)}{\frac{s(z)}{z} * h''(z)} \\ &= \frac{\frac{s(z)}{z} * p(z)h''(z)}{\frac{s(z)}{z} * h''(z)} \\ &= \frac{s(z) * p(z) \frac{zh''(z)}{2b_2}}{s(z) * \frac{zh''(z)}{2b_2}} \\ &\in \overline{co} p(\mathbb{D}) \end{aligned}$$

because $s(z) \in \mathcal{C}$ and $zh''(z)/(2b_2) \in \mathcal{S}^*$. □

The differential subordination theory provides another method of proving this type of results. For two functions $f(z)$ and $g(z)$ analytic in $\mathbb{D}$, we say that $ff(z)$ is subor-

ordinate to $g(z)$, written by $f(z) \prec g(z)$, if there exists a function $w(z)$, analytic in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

In [3, p.70] we can find the following result

Lemma 1.7 *Let $H(z)$ be convex univalent, $P(z) \in \mathcal{H}$ and $\Re\{P(z)\} > 0$ in $\mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$ with $p(0) = H(0)$, then*

$$p(z) + P(z)zp'(z) \prec H(z) \Rightarrow p(z) \prec H(z). \quad (1.17)$$

Theorem 1.8 *Let $H(z)$ be convex univalent in $\mathbb{D}$. Assume that*

$$\frac{f^{(s)}(z)}{g^{(s)}(z)}, \quad \frac{f^{(s-1)}(z)}{g^{(s-1)}(z)}, \quad \frac{g^{(s-1)}(z)}{zg^{(s)}(z)}$$

are analytic in $\mathbb{D}$ for some positive integer s and

$$\Re \left\{ \frac{g^{(s-1)}(z)}{zg^{(s)}(z)} \right\} > 0, \quad z \in \mathbb{D}. \quad (1.18)$$

If

$$\frac{f^{(s)}(z)}{g^{(s)}(z)} \prec H(z),$$

and with $f^{(s-1)}(0)/g^{(s-1)}(0) = H(0)$ then

$$\frac{f^{(s-1)}(z)}{g^{(s-1)}(z)} \prec H(z). \quad (1.19)$$

Proof Denote

$$p(z) = \frac{f^{(s-1)}(z)}{h^{(s-1)}(z)}.$$

Then we have $p(z)h^{(s-1)}(z) = f^{(s-1)}(z)$, by differentiation, we obtain

$$p(z) + \frac{h^{(s-1)}(z)}{zh^{(s)}(z)}zp'(z) = \frac{f^{(s)}(z)}{h^{(s)}(z)} \prec H(z).$$

Applying (1.17) and (1.18) give $p(z) \prec H(z)$, i. e. (1.19).

Corollary 1.9 *Assume that $\alpha \in (0, 1]$ and*

$$\frac{f''(z)}{h''(z)}, \quad \frac{f'(z)}{h'(z)}, \quad \frac{h(z)}{zh'(z)}, \quad \frac{h'(z)}{zh''(z)}$$

are analytic in $\mathbb{D}$ Then

$$\frac{f'(z)}{h'(z)} \prec \left\{ \frac{1+z}{1-z} \right\}^\alpha \quad \text{and} \quad \Re \left\{ \frac{h(z)}{zh'(z)} \right\} > 0, \quad z \in \mathbb{D}, \quad \Rightarrow \quad \frac{f(z)}{h(z)} \prec \left\{ \frac{1+z}{1-z} \right\}^\alpha \quad (1.20)$$

and

$$\frac{f''(z)}{h''(z)} \prec \left\{ \frac{1+z}{1-z} \right\}^\alpha \text{ and } \Re \left\{ \frac{h'(z)}{zh''(z)} \right\} > 0, \quad z \in \mathbb{D}, \Rightarrow \frac{f'(z)}{h'(z)} \prec \left\{ \frac{1+z}{1-z} \right\}^\alpha. \quad (1.21)$$

Consider, $h(z) = e^z - 1$, which satisfies

$$\Re \frac{zh''(z)}{h'(z)} = \Re \{z\} > -1, \quad z \in \mathbb{D}.$$

Therefore, $h(z) \in \mathcal{C}$ and so $h(z) \in \mathcal{S}^*$ which gives

$$\Re \left\{ \frac{h(z)}{zh'(z)} \right\} > 0, \quad z \in \mathbb{D}.$$

Applying (1.20) we obtain

$$\frac{f'(z)}{e^z} \prec \left\{ \frac{1+z}{1-z} \right\}^\alpha \Rightarrow \frac{f(z)}{e^z - 1} \prec \left\{ \frac{1+z}{1-z} \right\}^\alpha.$$

Corollary 1.10 Assume that $\alpha \in [0, 1)$ and

$$\frac{f''(z)}{h''(z)}, \quad \frac{f'(z)}{h'(z)}, \quad \frac{h(z)}{zh'(z)}, \quad \frac{h'(z)}{zh''(z)}$$

are analytic in $\mathbb{D}$. Then

$$\Re \left\{ \frac{h(z)}{zh'(z)} \right\} > 0 \text{ and } \Re \frac{f'(z)}{h'(z)} > \alpha, \quad z \in \mathbb{D} \Rightarrow \Re \frac{f(z)}{h(z)} > \alpha, \quad z \in \mathbb{D} \quad (1.22)$$

and

$$\Re \left\{ \frac{h'(z)}{zh''(z)} \right\} > 0 \text{ and } \Re \frac{f''(z)}{h''(z)} > \alpha, \quad z \in \mathbb{D} \Rightarrow \Re \frac{f'(z)}{h'(z)} > \alpha, \quad z \in \mathbb{D}. \quad (1.23)$$

Property (1.22) was known earlier, see Theorem 2.6f in [3, p.63].

Theorem 1.11 Assume that $h(z) \in \mathcal{H}$ and

$$\Re \frac{h'(z) - \gamma - z}{z(h''(z) - 1)} > 0, \quad z \in \mathbb{D} \quad (1.24)$$

for some $\gamma \in (1, \infty]$ and

$$|\arg\{h'(z) - \gamma - z\}| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}, \quad (1.25)$$

for some $\beta \in (0, 1]$. Assume that $f(z) \in \mathcal{H}$ and

$$\left| \arg \frac{f''(z) - 1}{h''(z) - 1} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \quad (1.26)$$

for some $\alpha \in (0, 1]$. Furthermore, assume that

$$F(z) = \frac{f'(z) - \gamma - z}{h'(z) - \gamma - z}$$

is analytic in $\mathbb{D}$ with $F(0) = 1$. Then

$$|\arg\{f'(z)\}| < \frac{(\alpha + \beta)\pi}{2}, \quad z \in \mathbb{D}. \quad (1.27)$$

Proof Consider Theorem 1.8 with $s = 2$, $H(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ and $g'(z) = h'(z) - \gamma - z$. Then $H(z)$ is convex univalent and condition (1.18) becomes (1.24). Applying this for (1.26) gives us

$$\left| \arg \left\{ \frac{f'(z) - \gamma - z}{h'(z) - \gamma - z} \right\} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}. \quad (1.28)$$

Now, applying (1.25), we have

$$\begin{aligned} & \left| \arg\{f'(z) - \gamma - z\} \right| - \left| \arg\{h'(z) - \gamma - z\} \right| \\ & \leq \left| \arg\{f'(z) - \gamma - z\} - \arg\{h'(z) - \gamma - z\} \right| \\ & = \left| \arg \left\{ \frac{f'(z) - \gamma - z}{h'(z) - \gamma - z} \right\} \right| \\ & < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \end{aligned}$$

because of (1.28). So we have

$$\begin{aligned} \left| \arg\{f'(z) - \gamma - z\} \right| & \leq \frac{\alpha\pi}{2} + \left| \arg\{h'(z) - \gamma - z\} \right| \\ & < \frac{(\alpha + \beta)\pi}{2}, \quad z \in \mathbb{D}. \end{aligned}$$

A simple geometric observation shows that

$$\left| \arg\{f'(z)\} \right| < \left| \arg\{f'(z) - \gamma - z\} \right|, \quad z \in \mathbb{D}, \quad \gamma \in (1, \infty].$$

Therefore,

$$\left| \arg\{f'(z)\} \right| < \left| \arg\{f'(z) - \gamma - z\} \right| < \frac{(\alpha + \beta)\pi}{2}, \quad z \in \mathbb{D}.$$

□

Condition (1.27) implies that $f(z)$ is strongly close-to-convex of order $\alpha + \beta$ with respect convex function $g(z) = z$. We obtain more when $\alpha + \beta \leq 1$ in the following corollary.

Corollary 1.12 *Assume that $h(z) \in \mathcal{H}$ and*

$$\Re \frac{h'(z) - \gamma - z}{z(h''(z) - 1)} > 0, \quad z \in \mathbb{D} \quad (1.29)$$

for some $\gamma \in (1, \infty]$ and

$$|\arg\{h'(z) - \gamma - z\}| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}, \quad (1.30)$$

for some $\beta \in (0, 1]$. Assume that $f(z) \in \mathcal{H}$ and

$$\left| \arg \frac{f''(z) - 1}{h''(z) - 1} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \quad (1.31)$$

for some $\alpha \in (0, 1 - \beta]$. Furthermore, assume that

$$F(z) = \frac{f'(z) - \gamma - z}{h'(z) - \gamma - z}$$

is analytic in $\mathbb{D}$ with $F(0) = 1$. Then

$$\Re\{f'(z)\} > 0, \quad z \in \mathbb{D}. \quad (1.32)$$

Proof We have $\alpha + \beta \leq 1$, then by Theorem 1.11, we have

$$|\arg\{f'(z)\}| < \frac{(\alpha + \beta)\pi}{2} \leq \frac{\pi}{2}, \quad z \in \mathbb{D}.$$

This gives (1.32). □

Condition (1.32) implies the univalence $f(z)$ in $\mathbb{D}$ by Noshiro–Warszawski’s theorem.

Data Availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Compliance with ethical standards

Conflict of interest The author declares no conflict of interest.

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References

1. Fukui, S., Sakaguchi, K.: An extension of theorem of Ruscheweyh. *Bull. Fac. Educ. Wakayama Univ. Nat. Sci.* **29**, 1–3 (1980)
2. Jack, I.S.: Functions starlike and convex of order α . *J. Lond. Math. Soc.* **3**, 469–474 (1971)
3. Miller, S.S., Mocanu, P.T.: *Differential Subordinations, Theory and Applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 225. Marcel Dekker Inc., New York (2000)
4. Nunokawa, M.: On properties of non-Carathéodory functions. *Proc. Jpn. Acad. Ser. A* **68**(6), 152–153 (1992)
5. Nunokawa, M.: On the order of strongly starlikeness of strongly convex functions. *Proc. Jpn. Acad. Ser. A* **69**(7), 234–237 (1992)
6. Robinson, R.M.: Univalent majorants. *Trans. Am. Math. Soc.* **61**, 1–35 (1947)
7. Pommerenke, Ch.: On close-to-convex analytic functions. *Trans. Am. Math. Soc.* **114**(1), 176–186 (1964)
8. Ruscheweyh, St.: *Convolution in geometric function theory*, Les Presses de l'Université de Montréal (1982)

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