On the Koebe Quarter Theorem for certain polynomials

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Abstract

We study problems similar to the Koebe Quarter Theorem for close-to-convex polynomials with all zeros of derivative in $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. We found minimal disc containing all images of $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and maximal disc contained in all images of $\mathbb{D}$ through polynomials of degree 3 and 4. Moreover we determine the extremal functions for both problems.

Keywords Univalent polynomials · Koebe radius · Critical points

Mathematics Subject Classification 30C10 · 30C15 · 30C25 · 30C45 · 30C75

Introduction

Let $\mathcal{S}$ be the class of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by $f(0) = f'(0) - 1 = 0$ which are univalent. Let $\mathcal{H}_d$ be the class of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by $f(0) = 1$ and such that $f \neq 0$ in $\mathbb{D}$. For $\alpha, \beta \geq 0$ the Kaplan class $K(\alpha, \beta)$ is the set of all functions $f \in \mathcal{H}_d$ satisfying the condition

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \leq \beta\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \quad (0.1)$$

for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ (see [9, pp. 32–33]). Kaplan classes were first defined by Sheil-Small in [10] and the restated version as in the condition (0.1) originally comes from [11]. The class $K(\alpha, \beta)$ is called the Kaplan class because using

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the Kaplan method [8], one can show that a normalized function f analytic in $\mathbb{D}$ is close-to-convex of order $\alpha \geq 0$ if and only if $f' \in K(\alpha, \alpha + 2)$. Let $\mathcal{C}$ be the class of functions in $\mathcal{S}$ that are close-to-convex (see [4]). In particular $f \in \mathcal{C}$ if and only if $f' \in K(1, 3)$. The analytical properties and nice geometric interpretation make Kaplan classes find many applications to this day. In [7] Jahangiri and Ponnusamy use Kaplan classes both as a tool in assumptions as well as part of the result. In this article we will need the following theorem by Sheil-Small (see [12, p. 248], in [5] and [6] one can find additional information about polynomials with all zeros of derivative in $\mathbb{T}$ in the context of Kaplan classes).

Theorem A (Sheil-Small) *Every polynomial of degree n with all zeros on $\mathbb{T}$ belongs to $K(1, 2\pi/\lambda - n + 1)$, where λ is the minimal arclength between each pair of zeros.*

Define $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$,

$$r(\mathcal{F}) := \sup \left\{ r > 0 : \mathbb{D}(r) \subset \bigcap_{f \in \mathcal{F}} f(\mathbb{D}) \right\}$$

and

$$R(\mathcal{F}) := \inf \left\{ r > 0 : \bigcup_{f \in \mathcal{F}} f(\mathbb{D}) \subset \mathbb{D}(r) \right\}$$

for any $\mathcal{F} \subset \mathcal{S}$. Bieberbach in 1916 proved the following theorem (see [1]).

Theorem B (Koebe Quarter Theorem) *For every $f \in \mathcal{S}$, $\mathbb{D}(1/4) \subset f(\mathbb{D})$, i.e. $r(\mathcal{S}) = 1/4$. Moreover the function*

$$\mathbb{D} \ni z \mapsto \frac{z}{(1-z)^2}$$

is extremal for this inclusion.

The problem of determining $r(\mathcal{F})$ or $R(\mathcal{F})$ for any class $\mathcal{F} \subset \mathcal{S}$ is well known in the literature. Goodman devoted an entire chapter (see [4, pp. 113–120]) for so called Koebe domains. The concept of the Koebe domain for a set $\mathcal{F}$ is similar to the concept of $r(\mathcal{F})$ (usually they are equivalent). Directly from Theorem B we conclude that $r(\mathcal{F}) = 1/4$ for $\mathcal{F}$ being the class of starlike functions as well as the class of close-to-convex functions since the Koebe function is starlike and in the consequences close-to-convex. Hence, the Koebe domain for starlike functions and close-to-convex functions is a disc $\mathbb{D}(1/4)$.

Let $n \in \mathbb{N}$. Denote by $\mathcal{U}_n$ the class of all polynomials of degree n belonging to $\mathcal{S}$. Additionally $\mathcal{U}_{n,\mathbb{R}}$ be the subclass of $\mathcal{U}_n$ containing only the polynomials with real coefficients. An interesting problem of determining $r(\mathcal{U}_n)$ and $r(\mathcal{U}_{n,\mathbb{R}})$ was stated in [2]. It is easy to show that $r(\mathcal{U}_{1,\mathbb{R}}) = r(\mathcal{U}_1) = 1$ and $r(\mathcal{U}_{2,\mathbb{R}}) = r(\mathcal{U}_2) = 1/2$, but cases

for $n \geq 3$ are far from trivial. Dmitrishin et al. [2] proved that

$$r(\mathcal{U}_{3,\mathbb{R}}) = \frac{3 - \sqrt{5}}{2} \approx 0,382$$

with the extremal function given by the formula

$$\mathbb{D} \ni z \mapsto z + \frac{2\sqrt{5}}{5}z^2 + \frac{5 - \sqrt{5}}{10}z^3.$$

Moreover they proposed a class of functions which can be extremal at the problem of determining $r(\mathcal{U}_{n,\mathbb{R}})$ for $n \geq 4$.

Let $n \in \mathbb{N}$. Denote by $\mathcal{C}_n(X)$ the class of all polynomials of degree n belonging to $\mathcal{C}$ with all zeros of derivative on $X \subset \mathbb{C} \setminus \mathbb{D}$. Inspired by [2] and [3] we state the problem of determining $r(\mathcal{C}_n(\mathbb{T}))$ and $R(\mathcal{C}_n(\mathbb{T}))$. The classes $\mathcal{C}_n(\mathbb{T})$ are fairly large, $\mathcal{C}_n(\mathbb{T}) \subset \mathcal{U}_n$ and $\mathcal{C}_n(\mathbb{T}) \not\subset \mathcal{U}_{n,\mathbb{R}}$. Let us notice that cases $n = 1$ and $n = 2$ are trivial. It is easy to show that $r(\mathcal{C}_1(\mathbb{T})) = R(\mathcal{C}_1(\mathbb{T})) = 1$, $r(\mathcal{C}_2(\mathbb{T})) = 1/2$ and $R(\mathcal{C}_2(\mathbb{T})) = 3/2$ with the extremal functions the same as in [2] for the problem of determining $r(\mathcal{U}_1)$ and $r(\mathcal{U}_2)$. In this article we determine $r(\mathcal{C}_n(\mathbb{T}))$ and $R(\mathcal{C}_n(\mathbb{T}))$ for $n = 3$ and $n = 4$.

1 Main results

In this section we solve the problem of determining $r(\mathcal{C}_n(\mathbb{T}))$ and $R(\mathcal{C}_n(\mathbb{T}))$ for $n = 3$ and $n = 4$.

Theorem 1.1 *The following equalities hold:*

$$r(\mathcal{C}_3(\mathbb{T})) = \frac{\sqrt{10}}{6} \quad (1.1)$$

and

$$R(\mathcal{C}_3(\mathbb{T})) = \frac{8 + 3\sqrt{2}}{6}. \quad (1.2)$$

Moreover the function

$$\mathbb{D} \ni z \mapsto z + \frac{-1 + i}{2}z^2 - \frac{i}{3}z^3$$

is extremal for (1.1) and (1.2).

Proof Let $P \in \mathcal{C}_3(\mathbb{T})$. Then by Theorem A,

$$\mathbb{D} \ni z \mapsto P(z) := \int_0^z (1 - ue^{-it_1})(1 - ue^{-it_2}) du$$

for certain $t_1, t_2 \in \mathbb{R}$ such that $\pi/2 \leq |t_1 - t_2| \leq \pi$. Without loss of generality we can assume that $t_1 := 0$ and $\pi/2 \leq t_2 \leq \pi$. Therefore

$$\mathbb{D} \ni z \mapsto P(z) := z - \frac{1}{2} \left(1 + e^{-it_2}\right) z^2 + \frac{1}{3} e^{-it_2} z^3.$$

Since P is a polynomial, so there exists the holomorphic extension of P on $\mathbb{T}$. Setting $z := e^{i\theta}$ for $\theta \in [0, 2\pi)$ we get

$$\begin{aligned} g(\theta) &:= \left| P(e^{i\theta}) \right|^2 = \left| 1 - \frac{1}{2} e^{i\theta} - \frac{1}{2} e^{i(\theta-t_2)} + \frac{1}{3} e^{i(2\theta-t_2)} \right|^2 \\ &= \frac{29}{18} - \frac{4}{3} \cos(\theta) + \frac{1}{2} \cos(t_2) - \frac{4}{3} \cos(\theta - t_2) + \frac{2}{3} \cos(2\theta - t_2). \end{aligned}$$

Therefore

$$\begin{aligned} g'(\theta) &= \frac{4}{3} (\sin(\theta) + \sin(\theta - t_2) - \sin(2\theta - t_2)) \\ &= \frac{16}{3} \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - t_2}{2}\right) \sin\left(\frac{2\theta - t_2}{2}\right) \end{aligned}$$

and as a consequence

$$g'(\theta) = 0 \iff \theta \in \left\{ 0, \frac{t_2}{2}, t_2, \frac{t_2 + 2\pi}{2} \right\}.$$

Function g has the minima at 0 and t_2 and the maxima at $t_2/2$ and $t_2/2 + \pi$. Since

$$g(0) = g(t_2) = \frac{5}{18} - \frac{1}{6} \cos(t_2) \in \left[\frac{5}{18}, \frac{4}{9} \right]$$

and

$$\begin{aligned} g\left(\frac{t_2}{2}\right) &= \left(\frac{4}{3} - \cos\left(\frac{t_2}{2}\right)\right)^2 \leq \left(\frac{4}{3} + \cos\left(\frac{t_2}{2}\right)\right)^2 \\ &= g\left(\frac{t_2}{2} + \pi\right) \in \left[\frac{16}{9}, \left(\frac{8 + 3\sqrt{2}}{6}\right)^2 \right], \end{aligned}$$

so for every $z \in \mathbb{T}$,

$$\frac{\sqrt{10}}{6} \leq |P(z)| \leq \frac{8 + 3\sqrt{2}}{6},$$

which leads to (1.1) and (1.2).

Now consider the function

$$\mathbb{C} \ni z \mapsto f(z) := z + \frac{-1+i}{2}z^2 - \frac{i}{3}z^3.$$

Then $f|_{\mathbb{D}} \in \mathcal{C}_3(\mathbb{T})$,

$$|f(1)| = \frac{\sqrt{10}}{6}$$

and

$$\left| f\left(\frac{-1-i}{\sqrt{2}}\right) \right| = \frac{8+3\sqrt{2}}{6}.$$

□

Fix $t_2 \in [2/5\pi, 2/3\pi]$ and $t_3 \in [2t_2, \pi + t_2/2]$. Define

$$\Omega_1 := \left[\max \left\{ 0, -\frac{2}{3} \arccos \left(-\frac{6}{11} \right) + \frac{t_2+t_3}{3} \right\}, -\frac{2}{3} \arccos \left(\frac{6}{11} \right) + \frac{t_2+t_3}{3} \right],$$

$$\Omega_2 := \left[\frac{2}{3} \arccos \left(\frac{6}{11} \right) + \frac{t_2+t_3}{3}, \frac{2}{3} \arccos \left(-\frac{6}{11} \right) + \frac{t_2+t_3}{3} \right],$$

$$\Omega_3 := \left[\frac{4}{3}\pi - \frac{2}{3} \arccos \left(-\frac{6}{11} \right) + \frac{t_2+t_3}{3}, \frac{4}{3}\pi - \frac{2}{3} \arccos \left(\frac{6}{11} \right) + \frac{t_2+t_3}{3} \right],$$

$$\Omega_4 := \left[\frac{4}{3}\pi + \frac{2}{3} \arccos \left(\frac{6}{11} \right) + \frac{t_2+t_3}{3}, 2\pi \right],$$

$$[0, 2\pi) \ni \theta \mapsto h_1(\theta) := \cos \left(\frac{3\theta - t_2 - t_3}{2} \right),$$

$$[0, 2\pi) \ni \theta \mapsto h_2(\theta) := \cos \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta - t_2}{2} \right) \cos \left(\frac{\theta - t_3}{2} \right),$$

$$[0, 2\pi) \ni \theta \mapsto h(\theta) := \frac{11}{8}h_1(\theta) - h_2(\theta).$$

Now we prove the following lemma.

Lemma 1.2 *The function h has exactly three zeros, one in each interval $(0, t_2)$, (t_2, t_3) and $(t_3, 2\pi)$.*

Proof Since

$$\begin{aligned} \left| \cos \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta - t_2}{2} \right) \cos \left(\frac{\theta - t_3}{2} \right) \right| &\leq \left| \cos \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta - t_3}{2} \right) \right| \\ &= \frac{1}{2} \left| \cos \left(\frac{2\theta - t_3}{2} \right) + \cos \left(\frac{t_3}{2} \right) \right| \leq \frac{1}{2} \left(1 + \left| \cos \left(\frac{t_3}{2} \right) \right| \right) \leq \frac{3}{4} \end{aligned} \quad (1.3)$$

for $\theta \in [0, 2\pi)$, so $h(\theta) \neq 0$ if

$$\left| \cos \left(\frac{3\theta - t_2 - t_3}{2} \right) \right| > \frac{6}{11} .$$

Therefore $h(\theta) \neq 0$ if $\theta \in [0, 2\pi) \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4)$. For $\theta \in [0, 2\pi)$ we obtain

$$h'(\theta) = -\frac{25}{16} \sin \left(\frac{3\theta - t_2 - t_3}{2} \right) + \frac{1}{2} \sin \left(\frac{\theta}{2} \right) \sin \left(\frac{\theta - t_2}{2} \right) \sin \left(\frac{\theta - t_3}{2} \right) .$$

Let us notice that

$$\frac{3}{10}\pi < \arccos \left(\frac{6}{11} \right) < \frac{3}{8}\pi \quad (1.4)$$

and

$$\frac{3}{5}\pi < \arccos \left(-\frac{6}{11} \right) < \frac{3}{4}\pi . \quad (1.5)$$

Assume that $\theta \in \Omega_1$. Since $t_2 \in [2/5\pi, 2/3\pi]$ and $t_3 \in [2t_2, \pi + t_2/2]$, so by using (1.4) and (1.5) we get

$$\begin{aligned} \frac{\theta}{2} &\in \left[0, \frac{7}{30}\pi \right] , \\ \frac{\theta - t_2}{2} &\in \left[-\frac{1}{4}\pi, -\frac{1}{30}\pi \right] , \\ \frac{\theta - t_3}{2} &\in \left[-\frac{7}{12}\pi, -\frac{3}{10}\pi \right] , \\ \frac{3\theta - t_2 - t_3}{2} &\in \left[-\frac{3}{4}\pi, -\frac{3}{10}\pi \right] . \end{aligned}$$

Therefore

$$\begin{aligned} \sin \left(\frac{\theta - t_2}{2} \right), \sin \left(\frac{\theta - t_3}{2} \right), \sin \left(\frac{3\theta - t_2 - t_3}{2} \right) &< 0 , \\ \sin \left(\frac{\theta}{2} \right) &\geq 0 . \end{aligned}$$

Hence $h'(\theta) > 0$ for $\theta \in \Omega_1$.

Assume that $\theta \in \Omega_2$. Since $t_2 \in [2/5\pi, 2/3\pi]$ and $t_3 \in [2t_2, \pi + t_2/2]$, so by using (1.4) and (1.5) we get

$$\frac{\theta}{2} \in \left[\frac{3}{10}\pi, \frac{7}{12}\pi \right] ,$$

$$\begin{aligned}\frac{\theta - t_2}{2} &\in \left[\frac{1}{10}\pi, \frac{19}{60}\pi \right], \\ \frac{\theta - t_3}{2} &\in \left[-\frac{7}{30}\pi, \frac{1}{20}\pi \right], \\ \frac{3\theta - t_2 - t_3}{2} &\in \left[\frac{3}{10}\pi, \frac{3}{4}\pi \right].\end{aligned}$$

Therefore

$$\begin{aligned}\sin\left(\frac{3\theta - t_2 - t_3}{2}\right) &\in \left[\frac{\sqrt{2}}{2}, 1 \right], \\ \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - t_3}{2}\right) &\in \left[\frac{1}{2} \left(\cos\left(\frac{2}{3}\pi\right) - 1 \right), \frac{1}{2} \left(\cos\left(\frac{2}{5}\pi\right) + 1 \right) \right] \\ &= \left[-\frac{3}{4}, \frac{3 + \sqrt{5}}{8} \right], \\ \sin\left(\frac{\theta - t_2}{2}\right) &\in \left(0, \sin\left(\frac{19}{60}\pi\right) \right].\end{aligned}$$

Hence $h'(\theta) < 0$ for $\theta \in \Omega_2$.

Assume that $\theta \in \Omega_3$. Since $t_2 \in [2/5\pi, 2/3\pi]$ and $t_3 \in [2t_2, \pi + t_2/2]$, so by using (1.4) and (1.5) we get

$$\begin{aligned}\frac{\theta}{2} &\in \left[\frac{37}{60}\pi, \frac{9}{10}\pi \right], \\ \frac{\theta - t_2}{2} &\in \left[\frac{5}{12}\pi, \frac{19}{30}\pi \right], \\ \frac{\theta - t_3}{2} &\in \left[\frac{1}{12}\pi, \frac{11}{30}\pi \right], \\ \frac{3\theta - t_2 - t_3}{2} &\in \left[\frac{5}{4}\pi, \frac{17}{10}\pi \right].\end{aligned}$$

Therefore

$$\begin{aligned}\sin\left(\frac{3\theta - t_2 - t_3}{2}\right) &< 0, \\ \sin\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta - t_2}{2}\right), \sin\left(\frac{\theta - t_3}{2}\right) &> 0.\end{aligned}$$

Hence $h'(\theta) > 0$ for $\theta \in \Omega_3$.

Assume that $\theta \in \Omega_4$. Since $t_2 \in [2/5\pi, 2/3\pi]$ and $t_3 \in [2t_2, \pi + t_2/2]$, so by using (1.4) and (1.5) we get

$$\begin{aligned}\frac{\theta}{2} &\in \left[\frac{29}{30}\pi, \pi \right], \\ \frac{\theta - t_2}{2} &\in \left[\frac{23}{30}\pi, \frac{4}{5}\pi \right], \\ \frac{\theta - t_3}{2} &\in \left[\frac{13}{30}\pi, \frac{3}{5}\pi \right], \\ \frac{3\theta - t_2 - t_3}{2} &\in \left[\frac{23}{10}\pi, \frac{12}{5}\pi \right].\end{aligned}$$

Therefore

$$\begin{aligned}\sin\left(\frac{3\theta - t_2 - t_3}{2}\right) &\in \left[\frac{1 + \sqrt{5}}{4}, \sqrt{\frac{5 + \sqrt{5}}{8}} \right], \\ \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - t_3}{2}\right) &\in \left[\frac{1}{2} \left(\cos\left(\frac{2}{3}\pi\right) - 1 \right), \frac{1}{2} \left(\cos\left(\frac{2}{5}\pi\right) + 1 \right) \right] \\ &= \left[-\frac{3}{4}, \frac{3 + \sqrt{5}}{8} \right], \\ \sin\left(\frac{\theta - t_2}{2}\right) &\in \left[\sqrt{\frac{5 - \sqrt{5}}{8}}, \sin\left(\frac{23}{30}\pi\right) \right].\end{aligned}$$

Hence $h'(\theta) < 0$ for $\theta \in \Omega_4$.

Let us notice that

$$\begin{aligned}h_1\left(-\frac{2}{3}\arccos\left(-\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) &= h_1\left(\frac{2}{3}\arccos\left(-\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) \\ &= h_1\left(\frac{4}{3}\pi - \frac{2}{3}\arccos\left(-\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) = -\frac{6}{11}, \\ h_1\left(-\frac{2}{3}\arccos\left(\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) &= h_1\left(\frac{2}{3}\arccos\left(\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) \\ &= h_1\left(\frac{4}{3}\pi - \frac{2}{3}\arccos\left(\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) \\ &= h_1\left(\frac{4}{3}\pi + \frac{2}{3}\arccos\left(\frac{6}{11}\right) + \frac{t_2 + t_3}{3}\right) = \frac{6}{11},\end{aligned}$$

Since $t_2 \in [2/5\pi, 2/3\pi]$ and $t_3 \in [2t_2, \pi + t_2/2]$, so $\Omega_1 \subset [0, t_2)$, $\Omega_3 \cup \Omega_4 \subset (t_3, 2\pi)$ and $t_2 < \min(\Omega_2) < t_3$. Since $t_2 \notin \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, so by (1.3) we obtain

$|h_2(\theta)| < 3/4$ for $\theta \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. Therefore

$$h(\min(\Omega_1)), h(\max(\Omega_2)), h(\min(\Omega_3)) < 0$$

and

$$h(\max(\Omega_1)), h(\min(\Omega_2)), h(\max(\Omega_3)), h(\min(\Omega_4)) > 0.$$

Moreover

$$\begin{aligned} h(0) &= \frac{7}{8} \cos\left(\frac{t_2 + t_3}{2}\right) - \frac{1}{2} \cos\left(\frac{t_3 - t_2}{2}\right) < 0, \\ h(t_3) &= \frac{7}{8} \cos\left(\frac{2t_3 - t_2}{2}\right) - \frac{1}{2} \cos\left(\frac{t_2}{2}\right) < 0, \\ h(2\pi) &= -\frac{7}{8} \cos\left(\frac{t_2 + t_3}{2}\right) + \frac{1}{2} \cos\left(\frac{t_3 - t_2}{2}\right) > 0. \end{aligned}$$

Hence and by monotonicity of h on intervals $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 we conclude that h has exactly three zeros, one in each interval $(0, t_2)$, (t_2, t_3) and $(t_3, 2\pi)$. $\square$

Using Lemma 1.2 we prove the following theorem.

Theorem 1.3 *The following equalities hold:*

$$r(\mathcal{C}_4(\mathbb{T})) = \frac{8 - \sqrt{5}}{12} \quad (1.6)$$

and

$$R(\mathcal{C}_4(\mathbb{T})) = \frac{20 + 5\sqrt{5}}{12}. \quad (1.7)$$

Moreover the function

$$\begin{aligned} \mathbb{D} \ni z \mapsto & z - \frac{1 - i\sqrt{5 + 2\sqrt{5}}}{4} z^2 - \frac{3 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}}{12} z^3 \\ & + \frac{1 + \sqrt{5} - i\sqrt{10 - 2\sqrt{5}}}{16} z^4 \end{aligned}$$

is extremal for (1.6) and (1.7).

Proof Let $P \in \mathcal{C}_4(\mathbb{T})$. Then by Theorem A,

$$\mathbb{D} \ni z \mapsto P(z) := \int_0^z \left(1 - ue^{-it_1}\right) \left(1 - ue^{-it_2}\right) \left(1 - ue^{-it_3}\right) du$$

for certain $t_1, t_2, t_3 \in \mathbb{R}$ such that $2\pi/5 \leq |t_1 - t_2| \leq |t_2 - t_3| \leq 4\pi/5$. Without loss of generality we can assume that $t_1 := 0$, $2\pi/5 \leq t_2 \leq 2\pi/3$ and $2t_2 \leq t_3 \leq t_2/2 + \pi$. Therefore P is given by the formula

$$P(z) := z - \frac{1}{2} \left(1 + e^{-it_2} + e^{-it_3} \right) z^2 + \frac{1}{3} \left(e^{-it_2} + e^{-it_3} + e^{-i(t_2+t_3)} \right) z^3 - \frac{1}{4} e^{-i(t_2+t_3)} z^4.$$

Since P is a polynomial, so there exists the holomorphic extension of P on $\mathbb{T}$. Setting $z := e^{i\theta}$ for $\theta \in [0, 2\pi)$ we get

$$\begin{aligned} g(\theta) &:= \left| P(e^{i\theta}) \right|^2 = \frac{103}{48} + \frac{13}{18} \cos(t_2) + \frac{13}{18} \cos(t_3) + \frac{13}{18} \cos(t_3 - t_2) \\ &\quad - \frac{11}{6} \cos(\theta) - \frac{11}{6} \cos(\theta - t_2) - \frac{11}{6} \cos(\theta - t_3) - \frac{1}{3} \cos(\theta - t_2 - t_3) \\ &\quad - \frac{1}{3} \cos(\theta + t_2 - t_3) - \frac{1}{3} \cos(\theta - t_2 + t_3) + \frac{11}{12} \cos(2\theta - t_2) \\ &\quad + \frac{11}{12} \cos(2\theta - t_3) \\ &\quad + \frac{11}{12} \cos(2\theta - t_2 - t_3) - \frac{1}{2} \cos(3\theta - t_2 - t_3) \end{aligned}$$

Therefore

$$g'(\theta) = -\frac{32}{3} h(\theta) \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - t_2}{2}\right) \sin\left(\frac{\theta - t_3}{2}\right).$$

The function

$$[0, 2\pi) \ni \theta \mapsto \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta - t_2}{2}\right) \sin\left(\frac{\theta - t_3}{2}\right)$$

has exactly three zeros at points 0, t_2 and t_3 . Hence by Lemma 1.2 we conclude that the function g' has exactly six zeros. Therefore g has three minima at points 0, t_2 and t_3 and three maxima one in each interval $(0, t_2)$, (t_2, t_3) and $(t_3, 2\pi)$. Moreover

$$\begin{aligned} g(0) &= \frac{5}{16} - \frac{7}{36} \cos(t_2) - \frac{7}{36} \cos(t_3) + \frac{1}{18} \cos(t_3 - t_2) + \frac{1}{12} \cos(t_2 + t_3) \\ &\geq \frac{5}{16} - \frac{7}{36} \cos\left(\frac{2}{5}\pi\right) - \frac{7}{36} \cos\left(\frac{4}{5}\pi\right) + \frac{1}{18} \cos\left(\frac{4}{5}\pi\right) \\ &\quad + \frac{1}{12} \cos\left(\frac{6}{5}\pi\right) = \frac{54 - 5\sqrt{5}}{144}, \end{aligned}$$

$$g(t_3) = \frac{5}{16} + \frac{1}{18} \cos(t_2) - \frac{7}{36} \cos(t_3) - \frac{7}{36} \cos(t_3 - t_2) + \frac{1}{12} \cos(2t_3 - t_2)$$

$$\begin{aligned} &\geq \frac{5}{16} + \frac{1}{18} \cos\left(\frac{4}{5}\pi\right) - \frac{7}{36} \cos\left(\frac{4}{5}\pi\right) - \frac{7}{36} \cos\left(\frac{2}{5}\pi\right) \\ &\quad + \frac{1}{12} \cos\left(\frac{6}{5}\pi\right) = \frac{54 - 5\sqrt{5}}{144}, \end{aligned}$$

and

$$g(t_2) = \frac{5}{16} - \frac{7}{36} \cos(t_2) + \frac{1}{18} \cos(t_3) - \frac{7}{36} \cos(t_3 - t_2) + \frac{1}{12} \cos(t_3 - 2t_2).$$

Since $g(t_2)$ is increasing with respect to t_2 and t_3 , so

$$\begin{aligned} g(t_2) &\geq \frac{5}{16} - \frac{7}{36} \cos\left(\frac{2}{5}\pi\right) + \frac{1}{18} \cos\left(\frac{4}{5}\pi\right) - \frac{7}{36} \cos\left(\frac{2}{5}\pi\right) + \frac{1}{12} \\ &= \left(\frac{8 - \sqrt{5}}{12}\right)^2 \end{aligned}$$

and the equality holds if and only if $t_2 = 2/5\pi$ and $t_3 = 4/5\pi$. This ends the proof of (1.6).

Now we prove (1.7). Let us notice that for $z \in \mathbb{D} \cup \mathbb{T}$,

$$\begin{aligned} |P(z)| &\leq \frac{5}{4} + \frac{1}{2} \left| 1 + e^{-it_2} + e^{-it_3} \right| + \frac{1}{3} \left| e^{-it_2} + e^{-it_3} + e^{-i(t_2+t_3)} \right| \\ &= \frac{5}{4} + \frac{5}{6} \left| 1 + e^{it_2} + e^{it_3} \right| = \frac{5}{4} + \frac{5}{6} \sqrt{3 + 2\cos(t_2) + 2\cos(t_3) + 2\cos(t_3 - t_2)} \\ &= \frac{5}{4} + \frac{5}{6} \sqrt{1 + 4\cos^2\left(\frac{t_2}{2}\right) + 4\cos\left(\frac{t_2}{2}\right)\cos\left(\frac{2t_3 - t_2}{2}\right)} \\ &\leq \frac{5}{4} + \frac{5}{6} \sqrt{1 + 4\cos^2\left(\frac{t_2}{2}\right) + 4\cos\left(\frac{t_2}{2}\right)\cos\left(\frac{3t_2}{2}\right)} = \frac{5}{4} + \frac{5}{6} (1 + 2\cos(t_2)) \\ &\leq \frac{5}{4} + \frac{5}{6} \left(1 + \frac{\sqrt{5} - 1}{2}\right) = \frac{20 + 5\sqrt{5}}{12}. \end{aligned}$$

and the equality holds if and only if $z = e^{\frac{7}{5}\pi i}$, $t_2 = 2/5\pi$ and $t_3 = 4/5\pi$. This ends the proof of (1.7).

Moreover from the above considerations we conclude that the extremal function for (1.6) and (1.7) is

$$\begin{aligned} \mathbb{D} \ni z &\mapsto z - \frac{1 + e^{-\frac{2}{5}\pi i} + e^{-\frac{4}{5}\pi i}}{2} z^2 + \frac{e^{-\frac{2}{5}\pi i} + e^{-\frac{4}{5}\pi i} + e^{-\frac{6}{5}\pi i}}{3} z^3 - \frac{e^{-\frac{6}{5}\pi i}}{4} z^4 \\ &= z - \frac{1 - i\sqrt{5 + 2\sqrt{5}}}{4} z^2 - \frac{3 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}}{12} z^3 \end{aligned}$$

$$+ \frac{1 + \sqrt{5} - i\sqrt{10 - 2\sqrt{5}}}{16} z^4.$$

□

Compliance with ethical standards

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