Integrability structures of the generalized Hunter–Saxton equation

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Abstract

We consider integrability structures of the generalized Hunter–Saxton equation. We obtain the Lax representation with non-removable spectral parameter, find local recursion operators for symmetries and cosymmetries, generate an infinite-dimensional Lie algebra of higher symmetries, and prove existence of infinite number of cosymmetries of higher order. Further, we give examples of employing the higher order symmetries to constructing exact globally defined solutions for the generalized Hunter–Saxton equation.

Keywords generalized Hunter–Saxton equation · Lax representation · Symmetry · Cosymmetry · Recursion operator · Conservation law

Mathematics Subject Classification 35G20 · 35Q60 · 17B50 · 22E70

1 Introduction

The Hunter–Saxton equation

$$u_{tx} = u u_{xx} + \frac{1}{2} u_x^2 \quad (1)$$

was introduced in [12] to describe the nonlinear instability of the director field in the nematic liquid crystal and then has been a subject of thorough investigation. As it was shown in [12], Eq. (1) admits a Lagrangian formulation with Lagrangian $L = (u_t - u u_x) u_x$. In [13] a bi-Hamiltonian structure, a Lax representation, a nonlocal recursion operator, and a series of conservation laws have been found. A tri-Hamiltonian formulation for (1) was proposed in [29]. Inverse scattering solutions

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for (1) were constructed in [2]. In [16] it has been proven that Eq. (1) can be understood as a geodesic equation associated to a right-invariant metric on an appropriate homogeneous space related to the Virasoro group. The pseudo-spherical formulation for Eq. (1) and quadratic pseudopotentials were proposed and used to find nonlocal symmetries and conservation laws in [33]. In [11], the nonlocal symmetries were used to construct exact solutions and a nonlocal recursion operator for (1). Nonlocal recursion operators, a fourth order local recursion operator, series of higher symmetries and conservation laws for Eq. (1) have been constructed in [38], see also [35].

The further discussion of the physical interpretation of Eq. (1) can be found in [4].

In this paper we consider the generalization

$$u_{tx} = u u_{xx} + \beta u_x^2, \quad \beta \neq 0, \quad (2)$$

of the Hunter–Saxton Eq. (1). This equation with $\beta \neq \frac{1}{2}$ has applications in geometry of Einstein–Weil structures [7,36], and in hydrodynamics [10]. In [5,31] a nonlocal transformation was used to construct a general solution for (2). In [26] we have shown that Eq. (2) is linearizable via the contact transformation $(t, x, u, u_t, u_x) \mapsto (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{t}}, \tilde{u}_{\tilde{x}})$ given by the formulae

$$\begin{cases} t = \beta^{-1} \tilde{t}, \\ x = -(\tilde{t} + \tilde{x})^{\frac{\beta-1}{\beta}} (\beta (\tilde{t} + \tilde{x}) \tilde{u}_{\tilde{x}} - \tilde{u}), \\ u = (\tilde{t} + \tilde{x})^{-\frac{1}{\beta}} (\beta (\tilde{t} + \tilde{x}) \tilde{u}_{\tilde{x}} + (\beta - 1) \tilde{u}), \\ u_t = \beta^2 (\tilde{t} + \tilde{x})^{-\frac{1}{\beta}} (\tilde{u}_{\tilde{t}} - \tilde{u}_{\tilde{x}}), \\ u_x = -(\tilde{t} + \tilde{x})^{-1}. \end{cases} \quad (3)$$

This transformation maps (2) to the Euler–Poisson equation

$$\tilde{u}_{\tilde{t}\tilde{x}} = \frac{1}{\beta (\tilde{t} + \tilde{x})} \tilde{u}_{\tilde{t}} + \frac{2(\beta - 1)}{\beta (\tilde{t} + \tilde{x})} \tilde{u}_{\tilde{x}} + \frac{2(\beta - 1)}{(\beta (\tilde{t} + \tilde{x}))^2} \tilde{u}. \quad (4)$$

Equation (4) is integrable by quadratures via Laplace’s method, [30, § 9.3]. The general solution to (4) combined with the inverse transformation to (3) provides the parametric formula for the general solution to Eq. (2), see details in [26]. This formula is locally defined and does not give global solutions to (2), while such solutions are of interest from the viewpoint of applications, see discussion in [4,12].

In the present paper we study integrability properties of Eq. (2). In Sect. 3 we find the Lax representation for (2) with arbitrary β . We show that this Lax representation includes the non-removable spectral parameter. We study contact symmetries of this equation in Sect. 4. We show that the Lie algebra of contact symmetries of Eq. (2) is the semi-direct sum $\mathfrak{s}_4 \ltimes \mathfrak{a}_\infty$ of the four-dimensional Lie algebra $\mathfrak{s}_4 \cong \mathfrak{gl}_2(\mathbb{R})$ and the infinite-dimensional Abelian ideal $\mathfrak{a}_\infty$. Then in Sect. 5 we apply the approach of [18–20,23] to find local and nonlocal recursion operators for symmetries of (2). In Sect. 6 we study the action of local recursion operators to the subalgebra $\mathfrak{s}_4$. This action generates a Lie subalgebra $\mathfrak{s}_\infty$ of the algebra of higher symmetries of Eq. (2). We show that $\mathfrak{s}_\infty$ has an interesting structure of the so-called Lie algebra of matrices of

complex size, [9]. Cosymmetries of (2) and recursion operators for cosymmetries are discussed in Sect. 7. Finally, in Sect. 8 we use higher symmetries from $\mathfrak{s}_\infty$ to construct globally defined invariant solutions of Eq. (2).

To simplify notation, we put $\beta = (\alpha + 2)^{-1}$, $\alpha \neq -2$, so Eq. (2) gets the form

$$u_{tx} = u u_{xx} + \frac{1}{\alpha + 2} u_x^2. \quad (5)$$

2 Preliminaries

The presentation in this section closely follows [21–25, 37]. Let $\pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\pi: (x^1, \dots, x^n, u^1, \dots, u^m) \mapsto (x^1, \dots, x^n)$, be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^i, u^\alpha, u_I^\alpha)$, where $I = (i_1, \dots, i_n)$ are multi-indices, and for every local section $f: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f): \mathbb{R}^n \rightarrow J^\infty(\pi)$ such that $u_I^\alpha(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$. We put $u^\alpha = u_{(0, \dots, 0)}^\alpha$. Also, we will simplify notation in the following way, e.g., in the case of $n = 2, m = 1$: we denote $x^1 = t, x^2 = x$ and $u_{(i,j)}^1 = u_{t \dots t x \dots x}$ with i times t and j times x , or u_{kx} , $k \in \mathbb{N}$, for $u_{xx \dots x}$ with k times x .

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u_{I+1_k}^\alpha \frac{\partial}{\partial u_I^\alpha}, \quad k \in \{1, \dots, n\},$$

$(i_1, \dots, i_k, \dots, i_n) + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$, are called *total derivatives*. They commute everywhere on $J^\infty(\pi)$.

The *evolutionary vector field* associated to an arbitrary vector-valued smooth function $\varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m$ is the vector field

$$\mathbf{E}_\varphi = \sum_{\#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u_I^\alpha} \quad (6)$$

with $D_I = D_{(i_1, \dots, i_n)} = D_{x^1}^{i_1} \circ \dots \circ D_{x^n}^{i_n}$. Notice that

$$[\mathbf{E}_\varphi, D_{x^i}] = 0 \quad (7)$$

for any φ and i .

A system of PDEs $F_r(x^i, u_I^\alpha) = 0$ of the order $s \geq 1$ with $\#I \leq s, r \in \{1, \dots, R\}$ for some $R \geq 1$, defines the submanifold $\mathcal{E} = \{(x^i, u_I^\alpha) \in J^\infty(\pi) \mid D_K(F_r(x^i, u_I^\alpha)) = 0, \#K \geq 0\}$ in $J^\infty(\pi)$.

A function $\varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m$ is called a (*generator of an infinitesimal*) *symmetry* of equation $\mathcal{E}$ when $\mathbf{E}_\varphi(F) = 0$ on $\mathcal{E}$. The symmetry φ is a solution to the *defining*

system

$$\ell_{\mathcal{E}}(\varphi) = 0, \quad (8)$$

where $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$ with the matrix differential operator

$$\ell_F = \left(\sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} D_I \right).$$

The *symmetry algebra* $\text{Sym}(\mathcal{E})$ of equation $\mathcal{E}$ is the linear space of solutions to (8) endowed with the structure of a Lie algebra over $\mathbb{R}$ by the *Jacobi bracket* $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$. The *algebra of contact symmetries* $\text{Sym}_0(\mathcal{E})$ is the Lie subalgebra of $\text{Sym}(\mathcal{E})$ defined as $\text{Sym}(\mathcal{E}) \cap C^\infty(J^1(\pi))$.

Let the linear space $\mathcal{W}$ be either $\mathbb{R}^N$ for some $N \geq 1$ or $\mathbb{R}^\infty$ endowed with local coordinates w^s , $s \in \{1, \dots, N\}$ or $s \in \mathbb{N}$, respectively. Locally, a *differential covering* of $\mathcal{E}$ is a trivial bundle $\tau: J^\infty(\pi) \times \mathcal{W} \rightarrow J^\infty(\pi)$ equipped with *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_s T_k^s(x^i, u_I^\alpha, w^j) \frac{\partial}{\partial w^s}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ if and only if $(x^i, u_I^\alpha) \in \mathcal{E}$. Define the partial derivatives of w^s by $w_{x^k}^s = \tilde{D}_{x^k}(w^s)$. This yields the system

$$w_{x^k}^s = T_k^s(x^i, u_I^\alpha, w^j) \quad (9)$$

that is compatible iff $(x^i, u_I^\alpha) \in \mathcal{E}$. System (9) is referred to as the *covering equations* or the *Lax representation* of equation $\mathcal{E}$.

Example 1 A differential covering for the Hunter–Saxton Eq. (1) has been presented in [33]. In a slightly different notation this is defined on $J^\infty(\pi) \times \mathbb{R}$ with $\pi: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$, $\pi: (t, x, u) \mapsto (t, x)$, by the vector fields

$$\begin{cases} \tilde{D}_t = D_t + \left(\left(\lambda u + \frac{1}{2} \right) w^2 + u_x w - u u_{xx} \right) \frac{\partial}{\partial w}, \\ \tilde{D}_x = D_x + \left(\lambda w^2 - u_{xx} \right) \frac{\partial}{\partial w}, \end{cases} \quad (10)$$

or by the system of the covering equations

$$\begin{cases} w_t = \left(\lambda u + \frac{1}{2} \right) w^2 + u_x w - u u_{xx}, \\ w_x = \lambda w^2 - u_{xx}. \end{cases} \quad (11)$$

The compatibility condition $(w_t)_x = (w_x)_t$ of this system coincides with Eq. (1). The τ -vertical parts of the right-hand sides of (10) are linear combinations of the vector

fields ∂_w , $w \partial_w$, and $w^2 \partial_w$. These vector fields generate the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ referred to as the *universal algebra* of the covering, [24]. $\diamond$

Consider operator $\tilde{\mathbf{E}}_\phi$ obtained by replacing D_{x^k} to $\tilde{D}_{x^k}$ in (6). Solutions $\phi = \phi(x^i, u_I^\alpha, w^j)$ to equation $\tilde{\mathbf{E}}_\phi(F) = 0$ are referred to as *shadows of symmetries in the covering* τ , or just *shadows*.

A PDE $\mathcal{E}$ has two important coverings: the tangent covering $\mathcal{T}\mathcal{E}$ and the cotangent covering $\mathcal{T}^*\mathcal{E}$. Their covering equations are given by systems $\ell_{\mathcal{E}}(q) = 0$ and $\ell_{\mathcal{E}}^*(p) = 0$, respectively, where $\ell_{\mathcal{E}}^*$ is the adjoint operator to $\ell_{\mathcal{E}}$. The local sections of the tangent covering are (generators of) symmetries, while the local sections of the cotangent covering are referred to as *cosymmetries*. The cosymmetries generate conservation laws for equation $\mathcal{E}$, see discussion in [23, Ch. 1]. and Example 4 below.

Example 2 The covering equations for the tangent and cotangent coverings of Eq. (5) have the form

$$\ell_{\mathcal{E}}(q) = q_{tx} - u q_{xx} - \frac{2}{\alpha + 2} u_x q_x - u_{xx} q = 0 \quad (12)$$

and

$$\ell_{\mathcal{E}}^*(p) = p_{tx} - u p_{xx} - \frac{2(\alpha + 1)}{\alpha + 2} (u_x p_x + u_{xx} p) = 0. \quad (13)$$

For $\alpha = 0$ Eqs. (12) and (13) coincide. This property holds for each PDE that admits a Lagrangian formulation, [23, Example 10.1]. $\diamond$

A *recursion operator for symmetries* of a PDE $\mathcal{E}$ is a Bäcklund autotransformation in the tangent covering $\mathcal{T}\mathcal{E}$. In other words, this is an operator $\mathcal{R}$ such that

$$\ell_{\mathcal{E}} \circ \mathcal{R} = \mathcal{S} \circ \ell_{\mathcal{E}} \quad (14)$$

for some operator $\mathcal{S}$. Likewise, a *recursion operator for cosymmetries* of a PDE $\mathcal{E}$ is a Bäcklund autotransformation in the cotangent covering $\mathcal{T}^*\mathcal{E}$. Taking adjoint operators to both sides of (14) we get

$$\mathcal{R}^* \circ \ell_{\mathcal{E}}^* = \ell_{\mathcal{E}}^* \circ \mathcal{S}^*. \quad (15)$$

Therefore operator $\mathcal{S}^*$ is a recursion operator for cosymmetries.

3 Lax representation

Based on Example 1 we conjecture that the generalized Hunter–Saxton Eq. (5) admits a Lax representation with the same universal algebra $\mathfrak{sl}_2(\mathbb{R})$. We assume also that the coefficients of the covering equations are functions of u , u_x , and u_{xx} , that is, there exists system

$$\begin{cases} w_t = T_2 w^2 + T_1 w + T_0, \\ w_x = X_2 w^2 + X_1 w + X_0, \end{cases} \quad (16)$$

with $T_i = T_i(u, u_x, u_{xx})$ and $X_i = X_i(u, u_x, u_{xx})$ such that (5) coincides with the integrability conditions of (16). Direct computations give such a system:

$$\begin{cases} w_t = \left(\lambda u u_x^\alpha + \frac{1}{\alpha + 2} \right) w^2 + \frac{2}{\alpha + 2} u_x w - u u_{xx}, \\ w_x = \lambda u_x^\alpha w^2 - u_{xx}. \end{cases} \quad (17)$$

When $\alpha = 0$, this system coincides with (11). The parameter $\lambda \neq 0$ in both systems (11) and (17) is non-removable. In accordance with [25, §§3.2, 3.6], [14, 17], to prove this assertion it is sufficient to note that symmetry $V = x \partial_x + u \partial_u$ of Eq. (5) does not admit a lift to a symmetry of system (17). Therefore the action $e^{\epsilon V} : (t, x, u, u_t, u_x, u_{xx}, w, w_t, w_x) \mapsto (t, e^\epsilon x, e^\epsilon u, e^\epsilon u_t, u_x, e^{-\epsilon} u_{xx}, w, w_t, e^{-\epsilon} w_x)$ of operator $e^{\epsilon V}$ transforms system (17) with $\lambda = 1$ to system

$$\begin{cases} w_t = \left(e^\epsilon u u_x^\alpha + \frac{1}{\alpha + 2} \right) w^2 + \frac{2}{\alpha + 2} u_x w - u u_{xx}, \\ e^{-\epsilon} w_x = u_x^\alpha w^2 - e^{-\epsilon} u_{xx}, \end{cases}$$

which coincides with (17) when $\lambda = e^\epsilon$.

The map

$$\partial_w \mapsto - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad w \partial_w \mapsto \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w^2 \partial_w \mapsto \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

rearranges the Lax representation (17) into the matrix form¹ $A_t - B_x = [A, B]$ with

$$A = \begin{pmatrix} 0 & u_{xx} \\ \lambda u u_x^\alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{\alpha + 2} u_x & u u_{xx} \\ \lambda u u_x^\alpha + \frac{1}{\alpha + 2} & -\frac{1}{\alpha + 2} u_x \end{pmatrix}.$$

System (17) can be written in the form of the pseudospherical type surface equations

$$\begin{cases} d\omega_1 = \omega_3 \wedge \omega_2, \\ d\omega_2 = \omega_1 \wedge \omega_3, \\ d\omega_3 = \omega_1 \wedge \omega_2, \end{cases}$$

with

$$\begin{aligned} \omega_1 &= \frac{2}{\alpha + 2} u_x dx, \\ \omega_2 &= \left(u u_{xx} + \frac{1}{\alpha + 2} + \lambda u u_x^\alpha \right) dt + (u_{xx} + \lambda u_x^\alpha) dx, \\ \omega_3 &= \left(u u_{xx} + \frac{1}{\alpha + 2} - \lambda u u_x^\alpha \right) dt + (u_{xx} - \lambda u_x^\alpha) dx, \end{aligned}$$

¹ This form of a Lax representation is called *the zero curvature representation*.

see discussion of the pseudospherical type equations in [6,11,33,34] and references therein.

4 Contact symmetries

The Lie algebra $\text{Sym}_0(\mathcal{E})$ of contact symmetries of Eq. (5) is generated by functions²

$$\begin{aligned}\phi_{0,0} &= x u_x - u, \\ \phi_{1,0} &= u_t, \\ \phi_{1,1} &= -2t u_t - (\alpha + 2)x u_x + \alpha u, \\ \phi_{1,2} &= -t^2 u_t - (\alpha + 2)t x u_x + \alpha t u - (\alpha + 2)x,\end{aligned}\tag{18}$$

and the family of solutions $U = U(t, u_x)$ to the linear PDE

$$U_{tu_x} = -\frac{1}{\alpha + 2} u_x^2 U_{u_x u_x} - u_x U_{u_x} + U.\tag{19}$$

The commutator table of $\text{Sym}_0(\mathcal{E})$ is given by equations

$$\begin{aligned}\{\phi_{0,0}, \phi_{1,i}\} &= 0, \\ \{\phi_{0,0}, U\} &= U, \\ \{\phi_{1,0}, \phi_{1,1}\} &= 2\phi_{1,0}, \\ \{\phi_{1,0}, \phi_{1,2}\} &= -\phi_{1,1}, \\ \{\phi_{1,0}, U\} &= -U_t, \\ \{\phi_{1,1}, \phi_{1,2}\} &= 2\phi_{1,2}, \\ \{\phi_{1,1}, U\} &= 2t U_t - 2u_x U_{u_x} - \alpha U, \\ \{\phi_{1,2}, U\} &= t^2 U_t - \alpha t U - (2t u_x + \alpha + 2) U_{u_x}, \\ \{U, \tilde{U}\} &= 0.\end{aligned}$$

This table implies that the contact symmetry algebra of (5) is the semi-direct sum $\text{Sym}_0(\mathcal{E}) = \mathfrak{s}_4 \ltimes \mathfrak{a}_\infty$ of the four-dimensional subalgebra $\mathfrak{s}_4 = \langle \phi_{0,0}, \phi_{1,0}, \phi_{1,1}, \phi_{1,2} \rangle$ that is isomorphic to $\mathfrak{gl}_2(\mathbb{R})$, and the infinite-dimensional Abelian ideal $\mathfrak{a}_\infty$ spanned by solutions to (19).

Equation (19) has solutions of the form $\psi(A) = A u_x + A'$, where $A = A(t)$ is an arbitrary function. These solutions generate a sub-ideal $\mathfrak{b}_\infty \subsetneq \mathfrak{a}_\infty$. The action of $\mathfrak{s}_4$ on $\mathfrak{b}_\infty$ is given by equations

$$\begin{aligned}\{\phi_{0,0}, \psi(A)\} &= \psi(A), \\ \{\phi_{1,0}, \psi(A)\} &= \psi(-A_t),\end{aligned}$$

² We carried out computations of generators of contact symmetries, their commutator tables, shadows of symmetries, and cosymmetries in the Jets software [1].

$$\begin{aligned}\{\phi_{1,1}, \psi(A)\} &= \psi(2t A_t - (\alpha + 2) A), \\ \{\phi_{1,2}, \psi(A)\} &= \psi(t^2 A_t - (\alpha + 2) t A).\end{aligned}$$

This action has the following reformulation: the vector space $\mathbb{A}$ of smooth functions $A = A(t)$ has a $\mathfrak{s}_4$ -module structure

$$\rho: \mathfrak{s}_4 \times \mathbb{A} \rightarrow \mathbb{A}, \quad \rho: (\phi, A) \mapsto \phi \cdot A \quad (20)$$

defined by formulae

$$\begin{aligned}\phi_{0,0} \cdot A &= A, & \phi_{1,1} \cdot A &= 2t A_t - (\alpha + 2) A, \\ \phi_{1,0} \cdot A &= -A_t, & \phi_{1,2} \cdot A &= t^2 A_t - (\alpha + 2) t A.\end{aligned} \quad (21)$$

Remark 1 Since Eqs. (4) and (5) are locally contact equivalent, their contact symmetry algebras are conjugated via the linearization of transformation (3). In particular, ideal $\mathfrak{a}_\infty$ is related to the part of the symmetry algebra of the linear Eq. (4) generated by the shifts on solutions. This implies that Eq. (19) is contact equivalent to (4); the direct proof of this statement mimics the proof of contact equivalence of Eqs. (5) and (4) presented in [26]. Therefore the problem to find all the local symmetries of the form $U(t, u_x)$ is as hard as the problem to find all solutions to Eq. (5). $\diamond$

5 Recursion operators

In this section we use the methods of [18–20,23] to find local and nonlocal recursion operators for symmetries of Eq. (5).

To construct local recursion operators of first order we search for shadows of symmetries of the form

$$\sigma = Q_1 q_t + Q_2 q_x + Q_3 q, \quad Q_i = Q_i(t, x, u, u_t, u_x, u_{tt}, u_{xx}),$$

where q is a solution to (12). Direct computations then give the following shadows:

$$\begin{aligned}\sigma_0 &= -q_t + \frac{E}{u_{xx}} q_x, \\ \sigma_1 &= 2t q_t - 2 \frac{t E + u_x}{u_{xx}} q_x - \alpha q, \\ \sigma_2 &= t^2 q_t - \frac{t^2 E + 2t u_x + \alpha + 2}{u_{xx}} q_x - \alpha t q,\end{aligned}$$

where E is the right hand side of Eq. (5). Therefore we have

Proposition 1 *Differential operators*

$$\mathcal{R}_0 = -D_t + \frac{E}{u_{xx}} D_x, \quad (22)$$

$$\mathcal{R}_1 = 2t D_t - 2 \frac{t E + u_x}{u_{xx}} D_x - \alpha, \quad (23)$$

$$\mathcal{R}_2 = t^2 D_t - \frac{t^2 E + 2t u_x + \alpha + 2}{u_{xx}} D_x - \alpha t \quad (24)$$

provide local recursion operators for symmetries of $\mathcal{E}$.

Proof Follows from the general results of [23], or from identities

$$\begin{aligned} [\ell_{\mathcal{E}}, \mathcal{R}_0] &= -\frac{u_x (u_x u_{xxx} - (\alpha + 4) u_{xx}^2)}{(\alpha + 2) u_{xx}^2} \ell_{\mathcal{E}}, \\ [\ell_{\mathcal{E}}, \mathcal{R}_1] &= \frac{2u_x ((t u_x + \alpha + 2) u_{xxx} - (\alpha + 4) t u_{xx}^2)}{(\alpha + 2) u_{xx}^2} \ell_{\mathcal{E}}, \\ [\ell_{\mathcal{E}}, \mathcal{R}_2] &= \frac{(t u_x + \alpha + 2)^2 u_{xxx} - (\alpha + 4) t^2 u_x u_{xx}^2}{(\alpha + 2) u_{xx}^2} \ell_{\mathcal{E}}. \end{aligned}$$

□

Notice that the local recursion operators have the following commutator table:

$$[\mathcal{R}_0, \mathcal{R}_1] = 2\mathcal{R}_0, \quad [\mathcal{R}_0, \mathcal{R}_2] = -\mathcal{R}_1, \quad [\mathcal{R}_1, \mathcal{R}_2] = 2\mathcal{R}_2, \quad (25)$$

in other words, they constitute the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

To find nonlocal recursion operators for symmetries we consider the Whitney product of the tangent covering (12) and the cotangent covering given by Eq. (13). Then Green's formula

$$\begin{aligned} (q \ell_{\mathcal{E}}^*(p) - p \ell_{\mathcal{E}}(q)) dt \wedge dx &= d(q p_x) \wedge dx \\ &+ dt \wedge d \left(p q_t - u p q_x - \left(\frac{\alpha}{\alpha + 2} p u_x - u p_x \right) q \right) \end{aligned}$$

provides the canonical conservation law [23, p. 22]

$$\begin{cases} S_t = p q_t - u p q_x - \left(\frac{\alpha}{\alpha + 2} p u_x - u p_x \right) q, \\ S_x = q p_x. \end{cases} \quad (26)$$

In particular, substituting for the solution $p = u_x^{-2}$ of (13) into (26) defines nonlocality s_1 by equations

$$\begin{cases} s_{1,t} = \frac{q_t - u q_x}{u_{xx}^2} - \frac{2(\alpha + 2) u u_{xx} - \alpha u_x^2}{(\alpha + 2) u_x^3} q, \\ s_{1,x} = -2 \frac{u_x}{u_{xx}^2} q. \end{cases} \quad (27)$$

Likewise, the cosymmetry $p = u_x^{\alpha+1}$ defines the nonlocality s_2 by system

$$\begin{cases} s_{2,t} = u_x^\alpha \left(u_x (q_t - u q_x) + (\alpha (\alpha + 2)^{-1} u_x^2 + (\alpha + 1) u u_{xx}) q \right), \\ s_{2,x} = (\alpha + 1) u_x^\alpha u_{xx} q. \end{cases} \quad (28)$$

We obtain four shadows of symmetries in the tangent covering of the form $\sigma = Q_1 q_t + Q_2 q_x + Q_3 q + Q_4 s_1 + Q_5 s_2$ with nontrivial functions Q_4 and Q_5 :

$$\begin{aligned} \sigma_4 &= u_x s_1 - \frac{t u_x + \alpha + 2}{(\alpha + 2) u_x} q, \\ \sigma_5 &= (t u_x + 1) s_1 + \frac{1}{(\alpha + 2)^2} t^3 q_t - \frac{t^3 u_x E + 3 t u_x (t u_x + \alpha + 2) + (\alpha + 2)^2}{(\alpha + 2)^2 u_x u_{xx}} q_x \\ &\quad - \frac{(2\alpha + 1) t^2 u_x^2 + (\alpha + 2)^2 (t u_x + 1)}{(\alpha + 2)^2 u_x^2} q, \\ \sigma_6 &= u_x^{-\alpha-2} s_2 - \frac{t u_x + \alpha + 3}{(\alpha + 2) u_x} q, \\ \sigma_7 &= \frac{(\alpha + 4) t u_x + (\alpha + 2)^2}{(\alpha + 2)^2 u_x^{\alpha+3}} s_2 + \frac{t u_x + 1}{\alpha + 2} s_1 - \frac{(\alpha + 3) (t u_x + \alpha + 2)^2}{(\alpha + 2)^3 u_x^2} q. \end{aligned}$$

From the second equation in (27) we have

$$s_1 = -2 D_x^{-1} \left(\frac{u_{xx}}{u_x^2} q \right),$$

therefore the nonlocal recursion operators

$$-2 u_x D_x^{-1} \circ \frac{u_{xx}}{u_x^2} - \frac{t u_x + \alpha + 2}{(\alpha + 2) u_x}$$

and

$$\begin{aligned} &-2 (t u_x + 1) D_x^{-1} \circ \frac{u_{xx}}{u_x^2} + \frac{t^3}{(\alpha + 2)^2} D_t - \frac{(2\alpha + 1) t^2 u_x^2 + (\alpha + 2)^2 (t u_x + 1)}{(\alpha + 2)^2 u_x^2} \\ &- \frac{t^3 u_x ((\alpha + 2) u u_{xx} + u_x^2) + 3 (\alpha + 2) t u_x (t u_x + \alpha + 2) + (\alpha + 2)^3}{(\alpha + 2)^3 u_x u_{xx}} D_x \end{aligned}$$

are associated with shadows σ_4 and σ_5 . In the same way from the second equation in (28) we have

$$s_2 = (\alpha + 1) D_x^{-1} \left(u_x^\alpha u_{xx} q \right),$$

therefore shadows σ_6 and σ_7 produce the nonlocal recursion operators

$$(\alpha + 1) u_x^{-\alpha-2} D_x^{-1} \circ u_x^\alpha u_{xx} - \frac{t u_x + \alpha + 3}{(\alpha + 2) u_x}$$

and

$$(\alpha + 1) \frac{(\alpha + 4) t u_x + (\alpha + 2)^2}{(\alpha + 2)^2 u_x^{\alpha+3}} D_x^{-1} \circ u_x^\alpha u_{xx} - 2 \frac{t u_x + 1}{\alpha + 2} D_x^{-1} \circ \frac{u_{xx}}{u_x^2} \\ - \frac{(\alpha + 3) (t u_x + \alpha + 2)^2}{(\alpha + 2)^3 u_x^2}$$

respectively.

6 Higher symmetries

The action of the local recursion operators (22)–(24) on the contact symmetries (18) produces the Lie subalgebra $\mathfrak{s}_\infty \subset \text{Sym}(\mathcal{E})$. In this section we study the structure of $\mathfrak{s}_\infty$. We have $\phi_{1,i} = \mathcal{R}_i(\phi_{0,0})$ for $i \in \{0, 1, 2\}$, hence

$$\mathfrak{s}_\infty = \langle \mathcal{R}^{(p,q,r)}(\phi_{0,0}) \mid p, q, r \in \mathbb{N} \cup \{0\} \rangle \quad (29)$$

for

$$\mathcal{R}^{(p,q,r)} = \mathcal{R}_0^p \circ \mathcal{R}_1^q \circ \mathcal{R}_2^r = \underbrace{\mathcal{R}_0 \circ \cdots \circ \mathcal{R}_0}_{p \text{ times}} \circ \underbrace{\mathcal{R}_1 \circ \cdots \circ \mathcal{R}_1}_{q \text{ times}} \circ \underbrace{\mathcal{R}_2 \circ \cdots \circ \mathcal{R}_2}_{r \text{ times}}$$

due to (25).

Lemma *For every $\phi \in C^\infty(\mathcal{E})$ and $i \in \{0, 1, 2\}$ there holds*

$$\{\mathcal{R}_i(\phi), \psi(A)\} = \mathcal{R}_i(\{\phi, \psi(A)\}).$$

Proof Suppose $i = 0$. Denote $W = \mathcal{R}_0(x)$, so $\mathcal{R}_0 = -D_t + W D_x$, and for arbitrary $A = A(t)$ denote $\psi = \psi(A)$ for short. Using (7) we have

$$\{\mathcal{R}_0(\phi), \psi\} - \mathcal{R}_0(\{\phi, \psi\}) = \mathbf{E}_{-D_t(\phi) + W D_x(\phi)}(\psi) - \mathbf{E}_\psi(-D_t(\phi) + W D_x(\phi)) \\ + D_t(\mathbf{E}_\phi(\psi) - \mathbf{E}_\psi(\phi)) - W D_x(\mathbf{E}_\phi(\psi) - \mathbf{E}_\psi(\phi)) \\ = A D_x(-D_t(\phi) + W D_x(\phi)) - \mathbf{E}_\psi(w) D_x(\phi) + [\mathbf{E}_\psi, D_t](\phi) - W [\mathbf{E}_\psi, D_x](\phi) \\ + D_t(A D_x(\phi)) - W A D_x^2(\phi) = (A D_x(W) + A' - \mathbf{E}_\psi(W)) D_x(\phi) = 0,$$

since direct computations give $\mathbf{E}_\psi(W) = A D_x(W) + A'$.

For $i = 1$ and $i = 2$ the proof is similar. $\square$

In particular, we have (recall notation of (20), (21))

$$\mathcal{R}_i(\psi(A)) = \mathcal{R}_i(\{\phi_{0,0}, \psi(A)\}) = \{\phi_{1,i}, \psi(A)\} = \psi(\phi_{1,i} \cdot A).$$

Combining this with (29) we obtain

Proposition 2 Representation (20), (21) admits a prolongation to the Lie algebra $\mathfrak{s}_\infty$ given by formula

$$\mathcal{R}^{(p,q,r)}(\phi_{0,0}) \cdot A = \underbrace{\phi_{1,0} \cdot (\dots (\phi_{1,0} \cdot}_{p \text{ times}} \underbrace{(\phi_{1,1} \cdot (\dots (\phi_{1,1} \cdot}_{q \text{ times}} \underbrace{(\phi_{1,2} \cdot (\dots (\phi_{1,2} \cdot A) \dots)}_{r \text{ times}}) \dots) \dots). \quad (30)$$

□

To show that $\mathfrak{s}_\infty$ has the structure of the so-called *Lie algebra of matrices of the complex size* introduced in [9] and studied in [15,32], we recall the constructions of the last paper. Following [9,15,32] we consider this Lie algebra over $\mathbb{C}$, while one can replace $\mathbb{C}$ by $\mathbb{R}$ in the next two paragraphs.

Let $\mathfrak{d}$ denote the Lie algebra of differential operators of the form $p_n(t) \partial_t^n + p_{n-1}(t) \partial_t^{n-1} + \dots + p_1(t) \partial_t + p_0(t)$ where $p_k \in \mathbb{C}[t]$ for $k \in \{0, \dots, n\}$ and $n \in \mathbb{N} \cup \{0\}$, with the Lie bracket defined by the commutator. For fixed $\lambda \in \mathbb{C}$ consider the subalgebra $\mathfrak{gl}(\lambda) \subset \mathfrak{d}$ generated by the differential operators 1,

$$T_0 = -\partial_t, \quad T_1 = 2t \partial_t - \lambda + 1, \quad T_2 = t^2 \partial_t - (\lambda - 1)t. \quad (31)$$

The Lie algebra $\mathfrak{gl}(\lambda)$ is isomorphic to $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))/I_\lambda$, where $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ is the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ and I_λ is the ideal in $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$ generated by the differential operator $2T_2 \circ T_0 + 2T_0 \circ T_2 + T_1 \circ T_1 + (\lambda - 1)^2$.

When λ is not integral, $\mathfrak{gl}(\lambda)$ is the sum of $\mathbb{C}$ and a simple infinite-dimensional Lie algebra; if $\lambda = \pm n$ for $n \in \mathbb{N}$, then $\mathfrak{gl}(\lambda)$ contains an infinite-dimensional ideal and the quotient is isomorphic to $\mathfrak{gl}(n)$, see [15] for discussion of the structure of $\mathfrak{gl}(\lambda)$, its extension, embedding in the Lie algebra $\mathfrak{gl}_\infty$ of infinite-dimensional matrices, and applications to integrable systems.

Comparing (21), (30), and (31), we obtain the following statement:

Theorem The Lie algebra $\mathfrak{s}_\infty \subset \text{Sym}(\mathcal{E})$ is isomorphic to $\mathfrak{gl}(\alpha + 3)$. □

The results of [32, § 4.1] yield

Corollary The Lie algebra $\mathfrak{s}_\infty$ has a basis given by symmetries $\phi_{0,0}, \phi_{1,0}, \phi_{1,1}, \phi_{1,2}$, and $\phi_{n,2n-k} = \text{ad}_{\phi_{1,0}}^k (\mathcal{R}_2^n(\phi_{0,0}))$, $n \geq 2$, $k \in \{0, \dots, 2n\}$. □

Example 3 Symmetries $\phi_{2,0}, \dots, \phi_{2,4}$ are given by equations

$$\begin{aligned} \phi_{2,0} &= 24 (u_{tt} - u_{xx}^{-1} E^2), \\ \phi_{2,1} &= -24 \left(t u_{tt} - \frac{1}{2} (\alpha - 1) u_t - \frac{E (t + u_x)}{u_{xx}} \right), \\ \phi_{2,2} &= 12 \left(t^2 u_{tt} - (\alpha - 1) t u_t - \frac{E (t^2 E + 2 t u_x + \alpha + 2)}{u_{xx}} \right) \\ &\quad + 2 ((\alpha + 1) (\alpha + 2) x u_x - (\alpha^2 + 3\alpha + 8) u), \\ \phi_{2,3} &= -4 t^3 u_{tt} + 6 (\alpha - 1) t^2 u_t - 2 (\alpha^2 + 3\alpha + 8) t u \end{aligned}$$

$$\begin{aligned}
& + 2(\alpha + 1)(\alpha + 2)x(tu_x + 1) \\
& + 4u_{xx}^{-1} \left(t^3 E^2 - 3t(tu_x + \alpha + 2)E + (\alpha + 2)u_x \right), \\
\phi_{2,4} = & t^4 u_{tt} + 2(\alpha - 1)t^3 u_t \\
& - \frac{t^2 E(t^2 E^3 + 2(2tu_x + 3(\alpha + 2))) + (\alpha + 2)(4tu_x + \alpha + 2)}{u_{xx}} \\
& - (\alpha + 1)(\alpha + 2)tx(tu_x + 2) + (\alpha^2 + 3\alpha + 8)t^2 u.
\end{aligned}$$

◇

Remark 2 When $\alpha = 0$, the Lie algebra $\mathfrak{s}_\infty$ is a proper subalgebra of $\text{Sym}(\mathcal{E})$. The family of symmetries of third order

$$\eta_m = \frac{u_{xx}^m}{u_x^{4m+7}} \left((m+2)u_x u_{xxx} - (2m+3)u_{xx}^2 \right), \quad m \in \mathbb{R}, \quad (32)$$

was found in [38]. We have $\eta_m \notin \mathfrak{s}_\infty$. The action of the local recursion operators $\mathcal{R}_i$ on η_m provides a family of higher symmetries of increasing order. This family is not included in $\mathfrak{s}_\infty$.

We have no examples of higher symmetries that are not included in $\mathfrak{s}_\infty$ when $\alpha \neq 0$.

◇

Remark 3 The local recursion operators $\mathcal{R}_i$ preserve the ideal $\mathfrak{a}_\infty$, since $\mathcal{R}_i$ map solutions of Eq. (19) to solutions of the same equation. ◇

Remark 4 When $\alpha = 0$, the family of local recursion operators of fourth order $\mathcal{P}_m = \mathcal{P}_{m,1} \circ \mathcal{P}_{m,2} \circ \mathcal{P}_{m,3} \circ D_x$ with

$$\begin{aligned}
\mathcal{P}_{m,1} &= D_x + \frac{u_{xxx}}{u_{xx}}, \quad \mathcal{P}_{m,2} = \frac{4u_{xx}^2 - u_x u_{xxx}}{u_x^{4m-5} u_{xx}^{5-m}} D_x + D_x \circ \frac{4u_{xx}^2 - u_x u_{xxx}}{u_x^{4m-5} u_{xx}^{5-m}}, \\
\mathcal{P}_{m,3} &= D_x - \frac{u_{xxx}}{u_{xx}}
\end{aligned}$$

was constructed in [38]. We have $\mathcal{P}_m(\psi(A)) = 0$, hence $\mathcal{P}_m$ is not a linear combination of the recursion operators of the form $\mathcal{R}^{(p,q,r)}$.

We have no examples of local recursion operators that do not belong to the span of $\mathcal{R}^{(p,q,r)}$ when $\alpha \neq 0$. ◇

7 Cosymmetries

Equations (12) and (13) coincide when $\alpha = 0$, hence in this case cosymmetries are the same as the generators of symmetries. For other values of α we have

Proposition 3 *All the cosymmetries of Eq. (5) with $\alpha \neq 0$ that belong to $C^\infty(J^1(\pi))$ have the form $\psi = V(t, u_x)$, where functions V are solutions to the PDE*

$$V_{tu_x} + \frac{1}{\alpha + 2} u_x^2 V_{u_x u_x} + \frac{2 - \alpha}{\alpha + 2} u_x V_{u_x} - \frac{2(\alpha + 1)}{\alpha + 2} V = 0. \quad (33)$$

□

Equation (33) is the adjoint equation for (19). This can be checked by a direct computation or inferred from the same considerations as in Remark 1. Indeed, the linear Eq. (4) coincides with the defining equation for the tangent covering thereof. The last equation is related by the linearization of transformation (3) to Eq. (19). The defining equations for the tangent and cotangent coverings of (4) are adjoint, and the last equation is mapped to Eq. (33) by the linearization of (3).

Notice that in general adjoint linear equations are not necessary contact equivalent. E.g., from results of [27] it follows that equation $u_{tx} = t^2 x^2 u_t + u$ is not contact equivalent to the adjoint equation. However Eq. (33) turns out to be contact equivalent to Eq. (19). The proof of this assertion is similar to the proof of contact equivalence of Eqs. (5) and (4) given in [26]. Therefore the problem to find all solutions to Eq. (33) is as hard as the problem to find all solutions to Eq. (5). Nevertheless, we can obtain some particular solutions of (33). For example, when $V_t = 0$, this equation get the form of a linear ordinary differential equation of second order. The general solution of this ODE is a linear combination with constant coefficients of two fundamental solutions $\psi_1 = u_x^{-2}$ and $\psi_2 = u_x^{\alpha+1}$.

Equation (5) has higher cosymmetries. E.g., a family of cosymmetries of third order is defined by formulae

$$\psi_H = u_{xx}^{-\frac{\alpha+6}{\alpha+4}} u_{xxx} \left(\int H'(z) z^{-\frac{\alpha+6}{\alpha+4}} dz \right) - (\alpha + 4) u_x^{\alpha+1} H(z), \quad z = u_{xx} u_x^{-\alpha-4} \quad (34)$$

when $\alpha \neq -4$ and

$$\psi_G = \frac{u_x G'(u_{xx}) u_{xxx} - 2 u_{xx} G(u_{xx})}{u_x^3 u_{xx}} \quad (35)$$

when $\alpha = -4$, where H and G are arbitrary functions of their arguments.

Remark 5 Since symmetries and cosymmetries coincide when $\alpha = 0$, Eq. (34) with $\alpha = 0$ provides a family of symmetries of third order for Eq. (1). This family generalizes (32). Indeed, η_m coincides with $\frac{1}{2} \psi_H$ when $\alpha = 0$ and $H(z) \equiv z^{m+2}$.
◇

Using (15) we find three local recursion operators for cosymmetries given by equations

$$\mathcal{S}_0 = -\mathcal{R}_0, \quad \mathcal{S}_1 = -\mathcal{R}_1 - 2\alpha, \quad \mathcal{S}_2 = -\mathcal{R}_2 - 2\alpha t. \quad (36)$$

We have $[\mathcal{S}_0, \mathcal{S}_1] = -2\mathcal{S}_0$, $[\mathcal{S}_0, \mathcal{S}_2] = \mathcal{S}_1$, $[\mathcal{S}_1, \mathcal{S}_2] = -2\mathcal{S}_2$, hence these operators constitute the Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. The representation of this Lie algebra

on the space of solutions to Eq. (33) is given by formulae

$$\begin{aligned} \mathcal{S}_0(V) &= \partial_t V, & \mathcal{S}_1(V) &= -2t \partial_t V + 2u_x \partial_{u_x} V - \alpha V, \\ \mathcal{S}_2(V) &= -t^2 \partial_t V + (2t u_x + \alpha + 2) \partial_{u_x} V - \alpha t V. \end{aligned}$$

Action of operators (36) on cosymmetries from families (34) or (35) produces cosymmetries of higher order.

Example 4 Function $\zeta = (u_x u_{xxx} - 2u_{xx}^2) u_x^{-3} u_{xx}^{-1}$ is a cosymmetry of Eq. (2) for each $\alpha \neq -2$. The related conservation law is

$$\Omega = \frac{u_x u_{xxx} - u_{xx}^2}{u_x^2 u_{xx}} (dx + u dt) - \frac{\ln |u_{xx}| - (\alpha + 4) \ln |u_x|}{\alpha + 2} dt.$$

The simple induction shows that

$$\mathcal{S}_0^k(\zeta) = \frac{u_x^{2(k-1)}}{2^k u_{xx}^{k+1}} u_{(k+3)x} + W_k(u, u_x, u_{xx}, \dots, u_{(k+2)x})$$

for certain functions W_k . Therefore for each $\alpha \neq -2$ Eq. (5) has cosymmetries of arbitrary higher order. $\diamond$

8 Invariant solutions

In this section we give examples of implementing higher symmetries for finding globally-defined solutions of the generalized Hunter–Saxton equation. For a symmetry $\phi \in \text{Sym}(\mathcal{E})$ the ϕ -invariant solutions satisfy the over-determined system given by (5) and $\phi = 0$. We analyze this system for symmetries $\phi_{2,0}$, $\phi_{2,1}$, and $\phi_{3,0}$ and construct exact invariant solutions for some values of parameter α .

8.1 $\phi_{2,0}$ -invariant solutions

These are solutions of (5) and equation

$$u_{tt} = \frac{(u u_{xx} + (\alpha + 2)^{-1} u_x^2)^2}{u_{xx}}. \quad (37)$$

The compatibility conditions for system (5), (37) get the form

$$u_t = u u_x - \frac{u_x^3}{(\alpha + 2)^2 u_{xx}^3} (u_x u_{xxx} - 2(\alpha + 3) u_{xx}^2), \quad (38)$$

$$u_{xxx} = \frac{3u_{xx}^2}{u_{xx}} - \frac{(\alpha + 5)(2u_x u_{xx} u_{xxx} - (\alpha + 4) u_{xx}^3)}{u_x^2}, \quad (39)$$

whence to obtain $\phi_{2,0}$ -invariant solutions of (5) we can proceed as follows: first, we find the general solution to ODE (39) in the form $u = S(x, c_1, c_2, c_3, c_4)$. This solution depends on four arbitrary constants $c_1, \dots, c_4$. Second, we replace these constants by unknown functions $s_1(t), \dots, s_4(t)$ and substitute the obtained function $u = S(x, s_1(t), s_2(t), s_3(t), s_4(t))$ into equation (38). This yields a system of ODEs that defines functions $s_i(t)$.

Equation (39) has four-dimensional solvable algebra of point symmetries generated by vectors $\partial_x, \partial_u, x \partial_x, u \partial_u$, therefore this equation is integrable by quadratures in accordance with the Lie–Bianchi theorem [3, § 167], [28, Th. 2.64], [8]. Indeed, substituting for $z = u_x, z_x = w(z)$ yields the ODE of second order

$$w_{zz} = \frac{2w_z^2}{w} - \frac{2(\alpha+5)w_z}{z} + \frac{(\alpha+4)(\alpha+5)w}{z^2}.$$

Then we put $w_z = p(z)w$ and reduce the order of this ODE by one:

$$p_z = \left(p - \frac{\alpha+5}{z}\right)^2 - \frac{\alpha+5}{z^2}.$$

New function $r(z) = p(z) - (\alpha+5)z^{-1}$ satisfies the separable ODE $r_z = r^2$. Its general solution $r = (c_0 - z)^{-1}$ provides

$$w_z = \left(\frac{1}{c_0 - z} + \frac{\alpha+5}{z}\right)w$$

and therefore $w = c_1 z^{\alpha+5} (c_0 - z)^{-1}$, which gives

$$u_{xx} = c_1 u_x^{\alpha+5} (c_0 - u_x)^{-1}. \quad (40)$$

While this equation is integrable by quadratures, its general solution is too complicated for arbitrary value of α . We consider case $\alpha = -\frac{7}{2}$ when the formula for the general solution of Eq. (40) simplifies enough to give explicit expression for the globally-defined $\phi_{2,0}$ -invariant solution of (5). We obtain

$$u = \frac{s_1}{s_3^2} \left((x + s_2)^2 + s_3 \right)^{\frac{3}{2}} - \frac{s_1}{s_3^2} (x + s_2)^3 - \frac{3s_1}{2s_3} x + s_4,$$

where $s_1(t), \dots, s_4(t)$ are solutions to system

$$\begin{cases} s_{1,t} = 0, \\ s_{2,t} = \frac{3s_1s_2 + 2s_3s_4}{2s_3}, \\ s_{3,t} = s_1, \\ s_{4,t} = -\frac{3s_1(s_1s_2 + 2s_3s_4)}{4s_3^2}. \end{cases}$$

The general solution of this system

$$s_1 = a_1, \quad s_2 = a_4 + a_3 t, \quad s_3 = a_2 + a_1 t, \quad s_4 = \frac{2 a_2 a_3 - a_1 (a_3 t + 3 a_4)}{2 (a_2 + a_1 t)}, \quad a_i \in \mathbb{R},$$

provides the four-parametric $\phi_{2,0}$ -invariant solution

$$u = \frac{a_1 \left(((x + a_3 t + a_4)^2 + a_1 t + a_2)^{3/2} - (x + a_3 t + a_4)^3 \right)}{(a_1 t + a_2)^2} - \frac{3 a_1 (x + a_4) + a_3 (a_1 t - 2 a_2)}{2 (a_1 t + a_2)}$$

to Eq. (5) with $\alpha = -\frac{7}{2}$.

8.2 $\phi_{2,1}$ -invariant solutions

The compatibility conditions for system (5), $\phi_{2,1} = 0$ get the form

$$u_t = -\frac{(t u_x + \alpha + 2) u_x^3}{(\alpha + 2)^2 t u_{xx}^3} u_{xxx} + \frac{(\alpha + 3) (2 t u_x + \alpha + 2) u_x^2}{(\alpha + 2)^2 t u_{xx}} + \frac{u (t u_x - 1)}{t}, \quad (41)$$

$$u_{xxx} = \frac{3 u_{xx}^2}{u_{xx}} + \left(2 (\alpha + 5) - \frac{(\alpha + 2) (\alpha + 4)}{t u_x + \alpha + 2} \right) \frac{u_{xx} u_{xxx}}{u_x} + (\alpha + 4) \left(\alpha + 5 + \frac{(\alpha + 2) (\alpha + 4)}{t u_x + \alpha + 2} \right) \frac{u_{xx}^3}{u_x^2}. \quad (42)$$

For each fixed value of t Eq. (42) can be considered as an ODE of fourth order. The structure of the point symmetry algebra of this ODE depends on parameter α . When $\alpha \neq -4$, this is the three-dimensional solvable Lie algebra generated by ∂_x , ∂_u , and $x \partial_x + u \partial_u$, therefore in this case Eq. (42) is reducible an ODE of first order. When $\alpha = -4$, the point symmetry algebra of the ODE (42) is the four-dimensional solvable Lie algebra generated by ∂_x , ∂_u , $x \partial_x$, and $u \partial_u$, so in this case (42) is integrable by quadratures. Indeed, for $\alpha = -4$ we have

$$u_{xxx} = \frac{3 u_x u_{xx} - 2 u_{xx}^2}{u_x u_{xx}} u_{xxx}, \quad (43)$$

then substituting for $u_{xx} = v u_x$ with $v = v(x)$ we obtain the ODE

$$v_{xx} = \frac{v_x (3 v_x + v^2)}{v}.$$

We put $v_x = H(v)$ and get

$$H_v = \frac{3 H + v^2}{v} \quad (44)$$

or $H = 0$. The last case corresponds to trivial solutions $u = u(t)$ of Eq. (5), while from (44) we find $H = c_0 v^3 - v^2$ with $c_0 = \text{const}$. The resulting equation $v_x = c_0 v^3 - v^2$ is the separable ODE. Its general solution with $c_0 \neq 0$ gives the general solution of (43) which is too complicated to produce an explicit expression for a solution of (5). The case $c_0 = 0$ corresponds to $u_{xxx} = 0$ and gives solution $u = c_1 + c_2 x + c_3 x^2$ with $c_i = \text{const}$. We replace the constants c_i by functions $s_i(t)$ and substitute the resulting expression $u = s_1(t) + s_2(t)x + s_3(t)x^2$ into (41). Integration of the resulting system of ODEs for functions s_i produces the following family of solutions

$$u = a_1 (x + a_2 \ln |t| + a_3)^2 + \frac{a_2}{t}, \quad a_i \in \mathbb{R},$$

for (5).

8.3 $\phi_{3,0}$ -invariant solutions

For symmetry

$$\begin{aligned} \phi_{3,0} = & -u_{ttt} + \frac{(\alpha + 2)^3 u^3 u_{xx}^3 + u_x^6}{(\alpha + 2)^3 u_{xx}^3} u_{xxx} + \frac{3 u_x}{\alpha + 2} (u_t u_x + 2 u^2) \\ & + 3 u u_t - \frac{3 \alpha u u_x^3}{(\alpha + 2)^3} - \frac{3 u_x^5}{(\alpha + 2)^2 u_{xx}} \end{aligned}$$

system (5), $\phi_{3,0} = 0$ is compatible by virtue of system

$$\begin{aligned} u_{tt} = & \frac{u_x^6}{(\alpha + 2)^3 u_{xx}^5} (u_{xx} u_{xxx} - 3 u_{xxx}^2) + \frac{3 (\alpha + 4) u_x^5 u_{xxx}}{(\alpha + 2)^3 u_{xx}^3} \\ & - \frac{3 (\alpha^2 + 6 \alpha + 10) u_x^4}{(\alpha + 2)^3 u_{xx}} - \frac{\alpha u u_x^2}{\alpha + 2} \\ & + u_t u_x + u^2 u_{xx}, \\ u_{xxxx} = & \left(\frac{10 u_{xxx}}{u_{xx}} - \frac{3 (\alpha + 6) u_{xx}}{u_x} \right) u_{xxx} - \frac{15 u_{xxx}^3}{u_{xx}^2} + \frac{9 (\alpha + 6) u_{xxx}^2}{u_x} \\ & - \frac{3 (\alpha + 5) (\alpha + 6) u_{xx}^2 u_{xxx}}{u_x^2} \\ & + \frac{(\alpha + 4) (\alpha + 5) (\alpha + 6) u_{xx}^4}{u_x^3}. \end{aligned} \quad (45)$$

The structure of the point symmetry algebra of the last ODE depends on α . When $\alpha \notin \{-6, -1\}$, this is the solvable Lie algebra generated by $1, u, u_x, x u_x$, therefore Eq. (45) is reducible to ODE of first order. For $\alpha = -1$ or $\alpha = -6$ the point symmetry algebra of (45) is $\langle 1, u, u_x, x u_x, u u_x \rangle$ or $\langle 1, x, u, u_x, x u_x \rangle$, respectively. Both algebras are solvable five-dimensional Lie algebras, so Eq. (45) is integrable by quadratures when $\alpha \in \{-6, -1\}$ in accordance with the Lie–Bianchi theorem. In both cases the general

solution of equation (45) is too complicated to allow one to construct an explicit expression for a solution of (5). Nevertheless it is possible to find particular solutions of (45) that generate exact solutions of (5). We have found two families of such solutions. For $\alpha = -1$ we have

$$u = a_1 |x + a_2 t^2 + a_3 t + a_4|^{1/2} + 2 a_3 t + a_4.$$

When $\alpha = -6$, we get

$$u = s_1 (x + s_2)^{4/3} + s_3 x + s_4,$$

where

$$\begin{aligned} s_1 &= a_1 (t + a_2)^{-2/3}, \quad s_2 = -27 a_1^{-3} (t + a_2)^{-1} + a_3 t^2 + a_4 t + a_5, \\ s_3 &= 4 (t + a_2)^{-1}, \\ s_4 &= 6 a_3 (t + a_2) - 81 a_1^{-3} (t + a_2)^{-2} + 4 (a_5 + a_2^2 a_3 - a_2 a_4) (t + a_2)^{-1} \\ &\quad + 5 (a_4 - 2 a_2 a_3). \end{aligned}$$

In both families $a_i \in \mathbb{R}$ are constants and $a_1 \neq 0$.

9 Conclusion

The results of the paper can be summarized as follows. We have found the Lax representation with non-removable spectral parameter for the generalized Hunter–Saxton equation and recursion operators for symmetries and cosymmetries. We employ the recursion operators to generate the infinite-dimensional Lie algebra of higher symmetries and then study the structure thereof, in particular we have found a basis of this Lie algebra. We have shown that the higher symmetries from the obtained Lie algebra can be used to construct global exact solutions for the generalized Hunter–Saxton equation. Furthermore, we have employed recursion operators to prove existence of an infinite number of cosymmetries of higher order, which indicates that the space of nontrivial conservation laws of higher order is infinite-dimensional as well.

We hope that the methods used in this paper are applicable to study other properties of the generalized Hunter–Saxton equation related to integrability such as variational symplectic and Poisson structures. These structures can be nonlocal, see, e.g., [23]. Since nonlocalities are not preserved by contact transformations, it is of interest to study integrability structures for the generalized Hunter–Saxton equation independently of the study of Eq. (4). We intend to address these issues in our future work.

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Data availability The author confirms that the data supporting the findings of this study are available within the article.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Ethical approval The author declares that he has adhered to the ethical standards of research execution.

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