A series expansion for generalized harmonic functions

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Abstract

We consider a class of generalized harmonic functions in the open unit disc in the complex plane. Our main results concern a canonical series expansion for such functions. Of particular interest is a certain individual generalized harmonic function which suitably normalized plays the role of an associated Poisson kernel.

Keywords Harmonic function · Power series · Poisson kernel · Hypergeometric function

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0 Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and denote by $\partial_z = \partial/\partial z$ and $\bar{\partial}_z = \partial/\partial \bar{z}$ the usual complex partial derivatives. This work is concerned with second order partial differential operators of the form

$$L_{p,q} = (1 - |z|^2)\partial_z\bar{\partial}_z + pz\partial_z + q\bar{z}\bar{\partial}_z - pq, \quad z \in \mathbb{D}, \quad (0.1)$$

where $p, q \in \mathbb{C}$ are complex parameters. Of particular interest are solutions of the associated homogeneous equation

$$L_{p,q}u = 0 \quad \text{in } \mathbb{D}.$$

We say that a function u is (p, q) -harmonic if u is twice continuously differentiable in $\mathbb{D}$ (in symbols $u \in C^2(\mathbb{D})$) and $L_{p,q}u = 0$ in $\mathbb{D}$, where $L_{p,q}$ is as in (0.1). Notice

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that a function u is (p, q) -harmonic if and only if its complex conjugate $\bar{u}$ is $(\bar{q}, \bar{p})$ -harmonic. Observe also that a $(0, 0)$ -harmonic function is a harmonic function in $\mathbb{D}$ in the usual sense.

An interesting example of a (p, q) -harmonic function is the function

$$u_{p,q}(z) = \frac{(1 - |z|^2)^{p+q+1}}{(1 - z)^{p+1}(1 - \bar{z})^{q+1}}, \quad z \in \mathbb{D} \quad (0.2)$$

(see Theorem 1.4). Here powers are defined in the usual way using the principal branch of the logarithm, that is, we require that $\log(1) = 0$. Notice that the above functions $u_{p,q}$ have the hermitian symmetry property that $\bar{u}_{p,q} = u_{\bar{q},\bar{p}}$ for $p, q \in \mathbb{C}$.

Recall that elements of the unit circle $\mathbb{T} = \partial\mathbb{D}$ act on the unit disc $\mathbb{D}$ as rotations. On a function level we consider rotation operators

$$R_{e^{i\theta}}u(z) = u(e^{i\theta}z), \quad z \in \mathbb{D}, \quad (0.3)$$

for $e^{i\theta} \in \mathbb{T}$ acting on functions u in $\mathbb{D}$. A basic observation concerning the differential operator $L_{p,q}$ is the commutativity relation

$$L_{p,q}R_{e^{i\theta}}u = R_{e^{i\theta}}L_{p,q}u, \quad u \in C^2(\mathbb{D}),$$

for $e^{i\theta} \in \mathbb{T}$. This latter commutativity relation suggests an analysis of (p, q) -harmonic functions using concepts natural to classical Fourier analysis on the unit circle.

Let $\mathbb{Z}$ be the set of integers. For a suitably smooth function u in $\mathbb{D}$ we define its m -th homogeneous part by the formula

$$u_m(z) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\theta} u(e^{i\theta}z) d\theta, \quad z \in \mathbb{D}, \quad (0.4)$$

for $m \in \mathbb{Z}$. Notice that the m -th homogeneous part u_m of u is the m -th Fourier coefficient of the vector-valued function

$$\mathbb{T} \ni e^{i\theta} \mapsto R_{e^{i\theta}}u,$$

where $R_{e^{i\theta}}$ is as in (0.3).

We set $C^\infty(\mathbb{D}) = \bigcap_{n=0}^\infty C^n(\mathbb{D})$, where $C^n(\mathbb{D})$ is the space of n -times continuously differentiable functions in $\mathbb{D}$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. We topologize these spaces in the usual way using the semi-norms

$$\|u\|_{j,k;K} = \max_{z \in K} |\partial^j \bar{\partial}^k u(z)|,$$

where $j, k \in \mathbb{N}$ are non-negative integers and $K \subset \mathbb{D}$ is a compact subset of $\mathbb{D}$.

Let us recall the classical hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad z \in \mathbb{D}, \quad (0.5)$$

for $a, b, c \in \mathbb{C}$ such that $c \neq -1, -2, \dots$. Here $(a)_0 = 1$ and

$$(a)_n = a(a+1) \dots (a+n-1)$$

for $n = 1, 2, \dots$ are Pochhammer symbols.

Let us return to a (p, q) -harmonic function u . We show that the m -th homogeneous part of u has the form

$$u_m(z) = c_m F(-p, m-q; m+1; |z|^2) z^m, \quad z \in \mathbb{D},$$

for some $c_m \in \mathbb{C}$ when $m \in \mathbb{N}$ is a non-negative integer (see Theorem 4.3). Here F is the hypergeometric function (0.5). Hermitian symmetry leads to a similar formula for u_m when $m \in \mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{N}$ is a negative integer (see Corollary 4.4). Notice that $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$.

A principal result of the present paper concerns the asymptotic behavior of the m -th homogeneous part u_m of a (p, q) -harmonic function u . We show that

$$\limsup_{|m| \rightarrow \infty} \|u_m\|_{j,k;K}^{1/|m|} < 1$$

for all $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact (see Theorem 4.6). This result enables us to use the classical root test to establish absolute convergence of the function series $\sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$ (see Corollary 4.7).

A further analysis leads to a function series characterization of (p, q) -harmonic functions. A function u in $\mathbb{D}$ is (p, q) -harmonic if and only if it has the form

$$\begin{aligned} u(z) &= \sum_{m=0}^{\infty} c_m F(-p, m-q; m+1; |z|^2) z^m \\ &\quad + \sum_{m=1}^{\infty} c_{-m} F(-q, m-p; m+1; |z|^2) \bar{z}^m, \quad z \in \mathbb{D}, \end{aligned} \quad (0.6)$$

for some sequence $\{c_m\}_{m=-\infty}^{\infty}$ of complex numbers such that

$$\limsup_{|m| \rightarrow \infty} |c_m|^{1/|m|} \leq 1$$

(see Theorem 5.1). As mentioned above, the sums in (0.6) are absolutely convergent in the space $C^\infty(\mathbb{D})$. As a consequence we have that $u \in C^\infty(\mathbb{D})$ if u is (p, q) -harmonic (see Corollary 4.9). This characterization of (p, q) -harmonic functions improves on a result by Ahern et al. [1, Theorem 2.1] when specialized to our setting.

We then turn to coefficient formulas for (p, q) -harmonic functions. We show that

$$c_m = \partial^m u(0)/m! \quad \text{and} \quad c_{-m} = \bar{\partial}^m u(0)/m!$$

for $m \in \mathbb{N}$, where the c_m 's are as in (0.6) (see Theorem 5.3). As a consequence, we have that

$$\begin{aligned} u(z) &= \sum_{m=0}^{\infty} \frac{\partial^m u(0)}{m!} F(-p, m-q; m+1; |z|^2) z^m \\ &\quad + \sum_{m=1}^{\infty} \frac{\bar{\partial}^m u(0)}{m!} F(-q, m-p; m+1; |z|^2) \bar{z}^m, \quad z \in \mathbb{D}, \end{aligned}$$

whenever u is a (p, q) -harmonic function (see Theorem 5.4) as well as a corresponding uniqueness result for such functions (see Corollary 5.5).

Let us return to the function $u_{p,q}$ in (0.2). A calculation of partial derivatives at the origin of the function $u_{p,q}$ leads to the series expansion

$$\begin{aligned} u_{p,q}(z) &= \sum_{m=0}^{\infty} \frac{(p+1)_m}{m!} F(-p, m-q; m+1; |z|^2) z^m \\ &\quad + \sum_{m=1}^{\infty} \frac{(q+1)_m}{m!} F(-q, m-p; m+1; |z|^2) \bar{z}^m, \quad z \in \mathbb{D} \end{aligned}$$

(see Theorem 6.3). This latter series expansion generalizes a well-known partial fraction decomposition formula for the classical Poisson kernel for $\mathbb{D}$ which is obtained for $p = q = 0$.

A main contribution of this paper concerns series expansion of (p, q) -harmonic functions. Of particular mention is a limit theorem for associated hypergeometric functions:

$$\lim_{m \rightarrow \infty} F(-p, m-q; m+1; z) = (1-z)^p, \quad z \in \mathbb{D}$$

(see Theorem 2.6). Apart from its intrinsic interest, this limit theorem provides an efficient tool for the study of limit properties of homogeneous parts of (p, q) -harmonic functions.

The results of this paper have applications to Poisson integral representations of (p, q) -harmonic functions which is possible when $p, q \in \mathbb{C} \setminus \mathbb{Z}^-$ are such that $\operatorname{Re}(p) + \operatorname{Re}(q) > -1$. In the final section of the paper we comment briefly on the connection to such theory.

We have traced the study of (p, q) -harmonic functions back to Daryl Geller [13]. Other significant contributions are those of Ahern and collaborators [1, 2]. Our interest in those topics [18–21] arose in connection to standard weights and earlier work of Garabedian [12]. Borichev and Hedenmalm [6] have established a connection to polyharmonic theory. Other recent related papers are [5, 8–10, 14, 16, 17, 22].

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1 The function $u_{p,q}$ is (p, q) -harmonic

Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Observe that the functions $u_{p,q}$ have the hermitian symmetry property $\bar{u}_{p,q} = u_{\bar{q},\bar{p}}$. Indeed,

$$\bar{u}_{p,q}(z) = \overline{u_{p,q}(z)} = \frac{(1 - |z|^2)^{\bar{p} + \bar{q} + 1}}{(1 - \bar{z})^{\bar{p} + 1} (1 - z)^{\bar{q} + 1}} = u_{\bar{q},\bar{p}}(z)$$

for $z \in \mathbb{D}$. We shall discuss in this section some differentiation formulas for the functions $u_{p,q}$. In particular, we shall show that the function $u_{p,q}$ is (p, q) -harmonic (see Theorem 1.4).

Lemma 1.1 *Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then*

$$z \partial u_{p,q}(z) = \left(-(p + q + 1) \frac{|z|^2}{1 - |z|^2} + (p + 1) \frac{z}{1 - z} \right) u_{p,q}(z)$$

for $z \in \mathbb{D}$.

Proof Differentiating using the product rule for differentiation we have that

$$\begin{aligned} \partial u_{p,q}(z) &= \frac{1}{(1 - \bar{z})^{q+1}} \left((p + q + 1)(1 - |z|^2)^{p+q} (-\bar{z}) \frac{1}{(1 - z)^{p+1}} \right. \\ &\quad \left. + (1 - |z|^2)^{p+q+1} (-(p + 1)) \frac{1}{(1 - z)^{p+2}} (-1) \right) \\ &= \left(-(p + q + 1) \frac{\bar{z}}{1 - |z|^2} + (p + 1) \frac{1}{1 - z} \right) u_{p,q}(z) \end{aligned}$$

for $z \in \mathbb{D}$, where the last equality is straightforward to check. This yields the conclusion of the lemma. $\square$

We next turn to the $\bar{\partial}$ -derivative of $u_{p,q}$.

Lemma 1.2 *Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then*

$$\bar{z} \bar{\partial} u_{p,q}(z) = \left(-(p + q + 1) \frac{|z|^2}{1 - |z|^2} + (q + 1) \frac{\bar{z}}{1 - \bar{z}} \right) u_{p,q}(z)$$

for $z \in \mathbb{D}$.

Proof We shall use the hermitian symmetry property $\bar{u}_{p,q} = u_{\bar{q},\bar{p}}$. From Lemma 1.1 we have that

$$z \partial u_{\bar{q},\bar{p}}(z) = \left(-(\bar{p} + \bar{q} + 1) \frac{|z|^2}{1 - |z|^2} + (\bar{q} + 1) \frac{z}{1 - z} \right) u_{\bar{q},\bar{p}}(z)$$

for $z \in \mathbb{D}$. A complex conjugation now yields the conclusion of the lemma. $\square$

Following earlier practice from [20] we denote by $A = z\partial - \bar{z}\bar{\partial}$ the angular derivative. We next calculate the angular derivative of $u_{p,q}$.

Corollary 1.3 *Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then*

$$(z\partial - \bar{z}\bar{\partial})u_{p,q}(z) = \left((p+1)\frac{z}{1-z} - (q+1)\frac{\bar{z}}{1-\bar{z}} \right) u_{p,q}(z)$$

for $z \in \mathbb{D}$.

Proof The result is evident from Lemmas 1.1 and 1.2. $\square$

We mention that Corollary 1.3 generalizes [20, Theorem 1.11].

Recall the partial fraction formula

$$\frac{1 - |z|^2}{|1 - z|^2} = \frac{z}{1 - z} + \frac{\bar{z}}{1 - \bar{z}} + 1 \quad (1.1)$$

for the classical Poisson kernel for the unit disc. Formula (1.1) is straightforward to check.

Theorem 1.4 *Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then $L_{p,q}u_{p,q} = 0$ in $\mathbb{D}$, where $L_{p,q}$ is as in (0.1).*

Proof Recall Lemma 1.2. Notice that the differential operator $z\partial$ satisfies the product rule for differentiation. It is straightforward to check that

$$z \frac{\partial}{\partial z} \left(\frac{|z|^2}{1 - |z|^2} \right) = \frac{|z|^2}{(1 - |z|^2)^2}$$

whenever the formula makes sense. Differentiating using the product rule we have that

$$\begin{aligned} |z|^2 \partial \bar{\partial} u_{p,q}(z) &= -(p+q+1) \frac{|z|^2}{(1 - |z|^2)^2} u_{p,q}(z) \\ &\quad + \left(-(p+q+1) \frac{|z|^2}{1 - |z|^2} + (q+1) \frac{\bar{z}}{1 - \bar{z}} \right) z \partial u_{p,q}(z) \end{aligned}$$

for $z \in \mathbb{D}$. From Lemma 1.1 we now have that

$$\begin{aligned} |z|^2 \partial \bar{\partial} u_{p,q}(z) &= -(p+q+1) \frac{|z|^2}{(1 - |z|^2)^2} u_{p,q}(z) \\ &\quad + \left(-(p+q+1) \frac{|z|^2}{1 - |z|^2} + (q+1) \frac{\bar{z}}{1 - \bar{z}} \right) \\ &\quad \times \left(-(p+q+1) \frac{|z|^2}{1 - |z|^2} + (p+1) \frac{z}{1 - z} \right) u_{p,q}(z) \end{aligned}$$

for $z \in \mathbb{D}$. Expanding the above product we see that

$$\begin{aligned} |z|^2 \partial \bar{\partial} u_{p,q}(z) &= \left(-(p+q+1) \frac{|z|^2}{(1-|z|^2)^2} + (p+q+1)^2 \frac{|z|^4}{(1-|z|^2)^2} \right) u_{p,q}(z) \\ &\quad - (p+q+1) \frac{|z|^2}{1-|z|^2} \left((p+1) \frac{z}{1-z} + (q+1) \frac{\bar{z}}{1-\bar{z}} \right) u_{p,q}(z) \\ &\quad + (p+1)(q+1) \frac{|z|^2}{|1-z|^2} u_{p,q}(z) \end{aligned}$$

for $z \in \mathbb{D}$. Multiplying by a factor $(1-|z|^2)/|z|^2$ we see that

$$\begin{aligned} (1-|z|^2) \partial \bar{\partial} u_{p,q}(z) &= \left((p+q+1)^2 \frac{|z|^2}{1-|z|^2} - (p+q+1) \frac{1}{1-|z|^2} \right) u_{p,q}(z) \\ &\quad - (p+q+1) \left((p+1) \frac{z}{1-z} + (q+1) \frac{\bar{z}}{1-\bar{z}} \right) u_{p,q}(z) \\ &\quad + (p+1)(q+1) \frac{1-|z|^2}{|1-z|^2} u_{p,q}(z) \end{aligned}$$

for $z \in \mathbb{D}$. Notice an appearance of the classical Poisson kernel in the rightmost term above. Using the partial fraction formula (1.1) we have that

$$\begin{aligned} (1-|z|^2) \partial \bar{\partial} u_{p,q}(z) &= (p+q+1) \left((p+q) \frac{|z|^2}{1-|z|^2} - 1 \right) u_{p,q}(z) \\ &\quad + ((p+1)(q+1) - (p+q+1)(p+1)) \frac{z}{1-z} u_{p,q}(z) \\ &\quad + ((p+1)(q+1) - (p+q+1)(q+1)) \frac{\bar{z}}{1-\bar{z}} u_{p,q}(z) \\ &\quad + (p+1)(q+1) u_{p,q}(z) \end{aligned}$$

for $z \in \mathbb{D}$. A simplification of terms now leads to the formula

$$\begin{aligned} (1-|z|^2) \partial \bar{\partial} u_{p,q}(z) &= (p+q+1)(p+q) \frac{|z|^2}{1-|z|^2} u_{p,q}(z) \\ &\quad - p(p+1) \frac{z}{1-z} u_{p,q}(z) - q(q+1) \frac{\bar{z}}{1-\bar{z}} u_{p,q}(z) + pq u_{p,q}(z) \quad (1.2) \end{aligned}$$

for $z \in \mathbb{D}$. Recall Lemmas 1.1 and 1.2. In view of these two lemmas our latter formula (1.2) says that $L_{p,q} u_{p,q} = 0$ in $\mathbb{D}$. $\square$

2 A sequence of hypergeometric functions

Let us first consider a second order partial differential operator of the form

$$L_{p,q;r} = (1 - |z|^2)\partial_z\bar{\partial}_z + pz\partial_z + q\bar{z}\bar{\partial}_z - r, \quad z \in \mathbb{D}, \quad (2.1)$$

where $p, q, r \in \mathbb{C}$ are complex parameters. A principal case is when $r = pq$. Notice that $L_{p,q;pq} = L_{p,q}$, where $L_{p,q}$ is as in (0.1). The introduction of an additional parameter $r \in \mathbb{C}$ allows for more general operators appearing in the study of conductivity problems, see for instance Calderón [7] or Astala and Päiväranta [4].

We shall evaluate the operator $L_{p,q;r}$ on a complex-valued function u in the punctured disc $\mathbb{D} \setminus \{0\}$ of the form

$$u(z) = f(|z|^2)z^m, \quad z \in \mathbb{D} \setminus \{0\}, \quad (2.2)$$

for some $f \in C^2(0, 1)$ and $m \in \mathbb{Z}$.

We introduce also the ordinary differential operator

$$H_{a,b;c} = (1-x)x\frac{d^2}{dx^2} + (c - [a+b+1]x)\frac{d}{dx} - ab, \quad (2.3)$$

where $a, b, c \in \mathbb{C}$ are complex parameters. Notice that the famous hypergeometric ordinary differential equation

$$(1-x)xy''(x) + (c - [a+b+1]x)y'(x) - aby(x) = 0$$

takes the form $H_{a,b;c}y = 0$ using the operator $H_{a,b;c}$.

Theorem 2.1 *Let $L_{p,q;r}$ be as in (2.1) for some $p, q, r \in \mathbb{C}$. Let u be a function of the form (2.2) for some $f \in C^2(0, 1)$ and $m \in \mathbb{Z}$. Then*

$$L_{p,q;r}u(z) = z^m H_{a,b;c}f(|z|^2), \quad z \in \mathbb{D} \setminus \{0\},$$

where $c = m + 1$ and

$$\begin{cases} a + b = m - p - q, \\ ab = r - pm. \end{cases} \quad (2.4)$$

Proof A differentiation shows that

$$\partial u(z) = mz^{m-1}f(|z|^2) + z^m\bar{z}f'(|z|^2)$$

for $z \in \mathbb{D} \setminus \{0\}$, and similarly that

$$\bar{\partial}u(z) = z^{m+1}f'(|z|^2)$$

for $z \in \mathbb{D} \setminus \{0\}$. Another differentiation gives that

$$\partial \bar{\partial} u(z) = (m+1)z^m f'(|z|^2) + z^{m+1} \bar{z} f''(|z|^2)$$

for $z \in \mathbb{D} \setminus \{0\}$. A calculation using these formulas gives that

$$\begin{aligned} L_{p,q;r} u(z) &= (1 - |z|^2)((m+1)z^m f'(|z|^2) + z^m |z|^2 f''(|z|^2)) + q \bar{z} z^{m+1} f'(|z|^2) \\ &\quad + pz(mz^{m-1} f(|z|^2) + z^m \bar{z} f'(|z|^2)) - rz^m f(|z|^2) \\ &= z^m \left(|z|^2(1 - |z|^2) f''(|z|^2) + (m+1 - [m+1-p-q]|z|^2) f'(|z|^2) \right. \\ &\quad \left. - (r - pm) f(|z|^2) \right) = z^m H_{a,b;c} f(|z|^2) \end{aligned}$$

for $z \in \mathbb{D} \setminus \{0\}$, where $c = m+1$, the numbers a and b are as in (2.4) and $H_{a,b;c}$ is as in (2.3). This yields the conclusion of the theorem. $\square$

Equations (2.4) say that a and b are the zeros of the quadratic polynomial

$$P_{p,q;r;m}(\lambda) = \lambda^2 - (m - p - q)\lambda + r - pm, \quad \lambda \in \mathbb{C}.$$

Notice that

$$P_{p,q;r;m}(\lambda) = (\lambda + p)(\lambda + q - m) + r - pq, \quad \lambda \in \mathbb{C}.$$

In particular, the zeros of $P_{p,q;r;m}$ are $-p$ and $m - q$ in the principal case when $r = pq$.

Theorem 2.1 suggests a natural construction of (p, q) -harmonic functions.

Proposition 2.2 *Let $p, q \in \mathbb{C}$. Consider the function*

$$u_m(z) = F(-p, m - q; m + 1; |z|^2) z^m, \quad z \in \mathbb{D},$$

where $m \in \mathbb{N}$ and F is the hypergeometric function (0.5). Then u_m is a (p, q) -harmonic function.

Proof Clearly $u_m \in C^\infty(\mathbb{D})$. It is well-known that the hypergeometric function $y = F(a, b; c; \cdot)$ satisfies the hypergeometric equation $H_{a,b;c} y = 0$ (see [3, Section 2.3]). The result now follows by Theorem 2.1. $\square$

Corollary 2.3 *Let $p, q \in \mathbb{C}$. Consider the function*

$$u_m(z) = F(-q, |m| - p; |m| + 1; |z|^2) \bar{z}^{|m|}, \quad z \in \mathbb{D},$$

where $m \in \mathbb{Z}^-$ and F is the hypergeometric function (0.5). Then u_m is a (p, q) -harmonic function.

Proof We consider the complex conjugate

$$\overline{u_m(z)} = F(-\bar{q}, |m| - \bar{p}; |m| + 1; |z|^2)z^{|m|}, \quad z \in \mathbb{D}.$$

By Proposition 2.2 we have that $\bar{u}_m$ is a $(\bar{q}, \bar{p})$ -harmonic function. From hermitian symmetry we conclude that u_m is a (p, q) -harmonic function. $\square$

Following earlier practice, a function u in $\mathbb{D} \setminus \{0\}$ is said to be homogeneous of order $m \in \mathbb{Z}$ with respect to rotations if it has the property that

$$u(e^{i\theta}z) = e^{im\theta}u(z), \quad z \in \mathbb{D} \setminus \{0\},$$

for $e^{i\theta} \in \mathbb{T}$. Notice that every function u of the form (2.2) is homogeneous of order m with respect to rotations.

Theorem 2.4 *Let $p, q \in \mathbb{C}$ and $m \in \mathbb{N}$. Let $u \in C^2(\mathbb{D})$ be homogeneous of order m with respect to rotations. Then u is (p, q) -harmonic if and only if it has the form*

$$u(z) = cF(-p, m - q; m + 1; |z|^2)z^m, \quad z \in \mathbb{D}, \quad (2.5)$$

for some $c \in \mathbb{C}$, where F is the hypergeometric function (0.5).

Proof From Proposition 2.2 we know that every function u of the form (2.5) is (p, q) -harmonic.

Assume next that u is (p, q) -harmonic. Since u is homogeneous of order m , we can put u on the form (2.2) for some $f \in C^2(0, 1)$. By Theorem 2.1 we have that

$$H_{-p, m-q, m+1}f(x) = 0, \quad 0 < x < 1,$$

where $H_{a,b;c}$ is as in (2.3). Below we shall check that

$$(m+1)f'(x) + p(m-q)f(x) = o(1/x^{m+1}) \quad (2.6)$$

as $x \rightarrow 0$. Condition (2.6) allows us to apply [18, Proposition 1.3] to conclude that f is a constant multiple of the hypergeometric function $F(-p, m - q; m + 1; \cdot)$. This will then complete the proof of the theorem.

We proceed to check (2.6). Recall formula (2.2). Since u is bounded near the origin we have that $f(x) = \mathcal{O}(1/x^{m/2})$ as $x \rightarrow 0$. A differentiation of (2.2) gives that

$$\bar{\partial}u(z) = z^{m+1}f'(|z|^2), \quad z \in \mathbb{D} \setminus \{0\}.$$

Since $\bar{\partial}u$ is bounded near the origin we have that $f'(x) = \mathcal{O}(1/x^{(m+1)/2})$ as $x \rightarrow 0$. We have now checked (2.6). $\square$

Theorem 2.4 and its proof are modeled on [18, Theorem 2.1]. We have merely supplied some details.

We shall use the fact that the hypergeometric functions are analytic in $\mathbb{D}$. Let $H(\mathbb{D})$ be the space of analytic functions in $\mathbb{D}$. The space $H(\mathbb{D})$ is topologized in the usual manner using the semi-norms

$$\|f\|_K = \max_{z \in K} |f(z)| \quad (2.7)$$

for $K \subset \mathbb{D}$ compact. Convergence in the space $H(\mathbb{D})$ is usually referred to as normal convergence in $\mathbb{D}$.

Recall the terminology that a subset $\mathcal{F}$ of $H(\mathbb{D})$ is called a normal family if every sequence of functions of $\mathcal{F}$ has a subsequence which converges in $H(\mathbb{D})$. The limit function is not required to belong to $\mathcal{F}$. We refer to Conway [11, Chapter VII] for background.

Recall also the binomial series:

$$F(a, b; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = \frac{1}{(1-z)^a} \quad (2.8)$$

for $z \in \mathbb{D}$.

Lemma 2.5 *Let $p, q \in \mathbb{C}$ and consider the functions*

$$f_m(z) = F(-p, m - q; m + 1; z), \quad z \in \mathbb{D},$$

for $m \in \mathbb{N}$, where F is the hypergeometric function (0.5). Then $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ is a normal family of analytic functions in $\mathbb{D}$.

Proof We shall prove that the functions in $\mathcal{F}$ are uniformly bounded on compact subsets of $\mathbb{D}$. The conclusion of the lemma then follows by a classical result of Montel (see Conway [11, Theorem VII.2.9]).

Let $K \subset \mathbb{D}$ be compact. Since $K \subset \mathbb{D}$ is compact there exists $0 < r < 1$ such that $\max_{z \in K} |z| < r$. Choose N such that

$$|1 - (q + 1)/(m + 1)| \leq 1/r$$

for $m > N$. Observe that

$$\frac{(m - q)_n}{(m + 1)_n} = \prod_{k=0}^{n-1} \frac{m - q + k}{m + 1 + k} = \prod_{k=0}^{n-1} \left(1 - \frac{q + 1}{m + 1 + k}\right).$$

Therefore $|(m - q)_n / (m + 1)_n| \leq 1/r^n$ for $n \in \mathbb{N}$ provided $m > N$.

We now estimate the f_m 's with $m > N$. From (0.5) we have that

$$f_m(z) = \sum_{n=0}^{\infty} \frac{(-p)_n (m - q)_n}{(m + 1)_n} \frac{z^n}{n!}, \quad z \in \mathbb{D}.$$

Notice in this sum an appearance of the quotient $(m - q)_n / (m + 1)_n$ considered in the previous paragraph. From the triangle inequality and the result of the previous paragraph we have that

$$|f_m(z)| \leq \sum_{n=0}^{\infty} \frac{(|p|)_n}{n!} \frac{|z|^n}{r^n} = \frac{1}{(1 - |z|/r)^{|p|}}$$

for $|z| < r$ and $m > N$, where the last equality follows by the binomial series (2.8). This proves that the functions in $\mathcal{F}$ are uniformly bounded on K . $\square$

The following limit theorem will be much useful.

Theorem 2.6 *Let $p, q \in \mathbb{C}$. Then*

$$\lim_{m \rightarrow \infty} F(-p, m - q; m + 1; z) = (1 - z)^p, \quad z \in \mathbb{D},$$

with normal convergence, where F is the hypergeometric function (0.5).

Proof Let the f_m 's be as in Lemma 2.5. From Lemma 2.5 we have that the set $\mathcal{F} = \{f_m : m \in \mathbb{N}\}$ is a normal family. From (0.5) we have that

$$f_m^{(n)}(0) = \frac{(-p)_n(m - q)_n}{(m + 1)_n}$$

for $m, n \in \mathbb{N}$. Recall that the Pochhammer symbol $(\cdot)_n$ is a monic polynomial of degree n . Passing to the limit we have that $\lim_{m \rightarrow \infty} f_m^{(n)}(0) = (-p)_n$ for $n \in \mathbb{N}$.

From (2.8) we have that $f^{(n)}(0) = (-p)_n$ for $n \in \mathbb{N}$, where

$$f(z) = (1 - z)^p, \quad z \in \mathbb{D}.$$

A standard argument now yields that $f_m \rightarrow f$ in $H(\mathbb{D})$ as $m \rightarrow \infty$. Assume to reach a contradiction that there exists a compact set $K \subset \mathbb{D}$ such that $\{f_m\}_{m=0}^{\infty}$ does not converge uniformly to f on K . Passing to a subsequence we can assume that

$$\max_{z \in K} |f_{m_k}(z) - f(z)| \geq \delta > 0 \quad (2.9)$$

for $k = 1, 2, \dots$. Since the set $\mathcal{F}$ is a normal family, we can after passage to another subsequence if necessary, assume that $f_{m_k} \rightarrow g$ in $H(\mathbb{D})$ as $k \rightarrow \infty$ for some $g \in H(\mathbb{D})$. From the first paragraph of the proof we have $g^{(n)}(0) = (-p)_n$ for $n \in \mathbb{N}$, and a uniqueness argument gives that $g = f$ in $\mathbb{D}$. Thus $f_{m_k} \rightarrow f$ in $H(\mathbb{D})$ as $k \rightarrow \infty$, which contradicts (2.9). $\square$

We emphasize that Theorem 2.6 appears much natural in view of the binomial series (2.8).

3 A generalized power series

From the product rule for differentiation we have that

$$(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(n-k)} g^{(k)}$$

for, say, $f, g \in C^n[0, 1)$.

Lemma 3.1 *Let $j, k, m \in \mathbb{N}$ be such that $m \geq j$. Let u be a function of the form (2.2) for some $f \in C^\infty[0, 1)$. Then*

$$\partial^j \bar{\partial}^k u(z) = z^{m+k-j} \sum_{l=0}^j \binom{j}{l} \frac{(m+k)!}{(m+k+l-j)!} |z|^{2l} f^{(k+l)}(|z|^2)$$

for $z \in \mathbb{D}$.

Proof Recall (2.2). Differentiating with respect to $\bar{z}$ we have that

$$\bar{\partial}^k u(z) = z^{m+k} f^{(k)}(|z|^2)$$

for $z \in \mathbb{D}$. Another differentiation using the product rule for differentiation gives that

$$\begin{aligned} \partial^j \bar{\partial}^k u(z) &= \sum_{l=0}^j \binom{j}{l} \frac{(m+k)!}{(m+k-(j-l))!} z^{m+k-(j-l)} f^{(k+l)}(|z|^2) \bar{z}^l \\ &= z^{m+k-j} \sum_{l=0}^j \binom{j}{l} \frac{(m+k)!}{(m+k+l-j)!} |z|^{2l} f^{(k+l)}(|z|^2) \end{aligned}$$

for $z \in \mathbb{D}$. This completes the proof of the lemma. $\square$

Let $C^\infty(\mathbb{D})$ be the set of smooth complex-valued functions in the unit disc $\mathbb{D}$. The space $C^\infty(\mathbb{D})$ is topologized by means of the family of semi-norms

$$\|u\|_{j,k;K} = \max_{z \in K} |\partial^j \bar{\partial}^k u(z)|, \quad (3.1)$$

where $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ is compact. Recall that $u_m \rightarrow u$ in $C^\infty(\mathbb{D})$ as $m \rightarrow \infty$ means that $\lim_{m \rightarrow \infty} \|u_m - u\|_{j,k;K} = 0$ for all $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact.

A (formal) series $\sum_{m=0}^\infty u_m$ of functions $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{N}$ is said to be absolutely convergent in $C^\infty(\mathbb{D})$ if

$$\sum_{m=0}^\infty \|u_m\|_{j,k;K} < +\infty$$

whenever $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ is compact. By completeness of $C^\infty(\mathbb{D})$ we have that every series absolutely convergent in $C^\infty(\mathbb{D})$ is convergent in $C^\infty(\mathbb{D})$.

Theorem 3.2 *Let $\{f_m\}_{m=0}^\infty$ be a sequence in $C^\infty[0, 1)$ such that*

$$\limsup_{m \rightarrow \infty} \left(\max_{0 \leq x \leq r} |f_m^{(n)}(x)| \right)^{1/m} \leq 1$$

for $n \in \mathbb{N}$ and $0 < r < 1$. Set

$$u_m(z) = f_m(|z|^2)z^m, \quad z \in \mathbb{D},$$

for $m = 0, 1, 2, \dots$. Then $\limsup_{m \rightarrow \infty} \|u_m\|_{j,k;K}^{1/m} < 1$ for all $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact, where $\|\cdot\|_{j,k;K}$ is as in (3.1).

Proof Fix $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact. Set $r = \max_{z \in K} |z| < 1$. From Lemma 3.1 we have that

$$\partial^j \bar{\partial}^k u_m(z) = z^{m+k-j} \sum_{l=0}^j \binom{j}{l} \frac{(m+k)!}{(m+k+l-j)!} |z|^{2l} f_m^{(k+l)}(|z|^2)$$

for $z \in \mathbb{D}$ provided $m \geq j$. We next apply the triangle inequality to see that

$$\begin{aligned} \|u_m\|_{j,k;K} &\leq r^{m+k-j} \left(\sum_{l=0}^j \binom{j}{l} \frac{(m+k)!}{(m+k+l-j)!} r^{2l} \right) \max_{k \leq n \leq j+k} \|f_m^{(n)}\|_{[0,r^2]} \\ &\leq r^{m+k-j} (m+k)^j (1+r^2)^j \max_{k \leq n \leq j+k} \|f_m^{(n)}\|_{[0,r^2]} \end{aligned} \quad (3.2)$$

for $m \geq j$, where we have used the notation (2.7). We shall next apply the m -th root to (3.2) and pass to the limit as $m \rightarrow \infty$. In view of the assumption on the sequence $\{f_m\}_{m=0}^\infty$ we conclude from (3.2) that

$$\limsup_{m \rightarrow \infty} \|u_m\|_{j,k;K}^{1/m} \leq r.$$

Since $0 < r < 1$, this yields the conclusion of the theorem. $\square$

From the conclusion of Theorem 3.2 we have that $\sum_{m=0}^\infty \|u_m\|_{j,k;K} < +\infty$ whenever $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ is compact. Thus the series $\sum_{m=0}^\infty u_m$ is absolutely convergent in $C^\infty(\mathbb{D})$.

We can think of a function series

$$\sum_{m=0}^\infty f_m(|z|^2)z^m \quad (3.3)$$

as a vector-valued power series. The sequence of coefficients $\{f_m\}_{m=0}^\infty$ in (3.3) is now a sequence of functions in $C^\infty[0, 1]$. From this point of view, Theorem 3.2 generalizes a well-known root criteria for power series (see for instance Conway [11, Theorem III.1.3]).

Of particular concern to us are sequences $\{f_m\}_{m=0}^\infty$ of the form

$$f_m(x) = F(-p, m - q; m + 1; x), \quad 0 \leq x < 1,$$

for $m \in \mathbb{N}$, where $p, q \in \mathbb{C}$ and F is the hypergeometric function (0.5). We next observe that Theorem 2.6 guarantees that every such sequence $\{f_m\}_{k=0}^\infty$ satisfies the assumption of Theorem 3.2.

Proposition 3.3 *Let $p, q \in \mathbb{C}$. Then*

$$\limsup_{m \rightarrow \infty} \left(\max_{0 \leq x \leq r} |F^{(n)}(-p, m - q; m + 1; x)| \right)^{1/m} \leq 1$$

for $n \in \mathbb{N}$ and $0 < r < 1$, where F is the hypergeometric function (0.5).

Proof Recall the fact that the complex derivative $f \mapsto f'$ is continuous in the topology of normal convergence of analytic functions (see Conway [11, Theorem VII.2.1]). Recall Montel's theorem characterizing normal families of analytic functions (see Conway [11, Theorem VII.2.9]). In view of these two results, the proposition follows from Theorem 2.6. $\square$

4 Analysis of homogeneous parts

Notice that the rotation operators $\mathbb{T} \ni e^{i\theta} \mapsto R_{e^{i\theta}}$ from (0.3) have the group properties that $R_1 = I$ is the identity and

$$R_{e^{i(\theta+\tau)}} = R_{e^{i\theta}} R_{e^{i\tau}}$$

for $e^{i\theta}, e^{i\tau} \in \mathbb{T}$.

Recall the notion of homogeneity with respect to rotations made precise in the paragraph before Theorem 2.4. Observe that a function u in $\mathbb{D}$ is homogeneous of order $m \in \mathbb{Z}$ with respect to rotations if and only if $R_{e^{i\theta}} u = e^{im\theta} u$ for $e^{i\theta} \in \mathbb{T}$.

Proposition 4.1 *Let u_m be the m -th homogeneous part of $u \in C^n(\mathbb{D})$ for some $m \in \mathbb{Z}$, where $n \in \mathbb{N} \cup \{\infty\}$. Then $u_m \in C^n(\mathbb{D})$ and u_m is homogeneous of order m with respect to rotations.*

Proof Differentiations under the integral in (0.4) show that $u_m \in C^n(\mathbb{D})$. From formula (0.4) we have that

$$u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\tau}} u \, d\tau \quad (4.1)$$

in a vector-valued sense. Let $e^{i\theta} \in \mathbb{T}$. From the group property of rotation operators we have that

$$R_{e^{i\theta}} u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\theta}} R_{e^{i\tau}} u \, d\tau = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i(\theta+\tau)}} u \, d\tau$$

in a vector-valued sense. A change of variables now gives that

$$R_{e^{i\theta}} u_m = \frac{e^{im\theta}}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\tau}} u \, d\tau = e^{im\theta} u_m.$$

Since $e^{i\theta} \in \mathbb{T}$ is arbitrary, this yields that u_m is homogeneous of order m with respect to rotations. $\square$

Below we shall make use of the commutativity relation

$$L_{p,q} R_{e^{i\theta}} u = R_{e^{i\theta}} L_{p,q} u, \quad u \in C^2(\mathbb{D}), \quad (4.2)$$

for $e^{i\theta} \in \mathbb{T}$. In order to prove (4.2) it suffices to check that the differential operators $z\partial$ and $\bar{z}\bar{\partial}$ commute with the rotations $R_{e^{i\theta}}$ which is evident.

We now return to (p, q) -harmonic functions.

Proposition 4.2 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function and denote by u_m its m -th homogeneous part for some $m \in \mathbb{Z}$. Then u_m is (p, q) -harmonic.*

Proof From Proposition 4.1 we have that $u_m \in C^2(\mathbb{D})$ since $u \in C^2(\mathbb{D})$. Recall formula (4.1). Applying the operator $L_{p,q}$ we have that

$$L_{p,q} u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} L_{p,q} R_{e^{i\tau}} u \, d\tau$$

in a vector-valued sense. We now use the commutativity relation (4.2) to conclude that

$$L_{p,q} u_m = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-im\tau} R_{e^{i\tau}} L_{p,q} u \, d\tau = 0,$$

where the last equality is evident since u is (p, q) -harmonic. $\square$

Let us next calculate the homogeneous parts of a (p, q) -harmonic function.

Theorem 4.3 *Let $p, q \in \mathbb{C}$. Let u be (p, q) -harmonic function and denote by u_m its m -th homogeneous part for some $m \in \mathbb{N}$. Then*

$$u_m(z) = c_m F(-p, m - q; m + 1; |z|^2) z^m, \quad z \in \mathbb{D}, \quad (4.3)$$

for some $c_m \in \mathbb{C}$, where F is the hypergeometric function (0.5).

Proof By Proposition 4.1, the function u_m is homogeneous of order m with respect to rotations. By Proposition 4.2, the function u_m is (p, q) -harmonic. The result now follows from Theorem 2.4. $\square$

Corollary 4.4 *Let $p, q \in \mathbb{C}$. Let u be (p, q) -harmonic function and denote by u_m its m -th homogeneous part for some $m \in \mathbb{Z} \setminus \mathbb{Z}^+$. Then*

$$u_m(z) = c_m F(-q, |m| - p; |m| + 1; |z|^2) \bar{z}^{|m|}, \quad z \in \mathbb{D}, \quad (4.4)$$

for some $c_m \in \mathbb{C}$, where F is the hypergeometric function (0.5).

Proof The complex conjugate $\bar{u}_m$ is the $-m = |m|$ -th homogeneous part of the function $\bar{u}$. By hermitian symmetry, the function $\bar{u}$ is $(\bar{q}, \bar{p})$ -harmonic. From Theorem 4.3 we have that

$$\overline{u_m(z)} = a_m F(-\bar{q}, |m| - \bar{p}; |m| + 1; |z|^2) z^{|m|}, \quad z \in \mathbb{D},$$

for some $a_m \in \mathbb{C}$. A complex conjugation now yields (4.4) with $c_m = \bar{a}_m$. $\square$

As a by-product from Theorem 4.3 and Corollary 4.4 we have that $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$ if u is (p, q) -harmonic.

Recall formula (0.4) for the homogeneous parts of a function u in $\mathbb{D}$. From the triangle inequality we have that

$$\sup_{|z|=r} |u_m(z)| \leq \sup_{|z|=r} |u(z)| \quad (4.5)$$

for $0 < r < 1$ and $m \in \mathbb{Z}$. Notice that (4.5) generalizes a well-known bound for Fourier coefficients.

We next estimate the constants c_m appearing in Theorem 4.3 or Corollary 4.4.

Lemma 4.5 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function and consider its m -th homogeneous part u_m for $m \in \mathbb{Z}$. Let c_m be as in (4.4) or (4.4) depending on whether $m \in \mathbb{N}$ or $m \in \mathbb{Z}^-$. Then $\limsup_{|m| \rightarrow \infty} |c_m|^{1/|m|} \leq 1$.*

Proof For simplicity we consider the case $m \in \mathbb{N}$. The case $m \in \mathbb{Z}^-$ is analogous or follows by hermitian symmetry. Let $0 < r < 1$. From (4.4) and (4.5) we have that

$$|c_m| |F(-p, m - q; m + 1; r^2)| r^m \leq \max_{|z|=r} |u(z)| \quad (4.6)$$

for $m \in \mathbb{N}$. From Theorem 2.6 we have that

$$\lim_{m \rightarrow \infty} F(-p, m - q; m + 1; r^2) = (1 - r^2)^p \neq 0.$$

Therefore, a passage to the limit in (4.6) gives that

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} \leq 1/r.$$

Since $0 < r < 1$ is arbitrary we conclude that $\limsup_{m \rightarrow \infty} |c_m|^{1/m} \leq 1$. This yields the conclusion of the lemma. $\square$

The following theorem is our main result about the asymptotic behavior of the homogeneous parts of a (p, q) -harmonic function.

Theorem 4.6 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function and denote by u_m its m -th homogeneous part for $m \in \mathbb{Z}$. Then $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$ and*

$$\limsup_{|m| \rightarrow \infty} \|u_m\|_{j,k;K}^{1/|m|} < 1$$

for all $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact.

Proof From Theorem 4.3 and Corollary 4.4 we have that $u_m \in C^\infty(\mathbb{D})$ for $m \in \mathbb{Z}$.

We consider first the case $m \in \mathbb{N}$. From Theorem 4.3 we have (4.4) for $m \in \mathbb{N}$, where $c_m \in \mathbb{C}$. Notice that the function u_m in (4.4) is constructed from the function

$$f_m(x) = c_m F(-p, m - q; m + 1; x), \quad 0 \leq x < 1,$$

as in Theorem 3.2. From Proposition 3.3 and Lemma 4.5 we have that the assumption of Theorem 3.2 is satisfied. An application of Theorem 3.2 yields the conclusion that $\limsup_{m \rightarrow \infty} \|u_m\|_{j,k;K}^{1/m} < 1$ for $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact.

We next consider the case $m \in \mathbb{Z}^-$. The complex conjugate $\bar{u}_m$ of u_m is the $(-m)$ -th homogeneous part of the $(\bar{q}, \bar{p})$ -harmonic function $\bar{u}$. From the result of the previous paragraph we thus have that

$$\limsup_{m \rightarrow -\infty} \|u_m\|_{j,k;K}^{1/|m|} = \limsup_{m \rightarrow \infty} \|(\bar{u})_m\|_{k,j;K}^{1/m} < 1$$

for $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact. This completes the proof of the theorem. $\square$

We point out that Theorem 2.6 forms an integral part in the proof of Theorem 4.6.

Recall the notion of absolute convergence in $C^\infty(\mathbb{D})$, see the paragraph just before Theorem 3.2.

Corollary 4.7 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function with m -th homogeneous part u_m for $m \in \mathbb{Z}$. Then the function series $\sum_{m=-\infty}^{\infty} u_m$ is absolutely convergent in $C^\infty(\mathbb{D})$.*

Proof Theorem 4.6 allows us to apply the root test to conclude that

$$\sum_{m=-\infty}^{\infty} \|u_m\|_{j,k;K} < +\infty$$

for $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact. $\square$

The following lemma is well-known but included for the sake of completeness.

Lemma 4.8 *Let $u \in C^n(\mathbb{D})$ for some $n \in \mathbb{N}$ and denote by u_m its m -th homogeneous part for $m \in \mathbb{Z}$. Then*

$$u = \lim_{N \rightarrow +\infty} \sum_{m=-N}^N \left(1 - \frac{|m|}{N+1}\right) u_m$$

in $C^n(\mathbb{D})$.

Proof It is straightforward to check that

$$\sum_{m=-N}^N \left(1 - \frac{|m|}{N+1}\right) u_m = \frac{1}{2\pi} \int_{\mathbb{T}} K_N(e^{i\theta}) R_{e^{i\theta}} u \, d\theta$$

in a vector-valued sense, where

$$K_N(e^{i\theta}) = \sum_{m=-N}^N \left(1 - \frac{|m|}{N+1}\right) e^{im\theta}, \quad e^{i\theta} \in \mathbb{T},$$

is the Fejér kernel. It is well-known that the K_N 's are non-negative and form an approximate identity as $N \rightarrow \infty$. Since $u \in C^n(\mathbb{D})$, it is well-known that the function

$$\mathbb{T} \ni e^{i\theta} \mapsto R_{e^{i\theta}} u \in C^n(\mathbb{D})$$

is continuous from $\mathbb{T}$ into $C^n(\mathbb{D})$. The proof is now completed by a standard argument. We refer to Katznelson [15, Section I.2] for details. $\square$

We think of Lemma 4.8 as a version of Fejér's theorem adapted to our context.

We next deduce that (p, q) -harmonic functions belong to the space $C^\infty(\mathbb{D})$.

Corollary 4.9 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function. Then $u \in C^\infty(\mathbb{D})$ and $u = \sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$, where u_m is the m -th homogeneous part of u for $m \in \mathbb{Z}$.*

Proof From Lemma 4.8 we have that

$$u = \lim_{N \rightarrow +\infty} \sum_{m=-N}^N \left(1 - \frac{|m|}{N+1}\right) u_m \quad (4.7)$$

in $C^2(\mathbb{D})$ since u has such regularity. From Corollary 4.7 we know that the function series $\sum_{m=-\infty}^{\infty} u_m$ is absolutely convergent in $C^\infty(\mathbb{D})$. We can thus omit the convergence factors in (4.7) and deduce that $u = \sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$. By completeness we have that $u \in C^\infty(\mathbb{D})$. $\square$

5 A series expansion of harmonic functions

The analysis from Sect. 4 leads to a natural function series description of (p, q) -harmonic functions.

Theorem 5.1 *Let $p, q \in \mathbb{C}$. Then u is a (p, q) -harmonic function if and only if it has the form*

$$u(z) = \sum_{m=0}^{\infty} c_m F(-p, m - q; m + 1; |z|^2) z^m + \sum_{m=1}^{\infty} c_{-m} F(-q, m - p; m + 1; |z|^2) \bar{z}^m, \quad z \in \mathbb{D}, \quad (5.1)$$

for some sequence $\{c_m\}_{m=-\infty}^{\infty}$ of complex numbers such that

$$\limsup_{|m| \rightarrow \infty} |c_m|^{1/|m|} \leq 1, \quad (5.2)$$

where F is the hypergeometric function (0.5). Moreover, the sums in (5.1) are absolutely convergent in the space $C^\infty(\mathbb{D})$ when (5.2) holds.

Proof Consider first a (formal) function series of the form (5.1) with coefficient sequence $\{c_m\}_{m=-\infty}^{\infty}$ satisfying (5.2). From Theorem 3.2 and Proposition 3.3 we have that

$$\limsup_{|m| \rightarrow \infty} \|u_m\|_{j,k;K}^{1/|m|} < 1$$

for $j, k \in \mathbb{N}$ and $K \subset \mathbb{D}$ compact, where

$$u_m(z) = c_m F(-p, m - q; m + 1; |z|^2) z^m, \quad z \in \mathbb{D},$$

for $m \in \mathbb{N}$ and

$$u_m(z) = c_m F(-q, |m| - p; |m| + 1; |z|^2) \bar{z}^{|m|}, \quad z \in \mathbb{D},$$

for $m \in \mathbb{Z}^-$. The root test now applies to show that the series $\sum_{m=-\infty}^{\infty} u_m$ in (5.1) is absolutely convergent in $C^\infty(\mathbb{D})$. This defines a function $u = \sum_{m=-\infty}^{\infty} u_m$ by (5.1). From Proposition 2.2 and Corollary 2.3 we have that each term u_m in (5.1) is (p, q) -harmonic. Since the series expansion (5.1) converges in the space $C^\infty(\mathbb{D})$, we have that $u \in C^\infty(\mathbb{D})$. Applying the operator $L_{p,q}$ we have $L_{p,q}u = \sum_{m=-\infty}^{\infty} L_{p,q}u_m = 0$ in $C^\infty(\mathbb{D})$, which shows that u is (p, q) -harmonic.

Consider next a (p, q) -harmonic function u . We proceed to derive (5.1). Denote by u_m the m -th homogeneous part of u for $m \in \mathbb{Z}$. From Theorem 4.3 we have (4.4) for $m \in \mathbb{N}$ and from Corollary 4.4 we have (4.4) for $m \in \mathbb{Z}^-$, where $c_m \in \mathbb{C}$ for $m \in \mathbb{Z}$. From Corollary 4.9 we have (5.1) in the form that $u = \sum_{m=-\infty}^{\infty} u_m$ in $C^\infty(\mathbb{D})$.

By Corollary 4.7 we have that the function series $\sum_{m=-\infty}^{\infty} u_m$ in (5.1) is absolutely convergent in $C^\infty(\mathbb{D})$. From Lemma 4.5 we have that (5.2) holds. $\square$

We emphasize that the function series (5.1) has an interpretation of homogeneous expansion of the (p, q) -harmonic function u .

We mention that Theorem 5.1 improves on a result by Ahern et al. [1, Theorem 2.1] when specialized to the present setting. Theorem 5.1 contains also some earlier results of Olofsson and Wittsten. Theorem 5.1 with $(p, q) = (0, \alpha)$, $\alpha \in \mathbb{R}$, yields [21, Theorem 1.2]. Theorem 5.1 with $(p, q) = (\alpha/2, \alpha/2)$, $\alpha \in \mathbb{R}$, yields [18, Theorem 2.2].

We shall next turn to formulas for the coefficients c_m appearing in (5.1).

Proposition 5.2 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function of the form (5.1) for some sequence $\{c_m\}_{m=-\infty}^{\infty}$ of complex numbers satisfying (5.2). Then*

$$c_m = \lim_{r \rightarrow 0} \frac{1}{2\pi r^{|m|}} \int_{\mathbb{T}} u(re^{i\theta}) e^{-im\theta} d\theta$$

for $m \in \mathbb{Z}$.

Proof We consider the case $m \in \mathbb{N}$. Using the expansion (5.1) it is straightforward to check that

$$\frac{1}{2\pi} \int_{\mathbb{T}} u(re^{i\theta}) e^{-im\theta} d\theta = c_m F(-p, m - q; m + 1; r^2) r^m$$

for $0 < r < 1$. We next divide by r^m and pass to the limit to see that

$$\lim_{r \rightarrow 0} \frac{1}{2\pi r^m} \int_{\mathbb{T}} u(re^{i\theta}) e^{-im\theta} d\theta = c_m F(-p, m - q; m + 1; 0) = c_m.$$

This yields the conclusion of the proposition when $m \in \mathbb{N}$.

The remaining case $m \in \mathbb{Z} \setminus \mathbb{N}$ is proved similarly or follows by hermitian symmetry. We omit the details. $\square$

The integral quantity

$$I_m(r) = \frac{1}{2\pi r^{|m|}} \int_{\mathbb{T}} u(re^{i\theta}) e^{-im\theta} d\theta \quad (5.3)$$

in Proposition 5.2 is naturally thought of as a line integral. In fact,

$$I_m(r) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{u(z)}{z^{m+1}} dz \quad \text{and} \quad I_{-m}(r) = -\frac{1}{2\pi i} \oint_{|z|=r} \frac{u(z)}{\bar{z}^{m+1}} d\bar{z}$$

for $m \in \mathbb{N}$ and $0 < r < 1$, where the circle of integration is traversed once in positive direction. These formulas are straightforward to check.

Theorem 5.3 Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function of the form (5.1) for some sequence $\{c_m\}_{m=-\infty}^{\infty}$ of complex numbers satisfying (5.2). Then

$$c_m = \partial^m u(0)/m! \quad \text{and} \quad c_{-m} = \bar{\partial}^m u(0)/m!$$

for $m \in \mathbb{N}$.

Proof We shall calculate the integral limit in Proposition 5.2. Let $I_m(r)$ be as in (5.3). Recall from Corollary 4.9 that $u \in C^\infty(\mathbb{D})$. Consider the Taylor expansion of u at the origin of degree $m \geq 0$:

$$u(z) = \sum_{\substack{j,k \geq 0 \\ j+k \leq m}} \frac{1}{j!k!} \partial^j \bar{\partial}^k u(0) z^j \bar{z}^k + \mathcal{O}(|z|^{m+1})$$

as $z \rightarrow 0$. We have that

$$\begin{aligned} I_m(r) &= \frac{1}{2\pi r^m} \int_{\mathbb{T}} \left(\sum_{\substack{j,k \geq 0 \\ j+k \leq m}} \frac{1}{j!k!} \partial^j \bar{\partial}^k u(0) r^{j+k} e^{i(j-k)\theta} \right) e^{-im\theta} d\theta + \mathcal{O}(r) \\ &= \partial^m u(0)/m! + \mathcal{O}(r) \end{aligned}$$

as $r \rightarrow 0$, where the last equality follows by cancellation. A passage to the limit as $r \rightarrow 0$ now yields that $c_m = \partial^m u(0)/m!$.

We now turn to the formula for c_{-m} . A similar analysis as in the previous paragraph shows that

$$I_{-m}(r) = \bar{\partial}^m u(0)/m! + \mathcal{O}(r)$$

as $r \rightarrow 0$. A passage to the limit as $r \rightarrow 0$ yields that $c_{-m} = \bar{\partial}^m u(0)/m!$. This completes the proof of the theorem. $\square$

The coefficient formulas in Theorem 5.3 leads to an addendum to Theorem 5.1.

Theorem 5.4 Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function. Then

$$\begin{aligned} u(z) &= \sum_{m=0}^{\infty} \frac{\partial^m u(0)}{m!} F(-p, m-q; m+1; |z|^2) z^m \\ &\quad + \sum_{m=1}^{\infty} \frac{\bar{\partial}^m u(0)}{m!} F(-q, m-p; m+1; |z|^2) \bar{z}^m \end{aligned}$$

for $z \in \mathbb{D}$, where F is the hypergeometric function (0.5).

Proof Recall the series expansion (5.1) and (5.2) established in Theorem 5.1. By Theorem 5.3 we have that $c_m = \partial^m u(0)/m!$ and $c_{-m} = \bar{\partial}^m u(0)/m!$ for $m \in \mathbb{N}$. This yields the conclusion of the theorem. $\square$

As a by-product of Theorem 5.4 we have a uniqueness result of classical type.

Corollary 5.5 *Let $p, q \in \mathbb{C}$. Let u be a (p, q) -harmonic function. Assume that $u(0) = 0$ and*

$$\partial^m u(0) = \bar{\partial}^m u(0) = 0$$

for $m = 1, 2, \dots$. Then $u(z) = 0$ for all $z \in \mathbb{D}$.

Proof The result is evident by Theorem 5.4. □

6 Further properties of the function $u_{p,q}$

Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. In this section we derive the series expansion of $u_{p,q}$ (see Theorem 6.3). Of interest are also properties of integral means of $u_{p,q}$ (see Theorems 6.4 and 6.6).

Lemma 6.1 *Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Let $m \in \mathbb{N}$. Then*

$$\partial^m u_{p,q}(z) = \frac{(p+1)_m}{(1-z)^m} u_{p,q}(z) + \bar{z} g_m(z), \quad z \in \mathbb{D}, \quad (6.1)$$

where $g_m \in C^\infty(\mathbb{D})$.

Proof We prove the lemma by induction on $m \in \mathbb{N}$. Clearly (6.1) holds for $m = 0$ with $g_0 = 0$. By Lemma 1.1 we have that

$$\partial u_{p,q}(z) = \frac{p+1}{1-z} u_{p,q}(z) + \bar{z} g_1(z), \quad z \in \mathbb{D}, \quad (6.2)$$

with

$$g_1(z) = -\frac{p+q+1}{1-|z|^2} u_{p,q}(z), \quad z \in \mathbb{D}.$$

This proves (6.1) for $m = 1$.

Assume next that (6.1) holds for some $m = n \geq 1$, that is,

$$\partial^n u_{p,q}(z) = \frac{(p+1)_n}{(1-z)^n} u_{p,q}(z) + \bar{z} g_n(z), \quad z \in \mathbb{D},$$

where $g_n \in C^\infty(\mathbb{D})$. Differentiating we have that

$$\partial^{n+1} u_{p,q}(z) = (p+1)_n \frac{n}{(1-z)^{n+1}} u_{p,q}(z) + \frac{(p+1)_n}{(1-z)^n} \partial u_{p,q}(z) + \bar{z} \partial g_n(z)$$

for $z \in \mathbb{D}$. We next use (6.2) to see that

$$\begin{aligned}\partial^{n+1}u_{p,q}(z) &= (p+1)_n \frac{n}{(1-z)^{n+1}} u_{p,q}(z) + \frac{(p+1)_n}{(1-z)^n} \left(\frac{p+1}{1-z} u_{p,q}(z) + \bar{z} g_1(z) \right) \\ &\quad + \bar{z} \partial g_n(z) = \frac{(p+1)_{n+1}}{(1-z)^{n+1}} u_{p,q}(z) + \bar{z} g_{n+1}(z)\end{aligned}$$

for $z \in \mathbb{D}$, where

$$g_{n+1}(z) = \frac{(p+1)_n}{(1-z)^n} g_1(z) + \partial g_n(z), \quad z \in \mathbb{D}.$$

This proves (6.1) for $m = n + 1$. The conclusion of the lemma now follows by the principle of induction. $\square$

Notice that formula (6.1) interprets naturally as a congruence in $C^\infty(\mathbb{D})$ modulo $\bar{z}C^\infty(\mathbb{D})$.

Theorem 6.2 *Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$. Then*

$$\partial^m u_{p,q}(0) = (p+1)_m \quad \text{and} \quad \bar{\partial}^m u_{p,q}(0) = (q+1)_m$$

for $m \in \mathbb{N}$.

Proof We shall use Lemma 6.1. A point evaluation at the origin in (6.1) shows that $\partial^m u_{p,q}(0) = (p+1)_m$ for $m \in \mathbb{N}$. Moreover, by hermitian symmetry $\bar{u}_{p,q} = u_{\bar{q},\bar{p}}$ we have that

$$\bar{\partial}^m u_{p,q}(0) = \bar{\partial}^m \bar{u}_{\bar{q},\bar{p}}(0) = \overline{\partial^m u_{\bar{q},\bar{p}}(0)} = \overline{(\bar{q}+1)_m} = (q+1)_m$$

for $m \in \mathbb{N}$. $\square$

We can now derive the series expansion of the function $u_{p,q}$.

Theorem 6.3 *Let $p, q \in \mathbb{C}$. Then*

$$\begin{aligned}\frac{(1-|z|^2)^{p+q+1}}{(1-z)^{p+1}(1-\bar{z})^{q+1}} &= \sum_{m=0}^{\infty} \frac{(p+1)_m}{m!} F(-p, m-q; m+1; |z|^2) z^m \\ &\quad + \sum_{m=1}^{\infty} \frac{(q+1)_m}{m!} F(-q, m-p; m+1; |z|^2) \bar{z}^m\end{aligned}$$

for $z \in \mathbb{D}$, where F is the hypergeometric function (0.5).

Proof Let $u_{p,q}$ be as in (0.2). From Theorem 1.4 we know that the function $u_{p,q}$ is (p, q) -harmonic. By Theorem 5.4 it has the function series expansion

$$u_{p,q}(z) = \sum_{m=0}^{\infty} \frac{\partial^m u_{p,q}(0)}{m!} F(-p, m-q; m+1; |z|^2) z^m \\ + \sum_{m=1}^{\infty} \frac{\bar{\partial}^m u_{p,q}(0)}{m!} F(-q, m-p; m+1; |z|^2) \bar{z}^m$$

for $z \in \mathbb{D}$. The result now follows from Theorem 6.2. $\square$

We point out that Theorem 6.3 generalizes a well-known partial fraction decomposition formula for the classical Poisson kernel for $\mathbb{D}$, see formula (1.1). Suitably specialized Theorem 6.3 yields [21, Theorem 2.5] and [18, Theorem 3.2].

Notice that the series expansion in Theorem 6.3 is absolutely convergent in $C^\infty(\mathbb{D})$.

We denote by $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ the real and imaginary parts of a complex number z , respectively. The standard Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

for $\operatorname{Re}(z) > 0$. It is well-known that Γ continues to a meromorphic function in $\mathbb{C}$ with simple poles at the points $z = 0, -1, -2, \dots$

We next turn to L^1 means of $u_{p,q}$.

Theorem 6.4 Let $u_{p,q}$ be as in (0.2) for some $p, q \in \mathbb{C}$ such that $\operatorname{Re}(p) + \operatorname{Re}(q) > -1$. Then

$$\frac{1}{2\pi} \int_{\mathbb{T}} |u_{p,q}(re^{i\theta})| d\theta \leq e^{\frac{\pi}{2}|\operatorname{Im}(p) - \operatorname{Im}(q)|} \frac{\Gamma(\operatorname{Re}(p) + \operatorname{Re}(q) + 1)}{\Gamma(\frac{\operatorname{Re}(p) + \operatorname{Re}(q)}{2} + 1)^2}$$

for $0 \leq r < 1$, where Γ is the Gamma function.

Proof From definition of powers we have that

$$u_{p,q}(z) = e^{(p+q+1)\log(1-|z|^2)} e^{-(p+q+2)\log|1-z|} e^{-i(p-q)\arg(1-z)}$$

for $z \in \mathbb{D}$. Notice that the complex number $1-z$ lies in the open right half-plane when $z \in \mathbb{D}$. Passing to absolute values we have that

$$|u_{p,q}(z)| \leq e^{\frac{\pi}{2}|\operatorname{Im}(p) - \operatorname{Im}(q)|} \frac{(1-|z|^2)^{\operatorname{Re}(p) + \operatorname{Re}(q) + 1}}{|1-z|^{\operatorname{Re}(p) + \operatorname{Re}(q) + 2}}$$

for $z \in \mathbb{D}$. We next apply [18, Theorem 3.1] with $\alpha = \operatorname{Re}(p) + \operatorname{Re}(q)$ to conclude that

$$\frac{1}{2\pi} \int_{\mathbb{T}} |u_{p,q}(re^{i\theta})| d\theta \leq e^{\frac{\pi}{2}|\operatorname{Im}(p) - \operatorname{Im}(q)|} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha/2 + 1)^2}$$

for $0 \leq r < 1$. This yields the conclusion of the theorem. $\square$

Remark 6.5 A similar analysis as in the proof of Theorem 6.4 shows that

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{\mathbb{T}} |u_{p,q}(re^{i\theta})| d\theta = +\infty$$

if $\operatorname{Re}(p) + \operatorname{Re}(q) \leq -1$. We omit the details.

A classical result known as Gauss summation formula says that

$$\lim_{x \rightarrow 1} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (6.3)$$

when $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ are such that $\operatorname{Re}(c) > \operatorname{Re}(a) + \operatorname{Re}(b)$ (see [3, Theorem 2.2.2]).

Theorem 6.6 *Let $p, q \in \mathbb{C}$ be such that $\operatorname{Re}(p) + \operatorname{Re}(q) > -1$. Then*

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_{\mathbb{T}} u_{p,q}(re^{i\theta}) d\theta = \frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)},$$

where Γ is the Gamma function.

Proof Recall the series expansion of the function $u_{p,q}$ established in Theorem 6.3. By cancellation of terms we have that

$$\frac{1}{2\pi} \int_{\mathbb{T}} u_{p,q}(re^{i\theta}) d\theta = F(-p, -q; 1; r^2)$$

for $0 \leq r < 1$. Since $\operatorname{Re}(p) + \operatorname{Re}(q) > -1$, we can apply Gauss summation formula (6.3) to calculate the limit of the above quantity as $r \rightarrow 1$. This yields the conclusion of the theorem. $\square$

7 The Dirichlet problem: concluding remarks

In a restricted range of parameters, the class of (p, q) -harmonic functions can be analyzed in terms of their boundary values. We shall discuss in this section some rudiments of such theory.

Let $p, q \in \mathbb{C} \setminus \mathbb{Z}^-$ be such that $\operatorname{Re}(p) + \operatorname{Re}(q) > -1$. The (p, q) -harmonic Poisson kernel is defined by

$$K_{p,q}(z) = c_{p,q} \frac{(1 - |z|^2)^{p+q+1}}{(1 - z)^{p+1}(1 - \bar{z})^{q+1}}, \quad z \in \mathbb{D},$$

where

$$c_{p,q} = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)}.$$

Notice that the constant $c_{p,q}$ is a non-zero complex number in this range of parameters. By Theorem 1.4 the function $K_{p,q}$ is (p, q) -harmonic. Theorems 6.4 and 6.6 ensure that the function $K_{p,q}$ satisfies some standard properties for approximate identities.

The (p, q) -harmonic Poisson integral is defined by

$$K_{p,q}[f](z) = \frac{1}{2\pi} \int_{\mathbb{T}} K_{p,q}(ze^{-i\theta}) f(e^{i\theta}) d\theta, \quad z \in \mathbb{D},$$

for integrable functions $f \in L^1(\mathbb{T})$ on $\mathbb{T}$.

Let $C(\mathbb{T})$ be the space of continuous functions on $\mathbb{T}$ and fix $\varphi \in C(\mathbb{T})$. By the (p, q) -harmonic Dirichlet problem for φ we understand the problem of finding a (p, q) -harmonic function u such that $\lim_{r \rightarrow 1} u_r = \varphi$ in $C(\mathbb{T})$, where

$$u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \mathbb{T},$$

for $0 < r < 1$. Following usual practice, we formulate this latter Dirichlet problem as

$$\begin{cases} L_{p,q}u = 0 & \text{in } \mathbb{D}, \\ u = \varphi & \text{on } \mathbb{T}, \end{cases} \quad (7.1)$$

where $L_{p,q}$ is as in (0.1).

Theorem 7.1 *Let $p, q \in \mathbb{C} \setminus \mathbb{Z}^-$ be such that $\operatorname{Re}(p) + \operatorname{Re}(q) > -1$. Let $\varphi \in C(\mathbb{T})$. Then a function u in $\mathbb{D}$ satisfies (7.1) if and only if it has the form*

$$u(z) = K_{p,q}[\varphi](z), \quad z \in \mathbb{D}.$$

The proof of Theorem 7.1 follows a standard scheme for such results and is therefore omitted, see [18, 21] for details.

The present paper suggests a finer study of (p, q) -harmonic functions. Earlier results of such type concern Poisson integral representations, pointwise boundary limits, Green functions and Lipschitz continuity of generalized harmonic functions, see [18–21]. We mention here also work of Ahern and collaborators [1, 2].

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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